PHYSICAL REVIEW E **76**, 026211 (2007)

Reducing colored noise for chaotic time series in the local phase space

Junfeng Sun,[1,*] Yi Zhao,[1] Jie Zhang,[1] Xiaodong Luo,[2] and Michael Small[1]
[1]*Department of Electronic and Information Engineering, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong*
[2]*OCIAM, Mathematical Institute, 24-29 St. Giles', Oxford OX1 3LB, United Kingdom*
(Received 18 December 2006; revised manuscript received 1 June 2007; published 17 August 2007)

A two step method is proposed to reduce colored noise for chaotic data in the local phase space. With the observation that the energy of colored noise is mainly distributed in a low dimensional subspace, a noise dominated subspace is first estimated by the energy distribution of colored noise. At step 1, for the reference phase point, the components projected into the noise dominated subspace are deleted and the enhanced data are reconstructed with the remaining components. The residual error of the output of step 1 tends to distribute on each direction uniformly. So at step 2, the local projection method is further applied to the output of step 1, treating the residual error as white noise. Experiments show that our method performs well in eliminating colored noise for chaotic data.

DOI: 10.1103/PhysRevE.76.026211 PACS number(s): 05.45.Tp, 05.10.−a

I. INTRODUCTION

The presence of noise can greatly affect the analysis of the observed data from chaotic systems. For example, noise may obscure or even destroy the fractal structure of chaotic attractor, may mislead the calculation of correlation dimension and Lyapunov exponents [1]. Therefore, it is desirable to reduce the noise level. However, most noise reduction methods are designed for signals that can be treated by a linear model and fail to eliminate noise from a contaminated chaotic time series because the spectra of the chaotic signal and the noise overlap [2]. Noise reduction based on time delay embedding, which has been widely studied, may be the most promising way to filter the noisy chaotic data [3–9]. Several methods were proposed independently in the local phase space [3–5], and further proved to be the special cases of an optimal one [6], which is named the local projection (LP) method. Encouraging results have been obtained with the LP method for both artificial data generated from chaotic systems (e.g., the Lorenz system) and noisy experimental data (e.g., NMR-laser data) [7]. Further, the LP method has been successfully adapted to ECG extraction [10] and speech enhancement [11,12]. However, the above algorithms are all for additive white noise. Time series with dynamical noise can be processed by "shadowing" [13].

Recently, one generalization (hereafter called the local subspace method) of the LP method, inspired from the linear subspace technique [14], was proposed by weighted projection in the local phase space, and the LP method described in Ref. [11] is proved to be its least square (LS) case [15]. By time delay embedding, state recurrences (an important feature of chaotic systems) of a reference phase point appear as the nearest neighbors which cover temporally scattered data segments with similar wave forms to that of the reference phase point [16]. The local subspace method, actually an extension of the linear subspace technique in the local phase space, utilizes the redundant information possessed by the neighbors appropriately and thus reduces noise for chaotic data successfully. A more general phase space projector has been further deduced with no independence assumption between the noise and the clean signal [17]. However, this generalization seems impossible to implement numerically, and only a reduced case with an additional independence assumption was implemented. All these methods decompose the reconstructed phase space into two orthogonal subspaces, called the signal subspace which contains most of the pure signal components plus some noise components and the noise subspace that contains the remaining noise components.

As far as we know, the existing noise reduction methods for chaotic data almost all assume the noise is white noise, and the case of chaotic data with colored noise has not yet been tackled. In the frequency domain, a random sequence is called white noise if its spectra are flat, otherwise it is called colored noise. Correspondingly, in the local phase space, the energy of white noise distributes uniformly on each direction, while the energy of colored noise mainly distributes in a low dimensional subspace. The LP method yields poor results for chaotic data contaminated with colored noise, because its estimated signal subspace is not appropriate (lots of noise components will be included into the signal subspace for this case).

In this paper, we propose a two step method to reduce colored noise for noisy chaotic data. This method assumes that the colored noise is stationary, and a segment of the colored noise or its covariance matrix can be obtained in advance (note that this assumption is widely adopted in signal processing; for example, in speech enhancement, a segment of pure noise can be obtained during a period of speech absence). At the first step, a noise dominated subspace can be obtained (spanned by the eigenvectors associated with the several largest eigenvalues) by performing eigenvalue decomposition to the covariance matrix of the colored noise. Then in each local phase space, the components of the reference phase point projected into the noise dominated subspace are deleted and the enhanced data are reconstructed with the remaining components. After the first step, most of the colored noise has been eliminated. The energy of residual error tends to distribute "uniformly" on each direction. Thus we can treat the residual error as white noise and further

*sun.junfeng@polyu.edu.hk

apply the LP method to the output of the first step.

The organization of this paper is as follows. In Sec. II, the principle of noise reduction for chaotic data in the local phase space is reviewed, and a two step strategy is proposed to eliminate colored noise from noisy chaotic data. In Sec. III, noise reduction is performed for chaotic data with colored noise. Finally, some discussions and conclusions are given in Sec. IV.

II. PRINCIPLE OF THE METHOD

Let $z_n = s_n + w_n$ denote the time series contaminated by noise, where s_n is the clean data generated by a dynamical system and w_n is the additive noise. For a time series $\{z_n\}_{n=1}^{L}$ with L samples, the phase points can be reconstructed by time delay embedding, i.e., $\{\mathbf{z}_n\}_{n=1}^{L-(d-1)\tau}$,

$$\mathbf{z}_n = [z_n, z_{n+\tau}, z_{n+2\tau}, \ldots, z_{n+(d-1)\tau}]^T,$$

where d is the embedding dimension, τ is time delay, and $(\cdot)^T$ denotes the transpose of a real matrix. The near neighborhood of the reference point $\mathbf{z}_n$ is defined as

$$\mathbf{N}_n \triangleq \{\mathbf{z}_k : \|\mathbf{z}_k - \mathbf{z}_n\| < \varepsilon, 1 \leq k \leq L-(d-1)\tau\},$$

where ε is the size of the neighborhood.

A. The local projection method

The LP method [6,11] assumes that the noise is white noise and the local phase space, i.e., the neighborhood $\mathbf{N}_n$ of the reference point $\mathbf{z}_n$, can be divided into an M-dimensional signal subspace and a $(d-M)$-dimensional noise subspace, where M is the minimum embedding dimension of the dynamical system [18]. The signal subspace contains most of the clean signal plus a certain amount of the noise components, while the noise subspace contains most of the noise components and a certain, small, amount of the signal components. The energy of white noise is almost uniformly distributed on each direction of the local phase space. For a preset M, the noise subspace can be estimated by minimizing the total energy that is distributed in it. The minimization turns out to be the standard eigenvalue decomposition for the covariance matrix $\mathbf{C}_n$ of the neighborhood $\mathbf{N}_n$, i.e.,

$$\mathbf{C}_n \cdot \mathbf{u}_i - \lambda_i \mathbf{u}_i = 0. \tag{1}$$

The matrix $\mathbf{C}_n$ is defined as $\mathbf{C}_n = \frac{1}{N}\Sigma_{\mathbf{z}_k \in \mathbf{N}_n} \mathbf{x}_k \cdot \mathbf{x}_k^T$ with notation $\mathbf{x}_k = \mathbf{z}_k - \overline{\mathbf{z}}_n$, where $\overline{\mathbf{z}}_n$ is the center of the neighborhood, i.e., $\overline{\mathbf{z}}_n = \frac{1}{N}\Sigma_{\mathbf{z}_k \in \mathbf{N}_n} \mathbf{z}_k$, and N is the number of neighbors in $\mathbf{N}_n$.

Sorting the eigenvalues $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ in descending order, the eigenvectors $\mathbf{U}_1 = [\mathbf{u}_1, \ldots, \mathbf{u}_M]$, associated with the M largest eigenvalues, span the signal subspace, and the eigenvectors $\mathbf{U}_2 = [\mathbf{u}_{M+1}, \ldots, \mathbf{u}_d]$, corresponding to the $(d-M)$ smallest eigenvalues, span the noise subspace, respectively. Then the phase vector $\mathbf{z}_n$ can be decomposed as

$$\mathbf{z}_n = \overline{\mathbf{z}}_n + \mathbf{U}_1 \cdot \mathbf{U}_1^T(\mathbf{z}_n - \overline{\mathbf{z}}_n) + \mathbf{U}_2 \cdot \mathbf{U}_2^T(\mathbf{z}_n - \overline{\mathbf{z}}_n) \tag{2}$$

in the local phase space, where $\mathbf{U}_1 \cdot \mathbf{U}_1^T(\mathbf{z}_n - \overline{\mathbf{z}}_n)$ and $\mathbf{U}_2 \cdot \mathbf{U}_2^T(\mathbf{z}_n - \overline{\mathbf{z}}_n)$ are the projections of $(\mathbf{z}_n - \overline{\mathbf{z}}_n)$ in the signal subspace and the noise subspace, respectively. Eliminating $\mathbf{U}_2 \cdot \mathbf{U}_2^T(\mathbf{z}_n - \overline{\mathbf{z}}_n)$, we obtain the enhanced signal vector

$$\hat{\mathbf{s}}_n = \overline{\mathbf{z}}_n + \mathbf{U}_1 \cdot \mathbf{U}_1^T(\mathbf{z}_n - \overline{\mathbf{z}}_n). \tag{3}$$

For chaotic data with colored noise, as the linear subspace technique suggests [14], first, the noisy data can be whitened by multiplying a whitening matrix $\mathbf{C}_w^{-1/2}$, where $\mathbf{C}_w$ is the covariance matrix of the colored noise. Then the whitened data can be processed as the case of white noise. Finally, a dewhitening strategy is performed. The enhanced signal vector can be expressed as

$$\hat{\mathbf{s}}_n = \overline{\mathbf{z}}_n + \mathbf{C}_w^{1/2} \cdot \mathbf{U}_1 \cdot \mathbf{U}_1^T \cdot \mathbf{C}_w^{-1/2}(\mathbf{z}_n - \overline{\mathbf{z}}_n). \tag{4}$$

As each element of the time series $\{z_n\}_{n=1}^{L}$ occurs as an entry of one of d successive phase vectors $\mathbf{z}_k$, $k = n-(d-1)\tau, \ldots, n$, there are d enhanced entries which may be different in values. The arithmetic mean over these values is taken as the enhanced element $\hat{s}_n$. More details about the LP method and its generalization can be found in Refs. [11,15].

B. Reducing colored noise in the local phase space

Before introducing our method, we will first demonstrate the difference of energy distribution in the local phase space for the cases of chaotic data with white noise and colored noise, respectively. Let $\mathbf{U}_t = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{m_t}]$ span a subspace of the local phase space that is to be investigated, where m_t is the dimension of this subspace and $\mathbf{u}_i$ is the eigenvector associated with the ith largest eigenvalue. For the reference phase point $\mathbf{z}_n$, we have $\mathbf{z}_n - \overline{\mathbf{z}}_n = (\mathbf{s}_n - \overline{\mathbf{z}}_n) + \mathbf{w}_n$. Then we can consider $\|\mathbf{u}_i \cdot \mathbf{u}_i^T(\mathbf{s}_n - \overline{\mathbf{z}}_n)\|^2$ as the energy of signal components projected onto the direction $\mathbf{u}_i$, and $\|\mathbf{u}_i \cdot \mathbf{u}_i^T \cdot \mathbf{w}_n\|^2$ as the energy of noise components projected onto the direction $\mathbf{u}_i$ in the local phase space (here we just investigate the energy of the components after removing the geometric center of the neighborhood in the local phase space, so $\|\mathbf{u}_i \cdot \mathbf{u}_i^T(\mathbf{s}_n - \overline{\mathbf{z}}_n)\|^2$ is not the absolute energy of the clean signal on direction $\mathbf{u}_i$).

Assume the clean signal $\{s_n\}$ and the noise $\{w_n\}$ are known. Here we take a 10 000 point sequence measured from the x component of the Lorenz system [19]

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = (r - z)x - y, \\ \dot{z} = xy - bz, \end{cases} \tag{5}$$

as the clean signal $\{s_n\}$, where $(\sigma, r, b) = (10, 28, 8/3)$ and sample time interval is 0.04.

For chaotic data with additive white noise $\{w_n\}$ [$w_n \sim N(0,1)$ follows the normal distribution], we study two cases.

Case 1: The subspace $\mathbf{U}_t$ is estimated by the energy of the clean signal; i.e., the covariance matrix $\mathbf{C}_n$ in Eq. (1) is estimated by $\mathbf{C}_n = \frac{1}{N}\Sigma_{\mathbf{z}_k \in \mathbf{N}_n}(\mathbf{s}_k - \overline{\mathbf{s}}_n)(\mathbf{s}_k - \overline{\mathbf{s}}_n)^T$, where $\overline{\mathbf{s}}_n = \frac{1}{N}\Sigma_{\mathbf{z}_k \in \mathbf{N}_n}\mathbf{s}_k$. As Fig. 1(a) indicates, the energy of the projection of the clean signal vector on the first several directions, i.e., $\|\mathbf{u}_i \cdot \mathbf{u}_i^T(\mathbf{s}_n - \overline{\mathbf{z}}_n)\|^2$, is much larger than that of white noise. And the energy of white noise is almost projected onto each direction uniformly.

Case 2: The subspace $\mathbf{U}_t$ is estimated by the energy of the noisy signal; i.e., the covariance matrix $\mathbf{C}_n$ in Eq. (1) is es-

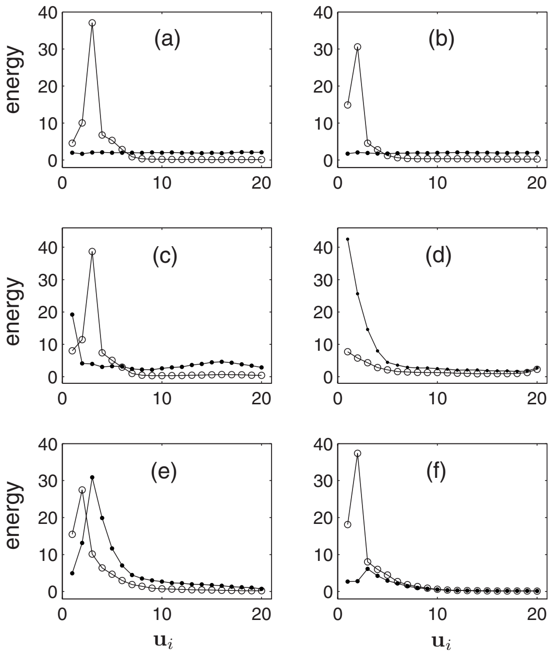

FIG. 1. Energy of the projection of clean signal vector and noise on each direction, respectively. For each case, the signal-to-noise ratio (SNR) is 15 dB, the 20 nearest neighbors of each reference phase point are utilized, and only the average energy of the components projected onto the first 20 directions is plotted. ○: $E(\|\mathbf{u}_i\cdot\mathbf{u}_i^T(\mathbf{s}_n-\bar{\mathbf{z}}_n)\|^2)$; ●: $E(\|\mathbf{u}_i\cdot\mathbf{u}_i^T\cdot\mathbf{w}_n\|^2)$, where $E(\|\mathbf{u}_i\cdot\mathbf{u}_i^T\cdot\mathbf{w}_n\|^2)$ denotes the mean of $\|\mathbf{u}_i\cdot\mathbf{u}_i^T\cdot\mathbf{w}_n\|^2$ over $n=1,\ldots,L-(d-1)\tau$. (a) Case 1, with white noise; (b) case 2, with white noise; (c) case 3, with AR(3) noise; (d) case 4, with AR(3) noise; (e) case 5, with AR(3) noise; (f) case 6, the output of step 1 of our method.

timated by $\mathbf{C}_n=\frac{1}{N}\Sigma_{\mathbf{z}_k\in\mathbf{N}_n}(\mathbf{z}_k-\bar{\mathbf{z}}_n)(\mathbf{z}_k-\bar{\mathbf{z}}_n)^T$. As Fig. 1(b) indicates, the energy of the projection of the clean signal vector on the first several directions is larger than that of white noise, which is similar to case 1. The LP method can appropriately estimate the signal subspace by the energy of noisy data in the local phase space, because the energy of white noise is almost uniformly distributed on each direction. This has been verified by the performance of LP for chaotic data with white noise.

Further, we study three cases for chaotic data with colored noise generated from a three-order autoregressive process [AR(3)], $w_n=0.8w_{n-1}-0.5w_{n-2}+0.6w_{n-3}+\epsilon_n$, where $\epsilon_n\sim N(0,1)$ follows the normal distribution.

Case 3: The subspace $\mathbf{U}_t$ is estimated by the energy of the clean signal, just as case 1 does. As Fig. 1(c) indicates, the energy of colored noise vector $\mathbf{w}_n$ is not uniformly projected onto each direction.

Case 4: The subspace $\mathbf{U}_t$ is estimated by the energy of colored noise; i.e., the covariance matrix $\mathbf{C}_n$ in Eq. (1) is estimated by $\mathbf{C}_n=\frac{1}{N}\Sigma_{\mathbf{z}_k\in\mathbf{N}_n}(\mathbf{w}_k-\bar{\mathbf{w}}_n)(\mathbf{w}_k-\bar{\mathbf{w}}_n)^T$, where $\bar{\mathbf{w}}_n=\frac{1}{N}\Sigma_{\mathbf{z}_k\in\mathbf{N}_n}\mathbf{w}_k$. As Fig. 1(d) indicates, the energy of the colored noise vector $\mathbf{w}_n$ is mainly projected onto the first several directions, and only a certain, relatively small, amount of signal components are projected onto these directions, respectively. So if we estimate a noise dominated subspace in this way, and delete the components projected into this subspace, then most of the noise components can be reduced at the price of introducing a relatively small signal distortion.

Case 5: The subspace $\mathbf{U}_t$ is estimated by the energy of the noisy signal, as case 2 does. As Fig. 1(e) indicates, a certain, large, amount of noise components are projected into the subspace spanned by the first several directions. If we adopt the strategy of the LP method, i.e., estimating the signal subspace from the energy of the noisy signal, then a large amount of noise components will be included into the signal subspace, and thus cannot be reduced by projection.

With the above observations, the first step of our method follows.

Step 1. First, estimate the noise dominated subspace $\mathbf{U}_{ND}$ by performing eigenvalue decomposition to the covariance matrix $\mathbf{C}_{\text{noise}}$ of colored noise (note that this covariance matrix is estimated from a noise sequence obtained in advance, but with the assumption that the noise process is stationary, this covariance matrix can be used to substitute the one in case 4). Then in each local phase space, i.e., the neighborhood, the components projected into the noise dominated subspace are deleted and the enhanced phase vector can be reconstructed with the remaining components, i.e.,

$$\hat{\mathbf{s}}_n=\bar{\mathbf{z}}_n+(\mathbf{I}-\mathbf{U}_{ND}\mathbf{U}_{ND}^T)(\mathbf{z}_n-\bar{\mathbf{z}}_n), \tag{6}$$

where $\mathbf{I}$ is the identity matrix.

After step 1, the noise components projected into the noise dominated subspace have been eliminated. The energy of residual error (the difference between the clean signal and the output of step 1) tends to distribute "uniformly" on each direction. This can be confirmed as follows.

Case 6. The subspace $\mathbf{U}_t$ is estimated by the energy of the output of step 1, as case 2 does. As Fig. 1(f) indicates, the energy of the projection of the clean signal vector on the first several directions is much larger than that of the residual error after step 1. The energy of the residual error is more "uniformly" distributed compared with case 5, and similar with case 2.

With this observation, the second step of our method follows.

Step 2. Treat the residual error after step 1 as white noise, and apply the LP method to the output of step 1.

III. NUMERICAL RESULTS

We apply our method to time series measured from the Lorenz system [Eq. (5)]. It has been argued that the LP method can obtain better results by over-embedding with time delay $\tau=1$ and an appropriately longer embedding window [11,20,21]. While the embedding window cannot be set too long, otherwise there are not enough appropriate neighbors for the reference phase point (here the appropriate neighbors mean that the wave forms of the data segments covered by the neighbors are well matched that of the reference phase point; see more discussions in Ref. [16]). Thus a tradeoff of the length of the embedding window should be made. In this paper, we set $d=80$ and $\tau=1$, and the first 20 nearest neighbors are used for each reference phase point. The covariance matrix $\mathbf{C}_{\text{noise}}$ of colored noise is estimated with 20 nonoverlapped noise sequences [each sequence has

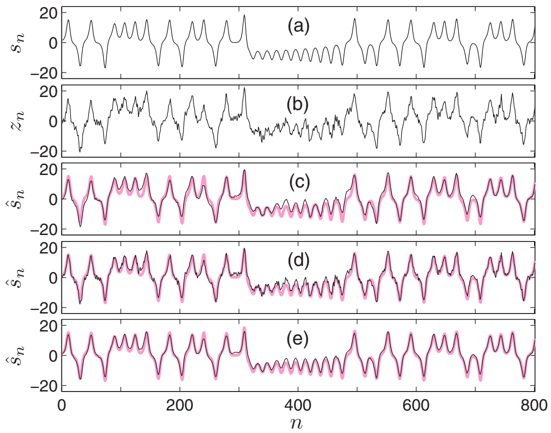

FIG. 2. (Color online) Wave forms of (noisy) Lorenz time series. The thin black curves in panels (a), (b), (c), (d), and (e) are the wave forms of the clean Lorenz time series, the noisy time series with 10 dB AR(3) noise, the enhanced data by the LP method, the output of step 1, and the output of step 2 of our method, respectively. For comparison, the wave form of clean data in panel (a) is plotted with thick curves in panels (c), (d), (e), respectively.

$(d-1)\tau+1$ points, equal to the length of the embedding window] generated from the AR(3) process (note that these noise sequences are not the noise sequences that are added to the Lorenz time series).

With eigenvalue decomposition to $\mathbf{C}_{\text{noise}}$, the noise dominated subspace $\mathbf{U}_{ND}=[\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{m_w}]$, spanned by the eigenvectors associated with the m_w largest eigenvalues, can be obtained, where m_w is the dimension of the noise dominated subspace. We have tried our method with different values of m_w. At the first step, as m_w decreases below 15, the performance becomes worse, because the energy of the projection of noise on each direction $\mathbf{u}_i$ $(1\leqslant i\leqslant 15)$ is bigger than that of the clean signal, while as m_w increases above 15, the performance varies little, because the energy of the projection of noise on each direction $\mathbf{u}_i$ $(15<i\leqslant 20)$ is almost equal to that of the clean signal. For data with high level noise, m_w should be set a little bigger. For simplicity, we set $m_n=20$ for all cases. We perform the LP method [the first 20 neighbors are used, $(d,\tau,M)=(80,1,8)$, where M is set a values relatively bigger than the minimum embedding dimension to control the introduced signal distortion at a small level] for comparison [note that Eq. (4) is difficult to be implemented, because with 20 segments of noise data and 80 dimensional embedding, the matrix $\mathbf{C}_w$ will be rank-deficient and matrix $\mathbf{C}_w^{-1/2}$ cannot be obtained]. We have also tried the local subspace method [15] to chaotic data with colored noise, but the results are not better than that of the LP method. The method proposed by Luo *et al.* [17] has been applied too, but as they had reported, the performance for chaotic data with colored noise is poor. So we do not present the results of these two methods here.

From Fig. 2, we can observe that most colored noise is deleted after step 2 of our method and the distortion with our method is much smaller than that of the enhanced data by the LP method. From Fig. 3, this can be more obviously observed in the reconstructed phase space. A more comprehensive comparison is summarized in Table I. We can see that

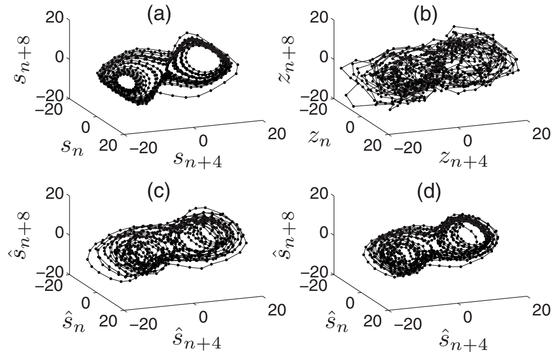

FIG. 3. Attractors reconstructed by time delay embedding with $\tau=4$. The data used in panels (a), (b), (c), and (d) are the same data that are used in Figs. 2(a)–2(c) and 2(e), respectively.

significant SNR gains are obtained by our method, outperforming the LP method much for the case the Lorenz time series is contaminated by the noise generated from the AR(3) process.

We further test our method with two other typical colored noise. One is pink noise, which is generated by a model proposed to explain the physics of $1/f$ noise [22,23]. Another noise is surrogate data generated by shuffling the phase of the Lorenz sequence [24], and thus with almost the same power spectra of the original Lorenz sequence. By phase shuffling, the deterministic structure is destroyed in the surrogate data, and thus it is used as noise in this paper. Their power spectra are plotted in Fig. 4. From the figure, we can see that it is difficult to separate the pink noise and the phase shuffled data, as well as the AR(3) noise, from the Lorenz time series in the frequency domain, because their spectra extensively overlap in the low frequency region. However, our method works well for the Lorenz time series with pink noise and phase shuffled surrogate data, as Tables II and III indicate. For the case with phase shuffled surrogate data, a good result can be obtained even with only the first step of our method. Our method has also been tested with the contaminated Rössler time series (measured from the x component of the Rössler system [25] with time interval 0.2), and about 3–5 dB SNR gains can be obtained for different noise level.

Noisy speech with additive white noise has been successfully enhanced by the LP method based on its deterministic

TABLE I. Performance of noise reduction for the noisy Lorenz time series with colored noise generated by AR(3). Ten sequences (each 10 000 points) are measured from x components of the Lorenz system with the same parameters but different initial condition. The four columns from left to right are the SNR of the original noisy data, the output of step 1, the output of step 2, and the enhanced data by the LP method, respectively.

Noisy data (dB)	step 1 (dB)	step 2 (dB)	LP (dB)
15	19.47±0.14	21.91±0.31	16.98±0.46
10	13.86±0.10	16.49±0.57	12.14±0.57
5	7.70±0.10	9.85±0.55	7.15±0.63

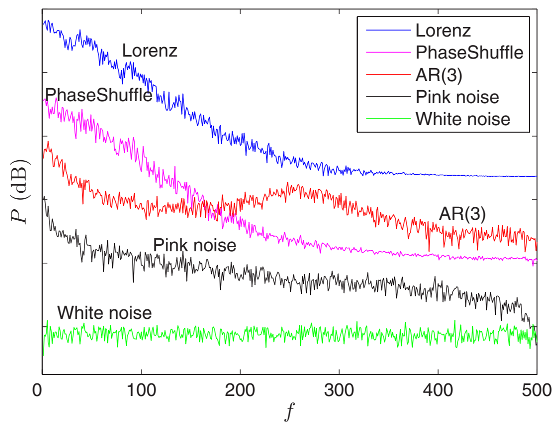

FIG. 4. (Color online) The power spectra of Lorenz time series, phase shuffled surrogate data of Lorenz times series, AR(3) noise, pink noise, and white noise. Each data have 10 000 samples, and their spectra are estimated by the periodogram averaging method [22]; i.e., the data are divided into 10 blocks (each block has 1000 samples), the spectra of each block are estimated by periodogram, and the average of the spectra of these blocks is taken as the final spectra. The spectra are offset vertically for clarity, and the scale in the vertical axis is therefore arbitrary.

feature [11,12]. To further verify our method, we apply it to ten speech sequences (five vowels, /*a*/, /*e*/, /*i*/, /*o*/, and /*u*/, articulated at normal speed by one male speaker and one female speaker, respectively, and recorded with 8 kHz sampling rate and 16 bits quantization) added with environmental noise measured in a running car. The results are summarized in Table IV.

IV. DISCUSSION AND CONCLUSION

We first investigated the pattern of energy distribution on each direction in the local phase space, and observed that for chaotic data with white noise, it is appropriate to estimate the signal subspace by the energy distribution of noisy data, while for the case with colored noise, the signal subspace estimated by the energy of noisy data may include considerable noise projection, because the energy of colored noise is mainly distributed in a low dimensional subspace. With these observations, we devised a two step strategy to delete colored noise for noisy chaotic data. At step 1, a noise dominated subspace which contains most of the noise components and a certain, small, amount of signal components is estimated by the energy distribution of colored noise. Then for a reference phase point, the components projected into the noise dominated subspace are eliminated and the enhanced data are reconstructed with the remaining components. After step 1, the energy of the residual error tends to distribute uniformly on each direction. So at step 2, we treat the residual error as white noise and apply the LP method to the output of step 1.

We applied this two step strategy to the noisy Lorenz time series, and the noisy Rössler time series, which are contaminated by noise generated by AR(3) process, pink noise, and phase shuffled surrogate data, respectively. We also enhanced the noisy speech contaminated by the environmental noise measured in a car. Experiments show that our method can reduce colored noise significantly, and is superior to the LP method in reducing colored noise for noisy chaotic data.

Generally, we can consider time delay embedding as a transform from time domain to phase space. If the colored noise is mainly distributed in a certain noise subspace, and the signal is mainly distributed in a signal subspace which is orthogonal to the noise subspace, we can delete the noise by nulling out the noise subspace, just as the frequency domain methods filter the out-band spectra of noise. Some noise components are also, possibly, distributed in the same subspace of the signal. We cannot reduce these noise components effectively, just as the frequency domain methods can not eliminate in-band noise well. So to say whether our method is applicable to a certain contaminated signal, the analysis of energy distribution in the local phase space should be performed first. But the representation of the signal in the local phase space is not so obvious as the representation in the frequency domain. And the energy

TABLE III. Performance of noise reduction for the noisy Lorenz time series (ten sequences, each 10 000 points) with its phase shuffled surrogate data.

Noisy data (dB)	Step 1 (dB)	Step 2 (dB)	LP (dB)
15	20.21±0.45	20.24±0.44	15.24±0.04
10	14.97±0.33	14.97±0.33	10.22±0.04
5	8.23±0.23	8.23±0.23	5.09±0.02

TABLE II. Performance of noise reduction for the noisy Lorenz time series (ten sequences, each 10 000 points) with pink noise.

Noisy data (dB)	Step 1 (dB)	Step 2 (dB)	LP (dB)
15	19.40±0.23	20.32±0.30	15.80±0.08
10	13.65±0.18	14.40±0.26	10.85±0.09
5	7.44±0.14	8.03±0.21	5.85±0.97

TABLE IV. Performance of noise reduction for the noisy speech with environmental noise measured in a car.

Noisy data (dB)	Step 1 (dB)	Step 2 (dB)	LP (dB)
15	18.72±0.80	19.51±1.28	17.92±0.66
10	13.73±0.60	15.80±0.75	13.20±0.58
5	8.10±0.54	10.68±0.97	8.21±0.61

distribution pattern may be complicated. As the results listed in Tables I–III for three typical noise, the amount of SNR gains at step 1 and step 2 are different, which is may due to the different energy distribution pattern in the local phase space.

ACKNOWLEDGMENT

This research was funded by a Hong Kong University Grants Council Grant Competitive Earmarked Research Grant (CERG) number PolyU 5269/06E.

[1] E. J. Kostelich and T. Schreiber, Phys. Rev. E **48**, 1752 (1993).
[2] H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis*, 2nd ed. (Cambridge University Press, Cambridge, 2003).
[3] R. Cawley and G. H. Hsu, Phys. Rev. A **46**, 3057 (1992).
[4] T. Sauer, Physica D **58**, 193 (1992).
[5] T. Schreiber and P. Grassberger, Phys. Lett. A **160**, 411 (1991).
[6] P. Grassberger, R. Heggler, H. Kantz, and C. Schaffrath, Chaos **3**, 127 (1993).
[7] H. Kantz, T. Schreiber, I. Hoffmann, T. Buzug, G. Pfister, L. G. Flepp, J. Simonet, R. Badii, and E. Brun, Phys. Rev. E **48**, 1529 (1993).
[8] A. I. Mees and K. Judd, Physica D **68**, 427 (1993).
[9] A. Leontitsisa, T. Bountis, and J. Pagge, Chaos **14**, 106 (2004).
[10] T. Schreiber and D. T. Kaplan, Chaos **6**, 87 (1996).
[11] R. Hegger, H. Kantz, and L. Matassini, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **48**, 1454 (2001).
[12] R. Hegger, H. Kantz, and L. Matassini, Phys. Rev. Lett. **84**, 3197 (2000).
[13] J. D. Farmer and J. J. Sidorowich, Physica D **47**, 373 (1991).
[14] Y. Ephraim and H. L. V. Trees, IEEE Trans. Speech Audio Process. **3**, 251 (1995).
[15] M. T. Johnson and R. J. Povinelli, Physica D **201**, 306 (2005).
[16] J. Sun, Y. Zhao, T. Nakamura, and M. Small, Phys. Rev. E **76**, 016220 (2007).
[17] X. Luo, J. Zhang, and M. Small, Phys. Rev. E **72**, 046710 (2005).
[18] M. B. Kennel, R. Brown, and H. D. I. Abarbanel, Phys. Rev. A **45**, 3403 (1992).
[19] E. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[20] R. Hegger, H. Kantz, L. Matassini, and T. Schreiber, Phys. Rev. Lett. **84**, 4092 (2000).
[21] J. Sun, N. Zheng, and X. Wang, Signal Process. **87**, 2431 (2007).
[22] S. J. Orfanidis, *Introduction to Signal Processing* (Prentice-Hall, New York, 1996).
[23] M. Schroeder, *Fractals, Chaos, Power Laws* (Freeman, New York, 1991).
[24] J. Theiler, S. Eubank, A. Longtin, B. Galdrikian, and J. D. Farmer, Physica D **58**, 77 (1992).
[25] O. E. Rossler, Phys. Lett. **57A**, 397 (1976).