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Stochastic Wave Equations With Cubic Nonlinearities in Two Dimensions

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STOCHASTIC WAVE EQUATIONS WITH CUBIC NONLINEARITIES

IN TWO DIMENSIONS

by

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Approved by:

Prof. Henri Schurz, Chair

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TITLE: STOCHASTIC WAVE EQUATION WITH CUBIC NONLINEARITIES IN TWO DIMENSIONS

MAJOR PROFESSOR: Prof. Dr. Henri Schurz

The main focus of my dissertation is the qualitative and quantative behavior of stochastic Wave equations with cubic nonlinearities in two dimensions. I evaluated the stochastic nonlinear wave equation in terms of its Fourier coefficients. I proved that the strong solution of that equation exists and is unique on an appropriate Hilbert space. Also, I studied the stability of N-dimensional truncations and give conclusions in three cases: stability in probability, estimates of $\mathbb{L}^p$-growth, and almost sure exponential stability. The main tool is the study of related Lyapunov-type functionals which admits to control the total energy of randomly vibrating membranes. Finally, I studied numerical methods for the Fourier coefficients. I focussed on the linear-implicit Euler method and the linear-implicit mid-point method. Their schemes have explicit representations. Eventually, I investigated their mean consistency and mean square consistency.

DEDICATION

TO MY PARENTS

Dad who died when I was three years old

Mom who waited me to complete my PhD and return to see her, but she died while I was in the progress.

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TABLE OF CONTENTS

INTRODUCTION

In this dissertation I study the stochastic wave equations with cubic nonlinearities in two dimensions in terms of all systems parameters, i.e., with non-global Lipschitz continuous nonlinearities. Our study focusses on existence, uniqueness, stability, energy of both analytic and numerical solutions under the geometric condition

$$\sigma^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2}) - a_1 = \frac{\sigma^2 \pi^2 (l_x^2 + l_y^2) - a_1 l_x^2 l_y^2}{l_x^2 l_y^2} := \gamma > 0,$$

where $0 \leq x \leq l_x$ and $0 \leq x \leq l_x$ such that $\mathbb{D} = [0, l_x] \times [0, l_x]$. $\Big($Note that l_x and l_y are dimension parameters of a vibrating plate or a membrane$\Big)$.

Many authors have treated stochastic wave equations. For example, Dalang and Frangos [5] treat the wave equation in two spatial dimensions driven by space-time Gaussian noise that is white in time, but it has a nondegenerate spatial covariance. Millet and Sanz-Sole [11] proved that the existence and uniqueness of a real-valued process solving a nonlinear stochastic wave equation driven by a Gaussian noise white in time and correlated in the two-dimensional space variable. Walsh [19] treats the stochastic wave equations in one dimension. Chow [4] studies the existence of solutions to some linear and semi-linear stochastic wave equation in a bounded domain in one dimension and he studied the additive noise case.

In chapter 1, I derive the eigenvalues solutions related to the Fourier coefficients of analytic solution. I shall start with the general wave equation, continue with nonlinear wave equation without noise and finally deal with nonlinear wave equation with additive and multiplicative noise. I find all $\mathbb{L}^2$-integrable strong solutions in terms of Fourier coefficients.

In both chapters 2 and 3, I prove the existence and uniqueness of strong solution for the stochastic wave equation with cubic nonlinearity in two dimensions and I shall find the total expected energy of the truncated system in chapter 2. For our analysis, we resort to finite-dimensional truncated systems approximating the original stochastic wave equation. At first, we study the existence, uniqueness of its finite-dimensional truncation. Thereafter, we carry over the main results of finite-dimensional to the infinite dimensional case in chapter 3.

In chapter 4, I study three cases of stability: stochastic stablility (i.e., stable in probability), growth estimates of $\mathbb{L}^p$- norm, and almost surely exponential stability. I verify that the trivial solution of the equation (1.33) is stochastically stable, and almost surely exponential stable under appropriate conditions.

Finally, in chapter 5, I study some cases of stable numerical methods for the Fourier coefficients like linear-implicit Euler method and linear-implicit midpoint method. We find explicit representations of linear-implicit Euler and linear-implicit midpoint methods. Also, I prove that these numerical solutions of the stochastic wave equation with cubic nonlinearity in two dimensions are locally mean consistent with rate $r_0 \geq 1.5$ and locally mean square consistent with rate $r_2 = 1.0$. This allows us to conclude mean square convergence of them with certain rates depending on dimension N of truncated and time-steps sizes h as long as $N^2 h \to 0$.

Eventually, chapter 6 reports on numerical experiments confirming our previous finding.

CHAPTER 1

EVALUATION IN TERMS OF FOURIER COEFFICIENTS

1.1 INTRODUCTION

Consider the wave equation

$$u_{tt} \,=\, \sigma^2\,(u_{xx} + u_{yy}) \tag{1.1}$$

where σ is a material dependent speed determining parameter of propagation (the wave speed) and $0 \,\le\, x \,\le\, l_x$, $0 \,\le\, y \,\le\, l_y$, and the boundary conditions are $u(x,0,t) \,=\, u(x,l_y,t) \,=\, 0$, $0 \,\le\, x \,\le\, l_x$, $t \,>\, 0$ and $u(0,y,t) \,=\, u(l_x,y,t) \,=\, 0$, $0 \,\le\, y \,\le l_y$, $t \,>\, 0$, also the initial conditions are $u(x,y,0) \,=\, f(x,y)$ with $f \,\in\, \mathbb{L}^2$ (initial position) and $u_t(x,y,0) \,=\, g(x,y)$ with $g \in \mathbb{L}^2$ (initial velocity). Recall that

$$\mathbb{L}^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{R} \mid \int_{\mathbb{D}} |f(x,y)|^2\, d\,\mu(x,y) < \infty\}.$$

1.2 FORMAL DERIVATION OF EIGENVALUES SOLUTIONS

Suppose that the series below in this chapter all converges absolutely. (This can be justified later by the use of energy or Lyapunov functional.) $\forall\,(x,\,y) \,\in\, \mathbb{D}$, let $g(x,y) \,=\, 0$. Use separation of variables, i.e, let $u(x,y,t) \,=\, X(x)\,Y(y)\,T(t)$. Then equation (1.1) becomes

$$X(x)\,Y(y)\,T''(t) \,=\, \sigma^2\Big(X''(x)\,Y(y)\,T(t) \,+\, X(x)\,Y''(y)\,T(t)\Big).$$

Thus

$$\frac{T''(t)}{\sigma^2 T(t)} \,-\, \frac{Y''(y)}{Y(y)} \,=\, \frac{X''(x)}{X(x)} \,=\, \lambda.$$

Now we can separate the last equation into two equations, first is

$$\frac{T''(t)}{\sigma^2 T(t)} - \frac{Y''(y)}{Y(y)} \,=\, \lambda \tag{1.2}$$

and therefore

$$\frac{T''(t)}{\sigma^2 T(t)} - \lambda = \frac{Y''(y)}{Y(y)} = \mu$$

which implies that

$$Y''(y) - \mu Y(y) = 0$$

and

$$T''(t) - \sigma^2 (\mu + \lambda) T(t) = 0$$

and the second equation is

$$X''(x) - \lambda X(x) = 0. \tag{1.3}$$

Now from the boundary condition we know that $u(0, y, t) = u(l_x, y, t) = 0$. So, we have $X(0) = X(l_x) = 0$. For the moment, we concentrate on X, i.e, we have to determine the values of λ for which $X''(x) - \lambda X(x) = 0$. We have three cases for λ.

1) If $\lambda = 0$, then $X''(x) = 0$, and the solution is $X(x) = A\, X(x) + B$ where A and B are real numbers. But $X(0) = B = 0$. Furthermore $X(l_x) = A\, l_x = 0$, $l_x \neq 0$. Therefore $A = 0$. Thus $X(x) = 0$, so this is the trivial solution.

2) If $\lambda > 0$, let $\lambda = a^2$. This implies that $X''(x) - a^2 X(x) = 0$, Solving this ordinary differential equation gives us $X(x) = A \exp(a\, x) + B \exp(-\, a\, x)$. But $X(0) = A + B = 0$, which implies that $A = -\, B$ and $X(l_x) = 0 = A \exp(a\, l_x) + B \exp(-\, a\, l_x)$. Thus $A \exp(a\, l_x) - A \exp(-\, a\, l_x) = 0$, i.e, $A\, (\exp(a\, l_x) - \exp(-\, a\, l_x)) = 0$ which implies that $A = 0$ and $B = 0$. Thus $X(x) = 0$. Again, it is a trivial solution.

3) If $\lambda < 0$, let $\lambda = -\, a^2$. *Then* $X''(x) + a^2 X(x) = 0$. The solution of the last equation is $X(x) = A\, cos(a\, x) + B\, sin(a\, x)$. But $X(0) = A = 0$, and $X(l_x) = B\, sin(a\, l_x) = 0$. Therefore, $a\, l_x = n\, \pi$ and thus $a = \frac{n\pi}{l_x}$ and the eigenvalues is $\lambda_n = -\, a^2 = -\frac{n^2\pi^2}{l_x^2}$. Therefore, the solution is $X(x) = B\, sin(\frac{n\pi\, x}{l_x})$ and then

$$X_n(x) = B_n\, sin(\frac{n\pi\, x}{l_x}) \tag{1.4}$$

Similarly, we can take the other boundary condition, $u(x, 0, t) = u(x, l_y, t) = 0$ (which implies that $Y(0) = Y(l_y) = 0$) to solve the equation $Y''(y) - \mu Y(y) = 0$. We have three cases for μ.

1) If $\mu = 0$, *then* $Y''(y) = 0$ which has a solution $Y(y) = AY(y) + B$ where A and B are real numbers. But $Y(0) = B = 0$. Furthermore, $Y(l_y) = A\, l_y = 0$, $l_y \neq 0$. Therefore $A = 0$. Thus $Y(y) = 0$, i.e, trivial solution.

2) If $\mu > 0$, let $\lambda = a^2$. This implies that $Y''(y) - a^2\, Y(y) = 0$. Solving this ordinary differential equation gives us $Y(y) = A\,\exp(a\,y) + B\,\exp(-\,a\,y)$. But $Y(0) = A + B = 0$ which implies that $A = -\,B$. and $Y(l_y) = 0 = A\,\exp(a\,l_y) + B\,\exp(-\,a\,l_y)$. Thus $A\,\exp(a\,l_y) - A\,\exp(-\,a\,l_y) = 0$, i.e, $A\Big(\exp(a\,l_y) - \exp(-\,a\,l_y)\Big) = 0$ which implies that $A = 0$ and $B = 0$. Thus $Y(y) = 0$. Again it is a trivial solution.

3) If $\mu < 0$, let $\mu = -\,b^2$. *Then* $Y''(y) + b^2\, Y(y) = 0$. The solution of the last equation is $Y(y) = A\,cos(b\,y) + B\,sin(b\,y)$. But $Y(0) = A = 0$, and $Y(l_y) = B\,sin(b\,l_y) = 0$. Therefore, $b\,l_y = m\,\pi$ and thus $b = \dfrac{m\pi}{l_y}$ i.e, the eigenvalues $\mu_m = -\,b^2 = -\,\dfrac{m^2\pi^2}{l_y^2}$. Therefore, the solution is $Y(y) = B\,sin(\frac{m\pi y}{l_y})$ and *then*

$$Y_m(x) = B_m\,sin(\frac{m\pi x}{l_y}) \tag{1.5}$$

Now take the last equation which is $T''(t) - (\lambda + \mu)\,\sigma^2\,T = 0$, and we know that the nontrivial solution occurs when $\lambda < 0$ i.e,$\lambda = -\,a^2$, and $\mu < 0$ which means $\mu = -\,b^2$. Then, $T''(t) + (a^2 + b^2)\sigma^2\,T = 0$, which has a solution $T(t) = A\,cos(\sqrt{a^2 + b^2}\,\sigma\,t) + B\,sin(\sqrt{a^2 + b^2}\,\sigma\,t)$. From the initial conditions, we have $T(0) = f(x, y)$ and $T'(0) = g(x, y)$ Let $g(x, y) = 0$, then $T'(0) = 0$. This leads to

$$T'(t) = -\sqrt{a^2+b^2}\,\sigma\,A\,sin(\sqrt{a^2+b^2}\sigma\,t) + \sqrt{a^2+b^2}\,\sigma\,B\,cos(\sqrt{a^2+b^2}\sigma\,t).$$

But $T'(0) = 0 = \sqrt{a^2+b^2}\sigma B$, which implies that $B = 0$. Thus

$T(t) = f(x,y)\,cos(\sqrt{a^2+b^2}\,\sigma\,t)$. Therefore, $T_{nm} = A\,cos(\sqrt{\frac{n^2\,\pi^2}{l_x^2} + \frac{m^2\pi^2}{l_y^2}}\,\sigma\,t)$ i.e.,

$$T_{nm} = A_{nm}cos(\sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}\,\pi\sigma\,t). \tag{1.6}$$

From equations (1.4), (1.5), and (1.6) we have

$$u_{nm}(x,y,t) = c_{nm}sin(\frac{n\pi x}{l_x})sin(\frac{m\pi y}{l_y})cos(\sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}\pi\sigma t)$$

which satisfies all the conditions except (may be) $u(x,y,0) = f(x,y)$. To satisfy this condition, let

$$\begin{aligned} u(x,y,t) &= \sum_{n,m=1}^{\infty} \bar{c}_{nm}\,sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})\,cos\Big(\sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}\,\pi\,\sigma\,t\Big) \\ &= \sum_{n,m=1}^{\infty} c_{nm}(t)\,sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\,\pi\,y}{l_y}). \end{aligned}$$

Choose c_{nm} so that $u(x,y,0) = f(x,y)$, then

$$f(x,y) = \sum_{n,m=1}^{\infty} c_{nm}(t)sin(\frac{n\pi x}{l_x})sin(\frac{m\pi y}{l_y}).$$

By Fourier sine expansion, we have

$$c_{nm} = \frac{4}{l_x l_y}\int_0^{l_y}\int_0^{l_x} f(x,y)sin(\frac{n\pi x}{l_x})sin(\frac{m\pi y}{l_y})dxdy,$$

because we know that $c_n = \frac{2}{l_x}\int_0^{l_x} f(x)sin(\frac{n\pi\,x}{l_x})\,dx$ and $f(x) = \sum_{n=0}^{\infty} c_n\,sin(\frac{n\pi x}{l_x})$. Thus

$$u(x,y,t) = \sum_{n,m=1}^{\infty}\Big[\frac{4}{l_x l_y}\int_0^{l_y}\int_0^{l_x} f(x,y)sin(\frac{n\pi\,x}{l_x})sin(\frac{m\pi\,y}{l_y})dx\,dy\Big]sin(\frac{n\pi\,x}{l_x})sin(\frac{m\pi\,y}{l_y}).$$

1.3 NONLINEAR WAVE EQUATIONS WITHOUT NOISE

Consider the stochastic wave equations with cubic potential,

$$u_{tt} = \sigma^2(u_{xx} + u_{yy}) + (a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})})u \,-\, \kappa\,v \tag{1.7}$$

where $u = u(x,y,t)$, $v = v(x,y,t) = \dot{u}(x,y,t) = \frac{\partial}{\partial t} u(x,y,t) = u_t$, and $\kappa \geq 0$ on bounded two-dimensional rectangular domain $\mathbb{D}$. But we know from the previous section that the Fourier solution of the wave equation (1.1) is

$$u(x,y,t) = \sum_{n,m=1}^{\infty} c_{n,m}(t)\, sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y}) \tag{1.8}$$

and we can find

$$u_t(x,y,t) = \sum_{n,m=1}^{\infty} \dot{c}_{n,m}(t)\, sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})$$

$$u_{tt}(x,y,t) = \sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t)\, sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})$$

$$u_x(x,y,t) = \sum_{n,m=1}^{\infty} \frac{n\pi}{l_x}\, c_{n,m}(t)\, cos(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})$$

$$u_{xx}(x,y,t) = -\sum_{n,m=1}^{\infty} \frac{n^2\pi^2}{l_x^2}\, c_{n,m}(t)\, sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})$$

$$u_y(x,y,t) = \sum_{n,m=1}^{\infty} \frac{m\pi}{l_y}\, c_{n,m}(t)\, sin(\frac{n\pi\, x}{l_x})\, cos(\frac{m\pi\, y}{l_y})$$

$$u_{yy}(x,y,t) = -\sum_{n,m=1}^{\infty} \frac{m^2\pi^2}{l_y^2}\, c_{n,m}(t)\, sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})$$

Substitute these equations into equation (1.7), and we have

$$\begin{aligned}\sum_{n,m=1}^{\infty} \ddot{c}_{n,m} sin(\frac{n\pi x}{l_x}) sin(\frac{m\pi y}{l_y}) &= -\sigma^2 \Big[\sum_{n,m=1}^{\infty} (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\pi^2 c_{n,m}(t) sin(\frac{n\pi x}{l_x}) sin(\frac{m\pi y}{l_y}) \Big] \\ &\quad + (a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})}) u(x,y,t) \,-\, \kappa\, v(x,y,t). \end{aligned} \tag{1.9}$$

Then multiply both sides by $sin(\frac{k\pi\, x}{l_x})\, sin(\frac{l\pi\, y}{l_y})$ and integrate with respect to x and y, consider $k = n$ and $l = m$. Then we have

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t) \int_0^{l_y}\int_0^{l_x} sin^2(\frac{n\pi\, x}{l_x}) sin^2(\frac{m\pi y}{l_y})\, dx\, dy$$
$$= \sum_{n,m=1}^{\infty} \Big[-\sigma^2 (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\, \pi^2\, c_{n,m}(t) + (a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})}) c_{n,m}(t) \; - \; \kappa\, \dot{c}_{n,m}(t)\Big]$$
$$\cdot \int_0^{l_y} sin^2(\frac{n\pi x}{l_x}) dx \int_0^{l_x} sin^2(\frac{m\pi y}{l_y})\, dy.$$

But we know from Calculus that $\int_0^{l_x} sin^2(\frac{n\pi x}{l_x}) dx = \frac{l_x}{2}$ and $\int_0^{l_y} sin^2(\frac{m\pi y}{l_y}) dy = \frac{l_y}{2}$. Thus last equation is equivalent to

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t)\, \frac{l_x}{2}\frac{l_y}{2} \;=\; \sum_{n,m=1}^{\infty} \Big[-\sigma^2 (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\, \pi^2\, c_{n,m}(t)$$
$$+ \,(a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})}) c_{n,m}(t) \; - \; \kappa\, \dot{c}_{n,m}(t)\Big] \frac{l_x}{2}\frac{l_y}{2}.$$

By cancelling $\frac{l_x}{2}\frac{l_y}{2}$ from both sides in last equation, we get

$$\sum_{n,m=1}^{\infty} \Big[\ddot{c}_{n,m} + \Big(\sigma^2 (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) \pi^2 - a_1 + a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big)\, c_{n,m} \; + \; \kappa\, \dot{c}_{n,m} \Big] \; = \; 0.$$

Therefore,

$$\ddot{c}_{nm} = \Big(-\sigma^2\pi^2 (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) + a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big)\, c_{n,m} \; - \; \kappa\, \dot{c}_{n,m}.$$

That is

$$\ddot{c}_{nm} = \Big(-\sigma^2\pi^2 (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) + a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big)\, c_{n,m} \; - \; \kappa\, v_{n,m}. \tag{1.10}$$

where $||u||^2_{\mathbb{L}(\mathbb{D})} = \sum_{k,l=1}^{\infty} [c_{k,l}]^2$.

Now if we take the boundary condition $u(0, y, t) = u(l_x, y, t) = 0$ and $u_y(x, 0, t) = u_y(x, l_y, t) = 0$, this implies that $Y'(0) \; = \; Y'(l_y) \; = \; 0$. Similarly, we get

$$Y(y) \; = \; A\, cos(a\, y) \; + \; B\, sin(a\, y).$$

Take the derivative, then

$$Y'(y) \;=\; -\,a\,A\,sin(a\,y) \;+\; a\,B\,sin(a\,y).$$

Substituting the boundary conditions, we get $Y'(0) \;=\; a\,B \;=\; 0$. This implies that $B = 0$ and

$$Y'(l_y) \;=\; -\,a\,A\,sin(a\,l_y) \;=\; 0.$$

Then $a\,l_y \;= m\,\pi$ i.e, $a \;=\; \dfrac{m\pi}{l_y}$. Thus $Y(y) \;=\; A_m\,cos(\frac{m\pi\,y}{l_y})$ and we know that $X_n(x) \;=\; A_n\,sin(\frac{n\pi\,x}{l_x})$ and $T_{n,m}(t) \;=\; A_{n,m}(t)\,cos(\sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}\sigma\pi\,t)$ which implies that

$$u_{nm}(x, y, t) \;=\; B_{n,m}\,sin(\frac{n\pi\,x}{l_x})\,cos(\frac{m\,\pi\,y}{l_y})\,A_{n,m}(t)\,cos(\sqrt{\frac{n^2}{l_x^2} \;+\; \frac{m^2}{l_y^2}}\sigma\,\pi\,t)$$

and therefore,

$$u(x, y, t) \;=\; \sum_{n,m=1}^{\infty} b_{n,m}(t) sin(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y}).$$

Now we find u_{tt}, u_{xx}, and u_{yy} as follows

$$u_{tt}(x, y, t) = \sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t)\,sin(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y})$$

$$u_{xx}(x, y, t) = -\sum_{n,m=1}^{\infty} \frac{n^2\pi^2}{l_x^2} c_{n,m}(t)\,sin(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y})$$

$$u_{yy}(x, y, t) = -\sum_{n,m=1}^{\infty} \frac{m^2\pi^2}{l_y^2}\,c_{n,m}(t)\,sin(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y}).$$

Substitute these equations in equation (1.7), we have

$$\begin{aligned}
&\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t)\,sin(\frac{n\pi\,x}{l_x}) sin(\frac{m\pi\,y}{l_y}) \\
&= \sum_{n,m=1}^{\infty} \Big[\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})c_{n,m}(t) + \Big(a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big)\,c_{n,m}(t) - \kappa\dot{c}_{n,m}(t)\Big] \\
&\cdot\; sin(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y}).
\end{aligned}$$

Then multiply both sides by $sin(\frac{k\pi\, x}{l_x})\, cos(\frac{l\pi\, y}{l_y})$ and integrate with respect to x and y, again consider $k = n$ and $l = m$, so we have

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t) \int_0^{l_y}\int_0^{l_x} sin^2(\frac{n\pi\, x}{l_x}) cos^2(\frac{m\pi\, y}{l_y})\, dx\, dy$$
$$= \sum_{n,m=1}^{\infty} \Big(-\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) c_{n,m}(t) + (a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})}) c_{n,m}(t) - \kappa \dot{c}_{n,m}(t) \Big)$$
$$\cdot \int_0^{l_x} sin^2(\frac{n\pi\, x}{l_x})\, dx \int_0^{l_y} cos^2(\frac{m\pi\, y}{l_y}) dy.$$

But we know from Calculus that $\int_0^{l_x} sin^2(\frac{n\pi x}{l_x}) dx = \frac{l_x}{2}$ and $\int_0^{l_y} cos^2(\frac{m\pi y}{l_y}) dy = \frac{l_y}{2}$. Thus last equation is equivalent to

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}\, \frac{l_x}{2}\frac{l_y}{2} \;=\; \sum_{n,m=1}^{\infty} \Big(\Big[-\, \sigma^2\, (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\, \pi^2 \,+\, a_1 \,-\, a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big] c_{nm} \,-\, \kappa\, \dot{c}_{n,m} \Big) \frac{l_x}{2}\frac{l_y}{2}.$$

By cancelling $\frac{l_x}{2}\frac{l_y}{2}$ from both sides in last equation, we get

$$\sum_{n,m=1}^{\infty} \Big[\ddot{c}_{n,m} + \Big(\sigma^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\pi^2 - a_1 + a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big)\, c_{n,m} \,+\, \kappa\, v_{n,m} \Big] = 0$$

Therefore,

$$\ddot{c}_{n,m} = \Big[-\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) + a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big] c_{n,m} \,-\, \kappa\, v_{n,m} \tag{1.11}$$

Take the boundary conditions $u_x(0, y, t) = u_x(l_x, y, t) = 0$ and $u(x, 0, t) = u(x, l_y, t) = 0$, then we have $X'(0) = X'(l_x) = 0$. Also, we know that if $\lambda < 0$ ($\lambda = -a^2$), then,

$$X''(x) \,+\, a^2 X(x) \,=\, 0$$

has a solution

$$X(x) \,=\, A\, cos(a\, x) \,+\, B\, sin(a\, x).$$

Then

$$X'(x) \,=\, -\, a\, A\, sin(a\, x) \,+\, a\, B\, sin(a\, x),$$

but $X'(0) = a\,B = 0$, which implies that $B = 0$ and $X'(l_x) = -\,a\,A\,sin(a\,l_x) = 0$ which implies that $a\,l_x = n\,\pi$. Thus $a = \dfrac{n\pi}{l_x}$ and we have $X_n(x) = A_n\,cos(\frac{n\pi\,x}{l_x})$. But we know from section (1.2) that $Y_m(y) = B_m\,sin(\frac{m\pi\,y}{l_y})$ and
$T_{n,m}(t) = A_{n,m}\,cos(\sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}\,\pi\,\sigma\,t)$ and by the last equations we have

$$u_{n,m}(x,y,t) = c_{n,m}\,cos(\frac{n\pi\,x}{l_x})sin(\frac{m\pi\,y}{l_y})\,T_{n,m}(t).$$

This means that

$$u(x,y,t) = \sum_{n,m=1}^{\infty} c_{n,m}(t)cos(\frac{n\pi x}{l_x})sin(\frac{m\pi y}{l_y}).$$

Then

$$u_{tt}(x,y,t) = -\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t)\,cos(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})$$

$$u_{xx}(x,y,t) = -\sum_{n,m=1}^{\infty} \frac{n^2\pi^2}{l_x^2}\,c_{n,m}(t)\,cos(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})$$

$$u_{yy}(x,y,t) = -\sum_{n,m=1}^{\infty} \frac{m^2\pi^2}{l_y^2}\,c_{n,m}(t)\,cos(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})$$

Similarly, substitute these equations in equation (1.7), we have

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t)cos(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})$$
$$= \sum_{n,m=1}^{\infty} \Big[-\sigma^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\,\pi^2\,c_{n,m}(t) + (a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})})c_{n,m}(t) - \kappa\,\dot{c}_{n,m}(t)\Big]$$
$$\cdot\ cos(\frac{n\pi x}{l_x})sin(\frac{m\pi y}{l_y}).$$

Then, by the same argument, we have

$$\ddot{c}_{n,m} = \Big[-\sigma^2\,\pi^2\,(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) + a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big]c_{n,m} - \kappa\,v_{n,m} \qquad (1.12)$$

Finally, take the boundary conditions $u_x(0,y,t) = u_x(l_x,y,t) = 0$, then $X'(0) = X'(l_x) = 0$, and $u_y(x,0,t) = u_y(x,l_y,t) = 0$, so $Y'(0) = Y'(l_y) = 0$. Also, we know that if $\lambda \leq 0$ $(\lambda = -a^2)$, then,

$$X''(x) + a^2 X(x) = 0$$

has a solution

$$X(x) = A\,cos(a\,x) + B\,sin(a\,x).$$

Using the previous arguments we have $X_n(x) = A_n\,cos(n\,\pi\,x/l_x)$ and $Y_m(y) = B_m\,cos(m\,\pi\,y/l_y)$. But we know from section (1.2) that $T_{n,m}(t) = A_{n,m}\,cos(\sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}\,\pi\,\sigma\,t)$, and by the last equations we have

$$u_{nm}(x,y,t) = c_{nm}(t)\,cos(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y})\,T_{n,m}(t).$$

This means that

$$u(x,y,t) = \sum_{n,m=1}^{\infty} c_{n,m}(t)\,cos(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y}).$$

Then

$$u_{tt}(x,y,t) = \sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t)\,cos(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y})$$

$$u_{xx}(x,y,t) = -\sum_{n,m=1}^{\infty} \frac{n^2\pi^2}{l_x^2}\,c_{n,m}(t)\,cos(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y})$$

$$u_{yy}(x,y,t) = -\sum_{n,m=1}^{\infty} \frac{m^2\pi^2}{l_y^2}\,c_{n,m}(t)\,cos(\frac{n\pi\,x}{l_x})\,cos(\frac{m\pi\,y}{l_y}).$$

Similarly, substitute these equations in equation (1.7), we have

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t)cos(\frac{n\pi\,x}{l_x})cos(\frac{m\pi\,y}{l_y})$$
$$= \sum_{n,m=1}^{\infty} \Big[-\sigma^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\,\pi^2\,c_{n,m}(t) + (a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})})\,c_{n,m}(t) - \kappa\,\dot{c}_{n,m}(t)\Big]$$
$$\cdot\, cos(\frac{n\pi x}{l_x})\,cos(\frac{m\pi y}{l_y}).$$

Then, by the same argument, we have

$$\ddot{c}_{n,m} \;=\; \Big[-\sigma^2\pi^2(\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}) + a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big]c_{n,m} \;-\; \kappa\, v_{n,m}. \tag{1.13}$$

Theorem 1.3.1. *The Fourier coefficients $\ddot{c}^{i,j}_{n,m}$ of series solution*

$$u(x,y,t) \;=\; \sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty} c^{i,j}_{n,m}(t)\, e^i_n(x)\, e^j_m(y) \tag{1.14}$$

of nonlinear PDEs (1.7) satisfies (1.13), where $c^{1,1}$ corresponding to $sin(.)sin(.)$ and $c^{1,2}$ corresponding to $sin(.)cos(.)$, while $c^{2,1}$ corresponding to $cos(.)sin(.)$ and $c^{2,2}$ corresponding to $cos(.)cos(.)$ and $e^i_n(z) = \sqrt{\frac{2}{l_z}}sin(\frac{n\pi z}{l_z}), e^j_m(z) = \sqrt{\frac{2}{l_z}}sin(\frac{m\pi z}{l_z})$ such that $z = x\; or\, y$.

Proof. It is clear from the above arguments. □

Note: u satisfies (1.7) if all related series of u_t, u_x, u_y, and u converge absolutely of Fourier coefficients satisfy (1.13).

1.4 STOCHASTIC NONLINEAR WAVE EQUATIONS WITH ADDITIVE NOISE

Consider the nonlinear stochastic wave equation with additive noise

$$u_{tt} = \sigma^2(u_{xx}+u_{yy}) + (a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})})u \;-\; \kappa\, v + b_0\,\frac{dW}{dt} \tag{1.15}$$

where

$$W(x,y,t) = \sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty} \alpha_{n,m} W^{i,j}_{n,m}(t) e^i_n(x) e^j_m(y) \tag{1.16}$$

and e_n and e_m are orthonormalized eigenfunctions of Laplace operator Δ on $\mathbb{D}$, driven by $i.i.d.$ standard Wiener processes $W^{i,j}_{n,m}$ with $\mathbb{E}[W^{i,j}_{n,m}(t)] = 0$, $\mathbb{E}[W^{i,j}_{n,m}(t)]^2 = t$. Recall that $e^i_n(z)$ and $e^j_m(z)$ are one of the functions $\sqrt{\frac{2}{l_z}}sin(\frac{k\pi z}{l})$ or $\sqrt{\frac{2}{l_z}}cos(\frac{k\pi z}{l})$, with $z = x$ or y and $k = n$ or m.

If we follow the same procedure as in section 1.3 and for simplicity we will take the case of $sin(.)sin(.)$, so we find that

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}\, sin(\frac{n\pi\, x}{l_x}) sin(\frac{m\pi\, y}{l_y})$$
$$= \sum_{n,m=1}^{\infty} \Big[-\, \sigma^2\, (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\, \pi^2 c_{n,m}(t) +\, (a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})})\, c_{n,m}(t)$$
$$-\, \kappa\, \dot{c}_{n,m}(t) \Big] sin(\frac{n\,\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y}) \,+\, b_0\, \frac{dW}{dt} \tag{1.17}$$

and we know the solution is

$$u(x, y, t) \,=\, \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} c^{i,j}_{n,m}(t)\, e^i_n(x)\, e^j_m(y). \tag{1.18}$$

Multiply both sides by $e^i_k(x)\, e^j_l(y)$ and integrate with respect to x and y. Recall the orthogonality relation, which says

$$\int\int_{(x,y)\in\,\mathbb{D}} e^{i_1}_{n_1}(x) e^{i_2}_{n_2}(x) e^{j_1}_{m_1}(y) e^{j_2}_{m_2}(y)\, dx\, dy$$
$$= \int_0^{l_y}\int_0^{l_x} e^{i_1}_{n_1}(x) e^{i_2}_{n_2}(x) e^{j_1}_{m_1}(y) e^{j_2}_{m_2}(y)\, dx\, dy$$
$$= \int_0^{l_x} e^{i_1}_{n_1}(x) e^{i_2}_{n_2}(x)\, dx \int_0^{l_y} e^{j_1}_{m_1}(y) e^{j_2}_{m_2}(y)\, dy$$
$$= \delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{i_1,i_2}\delta_{j_1,j_2}. \tag{1.19}$$

Then equation (1.17) is equivalent to

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t) \int_0^{l_y}\int_0^{l_x} (e^i_n(x))^2\, (e^j_m(y))^2\, dx\, dy$$
$$= \sum_{n,m=1}^{\infty} \Big[-\, \sigma^2 (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\, \pi^2\, c^{i,j}_{n,m}(t) +\, (a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})})\, c_{n,m}(t)$$
$$-\, \kappa\, \dot{c}_{n,m}(t) \Big] \int_0^{l_y} (e^j_m(y))^2\, dy \int_0^{l_x} (e^i_n(x))^2\, dx$$
$$+\, \sum_{n,m=1}^{\infty} \int_0^{l_y}\int_0^{l_x} b_0\, e_n\, (x)\, e_m(y)\, \frac{dW(x,y)}{dt}\, dx\, dy.$$

Integrating this identity leads to

$$\ddot{c}^{i,j}_{n,m}(t) \;=\; \Big[-\,\sigma^2\pi^2(\tfrac{n^2}{l_x^2}+\tfrac{m^2}{l_y^2}) \,+\, a_1 \,-\, a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big]\, c^{i,j}_{n,m}(t)$$
$$-\,\kappa\, v_{n,m}(t) \,+\, b_0\,\alpha^{i,j}_{n,m}\,\dot{W}^{i,j}_{n,m}(t). \tag{1.20}$$

Theorem 1.4.1. *Assume that* $\sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty}\big(\alpha^{i,j}_{n,m}\big)^2 \,<\, \infty$*, and* $u,\ u_t,\ u_x,$ *and* $u_y \,\in\, \mathbb{L}^2(\mathbb{D})$*. The Fourier coefficients of*

$$u(x,y,t) = \sum_{n,m=1}^{\infty} c^{i,j}_{n,m} e^i_n(x) e^j_m(y) \tag{1.21}$$

where $0 \le\, x \,\le\, l_x$ *and* $0 \le\, y \,\le\, l_y$ *satisfy (*$\mathbb{P}$*-a.s) the infinite-dimensional system of ordinary SDEs (*$n,\ m\,\in\,\mathbb{N}$*)*

$$\frac{d^2}{dt^2}c^{i,j}_{n,m}(t) \;=\; \Big[-\,\sigma^2\,\pi^2(\tfrac{n^2}{l_x^2}+\tfrac{m^2}{l_y^2}) \,+\, a_1 \,-\, a_2 \sum_{s,q=1}^{2}\sum_{k,l=1}^{\infty}(c^{s,q}_{k,l})^2\Big] c^{i,j}_{n,m}$$
$$-\,\kappa\, v_{n,m} \,+\, b_0\,\alpha^{i,j}_{n,m}\frac{dW^{i,j}_{n,m}}{dt} \tag{1.22}$$

Proof. Plug equation (1.20) into equation (1.15) and to simplify use the case $sin(.)sin(.)$ case, we have

$$u_{tt} = \sum_{n,m=1}^{\infty} sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})\,\ddot{c}^{1,1}_{n,m}(t) = \sum_{n,m=1}^{\infty} sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})$$
$$\Big[\Big(-\,\sigma^2\pi^2(\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2})+a_1\,-\,a_2\sum_{k,l=1}^{\infty}(c^{1,1}_{k,l})^2\Big)c^{1,1}_{n,m}\,-\,\kappa\,v_{n,m}\,+\,b_0\,\alpha^{1,1}_{n,m}\frac{dW^{1,1}_{n,m}}{dt}\Big] \tag{1.23}$$

which implies that

$$u_{tt} = \sum_{n,m=1}^{\infty} sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})\Big[\Big(-\,\sigma^2\,\pi^2\,(\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2})+a_1\,-a_2\sum_{k,l=1}^{\infty}(c^{1,1}_{k,l})^2\Big)c_{n,m}$$
$$-\,\kappa\,v_{n,m}\,+\,b_0\,\alpha_{n,m}\frac{dW^{1,1}_{n,m}}{dt}\Big]$$

for $0\,\le\,t\,\le\,T$, $0\,\le\,x\,\le\,l_x$, and $0\,\le\,y\,\le\,l_y$. Multiply the differential identity (1.23) by the eigenfunctions $\frac{2}{\sqrt{l_x\,l_y}}\,sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})$ and integrate both side with respect to x

and y over $\mathbb{D} = [0, l_x] \times [0, l_y]$ respectively. Thus, using orthonormality, we have

$$\frac{2}{\sqrt{l_x\, l_y}} \int_0^{l_y} \int_0^{l_x} u_{tt}\, sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})\, dx\, dy$$
$$= \frac{4}{l_x\, l_y} \sum_{n,m=1}^{\infty} \ddot{c}^{1,1}_{n,m} \int_0^{l_y} \int_0^{l_x} sin(\frac{n\pi\, x}{l_x})\, sin(\frac{k\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})\, sin(\frac{l\pi\, y}{l_y})\, dx\, dy$$
$$= \frac{4}{l_x\, l_y}\, \ddot{c}^{1,1}_{n,m} \int_0^{l_y} sin^2(\frac{m\pi\, y}{l_y})\, dy \int_0^{l_x} sin^2(\frac{n\pi\, x}{l_x})\, dx = \ddot{c}^{1,1}_{n,m}$$
$$= \Big[-\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{n^2}{l_x^2}) + a_1\; -\; a_2 \sum_{k,l=1}^{\infty} (c^{1,1}_{k,l})^2 \Big] c^{1,1}_{n,m}$$
$$-\; \kappa\, v^{1,1}_{n,m}\; +\; b_0\, \alpha^{1,1}_{n,m}\, \dot{W}^{1,1}_{n,m}.$$

The proof is complete. □

If we do the same procedure for the other cases, i.e., $sin(.)cos(.)$, $cos(.)sin(.)$, and $cos(.)cos(.)$, then we have similar scheme of equations for the solution. This means that

$$\ddot{c}^{i,j}_{n,m} \;=\; \Big[-\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{n^2}{l_x^2}) + a_1\; -\; a_2 \sum_{k,l=1}^{\infty} (c^{i,j}_{k,l})^2 \Big] c^{i,j}_{n,m}$$
$$-\, \kappa\, v^{i,j}_{n,m}\; +\; b_0\, \alpha^{i,j}_{n,m}\, \dot{W}^{i,j}_{n,m}.$$

Theorem 1.4.2. *Assume that* $\sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \big(\alpha^{i,j}_{n,m}\big)^2 < \infty$, $\forall\, u \in \mathbb{L}^2(\mathbb{D}) \cap C^1(\mathbb{D} \times \mathbb{R}^1_+)$ *with* u_x, u_y, *and* $u_t \in \mathbb{L}^2(\mathbb{D})$ *and* $W(x, y, t) = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \alpha^{i,j}_{n,m} W^{i,j}_{n,m}(t) e^i_n(x) e^j_m(y)$, *then for all* $t \geq 0$, $(x, y) \in \mathbb{D} = (0, l_x) \times (0, l_y)$, *the Fourier-series solutions(1.18) have Fourier coefficients* $c^{i,j}_{n,m}$ *satisfying (1.23)*

Proof. The main idea of the proof is based on 1) Multiply the SPDE by $e_k(x)\, e_l(y)$, 2) Integrate $\int_0^{l_y} \int_0^{l_x} (.) dxdy$ and 3) Simplify by orthogonality relations (1.19) for eigenfunctions $e^i_k(z) = \sqrt{\frac{2}{l_z}} sin(\frac{k\,\pi\, z}{l_z})$, or $\sqrt{\frac{2}{l_z}} cos(\frac{k\,\pi\, z}{l_z})$ where $k = n$ or m, $z = x$, or y, and $i = 1, 2$. Note we know that $||u||^2_{\mathbb{L}^2(\mathbb{D})} = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} [c^{i,j}_{n,m}]^2$, so we have

$$\sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \ddot{c}^{i,j}_{nm} \int_0^{l_y} \int_0^{l_x} (e^i_n(x))^2 \, (e^j_m(y)^2 \, dx \, dy$$
$$= \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \Big[\Big(- \sigma^2 \pi^2 \, (n^2/l_x^2 + m^2/l_y^2) + a_1 - a_2 \, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big) \, c^{i,j}_{n,m}(t)$$
$$- \kappa \, v^{i,j}_{n,m} + b_0 \, \alpha^{i,j}_{n,m} \frac{d\,W^{i,j}}{dt}\Big] \int_0^{l_y} (e^j_m(y)^2 \, dy \int_0^{l_x} (e^i_n(x))^2 \, dx.$$

Similarly, we have the equation

$$\ddot{c}^{i,j}_{n,m}(t) = \Big[- \sigma^2 \, \pi^2 (n^2/l_x^2 + m^2/l_y^2) + a_1 - a_2 \, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big] \, c^{i,j}_{n,m}(t)$$
$$- \kappa \, v^{i,j}_{n,m}(t) + b_0 \, \alpha^{i,j}_{n,m} \dot{W}^{i,j}_{n,m}(t) \qquad (1.24)$$

hence Theorem (1.4.2) is proven. □

1.5 STOCHASTIC NONLINEAR WAVE EQUATIONS WITH MULTIPLICATIVE NOISE

Consider the nonlinear stochastic wave equation with multiplicative noise

$$u_{tt} = \sigma^2 \, (u_{xx} + u_{yy}) + (a_1 - a_2 \, ||u||^2_{\mathbb{L}^2(\mathbb{D})}) u - \kappa \, v + b_1 \, ||u||_{\mathbb{L}^2(\mathbb{D})} \, \frac{dW}{dt} \qquad (1.25)$$

where

$$W(x, y, t) = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \alpha^{i,j}_{n,m} W^{i,j}_{n,m}(t) e^i_n(x) e^j_m(y)$$

and e^i_n and e^j_m are orthonormalized eigenfunctions of Laplace operator Δ on $\mathbb{D}$, driven by *i.i.d.* standard Wiener processes $W^{i,j}_{n,m}$ with $\mathbb{E}[W^{i,j}_{n,m}(t)] = 0$, $\mathbb{E}[W^{i,j}_{n,m}(t)]^2 = t$. Recall that $e^i_n(z)$ and $e^j_m(z)$ are one of the functions $\sqrt{\frac{2}{l_z}} sin(k \, \pi z/l_z)$ or $\sqrt{\frac{2}{l_z}} cos(k \, \pi z/l_z)$, with $z = x$ or y, $k = n$ or m, and $i, j = 1, 2$. If we follow the same procedure as in section

(1.5), we find that

$$\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}\, e_n^i(x) e_m^j(y)$$

$$= \sum_{n,m=1}^{\infty} \Big[-\,\sigma^2\, (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\, \pi^2 c_{n,m}(t) \;+\; (a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})})\, c^{i,j}_{n,m}(t)$$

$$-\;\kappa\, \dot{c}^{i,j}_{n,m}(t) \;+\; b_1\; ||u||_{\mathbb{L}^2(\mathbb{D})}\, \frac{d\,W^{i,j}}{dt} \Big] e_n^i(x) e_m^j(y)$$

and we know the solution is

$$u(x,y,t) \;=\; \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} c^{i,j}_{n,m}(t)\, e_n^i(x)\, e_m^j(y). \tag{1.26}$$

Multiply both sides by $e_k^i(x)\, e_l^j(y)$ and integrate with respect to x and y, and again by the orthogonality relation, we have

$$\sum_{n,m=1}^{\infty} \ddot{c}^{i,j}_{n,m}(t) \int_0^{l_y} \int_0^{l_x} (e_n^i(x))^2\, (e_m^j(y))^2\, dx\, dy$$

$$= \sum_{n,m=1}^{\infty} \Big[-\,\sigma^2 (\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\, \pi^2\, c^{i,j}_{n,m}(t) \;+\; (a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})})\, c^{i,j}_{n,m}(t) \;-\; \kappa\, \dot{c}^{i,j}_{n,m}(t)$$

$$+\, b_1\, ||u||_{\mathbb{L}^2(\mathbb{D})}\, \frac{d\,W^{i,j}(x,y)}{dt} \Big] \int_0^{l_y} (e_m^j(y))^2\, dy \int_0^{l_x} (e_n^i(x))^2\, dx.$$

Similarly, we have the equation

$$\ddot{c}^{i,j}_{n,m}(t) \;=\; \Big[-\,\sigma^2 \pi^2 (\tfrac{n^2}{l_x^2} + \tfrac{m^2}{l_y^2}) \;+\; a_1 \;-\; a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big]\, c^{i,j}_{n,m}(t)$$
$$-\,\kappa\, v_{n,m}(t) \;+\; b_1\, ||u||_{\mathbb{L}^2(\mathbb{D})}\; \alpha^{i,j}_{n,m}\, \dot{W}^{i,j}_{n,m}(t). \tag{1.27}$$

Theorem 1.5.1. *Assume that* $\sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \big(\alpha^{i,j}_{n,m}\big)^2 \,<\, \infty$, *and* $u,\, u_t,\, u_x,$ *and* $u_y \,\in\, \mathbb{L}^2(\mathbb{D})$. *The Fourier coefficients of*

$$u(x,y,t) = \sum_{n,m=1}^{\infty} c^{1,1}_{n,m} sin(\frac{n\pi x}{l_x}) sin(\frac{m\pi y}{l_y}) \tag{1.28}$$

where $0 \le\, x \,\le\, l_x$ *and* $0 \le\, y \,\le\, l_y$ *satisfy (*$\mathbb{P}$*-a.s) the infinite-dimensional system of* *ordinary SDEs*

$$\frac{d^2}{dt^2}c^{1,1}_{n,m}(t) \;=\; \Big[-\,\sigma^2\,\pi^2(\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2})\,+\,a_1\,-\,a_2\sum_{k,l=1}^{\infty}(c^{1,1}_{k,l})^2\Big]c^{1,1}_{n,m}$$
$$-\,\kappa\,v^{1,1}_{n,m}\,+\,b_1\,||u||_{\mathbb{L}^2(\mathbb{D})}\,\alpha^{1,1}_{n,m}\frac{dW^{1,1}_{n,m}}{dt}. \tag{1.29}$$

Proof. Substitute equation (1.28) into equation (1.25), and we have

$$u_{tt} = \sum_{n,m=1}^{\infty}\frac{2}{\sqrt{l_x\,l_y}}sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})\,\ddot{c}^{1,1}_{n,m}(t) = \sum_{n,m=1}^{\infty}\frac{2}{\sqrt{l_x\,l_y}}\,sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})$$
$$\cdot\,\Big[\Big(-\,\sigma^2\pi^2(\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2})+a_1\,-\,a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big)c^{1,1}_{n,m}\,-\,\kappa\,v_{n,m}\,+\,b_1\,||u||_{\mathbb{L}^2(\mathbb{D})}\,\alpha^{1,1}_{n,m}\frac{dW^{1,1}_{n,m}}{dt}\Big] \tag{1.30}$$

which implies that

$$u_{tt} = \sum_{n,m=1}^{\infty}sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})\Big[\Big(-\,\sigma^2\,\pi^2\,(\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2})+a_1\,-a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big)c_{n,m}$$
$$-\,\kappa\,v_{n,m}\,+\,b_1\,||u||_{\mathbb{L}^2(\mathbb{D})}\,\alpha_{n,m}\frac{dW^{1,1}_{n,m}}{dt}\Big]$$

for $0\,\le\,t\,\le\,T$, $0\,\le\,x\,\le\,l_x$, and $0\,\le\,y\,\le\,l_y$. Multiply the differential identity (1.30) by the eigenfunctions $\frac{2}{\sqrt{l_x\,l_y}}sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})$, and integrate both side with respect to x and y over $\mathbb{D} = [0, l_x] \times [0, l_y]$ respectively. Thus by using orthonormality, we have

$$\int_0^{l_y}\int_0^{l_x}u_{tt}\,sin(\frac{n\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})\,dx\,dy$$
$$= \frac{4}{l_x\,l_y}\sum_{n,m=1}^{\infty}\ddot{c}^{1,1}_{n,m}\int_0^{l_y}\int_0^{l_x}sin(\frac{n\pi\,x}{l_x})sin(\frac{k\pi\,x}{l_x})\,sin(\frac{m\pi\,y}{l_y})\,sin(\frac{l\pi\,y}{l_y})\,dx\,dy$$
$$= \frac{4}{l_x\,l_y}\ddot{c}^{1,1}_{n,m}\int_0^{l_y}sin^2(\frac{m\pi\,y}{l_y})\,dy\int_0^{l_x}sin^2(\frac{n\pi\,x}{l_x})\,dx\;=\;\ddot{c}^{1,1}_{n,m}$$
$$= \Big[-\,\sigma^2\pi^2(\frac{n^2}{l_x^2}+\frac{n^2}{l_x^2})+a_1\,-\,a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big]c^{1,1}_{n,m}$$
$$-\,\kappa\,v^{1,1}_{n,m}\,+\,b_1\,||u||_{\mathbb{L}^2(\mathbb{D})}\,\alpha^{1,1}_{n,m}\,\dot{W}^{1,1}_{n,m}.$$

The proof is complete. □

If we do the same procedure for the other cases, i.e., $sin(.)cos(.)$, $cos(.)sin(.)$, and $cos(.)cos(.)$, then we have similar scheme of equations for the solution. This means that

$$\begin{aligned}\ddot{c}^{i,j}_{n,m} &= \Big[-\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{n^2}{l_x^2}) + a_1 \, - \, a_2 \, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big] c^{i,j}_{n,m} \\ &\quad - \kappa \, v^{i,j}_{n,m} \, + \, b_1 \, ||u||_{\mathbb{L}^2(\mathbb{D})} \, \alpha^{i,j}_{n,m} \, \dot{W}^{i,j}_{n,m}. \end{aligned} \tag{1.31}$$

Theorem 1.5.2. *Assume that* $\sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \big(\alpha^{i,j}_{n,m}\big)^2 \, < \, \infty, \; \forall \, u \, \in \, \mathbb{L}^2(\mathbb{D}) \cap C^1(\mathbb{D} \times \mathbb{R}^1_+)$ *with* u_x, u_y, *and* $u_t \, \in \, \mathbb{L}^2(\mathbb{D})$ *and* $W(x,y,t) = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \alpha^{i,j}_{n,m} W^{i,j}_{n,m}(t) e^i_n(x) e^j_m(y)$, *then for all* $t \geq \, 0$, $(x,y) \in \mathbb{D} = (0,l_x) \times (0,l_y)$, *the Fourier-series solutions(1.26) have Fourier coefficients* $c^{i,j}_{n,m}$ *satisfying (1.31).*

Proof. The main idea of the proof is based on 1) Multiply the SPDE by $e^i_k(x) \, e^j_l(y)$, 2) Integrate $\int_0^{l_y} \int_0^{l_x} (.) dxdy$ and 3) Simplify by orthogonality relations (1.19) for eigenfunctions $e_k(.)$, where $k = n$ or m.

Note we know that $||u||^2_{\mathbb{L}^2(\mathbb{D})} = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} [c^{i,j}_{n,m}]^2$, so we have

$$\begin{aligned} &\sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \ddot{c}^{i,j}_{n,m} \int_0^{l_y} \int_0^{l_x} e^i_n(x) \tilde{e}^i_n(x) \, e^j_m(y) \tilde{e}^j_m(y) \, dx \, dy \\ &= \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \Big[\Big(- \, \sigma^2 \, \pi^2 \, (n^2/l_x^2 + m^2/l_y^2) \, + \, a_1 - a_2 \, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big) \, c^{i,j}_{n,m}(t) \\ &\quad - \, \kappa \, v^{i,j}_{n,m} \, + \, b_1 \, ||u||_{\mathbb{L}^2(\mathbb{D})} \, \alpha^{i,j}_{n,m} \frac{d \, W^{i,j}}{dt} \Big] \int_0^{l_y} (e^j_m(y))^2 \, dy \int_0^{l_x} (e^i_n(x))^2 \, dx. \end{aligned}$$

Integration using orthonormality leads to

$$\begin{aligned} \ddot{c}^{i,j}_{n,m}(t) &= \Big[- \sigma^2 \, \pi^2 (n^2/l_x^2 + m^2/l_y^2) + \, a_1 - \, a_2 \, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big] \, c^{i,j}_{n,m}(t) \\ &\quad - \kappa \, v^{i,j}_{n,m}(t) \, + \, b_1 \, ||u||_{\mathbb{L}^2(\mathbb{D})} \, \alpha^{i,j}_{n,m} \dot{W}^{i,j}_{n,m}(t) \end{aligned} \tag{1.32}$$

hence Theorem (1.5.2) is proven. □

1.6 STOCHASTIC NONLINEAR WAVE EQUATIONS WITH ADDITIVE AND MULTIPLICATIVE NOISE

Consider the nonlinear stochastic wave equation with noise

$$u_{tt} = \sigma^2(u_{xx} + u_{yy}) + (a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})})u \;-\; \kappa\, v + (b_0 + b_1\,||u||_{\mathbb{L}^2(\mathbb{D})})\,\frac{dW}{dt} \tag{1.33}$$

where

$$W(x,y,t) = \sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty} \alpha^{i,j}_{n,m} W^{i,j}_{n,m}(t) e^i_n(x) e^j_m(y)$$

and e_n and e_m are orthonormalized eigenfunctions of Laplace operator Δ on $\mathbb{D}$, driven by $i.i.d.$ standard Wiener processes $W^{i,j}_{n,m}$ with $\mathbb{E}[W^{i,j}_{n,m}(t)] = 0$, $\mathbb{E}[W^{i,j}_{n,m}(t)]^2 = t$. Recall that $e^i_n(z)$ and $e^j_m(z)$ are one of the functions $\sqrt{\frac{2}{l_z}} sin(k\pi z/l_z)$ or $\sqrt{\frac{2}{l_z}} cos(k\pi z/l_z)$, with $z = x$ or y, $k = n$ or m, and $i, j = 1, 2$.

Again, following the same procedure as in the previous sections give us that

$$\begin{aligned}
&\sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}\, e^i_n(x) e^j_m(y) \\
&= \sum_{n,m=1}^{\infty} \Big[-\,\sigma^2\,(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\,\pi^2 c^{i,j}_{n,m}(t) + \,(a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})})\,c^{i,j}_{n,m}(t) \;-\; \kappa\,\dot{c}^{i,j}_{n,m}(t) \\
&\quad + \,(b_0 + b_1)\,||u||_{\mathbb{L}^2(\mathbb{D})}\,\frac{d\,W^{i,j}}{dt}\Big] \cdot\, e^i_n(x)\, e^j_m(y)
\end{aligned}$$

and we know the solution is

$$u(x,y,t) \;=\; \sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty} c^{i,j}_{n,m}(t)\, e^i_n(x)\, e^j_m(y). \tag{1.34}$$

Multiply both sides by $e^i_k(x)\, e^j_l(y)$ and integrate with respect to x and y, and by the orthogonality relation, we have

$$\begin{aligned}
&\sum_{n,m=1}^{\infty} \ddot{c}_{n,m}(t) \int_0^{l_y}\!\int_0^{l_x} (e^i_n(x))^2\,(e^j_m(y))^2\,dx\,dy \\
&= \sum_{n,m=1}^{\infty} \Big[-\,\sigma^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})\,\pi^2\, c^{i,j}_{n,m}(t) \,+\, (a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})})\,c^{i,j}_{n,m}(t) \;-\; \kappa\,\dot{c}^{i,j}_{n,m}(t) \\
&\quad + (b_0 + b_1\,||u||_{\mathbb{L}^2(\mathbb{D})})\frac{d\,W^{i,j}(x,y)}{dt}\Big] \int_0^{l_y} (e^j_m(y))^2\,dy \int_0^{l_x} (e^i_n(x))^2\,dx.
\end{aligned}$$

Similarly, we have the equation

$$\ddot{c}^{i,j}_{n,m}(t) \;=\; \Big[-\,\sigma^2\pi^2(\tfrac{n^2}{l_x^2} + \tfrac{m^2}{l_y^2}) \,+\, a_1 \,-\, a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big]\, c^{i,j}_{n,m}(t)$$
$$-\,\kappa\, v_{n,m}(t) \,+\, (b_0 + b_1\,||u||_{\mathbb{L}^2(\mathbb{D})})\,\alpha^{i,j}_{n,m}\,\dot{W}^{i,j}_{n,m}(t). \tag{1.35}$$

Theorem 1.6.1. *Assume that* $\sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty}\big(\alpha^{i,j}_{n,m}\big)^2 < \infty,$ *and* $u,\, u_t,\, u_x,$ *and* $u_y \in \mathbb{L}^2(\mathbb{D})$*. The Fourier coefficients of*

$$u(x,y,t) = \sum_{n,m=1}^{\infty} c^{1,1}_{n,m} sin(\frac{n\pi x}{l_x}) sin(\frac{m\pi y}{l_y}) \tag{1.36}$$

where $0 \le x \le l_x$ *and* $0 \le y \le l_y$ *satisfy (*$\mathbb{P}$*-a.s) the infinite-dimensional system of ordinary SDEs*

$$\frac{d^2}{dt^2}c^{1,1}_{n,m}(t) \;=\; \Big[-\,\sigma^2\,\pi^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) \,+\, a_1 \,-\, a_2 \sum_{k,l=1}^{\infty}(c^{1,1}_{k,l})^2\Big] c^{1,1}_{n,m}$$
$$-\,\kappa\, v^{1,1}_{n,m} \,+\, (b_0 + b_1\,||u||_{\mathbb{L}^2(\mathbb{D})})\,\alpha^{1,1}_{n,m}\frac{dW^{1,1}_{n,m}}{dt}. \tag{1.37}$$

Proof. Substitute equation (1.37) into equation (1.33), and we have

$$u_{tt} = \sum_{n,m=1}^{\infty} \frac{2}{\sqrt{l_x\, l_y}} sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})\, \ddot{c}^{1,1}_{n,m}(t)$$
$$= \sum_{n,m=1}^{\infty} \frac{2}{\sqrt{l_x\, l_y}} sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})\Big[\Big(-\,\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2})$$
$$+\, a_1 \,-\, a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big) c^{1,1}_{n,m} \,-\, \kappa\, v_{n,m} \,+\, (b_0 + b_1\,||u||_{\mathbb{L}^2(\mathbb{D})})\,\alpha^{1,1}_{n,m}\frac{dW^{1,1}_{n,m}}{dt}\Big] \tag{1.38}$$

which implies that

$$u_{tt} = \sum_{n,m=1}^{\infty} sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})\Big[\Big(-\,\sigma^2\,\pi^2\,(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) + a_1 \,-\, a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big) c_{n,m}$$
$$-\,\kappa\, v_{n,m} \,+\, (b_0 + b_1\,||u||_{\mathbb{L}^2(\mathbb{D})})\,\alpha_{n,m}\frac{dW^{1,1}_{n,m}}{dt}\Big]$$

for $0 \le t \le T$, $0 \le x \le l_x$, and $0 \le y \le l_y$. Multiply the differential identity (1.38) by the eigenfunctions $\frac{2}{\sqrt{l_x\, l_y}} sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})$, and integrate both side with respect to x

and y over $\mathbb{D} = [0, l_x] \times [0, l_y]$ respectively. Thus, by using orthonormality, we have

$$
\begin{aligned}
&\frac{2}{\sqrt{l_x\, l_y}} \int_0^{l_y} \int_0^{l_x} u_{tt}\, sin(\frac{n\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})\, dx\, dy \\
&= \frac{4}{l_x\, l_y} \sum_{n,m=1}^{\infty} \ddot{c}^{1,1}_{n,m} \int_0^{l_y} \int_0^{l_x} sin(\frac{n\pi\, x}{l_x})\, sin(\frac{k\pi\, x}{l_x})\, sin(\frac{m\pi\, y}{l_y})\, sin(\frac{l\pi\, y}{l_y})\, dx\, dy \\
&= \frac{4}{l_x\, l_y} \ddot{c}^{1,1}_{n,m} \int_0^{l_y} sin^2(\frac{m\pi\, y}{l_y})\, dy \int_0^{l_x} sin^2(\frac{n\pi\, x}{l_x})\, dx \\
&= \ddot{c}^{1,1}_{n,m} \\
&= \Big[-\sigma^2 \pi^2 (\frac{n^2}{l_x^2} + \frac{n^2}{l_x^2}) + a_1 \,-\, a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big] c^{1,1}_{n,m} \\
&\quad -\, \kappa\, v^{1,1}_{n,m} \,+\, (b_0 + b_1\, ||u||_{\mathbb{L}^2(\mathbb{D})})\, \alpha^{1,1}_{n,m}\, \dot{W}^{1,1}_{n,m}.
\end{aligned}
$$

The proof is complete. □

If we do the same procedure for the other cases, i.e., $sin(.)cos(.)$, $cos(.)sin(.)$, and $cos(.)cos(.)$, then we have similar scheme of equations for the solution. This means that

$$
\begin{aligned}
\ddot{c}^{i,j}_{n,m} \;&=\; \Big[-\sigma^2 \pi^2 (\frac{n^2}{l_x^2} + \frac{n^2}{l_x^2}) + a_1 \,-\, a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big] c^{i,j}_{n,m} \\
&\quad -\, \kappa\, v^{i,j}_{n,m} \,+\, (b_0 + b_1\, ||u||_{\mathbb{L}^2(\mathbb{D})})\, \alpha^{i,j}_{n,m}\, \dot{W}^{i,j}_{n,m}.
\end{aligned} \tag{1.39}
$$

Theorem 1.6.2. *Assume that* $\sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \big(\alpha^{i,j}_{n,m}\big)^2 \,<\, \infty,\ \forall\, u \,\in\, \mathbb{L}^2(\mathbb{D}) \cap C^1(\mathbb{D} \times \mathbb{R}^1_+)$ *with* u_x, u_y, *and* $u_t \,\in\, \mathbb{L}^2(\mathbb{D})$ *and* $W(x, y, t) = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \alpha^{i,j}_{n,m} W^{i,j}_{n,m}(t) e^i_n(x) e^j_m(y)$, *then for all* $t \geq\, 0$, $(x, y) \in \mathbb{D} = (0, l_x) \times (0, l_y)$, *the Fourier-series solutions(1.34) have Fourier coefficients* $c^{i,j}_{n,m}$ *satisfying (1.39).*

Proof. The main idea of the proof is based on 1) Multiply the SPDE by $e^i_k(x)\, e^j_l(y)$, 2) Integrate $\int_0^{l_y} \int_0^{l_x} (.) dx dy$ and 3) Simplify by orthogonality relations (1.19) for eigenfunctions $e_k(.)$, where $k = n$ or m.

Note we know that $||u||^2_{\mathbb{L}^2(\mathbb{D})} = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} [c^{i,j}_{n,m}]^2$, so we have

$$\sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \ddot{c}^{i,j}_{n,m} \int_0^{l_y} \int_0^{l_x} e^i_n(x)\tilde{e}^i_n(x)\, e^j_m(y)\tilde{e}^j_m(y)\, dx\, dy$$

$$= \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \Big[\Big(-\, \sigma^2\, \pi^2\, (n^2/l_x^2 + m^2/l_y^2) \;+\; a_1 - a_2\; ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big)\, c^{i,j}_{n,m}(t)$$

$$-\; \kappa\, v^{i,j}_{n,m} \;+\; b_1\, ||u||_{\mathbb{L}^2(\mathbb{D})}\; \alpha^{i,j}_{n,m} \frac{d\, W^{i,j}_{n,m}}{dt} \Big] \int_0^{l_y} e^j_m(y)\tilde{e}^j_m(y)\, dy \int_0^{l_x} e^i_n(x)\tilde{e}^i_n(x)\, dx.$$

Integration using orthonormality reduces to

$$\begin{aligned} \ddot{c}^{i,j}_{n,m}(t) \;&=\; \Big[-\, \sigma^2\, \pi^2 (n^2/l_x^2 + m^2/l_y^2) +\; a_1 -\; a_2\; ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big]\, c^{i,j}_{n,m}(t) \\ &\qquad -\, \kappa\, v^{i,j}_{n,m}(t) \;+\; b_1\, ||u||_{\mathbb{L}^2(\mathbb{D})}\; \alpha^{i,j}_{n,m} \dot{W}^{i,j}_{n,m}(t) \end{aligned} \tag{1.40}$$

hence Theorem (1.6.2) is proven. □

CHAPTER 2

EXISTENCE AND UNIQUENESS OF TRUNCATED SYSTEM

2.1 EXISTENCE AND UNIQUENESS OF TRUNCATED FOURIER SERIES

We need to truncate the infinite series (1.14) for practical computations. So, we have to consider finite-dimensional truncations of the form

$$u_N(x,y,t) = \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} c_{n,m}^{i,j} e_n^i(x) e_m^j(y) \tag{2.1}$$

with Fourier coefficients $c_{n,m}^{i,j}$ satisfying the naturally truncated system of stochastic differential equations (from now on we use the abbreviation SDEs for wording stochastic differential equations.)

$$\begin{aligned} \frac{d^2\, c_{n,m}^{i,j}}{dt^2} &= \Big[-\sigma^2(\lambda_n+\beta_m) + a_1 - a_2 \sum_{i,j=1}^{2} \sum_{k,l=1}^{N} [c_{k,l}^{i,j}]^2 \Big] c_{n,m}^{i,j} \\ &\quad - \kappa\, v_{n,m}^{i,j} + \Big(b_0 + b_1 \sqrt{\sum_{i,j=1}^{2} \sum_{k,l=1}^{N} [c_{k,l}^{i,j}]^2} \Big) \alpha_{n,m}^{i,j} \dot{W}_{n,m}^{i,j} \end{aligned} \tag{2.2}$$

where $\lambda_n = (\frac{n\pi}{l_x})^2$ and $\beta_m = (\frac{m\pi}{l_y})^2$. Now, we will discuss a mathematical justification of this truncation procedure following ideas of Schurz [14]. Note that equation (2.2) is equivalent to the system (for $1 \le n,m \le N$, and $i,j = 1,2$.)

$$\dot{c}_{n,m}^{i,j} = v_{n,m}^{i,j} \tag{2.3}$$

$$\dot{v}_{n,m}^{i,j} = \Big[-\sigma^2(\lambda_n+\beta_m) + a_1 - a_2 \sum_{i,j}^{2} \sum_{k,l=1}^{N} [c_{k,l}^{i,j}]^2 \Big] c_{n,m}^{i,j} \tag{2.4}$$

$$- \kappa\, v_{n,m}^{i,j} + \Big(b_0 + b_1 \sqrt{\sum_{i,j}^{2} \sum_{k,l=1}^{N} [c_{k,l}^{i,j}]^2} \Big) \alpha_{n,m}^{i,j} \dot{W}_{n,m}^{i,j} \tag{2.5}$$

subject to the initial conditions $c^{i,j}_{n,m}(0) = c^{i,j}_{n,m,0}, v^{i,j}_{n,m}(0) = v^{i,j}_{n,m,0}, n, m \geq 1$, where $(c_{n,m},\, v_{n,m}) \in \mathbb{L}^2_N$ such that

$$\mathbb{L}^2_N \;=\; \{(c,\, v) \;\in\; l^2_{2N}|\mathbb{E}|c|^2_{l^2_N} + \mathbb{E}|v|^2_{l^2_N} \;<\; \infty\, \text{and}\, c,\, v\, \text{are independent of}\, \mathcal{F}_t = \sigma(W)\}.$$

Assume that $\sigma^2\pi^2\left(\frac{1}{l_x} + \frac{1}{l_y}\right)^2 > a_1$. Define the Lyapunov functional V_N as follows

$$\begin{aligned} V_N((c^{i,j}_{n,m}, v^{i,j}_{n,m})_{n,m=1,\dots,N}) \;&=\; \sum_{i,j}^{2}\sum_{n,m=1}^{N}\Big((v^{i,j}_{n,m})^2 + [\sigma^2(\lambda_n + \beta_m) - a_1](c^{i,j}_{n,m})^2\Big) \\ &+ \frac{a_2}{2}\Big(\sum_{i,j}^{2}\sum_{n,m=1}^{N}(c^{i,j})^2_{n,m}\Big)^2 \end{aligned} \tag{2.6}$$

for $N \in \mathbb{N}$. This functional is a modification of a functional appeared in Schurz[14].

It is clear that this function is of Lyapunov-type because it is nonnegative and smooth as large as $(a_2 \geq 0)$, radially unbounded if additionally $\sigma^2\,\pi^2\,(l_x^2 + l_y^2) \;>\; a_1\, l_x^2\, l_y^2$. To see that, let

$$(c^{i,j}, v^{i,j}) \;=\; (c^{i,j}_{1,1}, c^{i,j}_{1,2}, \dots, c^{i,j}_{N,N}, v^{i,j}_{1,1}, v^{i,j}_{1,2}, \dots, v^{i,j}_{N,N})$$

and

$$l^2_N := \{(c^{i,j}_{1,1}, \dots, c^{i,j}_{N,N}, v^{i,j}_{1,1}, \dots, v^{i,j}_{N,N}) : c^{i,j}, v^{i,j} \in \mathbb{R}, i, j = 1, 2\}$$

equipped with Euclidean norm

$$|(c, v)|_{l^2_N} = \sqrt{\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}\big((c^{i,j}_{n,m})^2 + (v^{i,j}_{n,m})^2\big)}.$$

Lemma 2.1.1. *Consider the Lyapunov functional defined in equation (2.4), and Let*

$$\sigma^2\pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2}) - a_1 = \frac{\sigma^2\pi^2(l_x^2 + l_y^2) - a_1\, l_x^2\, l_y^2}{l_x^2 l_y^2} =: \gamma.$$

Then 1) $\forall\, u \;\in\; \mathbb{L}^2(\mathbb{D})$:

$$V_N(u) \;\geq\; \gamma\; ||u||^2_{\mathbb{L}^2(\mathbb{D})} \tag{2.7}$$

2) $\forall\, u,\, \dot{u} \;\in\; \mathbb{L}^2(\mathbb{D})$:

$$V_N(u) \;\geq\; ||\dot{u}||^2_{\mathbb{L}^2(\mathbb{D})} \tag{2.8}$$

and 3) $\forall\, u \in \mathbb{L}^2(\mathbb{D})$ *with* $\dot{u} \in \mathbb{L}^2(\mathbb{D})$ *:*

$$V_N(u) \geq \min\{1,\, \gamma\} \left(||u||^2_{\mathbb{L}^2(\mathbb{D})} + ||\dot{u}||^2_{\mathbb{L}^2(\mathbb{D})} \right). \tag{2.9}$$

Proof. We know that

$$\begin{aligned}
&\min\{1, \sigma^2\pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2}) - a_1\}\ ||u||^2_{\mathbb{L}^2(\mathbb{D})} + \frac{a_2}{2}\, ||u||^4_{\mathbb{L}^2(\mathbb{D})} \\
&\leq\ V_N(u)(t) = c_0 + \sum_{n,m=1}^{N} \left[v^2_{n,m}\ +\ \left(\sigma^2(\lambda_n + \beta_m) - a_1 \right) c^2_{n,m} \right] + \frac{a_2}{2}\, ||u||^4_{\mathbb{L}^2(\mathbb{D})}\,,
\end{aligned}$$

then

$$\begin{aligned}
&\sum_{n,m=1}^{N} \left(v^2_{n,m} + (\sigma^2(\lambda_n + \beta_m) - a_1) c^2_{n,m} \right) \\
&\geq \sum_{n,m=1}^{N} v^2_{n,m} + \left(\sigma^2\pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2}) - a_1 \right) \sum_{n,m=1}^{N} c^2_{n,m} \\
&\geq\ \min\{1, \sigma^2\pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2}) - a_1\} \Big(\sum_{n,m=1}^{N} v^2_{n,m} + \sum_{n,m=1}^{N} c^2_{n,m} \Big) \\
&=\ \min\{1, \sigma^2\pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2}) - a_1\} \Big(\, ||\dot{u}_N||^2_{\mathbb{L}^2(\mathbb{D})} + ||u_N||^2_{\mathbb{L}^2(\mathbb{D})} \Big).
\end{aligned}$$

Let

$$\sigma^2\pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2}) - a_1 = \frac{\sigma^2\pi^2(l_x^2 + l_y^2) - a_1\, l_x^2\, l_y^2}{l_x^2 l_y^2} =: \gamma.$$

Then inequality (2.7) is proved and it is clear that

$$V_N(u)\ \geq\ \left[(\sigma^2\pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2})) - a_1 \right] ||u||^2_{\mathbb{L}^2} = \frac{\sigma^2\pi^2(l_x^2 + l_y^2) - a_1\, l_x^2\, l_y^2}{l_x^2 l_y^2}\, ||u||^2_{\mathbb{L}^2}\,.$$

Then

$$V_N(u)\ \geq\ \gamma\ ||u||^2_{\mathbb{L}^2}\,.$$

So inequality (2.5) is proved and we may use the same argument for inequality (2.6). The lemma is proved. □

Lemma 2.1.2. *Assume that* $a_2\ \geq\ 0$. Then, $\forall\, N \in \mathbb{N}$, *the functional* V_N *is* *(a)* nonnegative and positive semi-definite *on* l_N^2 *if* $(\sigma^2\pi^2(l_x^2 + l_y^2) \geq a_1 l_x^2 l_y^2)$ *or* $a_2 \geq 0$,

(b) positive-definite *on* l_N^2 *if* $(\sigma^2\pi^2(l_x^2 + l_y^2) > a_1 l_x^2 l_y^2)$,

and

(c) satisfies the condition of radial unboundedness

$$\lim_{|(c,v)|_{l_N^2} \to +\infty} V_N(c, v) = +\infty,$$

if $[\sigma^2\pi^2(l_x^2 + l_y^2) \; - \; a_1 \, l_x^2 \, l_y^2]_+ + a_2 \; > \; 0$.

Proof. (a) It is clear that $(v_{n,m}^{i,j})^2 \geq 0$, and $[\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) - a_1](c_{n,m}^{i,j})^2 \geq 0$, therefore $V_N \geq 0$. And thus it is positive semi-definite.

(b) From the definition of V_N, and using equation (2.5) which says that $V_N(u) \; \geq \; \gamma \; ||u||_{\mathbb{L}^2}^2, V_N$ is *indeed* positive-definite.

$$(c) \lim_{|(c,v)|_{l_N^2} \to +\infty} V_N(c^{i,j}, v^{i,j})$$

$$= \lim_{|(c,v)|_{l_N^2} \to +\infty} \Big(\sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \Big[(v_{n,m}^{i,j})^2 + \Big(\sigma^2\pi^2(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}) - a_1\Big)(c_{n,m}^{i,j})^2 \Big]$$

$$+ \frac{a_2}{2}(\sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (c_{n,m}^{i,j})^2)^2 \Big).$$

This implies that

$$\lim_{|(c,v)|_{l_N^2} \to +\infty} V_N(c^{i,j}, v^{i,j})$$

$$\geq \; min(1, \sigma^2 \, \pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2}) - a_1) \lim_{|(c,v)|_{l_N^2} \to +\infty} \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \Big[(v_{n,m}^{i,j})^2 + (c_{n,m}^{i,j})^2 \Big]$$

$$+ \lim_{|(c,v)|_{l_N^2} \to +\infty} \frac{a_2}{2} \left(\sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (c_{n,m}^{i,j})^2 \right)^2$$

$$= \; \infty.$$

□

2.2 EXISTENCE/UNIQUENESS OF STRONG SOLUTION OF TRUNCATED SYSTEM (2.3)

In this section,we will establish existence of unique strong solutions of the truncated Fourier coefficient with uniformly bounded expected energy in time t.

Theorem 2.2.1. *Assume that* $\mathbb{E}[V_N((c_{n,m}(0), v_{n,m}(0))_{1\le n,m\le N})] < +\infty$ *where the functional* V_N *is defined by (2.4), the initial values* $(c_{n,m}(0), v_{n,m}(0))_{1\le n,m\le N}$ *are independent of the* $\sigma-$*algebra*

$$\mathcal{F}_T^W = \sigma\{W_{n,m}^{i,j}(t) : 0 \le t \le T; 1 \le n,m \le N\}$$

and $a_2 \ge 0$, $[\sigma^2\pi^2(l_x^2 + l_y^2) - a_1 l_x^2 l_y^2]_+ + a_2 > 0$. *Then the truncated Fourier system (2.3) has an almost surely unique, Markovian, continuous strong solution* (c, v) *satisfying*

$$0 \le \mathbb{E}\Big[V_N((c_{n,m}(t), v_{n,m}(t))_{1\le n,m\le N})\Big] < +\infty. \tag{2.10}$$

Proof. Let

$$B_r = \{(c,v) \in \mathbb{R}^{2N\times 2N} : \sum_{n,m=1}^{2N} (c_{n,m}^2 + v_{n,m}^2) < r^2\}$$

denote the open ball with radius $r > 0$ in $\mathbb{R}^{2N\times N}$. Many authors proved that the local unique, Markovian, continuous solution exists up to the first exit from B_r (L. Arnold [3], and T.C. Gard [8]). Define the stopping time

$$\tau_r = \inf\{t \ge 0 : (c_{n,m}(t), v_{n,m}(t))_{1\le n,m\le N} \notin B_r\}.$$

Let $\tau_r(t) = min(t, \tau_r)$ for $t \ge 0$. Applying Dynkin's formula to our case tells us that

$$\mathbb{E}\Big[V_N\Big((c_{n,m}(\tau_r(t)), v_{n,m}(\tau_r(t)))_{1\le n,m\le N}\Big)\Big]$$
$$= \mathbb{E}\Big[V_N\Big((c_{n,m}(0), v_{n,m}(0))\Big)_{1\le n,m\le N}\Big] + \mathbb{E}\int_0^{\tau_r(t)} \mathcal{L}V_N\Big((c_{n,m}(s), v_{n,m}(s))_{1\le n,m\le N}\Big)\, ds,$$

where $\mathcal{L}$ is the infinitesimal generator of (c, v). The infinitesimal generator of the system (2.3) is defined by the expression

$$\begin{aligned}
\mathcal{L} &= \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} v_{n,m}^{i,j} \frac{\partial}{\partial c_{n,m}^{i,j}} \\
&+ \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \Big(\Big[-\sigma^2(\lambda_n + \beta_m) + a_1 - a_2 \sum_{r,s=1}^{2} \sum_{k,l=1}^{N} (c_{k,l}^{r,s})^2 \Big] c_{n,m}^{i,j} - \kappa\, v_{n,m}^{i,j} \Big) \frac{\partial}{\partial v_{n,m}^{i,j}} \\
&+ \frac{1}{2} \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \Big(b_0 + b_1 \sqrt{\sum_{r,s=1}^{2} \sum_{k,l=1}^{N} (c_{k,l}^{r,s})^2} \Big)^2 (\alpha_{n,m}^{i,j})^2 \frac{\partial^2}{\partial (v_{n,m}^{i,j})^2}
\end{aligned}$$

mapping from $\mathcal{C}^2$ to $\mathcal{C}^0$, where $\lambda_n = (n\pi/l_x)^2$ and $\beta_m = (m\pi/l_y)^2$. But

$$\begin{aligned}
\frac{\partial V_N}{\partial c_{n,m}^{i,j}} &= 2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \Big[\sigma^2(\lambda_n + \beta_m) - a_1 \Big] c_{n,m}^{i,j} + 2\, a_2 \Big(\sum_{r,s=1}^{2} \sum_{k,l=1}^{N} (c_{k,l}^{r,s})^2 \Big) \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} c_{n,m}^{i,j} \\
&= 2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \Big[\sigma^2(\lambda_n + \beta_m) + a_1 - a_2 \sum_{r,s=1}^{2} \sum_{k,l=1}^{N} (c_{k,l}^{r,s})^2 \Big] c_{n,m}^{i,j},
\end{aligned}$$

and $\dfrac{\partial V_N}{\partial v_{n,m}^{i,j}} = 2 v_{n,m}^{i,j}$, also we have $\dfrac{\partial^2 V_N}{\partial (v_{n,m}^{i,j})^2} = 2\, \delta_{n,m} \delta_{i,j}$. ($\delta-$Kronecker.) Then we find that

$$\begin{aligned}
\mathcal{L}V &= \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} v_{n,m}^{i,j} \Big[2 \Big(\sigma^2(\lambda_n + \beta_m) - a_1 + a_2 \sum_{r,s=1}^{2} \sum_{k,l=1}^{N} (c_{k,l}^{r,s})^2 \Big) \Big] c_{n,m}^{i,j} \\
&+ \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} 2 \Big[\Big(-\sigma^2(\lambda_n + \beta_m) + a_1 - a_2 \sum_{r,s=1}^{2} \sum_{k,l=1}^{N} (c_{k,l}^{r,s})^2 \Big) c_{n,m}^{i,j} v_{n,m}^{i,j} - \kappa\, (v_{n,m}^{i,j})^2 \Big] \\
&+ \frac{1}{2} \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \Big(b_0 + b_1 \|u\|_{\mathbb{L}^2(\mathbb{D})} \Big)^2 (\alpha_{n,m}^{i,j})^2.
\end{aligned}$$

This leads to

$$\mathcal{L}V = \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (b_0 + b_1 \|u\|_{\mathbb{L}^2(\mathbb{D})})^2 (\alpha_{n,m}^{i,j})^2 - \kappa \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2. \tag{2.11}$$

First, let $b_1 = 0$, then

$$\mathcal{L}V = b_0^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2 - \kappa \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2,$$

and we know that $u = (c_{n,m}, v_{n,m})$. Substituting the above computation into Dynkin's formula

$$\mathbb{E}[V_N(u(\tau_r(t)))] = \mathbb{E}[V_N(u(0))] + \mathbb{E}\int_0^{\tau_r(t)} \mathcal{L}V(u(s))ds \tag{2.12}$$

and using $\tau_r(t) \leq t$ ($\mathbb{P} - a.s.$) by definition, then we find that

$$\mathbb{E}[V_N(u(t))] = \mathbb{E}[V_N(u(0))] + \Big(b_0^2 \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2 \,-\, \kappa \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2\Big)\mathbb{E}[\tau_r(t)],$$

Since $\tau_r(t) \to t$ as $r \to +\infty$,

$$\mathbb{E}[V_N(u(t))] \leq\, \mathbb{E}[V_N(u(0))] \,+\, t\, b_0^2 \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2 \,<\, \infty. \tag{2.13}$$

Now, if $b_0 = 0$ and $b_1 \neq 0$, then

$$\mathcal{L}V = \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} b_1^2\, ||u||_{\mathbb{L}^2(\mathbb{D})}^2\, (\alpha_{n,m}^{i,j})^2 \,-\, \kappa \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2.$$

Applying Dynkin's formula, we have

$$\begin{aligned}
\mathbb{E}V_N(u(\tau_r(t))) &= \mathbb{E}V_N(u(0)) + \mathbb{E}\int_0^{\tau_r(t)} \Big(b_1^2 \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} ||u(.,.,s)||_{\mathbb{L}^2(\mathbb{D})}^2\, (\alpha_{n,m}^{i,j})^2 \\
&\quad -\, \kappa \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2\Big)\, d\, s \\
&\leq\, \mathbb{E}V[u(0)] + b_1^2 \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2 \mathbb{E}\int_0^{\tau_r(t)} ||u(.,.,s)||_{\mathbb{L}^2(\mathbb{D})}^2\, d\, s \\
&\leq\, \mathbb{E}V(u(0)) + \frac{b_1^2 \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2}{\gamma} \mathbb{E}\int_0^{\tau_r(t)} V(u(s))\, d\, s.
\end{aligned}$$

By using $\tau_r \leq t$ ($\mathbb{P} - a.s.$) and applying the Gronwall inequality, then we have

$$\mathbb{E}V_N(u(t)) \leq\, \mathbb{E}V(u(0)) \times\, exp\Big(\frac{b_1^2 \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2}{\gamma}\, t\Big). \tag{2.14}$$

Also, we know that from equation (2.5) that

$$\gamma\, ||u||_{\mathbb{L}^2}^2\, (\mathbb{D}) \leq V_N(u(t)),$$

therefore

$$0 \leq \mathbb{E}[\gamma \; ||u||^2_{\mathbb{L}^2} (\mathbb{D})] \leq \; \mathbb{E}V_N(u(t)) \leq \mathbb{E}V(u(0)) \times \; exp\Big(\frac{b_1^2 \sum\limits_{i,j=1}^{2} \sum\limits_{n,m=1}^{N} (\alpha^{i,j}_{n,m})^2}{\gamma} \, t \Big). \qquad (2.15)$$

Now, we study both cases, that is, additive and multiplicative noise,

$$\mathcal{L}V_N = \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (b_0 + b_1 \, ||u||_{\mathbb{L}^2(\mathbb{D})})^2 (\alpha^{i,j}_{n,m})^2 \; - \; \kappa \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (v^{i,j}_{n,m})^2.$$

Applying Dynkin's formula again, we have

$$\begin{aligned} \mathbb{E}V_N(u(\tau_r(t))) &= \mathbb{E}V_N(u(0)) + \mathbb{E} \int_0^{\tau_r(t)} \Big(\sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \Big(b_0 + b_1 \; ||u(s)||_{\mathbb{L}^2(\mathbb{D})} \Big)^2 (\alpha^{i,j}_{n,m})^2 \\ &\qquad - \; \kappa \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (v^{i,j}_{n,m})^2 \Big) \, d\, s \\ &\leq \; \mathbb{E}V_N(u(0)) + \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \mathbb{E} \int_0^{\tau_r(t)} \Big(2b_0^2 + 2b_1^2 \, ||u(s)||^2_{\mathbb{L}^2(\mathbb{D})} \Big) (\alpha^{i,j}_{n,m})^2 \, d\, s, \end{aligned}$$

thus

$$\begin{aligned} \mathbb{E}V_N(u(\tau_r(t))) \;\; &\leq \; \mathbb{E}V_N(u(0)) + 2\, b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} \mathbb{E} \int_0^{\tau_r(t)} \; ||u||^2_{\mathbb{L}^2(\mathbb{D})} \, (\alpha^{i,j}_{n,m})^2 ds \\ &\qquad + 2\, b_0^2 \, T \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha^{i,j}_{n,m})^2. \end{aligned} \qquad (2.16)$$

So, by using the Gronwall inequality and equation (2.5) and letting $r \; \rightarrow \; \infty$ gives $\tau_r(t) \; \rightarrow \; t \; (\mathbb{P} - a.s.)$, we have

$$\begin{aligned} \mathbb{E}V_N(u(t)) \;\; &\leq \; k \; + \; \frac{2\, b_1^2 \sum\limits_{i,j=1}^{2} \sum\limits_{n,m=1}^{N} (\alpha^{i,j}_{n,m})^2}{\gamma} \int_0^t \mathbb{E}V(u(s)) \, d\, s \\ &\leq \; k \; \cdot \; exp\Big(\frac{2\, b_1^2 \sum\limits_{i,j=1}^{2} \sum\limits_{n,m=1}^{N} (\alpha^{i,j}_{n,m})^2}{\gamma} \, t \Big) \end{aligned} \qquad (2.17)$$

where

$$k = \mathbb{E}V_N(u(0)) + \; 2\, b_0^2 \, T \; \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha^{i,j}_{n,m})^2.$$

On the other hand, if

$$[\sigma^2\,\pi^2\,(l_x^2+l_y^2)-a_1\,l_x^2\,l_y^2]_+ + a_2 \;>\; 0,$$

then we have

$$\mathbb{P}(\{\exists : 0\le s<t, (c_{n,m}(s), v_{n,m}(s))_{1\le n,m\le N \notin B_r}\})$$
$$\le \frac{\mathbb{E}V(u(0))\,exp(\frac{2b_1^2}{\gamma}\,t)+2\,b_0^2\,t\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}(\alpha_{n,m}^{i,j})^2}{r^2}.$$

If we take the limit $r\to\infty$, then we get to $\mathbb{P}(\{\tau<T\})=0$ for the first exist time τ of the process $\{(c_{n,m}(t), v_{n,m}(t))_{1\le n,m\le N}, t\ge 0\}$ from the open set $\mathbb{R}^{4N\times 4N}$. But we know that the Lyapunov-type functional V_N is radially unbounded and nonnegative as we proved in (Lemma 2.1.2) under

$$[\sigma^2\pi^2(l_x^2+l_y^2)-a_1 l_x^2 l_y^2]_+ + a_2 > 0, \text{ or } a_2 >\; 0.$$

Thus the local solution u_N with $||u||^2_{\mathbb{L}^2(\mathbb{D})} < \infty$ can never explode at finite terminal time T and the unique continuation to a global solution must exist. This concludes the proof of Theorem 2.2.1. □

2.3 TOTAL ENERGY OF TRUNCATED SYSTEM

Let $\sigma^2\pi^2\left(\frac{1}{l_x^2}+\frac{1}{l_y^2}\right)^2 >\; a_1$. The total expected energy is the average sum of kinetic, potential, and nonlinear restoring energy defined by

$$e(t) := \frac{1}{2}\mathbb{E}\left[||u_t||^2_{\mathbb{L}^2(\mathbb{D})}+\sigma^2\,||\nabla u||^2_{\mathbb{L}^2(\mathbb{D})}-a_1\,||u||^2_{\mathbb{L}^2(\mathbb{D})}+\frac{a_2}{2}\,||u||^4_{\mathbb{L}^2(\mathbb{D})}\right]. \qquad (2.18)$$

and therefore,

$$e_N(t) := \frac{1}{2}\mathbb{E}\left[||(u_t)_N||^2_{\mathbb{L}^2(\mathbb{D})}+\sigma^2\,||\nabla u_N||^2_{\mathbb{L}^2(\mathbb{D})}-a_1\,||u_N||^2_{\mathbb{L}^2(\mathbb{D})}+\frac{a_2}{2}\,||u_N||^4_{\mathbb{L}^2(\mathbb{D})}\right]. \qquad (2.19)$$

Theorem 2.3.1. *Assume that* $u(x,y,0)=f(x,y)$, *where* $f\in\mathbb{L}^2(\mathbb{D})$,
$u_t(x,y,0)=g(x,y)$, *where* $g\in\mathbb{L}^2(\mathbb{D})$, *and* $[\sigma^2\pi^2(l_x^2+l_y^2)_{\mathbb{R}^2}-l_x^2l_y^2a_1]_+ + a_2>0$, $a_2\ge 0$,

$u(.,.,t)|_{\partial\mathbb{D}} = 0$, *then* $\forall\ 0 \le t \le T$, $\forall\ u \in \mathbb{L}^2(\mathbb{D}\times[0,T])$ *with* $e(0) < \infty$, *the expected energy* $e_N(t)$ *satisfies*

$$e_N(t) \le \left[e_N(0) + 2b_0^2 T \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2 \right] \cdot exp\Big(\frac{2b_1^2}{\gamma} t \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2 \Big). \qquad (2.20)$$

Proof. From Theorem 2.2.1, we know that $\mathbb{E}[V_N(u(t))] < +\infty$ and from (2.4) we have

$$V_N(u(t)) = ||(u_t)_N||_{\mathbb{L}^2}^2 + \sigma^2\, ||\nabla u_N||_{\mathbb{L}^2}^2 - a_1\, ||u_N||_{\mathbb{L}^2}^2 + \frac{a_2}{2}\, ||u_N||_{\mathbb{L}^2}^4 . \qquad (2.21)$$

Therefore,

$$\begin{aligned} e_N(t) =&\ \frac{1}{2}\, \mathbb{E} V_N(u(t)) \\ =&\ \frac{1}{2}\mathbb{E}\Big[\, ||(u_N)_t||_{\mathbb{L}^2}^2 + \sigma^2\, ||\nabla u_N||_{\mathbb{L}^2}^2 - a_1\, ||u_N||_{\mathbb{L}^2}^2 + \frac{a_2}{2}\, ||u_N||_{\mathbb{L}^2}^4 \,\Big]. \end{aligned} \qquad (2.22)$$

Thus

$$e_N(t) = \frac{1}{2}\mathbb{E}\Big[\, ||(u_N)_t||_{\mathbb{L}^2}^2 + \sigma^2\, ||\nabla u_N||_{\mathbb{L}^2}^2 - a_1\, ||u_N||_{\mathbb{L}^2}^2 + \frac{a_2}{2}\, ||u_N||_{\mathbb{L}^2}^4 \,\Big]. \qquad (2.23)$$

If $b_1 = 0$ and $b_0 \neq 0$, then we have

$$e_N(t)\ = e_N(0) + \frac{1}{2}\, b_0^2\, t \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2.$$

In this case, the trace formula and conservation law of expected energy holds (c.f. Schurz [14]), i.e.,

$$e_N(t)\ = e_N(0) + \frac{b_0^2}{2} trace(Q)\, t\ = e_N(0) + \frac{1}{2}\, b_0^2\, t \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2. \qquad (2.24)$$

with $trace(Q)\ = \displaystyle\sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2$. Now, if $b_0 = 0$ and $b_1 \neq 0$, then

$$e_N(t) = \frac{1}{2}\mathbb{E} V_N(u(t)) \le \frac{1}{2}\mathbb{E} V_N(u(0)) exp\Big(\frac{b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2}{\gamma}\, t \Big).$$

Thus

$$e_N(t) \leq e_N(0)\Big[exp\Big(\frac{b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2}{\gamma}\, t\Big)\Big] \tag{2.25}$$

Finally, if $b_0 \neq 0$, $b_1 \neq 0$ (additive and multiplication noise), then

$$e_N(t) = \frac{1}{2}\mathbb{E}V_N(u(t)) \leq\; (e_N(0) + k) \;\cdot\; exp\Big(\frac{2b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2}{\gamma}\, t\Big)$$

where $k = 2\, b_0^2 T \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2$. Therefore,

$$e_N(t) \leq \Big[e_N(0) + b_0\, T\, \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2\Big] exp\Big(\frac{2b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2}{\gamma}\, t\Big). \tag{2.26}$$

□

CHAPTER 3

EXISTENCE AND UNIQUENESS OF ORIGINAL SYSTEM (2.3)

In chapter two, I discussed the existence and uniqueness of truncated system. Now I will state the existence and uniqueness of approximate strong solutions u of the original system (1.66). Define the Lyapunov Functional $V : H \to \mathbb{R}_+$ with $H = \mathbb{L}^2(\mathbb{D} \times [0,T])$ where

$$H = \{u \in \mathbb{L}^2(\mathbb{D}) | u_t \in \mathbb{L}^2, \text{ and } \nabla u \in \mathbb{L}^2\}$$

by

$$V(u(t)) = ||u_t||^2_{\mathbb{L}^2} + \sigma^2 ||\nabla u||^2_{\mathbb{L}^2} - a_1 ||u||^2_{\mathbb{L}^2} + \frac{a_2}{2} ||u||^4_{\mathbb{L}^2} . \tag{3.1}$$

Note that $V = \lim_{N\to\infty} V_N$, $||u(.,.,t)||_{\mathbb{L}^2(\mathbb{D})} = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (c^{i,j}_{n,m}(t))^2$, $||u_t(.,.,t)||_{\mathbb{L}^2(\mathbb{D})} = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (v^{i,j}_{n,m}(t))^2$, $|(l_x,l_y)|^2_{\mathbb{R}^2} = l_x^2 + l_y^2$ and $Area(\mathbb{D}) = l_x\, l_y$.

3.1 EXISTENCE AND UNIQUENESS OF ORIGINAL SYSTEM (2.2)

Theorem 3.1.1. *Assume that the initial values* $(c^{i,j}_{n,m}(0), v^{i,j}_{n,m}(0))_{n,m\in\mathbb{N}} \in l^2$ *are independent of the* $\sigma-$*algebra*

$$\mathcal{F}^W_T = \sigma\{W^{i,j}_{n,m}(t) : 0 \le t \le T; n,m \in \mathbb{N}\},$$

$V : \mathbb{L}^2(\mathbb{D}\times[0,T]) \longrightarrow \mathbb{R}^1_+$ *satisfies* $\mathbb{E}[V(u(0))] < +\infty$, *and*

$$u(x,y,0) = f(x,y) \in \mathbb{L}^2(\mathbb{D}), u_t(x,y,0) = g(x,y) \in \mathbb{L}^2(\mathbb{D}) \text{ (IC)}$$

$a_2 \ge 0$, $[\sigma^2\pi^2(l_x^2 + l_y^2) - a_1 l_x^2 l_y^2]_+ + a_2 > 0$, *and*

$$\forall\, 0 \le t \le T,\ u(.,.,t)|_{\partial\mathbb{D}} = 0 \text{ (BC)}.$$

Then there exists a unique $u \in \mathbb{L}^2(\mathbb{D}\times[0,T])(a.s.)$ *of the form*

$$u(x,y,t) = \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} c^{i,j}_{n,m}(t) e^i_n(x) e^j_m(y) \tag{3.2}$$

with u_t, $\nabla u \in \mathbb{L}^2(\mathbb{D}\times[0,T])$ *and* $\forall\ 0\le t\le T: V(u)(t) < +\infty(\mathbb{P}-a.s.)$,

$$\mathbb{E}[V(u(0))] \le \mathbb{E}[V(u(t))] \le\ k\ \times exp\Big(\frac{2b_1^2\sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty}(\alpha_{n,m}^{i,j})^2}{\gamma}t\Big) \tag{3.3}$$

where $k = \mathbb{E}V(u(0)) + 2b_0^2 T\sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty}(\alpha_{n,m}^{i,j})^2$, *i.e.,* $\mathbb{E}[V]$ *does not grow faster than an exponential function of* t *and its Fourier coefficient* $c_{n,m}^{i,j}\in C^1$ *are unique, Markovian, continuous strong solutions of (2.2).*

Proof. Note that the strong solution (c,v) of the infinite dimensional system of ordinary SDEs (2.2) exists, it is Markovian, almost surely unique and continuous on the set

$$H(r) = \{(c,v) : c, v \in C^0\big([0,T]\big), \forall t\in[0,T], ||(c,v)||_{l^2} < r\}.$$

Since the system has local Lipschitz-continuous drift and diffusion coefficients (see Da Prato and Zabczyk[6] and Grecksch and Tudor[9]), it suffices to show that $u \simeq (c,v)$ cannot explode as the radius r tends to infinity. To show that, we consider the truncated N-dimensional system with solution $(c^N, v^N)\in H_N(r)$ where

$$H_N(r) = \{(c,v)\in H(r) : \forall t\in[0,T], \forall n,m > N, c_{n,m} = v_{n,m} = 0\}.$$

It is clear that $H(r) = \lim_{N\to\infty} H_N(r)$ $(a.s.)$. Note that (c,v) and (c^N, v^N) coincide up to stopping time τ_r provided that $(c_{n,m}(0), v_{n,m}(0)) = (c_{n,m}^N(0), v_{n,m}^N(0)) \in H_N(r)$ for $i,j = 1,2,...,N$ and $(c_{n,m}(0), v_{n,m}(0)) = (0,0)$ for $n,m > N$. If we use the same procedure of Theorem 2.2.1, we have the same inequality for the functional $V = V_N\mid_{H_N}$. Now, taking the limit $N\to\infty$, one shows that the infinite-dimensional limit $(c,v)\in l^2$ cannot explode at a finite time $0\le t\le T$ due to the uniformity of all of our previous estimates, hence the global approximation strong solution of the original infinite-dimensional system (2.2) must exist. These solutions are unique, approximately Markovian and continuous on any finite, nonrandom interval $[0,T]$. Note that

$\lim_{N\to\infty}(c^N, v^N) = (c, v)$ $(\mathbb{P} - a.s)$ because V is increasing at N. For all non-random constant $K > 0$, we have

$$\mathbb{P}\Big(|V(c,v) - V_N(c^N, v^N)| \geq K\Big) \leq \frac{\mathbb{E}[|V(c,v) - V_N(c^N, v^N)|]}{K}$$
$$\leq \frac{\mathbb{E}[V(c,v)] + \mathbb{E}[V_N(c^N, v^N)]}{K} \times \mathbb{P}(\{\tau_r < T\}) \to 0.$$

By the Chebyshev-Markov and $(L^1 - L^\infty)$ Hölder inequality (see Shiryaev[17]), where

$$(c^N, v^N) \in H_N(r),\ (c_{i,j}(0), v_{i,j}(0)) = (c^N_{i,j}(0), v^N_{i,j}(0))\, for\, i,j = 1,2,...,N,$$

$$(c_{n,m}(0), v_{n,m}(0)) = (0,0), \text{for}\, n, m > N$$

as N and r tend to $+\infty$. Now, we use the fact that $\mathbb{E}[V(c,v)]$ and $\mathbb{E}[V_N(c^N, v^N)]$ are uniformly bounded with respect to r and N on any finite nonrandom time-interval $[0,T]$ by Theorem (2.2.1) under $\mathbb{L}^2$-regular noise with $trace(Q) = \sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty}(\alpha_{n,m})^2 < +\infty$. Moreover, convergence in probability along the nonnegative (also increasing in N if at least N large enough, $\sigma^2 > 0$ and $a_2 \geq 0$, $[\sigma^2\pi^2(l_x^2 + l_y^2) - a_1 l_x^2 l_y^2]_+ + a_2 > 0$) functional V_N implies convergence almost surely along the set $H_N(r)$ here. To prove this fact, we may use the Borel-Cantelli lemma as r and N tend to infinity, a probabilistic technique which is also known as the technique of fast $\mathbb{L}^p$-convergence since $\mathbb{P}(\{\tau_r < T\}) \leq \frac{C}{r^2}$ with an appropriate constant C is summable over integer-valued indexes r (see proof of Theorem (2.2.1). Thus, existence, uniqueness, Markovianity and continuity of truncated solutions carry over to strong solutions of original system (2.2) with representation $u(x,y,t) = \sum_{i,j=1}^{2}\sum_{n,m=1}^{\infty} c^{i,j}_{n,m}(t)e^i_n(x)e^j_m(y)$. Finally, (3.3) can be extracted from the proof of Theorem 2.2.1 by taking $N \to +\infty$. □

3.2 TOTAL ENERGY OF ORIGINAL SYSTEM (2.3)

The total expected energy is the averaged sum of kinetic, potential, and nonlinear restoring energy defined by

$$e(t) := \frac{1}{2}\mathbb{E}[||u_t||^2_{\mathbb{L}^2(\mathbb{D})} + \sigma^2\,||\nabla u||^2_{\mathbb{L}^2(\mathbb{D})} - a_1\,||u||^2_{\mathbb{L}^2(\mathbb{D})} + \frac{a_2}{2}\,||u||^4_{\mathbb{L}^2(\mathbb{D})}]. \tag{3.4}$$

Theorem 3.2.1. *Assume that* $u(x,y,0) = f(x,y) \in \mathbb{L}^2(\mathbb{D})$, $\sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} \infty\, (\alpha_{n,m}^{i,j})^2 < \infty$,
$\sigma^2\pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2}\right)^2 > a_1$, $u_t(x,y,0) = g(x,y) \in \mathbb{L}^2(\mathbb{D})$, *and*
$[\sigma^2\pi^2(l_x^2 + l_y^2)_{\mathbb{R}^2} - l_x^2 l_y^2 a_1]_+ + a_2 > 0$, $a_2 \geq 0$, $u(.,.,t)|_{\partial\mathbb{D}} = 0$, *then* $\forall\ 0 \leq t \leq T$,
$\forall u \in \mathbb{L}^2(\mathbb{D} \times [0,T])$ *with* $e(0) < \infty$, *the expected energy* $e(t)$ *is governed by*

$$e(t) \leq \left[e(0) + 2b_0^2 T \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2\right] \cdot exp\Big(\frac{2b_1^2}{\gamma} t \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2\Big). \tag{3.5}$$

Proof. From Theorem 3.1.1, we know that $\mathbb{E}[V_N(u(t))] < +\infty$ and from (3.4) we have

$$V(u(t)) = ||u_t||^2_{\mathbb{L}^2(\mathbb{D})} + \sigma^2 ||\nabla u||^2_{\mathbb{L}^2(\mathbb{D})} - a_1 ||u||^2_{\mathbb{L}^2(\mathbb{D})} + \frac{a_2}{2} ||u||^4_{\mathbb{L}^2(\mathbb{D})}. \tag{3.6}$$

Therefore,

$$e(t) = \frac{1}{2}\, \mathbb{E}V(u(t)) = \frac{1}{2}\mathbb{E}\Big[\, ||u_t||^2_{\mathbb{L}^2(\mathbb{D})} + \sigma^2 ||\nabla u||^2_{\mathbb{L}^2(\mathbb{D})} - a_1 ||u||^2_{\mathbb{L}^2(\mathbb{D})} + \frac{a_2}{2} ||u||^4_{\mathbb{L}^2(\mathbb{D})} \Big]$$

Thus

$$e(t) = \frac{1}{2}\mathbb{E}[||u_t||^2_{\mathbb{L}^2(\mathbb{D})} + \sigma^2 ||\nabla u||^2_{\mathbb{L}^2(\mathbb{D})} - a_1 ||u||^2_{\mathbb{L}^2(\mathbb{D})} + \frac{a_2}{2} ||u||^4_{\mathbb{L}^2(\mathbb{D})}]. \tag{3.7}$$

If $b_1 = 0$ and $b_0 \neq 0$, then we have

$$e(t) = e(0) + \frac{1}{2} b_0^2 t \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2.$$

In this case, the trace formula and conservation law of expected energy holds (as noted by Schurz [14]), i.e.,

$$e(t) = e(0) + \frac{b_0^2}{2}\, trace(Q)\, t = e(0) + \frac{1}{2}\, b_0^2\, t \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2. \tag{3.8}$$

with $trace(Q) = \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2$. Now, if $b_0 = 0$ and $b_1 \neq 0$, then

$$e(t) = \frac{1}{2}\mathbb{E}V(u(t)) \leq e(0)\, exp\Big(\frac{b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2}{\gamma} t\Big).$$

Thus

$$e(t) \leq e(0) \left[exp\Big(\frac{b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2}{\gamma} t \Big) \right]. \tag{3.9}$$

Finally, if $b_0 \neq 0$, $b_1 \neq 0$ (additive and multiplication noise), then

$$e(t) = \frac{1}{2} \mathbb{E} V_N(u(t)) \leq \; (e(0) + c) \; \times exp\Big(\frac{2b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2}{\gamma} \, t \Big),$$

where

$$c = 2 \, b_0^2 \, T \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2.$$

Therefore,

$$e(t) \leq \Big[e(0) + b_0^2 \, T \; \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2 \Big] exp\Big(\frac{2b_1^2 \sum_{i,j=1}^{2} \sum_{n,m=1}^{\infty} (\alpha_{n,m}^{i,j})^2}{\gamma} \, t \Big). \tag{3.10}$$

□

CHAPTER 4

STABILITY OF N-DIMENSIONAL TRUNCATION AND CONCLUSIONS

Recall the system (2.3) which is

$$\begin{aligned} \dot{c}^{i,j}_{n,m} &= v^{i,j}_{n,m} \\ \dot{v}^{i,j}_{n,m} &= \Big[-\sigma^2(\lambda_n + \beta_m) + a_1 - a_2\,||u_N||^2_{\mathbb{L}^2(\mathbb{D})} \Big] c^{i,j}_{n,m} \;-\; \kappa\, v^{i,j}_{n,m} \\ &\quad + \Big(b_0 + b_1\,||u_N||_{\mathbb{L}^2(\mathbb{D})} \,+\, b_2\,||(u_N)_t||_{\mathbb{L}^2(\mathbb{D})} \Big) \alpha^{i,j}_{n,m} \dot{W}^{i,j}_{n,m} \end{aligned} \tag{4.1}$$

where $\kappa \,\geq\, 0$. To simplify, let

$$f(u_N) = \begin{bmatrix} v^{i,j}_{n,m} \\ -\sigma^2(\lambda_n + \beta_m) + a_1 - a_2\,||u_N||^2 \end{bmatrix}$$

and

$$g(u(t)) = \begin{bmatrix} 0 \\ \Big(b_0 + b_1\,||u_N|| \,+\, b_2\,||(u_N)_t|| \Big) \alpha^{i,j}_{n,m} \end{bmatrix}$$

4.1 STABILITY IN PROBABILITY

Definition. The trivial solution of equation (4.1) (in terms of norm $||u||_{\mathbb{L}^2}$) is said to be *stochastically stable* or *stable in probability*, if for $0 < \,\epsilon\, < \,1$ and $r > \,0$, $\exists$ a $\delta \,=\, \delta(\epsilon,\, r)$ such that, $\forall\, t \,\geq\, \delta$, we have

$$\mathbb{P}\Big\{ \,||u(t)||_{\mathbb{L}^2} \,<\, r \Big\} \,\geq\, 1 \,-\, \epsilon. \tag{4.2}$$

whenever $\delta \,>\, 0$.

Lemma 4.1.1. *If $\exists$ a positive-definite function $V(x(t),\, t) \,\in\, \mathcal{C}^{2,1}(\mathbb{R}^2 \,\times\, [0,\,\infty),\, \mathbb{R}_+)$ such that $\mathcal{L}V(x(t),\, t) \,\leq\, 0$ and $\forall\,(x(t),\, t) \,\in\, \mathbb{R}^2 \times\, [0,\,\infty)$, then the trivial solution of the equation*

$$dX(t) \,=\, f(x(t),\, t)\, dt \,+\, g(x(t),\, t)\, dW(t) \tag{4.3}$$

is stochastically stable.

Proof. See Arnold [3]. □

Theorem 4.1.2. *Let $b_0 = b_1 = 0$ and*

$$V(u(t),t) = c_0 + ||u_t||^2_{\mathbb{L}^2(\mathbb{D})} + ||u||^2_{\mathbb{L}^2(\mathbb{D})}\left(\sigma^2 \sum_{n,m=1}^{N} (\lambda_n + \beta_m) - a_1\right) + \frac{a_2}{2}\,||u||^4_{\mathbb{L}^2(\mathbb{D})}\,.$$

If $\kappa \geq \dfrac{b_2^2 \sum_{i,j}^{2} \sum_{n,m=1}^{N} (\alpha^{i,j}_{n,m})^2}{2}$, *then the trivial solution of equation (4.1) is stochastically stable i.e., stable in probability.*

Proof. From Lemma 2.1.2, we know that $V(u(t),t)$ is positive-definite if $\forall\, n,m \in \mathbb{N}$, $\sigma^2(\lambda_n + \beta_m) - a_1 \geq 0$, and from equation (2.7) we know that

$$\mathcal{L}V(u(t),t) = \sum_{i,j}^{2} \sum_{n,m=1}^{N} b_2^2\, ||v||^2_{\mathbb{L}^2(\mathbb{D})}\, (\alpha^{i,j}_{n,m})^2 \;-\; 2\,\kappa \sum_{i,j}^{2} \sum_{n,m=1}^{N} (v^{i,j}_{n,m})^2. \tag{4.4}$$

But by our assumption that

$$\kappa \geq \frac{b_2^2 \sum_{i,j}^{2} \sum_{n,m=1}^{N} (\alpha^{i,j}_{n,m})^2}{2},$$

then

$$\mathcal{L}V(u(t),t) \leq 0.$$

So by Lemma 4.1.1, the trivial solution of equation (4.1) is *stochastically stable.* □

4.2 ESTIMATES OF $\mathbb{L}^P$- GROWTH

Theorem 4.2.1. *Let $p \geq 2$ and*

$$u_{n,m}(0) \in \mathbb{L}^2(\mathbb{D}, \mathbb{R})$$

is independent of the $\sigma-$algebra

$$\mathcal{F}^W_T = \sigma\{W^{i,j}_{n,m}(t) : 0 \leq t \leq T; 1 \leq n,m \leq N,\, i,j = 1,2\}$$

and $a_2 \geq 0$, $\sigma^2\,\pi^2(l_x^2 + l_y^2) \,-\, a_1\, l_x^2\, l_y^2 \,>\, 0$, *with* $\mathbb{E}\Big[V^{\frac{p}{2}}(u(.,.,0))\Big] \,<\, \infty$, *and If* $\kappa \,>\, \dfrac{(p-1)\, b_2^2\, ||\alpha||_N^2}{2}$, *then we have* $\forall\, t \geq 0$

$$\mathbb{E}\Big[V^{\frac{p}{2}}(u)(t)\Big] \,\leq\, \mathbb{E}\Big[V^{\frac{p}{2}}(u)(0)\Big] \tag{4.5}$$

where $V(u(t))$ *is defined as in Theorem 4.1.2.*

Proof. We know from equation (3.6) that

$$V(u(t)) = ||u_t||^2_{\mathbb{L}^2} + \sigma^2\, ||\nabla\, u||^2_{\mathbb{L}^2} \,-\, a_1\, ||u||^2_{\mathbb{L}^2} \,+\, \frac{a_2}{2}\, ||u||^4_{\mathbb{L}^2}\,,$$

i.e., in terms of Fourier coefficients

$$\begin{aligned} V(u(t)) = &\sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2 + \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} \Big(\sigma^2(\lambda_n + \beta_m) \,-\, a_1\Big)(c_{n,m}^{i,j})^2 \\ &+\, \frac{a_2}{2}\left(\sum_{i,j=1}^{2}\sum_{n,m=1}^{N} (c_{n,m}^{i,j})^2\right)^2 . \end{aligned}$$

We proved in Theorem 4.1.2 that if $\kappa \,\geq\, \dfrac{b_2^2 \displaystyle\sum_{i,j}^{2}\sum_{n,m=1}^{N} (\alpha_{n,m}^{i,j})^2}{2}$, then

$$\mathcal{L}V(u(t),t) \,=\, \sum_{i,j}^{2}\sum_{n,m=1}^{N} b_2^2\; ||v||^2_{\mathbb{L}^2(\mathbb{D})}\,(\alpha_{n,m}^{i,j})^2 \,-\, 2\,\kappa \sum_{i,j}^{2}\sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2$$

thus $\mathcal{L}V \,\leq\, 0$. Now redo some calculations to find $\mathcal{L}V^{\frac{p}{2}}(u)(t)$. Using calculations from Theorem 2.2.1 and the chain rule formula for derivatives, we have

$$\begin{aligned} \mathcal{L}V^{\frac{p}{2}}(u)(t) = &\sum_{i,j}^{2}\sum_{n,m=1}^{N} v_{n,m}^{i,j}\, p\, V^{\frac{p}{2}-1}\Big[\sigma^2(\lambda_n + \beta_m) \,-\, a_1 + a_2\; ||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big] c_{n,m}^{i,j} \\ &+ \sum_{i,j}^{2}\sum_{n,m=1}^{N}\; \Big[-\,\sigma^2(\lambda_n + \beta_m) \,+\, a_1 - a_2\; ||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big] v_{n,m}^{i,j} c_{n,m}^{i,j}\, V^{\frac{p}{2}-1} \\ &-\, \kappa \sum_{i,j}^{2}\sum_{n,m=1}^{N} v_{n,m}^{i,j}\, p\, V^{\frac{p}{2}-1} \,+\, \frac{b_2^2}{2}\sum_{i,j}^{2}\sum_{n,m=1}^{N} ||v||^2_{\mathbb{L}^2(\mathbb{D})}\,\Big(\alpha_{n,m}^{i,j}\Big)^2 \\ &\cdot\, \Big[p(\frac{p}{2} \,-\, 1)V^{\frac{p}{2}-1} \,\cdot\, 2\, v_{n,m}^{i,j} v_{k,l}^{i,j} + \,\delta_{k,n}\delta_{l,m} V^{\frac{p}{2}-1}\Big]. \end{aligned}$$

Now, if $k = n$ and $l = m$, then

$$\begin{aligned}
\mathcal{L}V^{\frac{p}{2}}(u)(t) = & -\kappa\, p \sum_{i,j}^{2} \sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2\, V^{\frac{p}{2}-1} \\
& + \frac{b_2^2}{2} \sum_{i,j}^{2} \sum_{n,m=1}^{N} ||v||_{\mathbb{L}^2(\mathbb{D})}^2 \left(\alpha_{n,m}^{i,j}\right)^2 p\,(p\,-\,2)\left(v_{n,m}^{i,j}\right)^2 V^{\frac{p}{2}-1} \\
& + p\,\frac{b_2^2}{2} \sum_{i,j}^{2} \sum_{n,m=1}^{N} ||v||_{\mathbb{L}^2(\mathbb{D})}^2 \left(\alpha_{n,m}^{i,j}\right)^2 V^{\frac{p}{2}-1},
\end{aligned}$$

hence

$$\begin{aligned}
\mathcal{L}V^{\frac{p}{2}}(u)(t) \le & -\kappa\, p \sum_{i,j}^{2} \sum_{n,m=1}^{N} (v_{n,m}^{i,j})^2\, V^{\frac{p}{2}-1} + \frac{p\, b_2^2}{2} \sum_{n,m=1}^{N} \left(\alpha_{n,m}^{i,j}\right)^2 ||v||_{\mathbb{L}^2}^2\, V^{\frac{p}{2}-1} \\
& + \frac{p\,(p\,-\,2)\, b_2^2}{2} \sum_{i,j}^{2} \sum_{n,m=1}^{N} \left(\alpha_{n,m}^{i,j}\right)^2 \left(v_{n,m}^{i,j}\right)^2 V^{\frac{p}{2}-1} \\
\le & -\kappa\, p\, ||v||_{\mathbb{L}^2}^2\, V^{\frac{p}{2}-1} + \frac{p\, b_2^2}{2} ||v||_{\mathbb{L}^2}^2\, ||\alpha||_N^2\, V^{\frac{p}{2}-1} + \frac{p\,(p\,-\,2)\, b_2^2}{2} ||v||_{\mathbb{L}^2}^2\, ||\alpha||_N^2\, V^{\frac{p}{2}-1} \\
\le & \; p\, ||v||_{\mathbb{L}^2}^2\, V^{\frac{p}{2}-1} \left[-\kappa + \frac{b_2^2}{2}(1 + p - 2)\, ||\alpha||_N^2\right] \\
= & \; p\, ||v||_{\mathbb{L}^2}^2\, V^{\frac{p}{2}-1} \left[-\kappa + \frac{b_2^2}{2}(p - 1)\, ||\alpha||_N^2\right]
\end{aligned}$$

but we know that $||v||_{\mathbb{L}^2(\mathbb{D})}^2 \le V, (v = u_t)$, so

$$\mathcal{L}V^{\frac{p}{2}}(u)(t) \le p\, V^{\frac{p}{2}} \left[-\kappa + \frac{b_2^2}{2}(p - 1)\, ||\alpha||_N^2\right]$$

using the assumption that $\kappa > \dfrac{(p-1)\, b_2^2\, ||\alpha||_N^2}{2}$, then

$$\mathcal{L}V^{\frac{p}{2}}(u)(t) \le 0.$$

Applying the Dynkin's lemma, we have

$$\begin{aligned}
\mathbb{E}V^{\frac{p}{2}}(c, v)(t) = & \; \mathbb{E}V^{\frac{p}{2}}(c, v)(0) + \mathbb{E}\int_0^t \mathcal{L}V^{\frac{p}{2}}(u)(s)\, d\, s \\
\le & \; \mathbb{E}V^{\frac{p}{2}}(u)(0).
\end{aligned}$$

Which means that the $p-$th moment estimates of V are uniformly bounded. □

Corollary 4.2.2. *Let the same assumption as in Theorem 4.2.1 be Valid when $N \to \infty$, then we have*

$$\mathbb{E}V^{\frac{p}{2}}(u)(t) \leq \mathbb{E}V^{\frac{p}{2}}(u)(0).$$

Proof. Take $N \to \infty$ in proof of Theorem 4.2.1, then assumption is clear. □

Corollary 4.2.3. *Let $p \geq 2$ and let V be as above. Imposing the same assumptions as in Theorem 4.2.1, then we have $\forall\, 0 \leq t \leq 1$,*

$$\mathbb{E}\,||v(t)||^p_{\mathbb{L}^2(\mathbb{D})} \leq \mathbb{E}V^{\frac{p}{2}}(u)(0).$$

Proof. We know, from the definition of V, that $||v||^2_{\mathbb{L}^2(\mathbb{D})} \leq V(u)$. This implies that $||v(t)||^p_{\mathbb{L}^2(\mathbb{D})} \leq V^{\frac{p}{2}}(u(t))$ and from Theorem 4.2.1 it is easily to show that

$$\mathbb{E}\,||v(t)||^p_{\mathbb{L}^2(\mathbb{D})} \leq \mathbb{E}V^{\frac{p}{2}}(u)(0).$$

□

Corollary 4.2.4. *$\forall p \geq 2$ and $\forall\, 0 \leq t \leq T$, with $\sigma^2(\lambda_1 + \beta_1) - a_1 > 0$, we have $\forall\, 0 \leq t \leq T$,*

(1) If $a_2 = 0$, then

$$\mathbb{E}\,||u(t)||^p_{\mathbb{L}^2(\mathbb{D})} \leq \frac{\mathbb{E}V^{\frac{p}{2}}(u(0))}{\Big[\sigma^2(\lambda_1 + \beta_1) - a_1\Big]^{\frac{p}{2}}}.$$

(2) If $a_2 > 0$, then

$$\mathbb{E}\,||u(t)||^p_{\mathbb{L}^2(\mathbb{D})} \leq \Big(\frac{2}{a_2}\Big)^{\frac{p}{4}}\mathbb{E}V^{\frac{p}{4}}(u(0)).$$

Proof. (1) Note that we have $\Big(\sigma^2(\lambda_1 + \beta_1) - a_1\Big)\, ||u(.,.,t)||^2_{\mathbb{L}^2(\mathbb{D})} \leq V(.,.,t)$. Since λ_n and β_m are increasing in n and m,

$$\Big[\sigma^2(\lambda_1 + \beta_1) - a_1\Big]^{\frac{p}{2}}\, ||u(t)||^p_{\mathbb{L}^2(\mathbb{D})} \leq V^{\frac{p}{2}}(u(t)).$$

Pull over expectation, then

$$\Big[\sigma^2(\lambda_1 + \beta_1) - a_1\Big]^{\frac{p}{2}}\mathbb{E}\,||u(t)||^p_{\mathbb{L}^2(\mathbb{D})} \leq \mathbb{E}V^{\frac{p}{2}}(u(t)).$$

By using Theorem 4.2.1, we have

$$\Big[\sigma^2(\lambda_1+\beta_1)\,-\,a_1\Big]^{\frac{p}{2}}\mathbb{E}\,||u(t)||^p_{\mathbb{L}^2(\mathbb{D})}\,\le\,\mathbb{E}V^{\frac{p}{2}}(u(0)).$$

So we have

$$\mathbb{E}\,||u(t)||^p_{\mathbb{L}^2(\mathbb{D})}\,\le\,\frac{\mathbb{E}V^{\frac{p}{2}}(u(0))}{\Big[\sigma^2(\lambda_1+\beta_1)\,-\,a_1\Big]^{\frac{p}{2}}}.$$

(2) From the definition of $V(u(t))$, it is clear that $\frac{a_2}{2}\,||u(t)||^4_{\mathbb{L}^2(\mathbb{D})}\,\le\,V(u(t))$, so

$$\Big(\frac{a_2}{2}\Big)^{\frac{p}{4}}\,||u(t)||^4\,\le\,V^{\frac{p}{4}}(u(0)).$$

Now, take the expectation to both sides, and we get

$$\Big(\frac{a_2}{2}\Big)^{\frac{p}{4}}\mathbb{E}\,||u(t)||^4_{\mathbb{L}^2(\mathbb{D})}\,\le\,\mathbb{E}V^{\frac{p}{4}}(u(0)),$$

i.e., $\forall\,0\le\,t\,\le\,T$,

$$\mathbb{E}\,||u(t)||^p_{\mathbb{L}^2(\mathbb{D})}\,\le\,\Big(\frac{2}{a_2}\Big)^{\frac{p}{4}}\mathbb{E}V^{\frac{p}{4}}(u(0)).$$

□

Remark. The previous corollary means that

$$\mathbb{E}\,||u(t)||^p_{\mathbb{L}^2(\mathbb{D})}\,\le\,\min\Big\{\frac{\mathbb{E}V^{\frac{p}{2}}(u(0))}{\Big[\sigma^2(\lambda_1+\beta_1)\,-\,a_1\Big]^{\frac{p}{2}}},\,\Big(\frac{2}{a_2}\Big)^{\frac{p}{4}}\mathbb{E}V^{\frac{p}{4}}(u(0))\Big\}$$

Remark. From the previous corollary, we can prove very quickly that

$$\limsup_{t\to\infty}\,\frac{1}{t}\,\log(\mathbb{E}\,||u(t)||^p_{\mathbb{L}^2(\mathbb{D})})\,\le\,0.$$

Corollary 4.2.5. *Assume that the assumptions of Corollary 4.2.3 hold with* $p=\,2$, *then we have*

$$\limsup_{t\to\infty}\,\frac{1}{t}\,\log(\mathbb{E}\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\,\le\,0. \tag{4.6}$$

Proof. From Corollary 4.2.3,let $p = 2$, then we have

$$\mathbb{E}\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})} \,\leq\, \mathbb{E}V(u(0)).$$

Thus

$$\log(\mathbb{E}\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}) \,\leq\, \log(\mathbb{E}V(u(0))).$$

So, if we divide both sides by t and take the lim sup as $t \to \infty$, then

$$\limsup_{t\to\infty} \frac{1}{t}\,\log(\mathbb{E}\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}) \,\leq\, 0$$

□

Lemma 4.2.6. *Let $g_{n,m}(u(t))$ in the system (4.1) be defined as*

$$g_{n,m}(u(t)) = \begin{bmatrix} 0 \\ \\ b_3\,\sqrt{V}\,\alpha^{i,j}_{n,m}, \end{bmatrix}$$

where b_3 is constant and $f_{n,m}(u(t))$ is defined as the same above. Let $p \geq 2$ and $u_{n,m}(0) \in \mathbb{L}^2\,(\mathbb{D},\,\mathbb{R})$ is independent of the $\sigma-$ algebra

$$\mathcal{F}^W_T \,=\, \sigma\{W^{i,j}_{n,m} : 0 \leq\, t \,\leq\, T; 1 \leq\, n, m \,\leq\, N\}$$

and $a_2 \,\geq\, 0$, $\sigma^2\,\pi^2\,(l_x^2 + l_y^2) \,-\, a_1\,l_x^2\,l_y^2 \,>\, 0$, with $\mathbb{E}\Big[V^{\frac{p}{2}}(u(.,.,0))\Big] \,\leq\, \infty$, and $\kappa \,\geq\, \dfrac{(p-1)\,b_3^2\,||\alpha||^2_{l^2_N}}{2}$, then we have $\forall\, t \,\geq\, 0$

$$\mathbb{E}\Big[V^{\frac{p}{2}}(c,v)(t)\Big] \,\leq\, \mathbb{E}\Big[V^{\frac{p}{2}}(c,v)(0)\Big]. \tag{4.7}$$

where $V(u(t))$ is defined as in Theorem 4.1.2.

Proof. We know from equation (3.6) that

$$V(u) \,=\, ||u_t||^2_{\mathbb{L}^2} \,+\, \sigma^2\,||\nabla\,u||^2_{\mathbb{L}^2} \,-\, a_1\,||u||^2_{\mathbb{L}^2} \,+\, \frac{a_2}{2}\,||u||^4_{\mathbb{L}^2}\,,$$

i.e., in terms of fourier coefficients,

$$V(u(t)) = \sum_{i,j=1}^{2}\sum_{n,m=1}^{N}(v_{n,m}^{i,j})^2 + \sum_{i,j=1}^{2}\sum_{n,m=1}^{N}\Big(\sigma^2(\lambda_n+\beta_m)\;-\;a_1\Big)(c_{n,m}^{i,j})^2 + \frac{a_2}{2}\left(\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}(c_{n,m}^{i,j})^2\right)^2.$$

We proved in Theorem 4.1.2 that

$$\mathcal{L}V(u(t),t) = \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} b_2^2\,||v||^2_{\mathbb{L}^2(\mathbb{D})}\,(\alpha_{n,m}^{i,j})^2 \;-\; 2\,\kappa\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}(v_{n,m}^{i,j})^2.$$

But $\kappa \ge \dfrac{b_3^2\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}(\alpha_{n,m}^{i,j})^2}{2}$, then $\mathcal{L}V \le 0$. Now we redo some calculations to find $\mathcal{L}V^{\frac{p}{2}}(u)(t)$. Using Theorem 2.2.1 and the chain rule formula for derivatives, we have

$$\begin{aligned}
\mathcal{L}V^{\frac{p}{2}}(u)(t) &= \sum_{i,j=1}^{2}\sum_{n,m=1}^{N} v_{n,m}^{i,j}\,p\,V^{\frac{p}{2}-1}\Big[\sigma^2(\lambda_n+\beta_m)\;-\;a_1\;+\;a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big]c_{n,m}^{i,j}\\
&\quad+\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}\Big[-\sigma^2(\lambda_n+\beta_m)\;+\;a_1\;-\;a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big]c_{n,m}^{i,j}\,p\,V^{\frac{p}{2}-1}v_{n,m}^{i,j}\\
&\quad-\;\kappa\sum_{i,j=1}^{2}\sum_{n,m=1}^{N} v_{n,m}^{i,j}\,p\,V^{\frac{p}{2}-1}\,v_{n,m}^{i,j}\\
&\quad+\frac{b_3^2}{2}\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}(\alpha_{n,m}^{i,j})^2\,V\Big[p\,(\frac{p}{2}-1)V^{\frac{p}{2}-1}\,2\,v_{n,m}^{i,j}v_{k,l}^{i,j}+\delta_{k,n}\delta_{l,m}\,p\,V^{\frac{p}{2}-1}\Big]\\
&= -\,\kappa\,p\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}V^{\frac{p}{2}-1}\,(v_{n,m}^{i,j})^2\;+\;\frac{b_3^2}{2}\,p\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}v_{n,m}^{i,j}\,V^{\frac{p}{2}}\,\delta_{k,n}\delta_{l,m}\\
&\quad+\;b_3^2\,p\,(\frac{p}{2}-1)\sum_{i,j=1}^{2}\sum_{n,m=1}^{N}v_{n,m}^{i,j}\,V^{\frac{p}{2}-1}(\alpha_{n,m}^{i,j})^2\,v_{n,m}^{i,j}v_{k,l}^{i,j}
\end{aligned}$$

Now,if $k=n$ and $l=m$, then

$$\begin{aligned}
\mathcal{L}V^{\frac{p}{2}}(u)(t) &= \;p\;||v||^2_{\mathbb{L}^2}\,V^{\frac{p}{2}-1}\Big[-\;\kappa+\frac{b_3^2}{2}\,(1+p-2)\,||\alpha||^2_{\infty}\Big]\\
&\le\;p\,V^{\frac{p}{2}}\Big[-\;\kappa+\frac{b_3^2}{2}\,(p-1)\,||\alpha||^2_{\infty}\Big]
\end{aligned}$$

again let $\kappa \geq \dfrac{(p-1)\, b_3^2\, ||\alpha||^2_{l^2_N}}{2}$, then

$$\mathcal{L}V^{\frac{p}{2}}(c,v)(t) \leq 0.$$

Applying the Dynkin's lemma we have

$$\begin{aligned}\mathbb{E}\mathcal{L}V^{\frac{p}{2}}(u)(t) &= \mathbb{E}\mathcal{L}V^{\frac{p}{2}}(c,v)(0) + \mathbb{E}\int_0^t \mathcal{L}V^{\frac{p}{2}}(c,v)(s)\,d\,s \\ &\leq \mathbb{E}\mathcal{L}V^{\frac{p}{2}}(u)(0).\end{aligned}$$

which means that the $p-$th moment estimates of V are uniformly bounded. □

4.3 ALMOST SURELY EXPONENTIALLY STABLE

Consider Consider the nonlinear stochastic wave equation with noise

$$u_{tt} = \sigma^2(u_{xx}+u_{yy}) + (a_1 - a_2\,||u||^2_{\mathbb{L}^2(\mathbb{D})})u - \kappa\, v + b_2\,||v||_{\mathbb{L}^2(\mathbb{D})}\,\frac{dW}{dt} \tag{4.8}$$

Definition. The trivial solution of equation (4.1) is said to be *a.s. exponentially stable* if

$$\limsup_{t\to\infty} \frac{1}{t}\log||u(t)||_{\mathbb{L}^2(\mathbb{D})} < 0 \qquad\qquad (a.s.) \tag{4.9}$$

$\forall\, u(0) \in \mathbb{D}$. The quantity of the left hand side of (4.8) is called *the sample top Lyapunov exponent.*

Theorem 4.3.1. *[Schurz and Hazaimeh [17] Let $V(u(t))$ as before in Theorem 4.1.2. If $\kappa > \frac{b_2^2||\alpha||^2_{l^2}}{2}$, then the v component of the trivial solution of $N-$dimensional equation (4.1) and original system (4.8) with $b_0 = 0 = b_1$ is a.s. exponentially stable.*

Proof. Return to the analysis of finite $N-$dimensional system (2.3). Recall that

$$\begin{aligned}&V_N(u(t),t) \\ &= c_0 + ||u_t||^2_{\mathbb{L}^2(\mathbb{D})} + ||u||^2_{\mathbb{L}^2(\mathbb{D})}\left(\sigma^2\sum_{n,m=1}^{N}(\lambda_n+\beta_m) - a_1\right) + \frac{a_2}{2}\,||u||^4_{\mathbb{L}^2(\mathbb{D})} \\ &= \sum_{n,m=1}^{N} v^2_{n,m}(t) + \sum_{n,m=1}^{N}\left[\sigma^2(\lambda_n+\beta_m) - a_1\right]c^2_{n,m} + \frac{a_2}{2}\left(\sum_{n,m=1}^{N} c^2_{n,m}\right)^2\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{L}V_N(c,v)(t)\\
&= \sum_{n,m=1}^{N} v_{n,m}\frac{\partial V_N}{\partial c_{n,m}} + \sum_{n,m=1}^{N}\left(\left[-\sigma^2(\lambda_n+\beta_m)+a_1-a_2\sum_{k,l=1}^{N}(c_{k,l})^2\right]\, c_{n,m}\right)\frac{\partial V_N}{\partial v_{n,m}}\\
&\quad - \sum_{k,l=1}^{N}\kappa\, v_{n,m}\frac{\partial V_N}{\partial v_{n,m}} + \frac{1}{2}\sum_{n,m=1}^{N}\Big(b_2\,||v||_{\mathbb{L}^2(\mathbb{D})}\Big)^2 (\alpha_{n,m})^2\frac{\partial^2 V_N}{\partial\, v_{n,m}^2}
\end{aligned}
$$

hence,

$$
\begin{aligned}
&\mathcal{L}V_N(c,v)(t)\\
&= \sum_{n,m=1}^{N} 2\, v_{n,m}\left[\sigma^2(\lambda_n+\beta_m)-a_1+a_2\sum_{k,l=1}^{N}(c_{k,l})^2\right]\, c_{n,m}\\
&\quad + \sum_{n,m=1}^{N} 2\left[-\sigma^2(\lambda_n+\beta_m)+a_1-a_2\sum_{k,l=1}^{N}(c_{k,l})^2\right]\, c_{n,m}\, v_{n,m}\\
&\quad - \sum_{n,m=1}^{N} 2\,\kappa\, v_{n,m}\, v_{n,m} + \frac{1}{2}\sum_{n,m=1}^{N} 2\, b_2^2\, ||v||^2_{\mathbb{L}^2(\mathbb{D})}\,(\alpha_{n,m})^2\\
&= \sum_{n,m=1}^{N} b_2^2\, ||v||^2_{\mathbb{L}^2(\mathbb{D})}\,(\alpha_{n,m})^2 \;-\; 2\,\kappa\sum_{n,m=1}^{N}(v_{n,m})^2\\
&= \Big(-\,2\,\kappa\,+b_2^2\,||\alpha||^2_{l^2_N}\Big)\,||v||^2_{\mathbb{L}^2(\mathbb{D})}
\end{aligned}
$$

but by the assumption that

$$
\kappa\;\geq\;\frac{b_2^2\,||\alpha||^2_{l^2_N}}{2},
$$

and using Dynkin's formula, we find that

$$
\begin{aligned}
\mathbb{E}V_N(c,v)(t) &= \mathbb{E}V_N(c,v)(0)\;+\;\mathbb{E}\int_0^t\,\big(-\,2\,\kappa\,+b_2^2\,||\alpha||^2_{l^2}\big)\,||v(s)||^2_{\mathbb{L}^2}\;d\,s\\
&\leq\;\mathbb{E}V_N(c,v)(0)\;+\Big(b_2^2\,||\alpha||^2_{l^2_N}\,-\,2\,\kappa\Big)\int_0^t\,\mathbb{E}\,||v(s)||^2_{\mathbb{L}^2}\;d\,s
\end{aligned}
$$

but $||v_N(t)||^2_{\mathbb{L}^2} \leq V_N(c,v)(t)$, so

$$\begin{aligned} \mathbb{E}\,||v_N(t)||^2_{\mathbb{L}^2} &\leq \mathbb{E}V_N(c,v)(t) \\ &\leq \mathbb{E}V_N(c,v)(0) + \Big(b_2^2\,||\alpha||^2_{l^2_N} - 2\,\kappa\Big) \int_0^t \mathbb{E}\,||v(s)||^2_{\mathbb{L}^2}\,d\,s \end{aligned}$$

using extended Gronwall lemma gives us

$$\mathbb{E}\,||v(t)||^2_{\mathbb{L}^2} \leq \mathbb{E}V(c,v)(0)\,e^{\left(b_2^2\,||\alpha||^2_{l^2} - 2\,\kappa\right)t}$$

hence

$$\log\mathbb{E}\,||v_N(t)||^2_{\mathbb{L}^2} \leq \log\mathbb{E}V_N(c,v)(0) + \Big(b_2^2\,||\alpha||^2_{l^2_N} - 2\,\kappa\Big)\,t$$

thus

$$\begin{aligned} \limsup_{t\to\infty} \frac{log\mathbb{E}\,||v_N(t)||^2_{\mathbb{L}^2}}{t} &\leq \limsup_{t\to\infty} \frac{log\mathbb{E}V_N(c,v)(0)}{t} + \Big(b_2^2\,||\alpha||^2_{l^2_N} - 2\,\kappa\Big) \\ &\leq b_2^2\,||\alpha||^2_{l^2_N} - 2\,\kappa. \end{aligned} \tag{4.10}$$

If $\kappa > \dfrac{b_2^2\,||\alpha||^2_{l^2_N}}{2}$, then the left side of identity (4.9) is negative and the trivial solution of $N-$dimensional equation (4.1) is *a.s.* exponential stable.

Finally, we observe that all the previous estimates are uniformly bounded as $N \to \infty$. Hence, under $||\alpha||^2_{l^2} \leq \infty$, we arrive at

$$\limsup_{t\to\infty} \frac{log\mathbb{E}\,||v(t)||^2_{\mathbb{L}^2}}{t} \leq b_2^2\,||\alpha||^2_{l^2} - 2\,\kappa. \tag{4.11}$$

□

Lemma 4.3.2. *Let $v(t)$ be a nonnegative integrable function such that*

$$v(t) \leq C + A\int_0^t v(s)\,d\,s, \qquad 0 \leq t \leq T \tag{4.12}$$

for some constants C, A. Then $C \geq 0$ and

$$v(t) \leq C\,\exp(A\,t), \qquad 0 \leq t \leq T. \tag{4.13}$$

CHAPTER 5

NUMERICAL METHODS FOR FOURIER COEFFICIENTS

Recall the form of Fourier solutions u and its approximate Fourier solutions u_N given by

$$u_N(x,y,t) = \sum_{i,j=1}^{2} \sum_{n,m=1}^{N} c_{n,m}^{i,j}(t) e_n^i(x) e_m^j(y) \tag{5.1}$$

with its coefficients $c_{n,m}^{i,j}$ satisfying

$$\begin{aligned} \ddot{c}_{n,m}^{i,j}(t) \;&=\; \Big[-\sigma^2 \Big(\lambda_n + \beta_m\Big) + a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big] c_{n,m}^{i,j}(t) \\ &\quad -\kappa\, v_{n,m}^{i,j}(t) + \Big(b_0 + b_1\, ||u||_{\mathbb{L}^2(\mathbb{D})} + b_2\, ||\dot{u}||_{\mathbb{L}^2(\mathbb{D})}\Big) \alpha_{n,m}^{i,j} \dot{W}_{n,m}^{i,j}(t) \end{aligned} \tag{5.2}$$

where $\lambda_n = \dfrac{\pi^2 n^2}{l_x^2}$ $\beta_m = \dfrac{\pi^2 m^2}{l_y^2}$, $||u||_{\mathbb{L}^2(\mathbb{D})} = \sqrt{\sum_{i,j=1}^{2} \sum_{n,m=1}^{N} [c_{k,l}^{i,j}]^2}$, and $||\dot{u}||_{\mathbb{L}^2(\mathbb{D})} = \sqrt{\sum_{i,j=1}^{2} \sum_{n,m=1}^{N} [\dot{c}_{k,l}^{i,j}]^2}$.

Note that equation (5.2) is equivalent to the system

$$\dot{c}_{n,m}^{i,j} = v_{n,m}^{i,j}$$

$$\begin{aligned} \dot{v}_{n,m}^{i,j} \;&=\; \Big[-\sigma^2(\lambda_n + \beta_m) + a_1 - a_2 \sum_{i,j}^{2} \sum_{k,l=1}^{N} [c_{k,l}^{i,j}]^2 \Big] c_{n,m}^{i,j} - \kappa\, v_{n,m}^{i,j}(t) \\ &\quad + \Big(b_0 + b_1 \sqrt{\sum_{i,j}^{2} \sum_{k,l=1}^{N} [c_{k,l}^{i,j}]^2} + \sqrt{\sum_{i,j}^{2} \sum_{k,l=1}^{N} [\dot{c}_{k,l}^{i,j}]^2}\Big) \alpha_{n,m}^{i,j} \dot{W}_{n,m}^{i,j}. \end{aligned} \tag{5.3}$$

subject to the initial conditions

$$\Big(c_{n,m}^{i,j}(0), v_{n,m}^{i,j}(0)\Big) \;=\; \Big(c_{n,m,0}^{i,j}, v_{n,m,0}^{i,j}\Big)_{n,m\geq 1,\, i,j=1,2}.$$

Now,we introduce the following standard definitions.

Definition. For $k \in \mathbb{N}$, take the partition

$$0 = t_0 < t_1 < t_2 < \ldots < t_k = T$$

of $[0,T]$ with current step sizes

$$h_k \,=\, t_{k+1} \,-\, t_k \,>\, 0,$$

the **forward Euler method (FEM)** is defined by the iterative scheme

$$c^{i,j}_{n,m}(t_{k+1}) \,=\, c^{i,j}_{n,m}(t_k) \,+\, h_k \, v^{i,j}_{n,m}(t_k) \tag{5.4}$$

$$\begin{aligned} v^{i,j}_{n,m}(t_{k+1}) \;&=\; v^{i,j}_{n,m}(t_k) \,+\, h_k \left[f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_k) \,-\, \kappa\, v^{i,j}_{n,m}(t_k) \right] \\ &\quad + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m} \end{aligned} \tag{5.5}$$

where

$$f_{n,m}(u(t_k)) \,=\, -\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c^{i,j}_{q,l}(t_k)]^2,$$

and

$$g_{n,m}(u(t_k)) \,=\, \left(b_0 \,+\, b_1 \sqrt{\sum_{i,j=1}^{2} \sum_{q,l=1}^{N} (c^{i,j}_{q,l}(t_k))^2} \,+\, b_2\, ||v||_{\mathbb{L}^2(\mathbb{D})} \right) \alpha^{i,j}_{n,m},$$

and

$$\triangle_k W^{i,j}_{n,m} \,=\, W^{i,j}_{n,m}(t_{k+1}) \,-\, W^{i,j}_{n,m}(t_k) \in \mathcal{N}(0, h_k),\; h_k \,=\, t_{k+1} - t_k$$

Definition. For $k \in \mathbb{N}$, take the partition

$$0 \,=\, t_0 \,<\, t_1 \,<\, t_2 \,<\, \ldots \,<\, t_k \,=\, T$$

of $[0,T]$ with current step sizes

$$h_k \,=\, t_{k+1} \,-\, t_k \,>\, 0,$$

the **backward Euler method (BEM)** follows the iterative scheme

$$c^{i,j}_{n,m}(t_{k+1}) \,=\, c^{i,j}_{n,m}(t_k) \,+\, h_k \, v^{i,j}_{n,m}(t_{k+1}) \tag{5.6}$$

$$\begin{aligned} v^{i,j}_{n,m}(t_{k+1}) \;&=\; v^{i,j}_{n,m}(t_k) \,+\, h_k \left[f_{n,m}(u(t_{k+1}))\, c^{i,j}_{n,m}(t_{k+1}) \,-\, \kappa\, v^{i,j}_{n,m}(t_{k+1}) \right] \\ &\quad + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m} \end{aligned} \tag{5.7}$$

where

$$f_{n,m}(u(t_{k+1})) \,=\, -\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c_{q,l}^{i,j}(t_{k+1})]^2,$$

and

$$g_{n,m}(u(t_{k+1})) \,=\, \Big(b_0 \,+\, b_1 \sqrt{\sum_{i,j=1}^{2} \sum_{q,l=1}^{N} (c_{q,l}^{i,j}(t_{k+1}))^2} \,+\, b_2\, ||v||_{\mathbb{L}^2(\mathbb{D})} \Big)\, \alpha_{n,m}^{i,j},$$

and

$$\triangle_k\, W_{n,m}^{i,j} \,=\, W_{n,m}^{i,j}(t_{k+1}) \,-\, W_{n,m}^{i,j}(t_k) \in \mathcal{N}(0, h_k),\; h_k \,=\, t_{k+1} - t_k$$

Now, as in Schurz[15], the linear-implicit Euler-type methods are introduced as follows:

Definition. For $k \in \mathbb{N}$, take the partition

$$0 \,=\, t_0 \,<\, t_1 \,<\, t_2 \,<\, ... \,<\, t_k \,=\, T$$

of $[0, T]$ with current step sizes

$$h_k \,=\, t_{k+1} \,-\, t_k \,>\, 0,$$

the **linear-implicit Euler-type method (LIEM)** is governed by the iterative scheme

$$c_{n,m}^{i,j}(t_{k+1}) \,=\, c_{n,m}^{i,j}(t_k) \,+\, h_k\, v_{n,m}^{i,j}(t_{k+1}) \tag{5.8}$$

$$\begin{aligned} v_{n,m}^{i,j}(t_{k+1}) \;&=\; v_{n,m}^{i,j}(t_k) \,+\, h_k \Big[f_{n,m}(u(t_k))\, c_{n,m}^{i,j}(t_{k+1}) \,-\, \kappa\, v_{n,m}^{i,j}(t_{k+1}) \Big] \\ &\quad + g_{n,m}(u(t_k)) \triangle_k\, W_{n,m}^{i,j} \end{aligned} \tag{5.9}$$

where

$$f_{n,m}(u(t_k)) \,=\, -\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2,$$

and

$$g_{n,m}(u(t_k)) \,=\, \Big(b_0 \,+\, b_1 \sqrt{\sum_{i,j=1}^{2} \sum_{q,l=1}^{N} (c_{q,l}^{i,j}(t_k))^2} \,+\, b_2\, ||v||_{\mathbb{L}^2(\mathbb{D})} \Big)\, \alpha_{n,m}^{i,j},$$

and

$$\triangle_k\, W_{n,m}^{i,j} \,=\, W_{n,m}^{i,j}(t_{k+1}) \,-\, W_{n,m}^{i,j}(t_k) \in \mathcal{N}(0, h_k),\; h_k \,=\, t_{k+1} - t_k$$

And finally, we introduce the definition of the linear-implicit midpoint method

Definition. For $k \in \mathbb{N}$, take the partition

$$0 = t_0 < t_1 < t_2 < \ldots < t_k = T$$

of $[0, T]$ with current step sizes

$$h_k = t_{k+1} - t_k > 0,$$

the **linear-implicit midpoint method (LIMM)** is governed by the iterative scheme

$$c^{i,j}_{n,m}(t_{k+1}) = c^{i,j}_{n,m}(t_k) + h_k \frac{v^{i,j}_{n,m}(t_k) + v^{i,j}_{n,m}(t_{k+1})}{2} \tag{5.10}$$

$$\begin{aligned} v^{i,j}_{n,m}(t_{k+1}) &= v^{i,j}_{n,m}(t_k) + h_k \Big[f_{n,m}(u(t_k)) \frac{c^{i,j}_{n,m}(t_k) + c^{i,j}_{n,m}(t_{k+1})}{2} \\ &\quad - \kappa \frac{v^{i,j}_{n,m}(t_k) + v^{i,j}_{n,m}(t_{k+1})}{2} \Big] + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m} \end{aligned} \tag{5.11}$$

where

$$f_{n,m}(u(t_k)) = -\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c^{i,j}_{q,l}(t_k)]^2,$$

and

$$g_{n,m}(u(t_k)) = \Big(b_0 + b_1 \sqrt{\sum_{i,j=1}^{2} \sum_{q,l=1}^{N} (c^{i,j}_{q,l}(t_k))^2} + b_2 \, ||u_t|| \Big) \alpha^{i,j}_{n,m},$$

and

$$\triangle_k W^{i,j}_{n,m} = W^{i,j}_{n,m}(t_{k+1}) - W^{i,j}_{n,m}(t_k) \in \mathcal{N}(0, h_k), \; h_k = t_{k+1} - t_k$$

5.1 EXPLICIT REPRESENTATION OF METHODS (LIEM) AND (LIMM)

If we choose the linear-implicit Euler-type method (LIEM) and the linear-implicit midpoint method (LIMM), then we have the following results.

Theorem 5.1.1. *(Explicit Representation of method (LIEM)). Assume that* $\left|c^{i,j}_{n,m}(t_k)\right| < \infty$, $\left|v^{i,j}_{n,m}(t_k)\right| < \infty$, *and*

$$a_2 \geq 0,\, \kappa \geq 0,\, and \,\forall\, n, m \in \mathbb{N} : \Big[-\sigma^2 (\lambda_n + \beta_m) + a_1 \Big] h_k^2 < 1 + h_k\, \kappa.$$

Then the method (LIEM) governed by equation (5.8) and equation (5.9) has the non-exploding explicit representation

$$1)\, c^{i,j}_{n,m}(t_{k+1}) = \frac{\Big(1 + h_k\, \kappa\Big) c^{i,j}_{n,m}(t_k) + h_k\, v^{i,j}_{n,m}(t_k) + h_k\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{1 + h_k\, \kappa - h_k^2\, f_{n,m}(u(t_k))} \tag{5.12}$$

and

$$2)\, v^{i,j}_{n,m}(t_{k+1}) = \frac{v^{i,j}_{n,m}(t_k) + h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_k) + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{1 + h_k\, \kappa - h_k^2\, f_{n,m}(u(t_k))} \tag{5.13}$$

where

$$f_{n,m}(u(t_k)) = -\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c^{i,j}_{q,l}(t_k)]^2,$$

and

$$g_{n,m}(u(t_k)) = \Bigg(b_0 + b_1 \sqrt{\sum_{i,j=1}^{2} \sum_{q,l=1}^{N} (c^{i,j}_{q,l}(t_k))^2} + b_2\, ||v||_{\mathbb{L}^2(\mathbb{D})} \Bigg) \alpha^{i,j}_{n,m},$$

and

$$\triangle_k W^{i,j}_{n,m} = W^{i,j}_{n,m}(t_{k+1}) - W^{i,j}_{n,m}(t_k) \in \mathcal{N}(0, h_k),\, h_k = t_{k+1} - t_k$$

Proof. Suppose that $a_2 \geq 0$, $\kappa \geq 0$ and $1 + \Big[\sigma^2(\lambda_n + \beta_m) - a_1\Big] h_k^2 + h_k\, \kappa > 0$, where $\lambda_n = \left(\frac{n\,\pi}{l_x}\right)^2$ and $\beta_m = \left(\frac{m\,\pi}{l_y}\right)^2$. Then the explicit representation in equation (5.12) is finite. That is,

$$\left| c^{i,j}_{n,m}(t_{k+1}) \right| < \infty.$$

To prove last inequality and equation (5.12), we will rewrite equation (5.9) as

$$\Big[1 + h_k\, \kappa\Big] v^{i,j}_{n,m}(t_{k+1}) = v^{i,j}_{n,m}(t_k) + h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_{k+1}) + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}.$$

Then plug $v^{i,j}_{n,m}(tk+1)$ from last equation into equation (5.8), we have

$$c^{i,j}_{n,m}(t_{k+1}) = c^{i,j}_{n,m}(t_k) + h_k\, \frac{v^{i,j}_{n,m}(t_k) + h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_{k+1}) + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{1 + h_k\, \kappa}.$$

Thus

$$\Big(1 \,+\, h_k\,\kappa\Big)\,c^{i,j}_{n,m}(t_{k+1}) \;=\; \Big(1 \,+\, h_k\,\kappa\Big)\,c^{i,j}_{n,m}(t_k) \,+\, h_k\,v^{i,j}_{n,m}(t_k) \\ +\,h_k^2\,f_{n,m}(u(t_k))\,c^{i,j}_{n,m}(t_{k+1}) \,+\, h_k\,g_{n,m}(u(t_k))\triangle_k\,W^{i,j}_{n,m}.$$

Hence

$$\Big(1 + h_k\,\kappa -\, h_k^2\,f_{n,m}(u(t_k))\Big)c^{i,j}_{n,m}(t_{k+1}) \;=\; \Big(1 + h_k\,\kappa\Big)\,c^{i,j}_{n,m}(t_k) + h_k\,v^{i,j}_{n,m}(t_k) \\ +\,h_k\,g_{n,m}(u(t_k))\triangle_k\,W^{i,j}_{n,m}.$$

Therefore

$$c^{i,j}_{n,m}(t_{k+1}) \;=\; \frac{\Big(1 + h_k\,\kappa\Big)\,c^{i,j}_{n,m}(t_k) + h_k\,v^{i,j}_{n,m}(t_k) \,+\, h_k\,g_{n,m}(u(t_k))\triangle_k\,W^{i,j}_{n,m}}{1 \,+\, h_k\,\kappa \,-\, h_k^2\,f_{n,m}(u(t_k))}.$$

but we know that

$$f_{n,m}(u(t_k)) \;=\; -\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2}\,\sum_{q,l=1}^{N}\,[c^{i,j}_{q,l}(t_k)]^2,$$

and so by the assumptions that $a_2 \,\geq\, 0$, $\kappa \,\geq\, 0$ and

$$\forall\,n, m\,\in\mathbb{N} : \Big[-\,\sigma^2(\lambda_n + \beta_m) \,+\, a_1\Big]\,h_k^2 \,<\, 1 \,+\, h_k\,\kappa,$$

we have

$$1 \,+\, h_k\,\kappa \,-\, h_k^2\,\Big[-\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2}\,\sum_{q,l=1}^{N}\,[c^{i,j}_{q,l}(t_k)]^2\Big]$$

$$=\, 1 \,+\, h_k\,\kappa \,+\, h_k^2\,\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2 \sum_{i,j=1}^{2}\,\sum_{q,l=1}^{N}\,[c^{i,j}_{q,l}(t_k)]^2\Big]$$

$$>\, 0\;\;\text{(by assumption)}.$$

Which implies that

$$\Big|c^{i,j}_{n,m}(t_{k+1})\Big| \,<\, \infty$$

2) Now we want to prove the explicit representation of equation (5.13) and also it is finite. To prove that, we will plug $c^{i,j}_{n,m}(t_{k+1})$ from equation (5.8) into equation (5.9), and then we have

$$\begin{aligned} v^{i,j}_{n,m}(t_{k+1}) \;&=\; v^{i,j}_{n,m}(t_k) \,+\, h_k\, f_{n,m}(u(t_k))\Big[c^{i,j}_{n,m}(t_k) \,+\, h_k\, v^{i,j}_{n,m}(t_k)\Big] \\ &\quad-\, h_k\, \kappa\, v^{i,j}_{n,m}(t_{k+1}) \,+\, g_{n,m}(u(t_k))\, \triangle_k\, W^{i,j}_{n,m} \\ &=\; v^{i,j}_{n,m}(t_k) \,+\, h_k\, f_{n,m}(u(t_k)) c^{i,j}_{n,m}(t_k) \,+\, h_k^2\, f_{n,m}(u(t_k))\, v^{i,j}_{n,m}(t_{k+1}) \\ &\quad-\, h_k\, \kappa\, v^{i,j}_{n,m}(t_{k+1}) \,+\, g_{n,m}(u(t_k))\, \triangle_k\, W^{i,j}_{n,m} \end{aligned}$$

which implies that

$$\begin{aligned} \Big[1 \,+\, h_k\, \kappa \,-\, h_k^2\, f_{n,m}(u(t_k))\Big] v^{i,j}_{n,m}(t_{k+1}) \;&=\; v^{i,j}_{n,m}(t_k) \,+\, h_k\, f_{n,m}(u(t_k)) c^{i,j}_{n,m}(t_k) \\ &\quad+\, g_{n,m}(u(t_k))\, \triangle_k\, W^{i,j}_{n,m}. \end{aligned}$$

Thus

$$v^{i,j}_{n,m}(t_{k+1}) \;=\; \frac{v^{i,j}_{n,m}(t_k) \,+\, h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_k) \,+\, g_{n,m}(u(t_k)) \triangle_k\, W^{i,j}_{n,m}}{1 \,+\, h_k\, \kappa \,-\, h_k^2\, f_{n,m}(u(t_k))} \tag{5.14}$$

And the same argument as above, we have

$$\Big| v^{i,j}_{n,m}(t_{k+1}) \Big| \;<\; \infty,$$

if all $\Big|c^{i,j}_{n,m}(t_k)\Big| \,<\, \infty$ and $\Big|v^{i,j}_{n,m}(t_k)\Big| \,<\, \infty$ that is, the explicit representation of $v^{i,j}_{n,m}(t_{k+1})$ in equation (5.13) is finite. □

Theorem 5.1.2. *(Explicit Representation + Boundedness of method (LIMM)). Assume that* $\big|c^{i,j}_{n,m}(t_k)\big| \,<\, \infty$, $\big|v^{i,j}_{n,m}(t_k)\big| \,<\, \infty, a_2 \,\geq\, 0,\, \kappa \,\geq\, 0,$ *and*

$$\forall\, n, m \in \mathbb{N} : \frac{\Big[-\, \sigma^2\, (\lambda_n \,+\, \beta_m) \,+\, a_1\Big]\, h_k^2}{4} \;<\; 1 \,+\, \frac{h_k\, \kappa}{2}.$$

Then the method (LIMM) governed by equation (5.10) and equation (5.11) has the non-exploding explicit representation

$$1)\, c^{i,j}_{n,m}(t_{k+1}) \;=\; \frac{\Big(1 \,+\, \dfrac{h_k\,\kappa}{2} \,+\, \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) c^{i,j}_{n,m}(t_k) \,+\, h_k\, v^{i,j}_{n,m}(t_k)}{1 \,+\, \dfrac{h_k\,\kappa}{2} \,-\, \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}} + \frac{\dfrac{h_k}{2}\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{1 \,+\, \dfrac{h_k\,\kappa}{2} \,-\, \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}} \tag{5.15}$$

and

$$2)\, v^{i,j}_{n,m}(t_{k+1}) \;=\; \frac{\Big(1 \,-\, \dfrac{h_k\,\kappa}{2} \,+\, \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v^{i,j}_{n,m}(t_k) \,+\, h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_k)}{1 \,+\, \dfrac{h_k\,\kappa}{2} \,-\, \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}} + \frac{g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{1 \,+\, \dfrac{h_k\,\kappa}{2} \,-\, \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}} \tag{5.16}$$

where

$$f_{n,m}(u(t_k)) \;=\; -\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c^{i,j}_{q,l}(t_k)]^2,$$

and

$$g_{n,m}(u(t_k)) \;=\; \Big(b_0 \,+\, b_1 \sqrt{\sum_{i,j=1}^{2} \sum_{q,l=1}^{N} (c^{i,j}_{q,l}(t_k))^2} \,+\, b_2\, ||v||_{\mathbb{L}^2(\mathbb{D})}\Big)\, \alpha^{i,j}_{n,m},$$

and

$$\triangle_k W^{i,j}_{n,m} \;=\; W^{i,j}_{n,m}(t_{k+1}) \,-\, W^{i,j}_{n,m}(t_k) \in \mathcal{N}(0, h_k),\; h_k \;=\; t_{k+1} - t_k$$

Proof. Suppose that $a_2 \,\geq\, 0$, $\kappa \,\geq\, 0$ and

$$1 \,+\, \frac{h_k\,\kappa}{2} \,+\, \frac{\Big[\sigma^2(\lambda_n + \beta_m) \,-\, a_1\Big]\, h_k^2}{4} \;>\; 0,$$

where $\lambda_n \;=\; \Big(\dfrac{n\,\pi}{l_x}\Big)^2$ and $\beta_m \;=\; \Big(\dfrac{m\,\pi}{l_y}\Big)^2$. Then the explicit representation in equation (5.15) is finite. That is,

$$\Big|c^{i,j}_{n,m}(t_{k+1})\Big| \;<\; \infty.$$

To prove last inequality and equation (5.15), we will rewrite equation (5.11) as

$$\begin{aligned}\Big[1 + \frac{h_k\,\kappa}{2}\Big]\, v^{i,j}_{n,m}(t_{k+1}) \qquad &= \Big[1 - \frac{h_k\,\kappa}{2}\Big] v^{i,j}_{n,m}(t_k) \,+\, \frac{h_k\, f_{n,m}(u(t_k))}{2}\, c^{i,j}_{n,m}(t_k)\\ &+ \frac{h_k\, f_{n,m}(u(t_k))}{2}\, c^{i,j}_{n,m}(t_{k+1}) \,+\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}.\end{aligned}$$

Hence

$$\begin{aligned}\Big(2 + h_k\,\kappa\Big)\, v^{i,j}_{n,m}(t_{k+1}) \qquad &= \Big(2 - h_k\,\kappa\Big) v^{i,j}_{n,m}(t_k) \,+\, h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_k)\\ &+ h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_{k+1}) \,+\, 2\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m},\end{aligned}$$

and therefore,

$$\begin{aligned}v^{i,j}_{n,m}(t_{k+1}) \qquad &= \frac{\Big(2 - h_k\,\kappa\Big) v^{i,j}_{n,m}(t_k) \,+\, h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_k)}{2 + h_k\,\kappa}\\ &+ \frac{h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_{k+1}) \,+\, 2\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{2 + h_k\,\kappa}.\end{aligned}$$

Now plug $v^{i,j}_{n,m}(t_{k+1})$ from last equation into equation (5.10), we have

$$\begin{aligned}c^{i,j}_{n,m}(t_{k+1}) \quad &= c^{i,j}_{n,m}(t_k) + \frac{h_k}{2}\, v^{i,j}_{n,m}(t_k) + \frac{h_k}{2}\Big[\frac{\Big(2 - h_k\kappa\Big) v^{i,j}_{n,m}(t_k) + h_k\, f_{n,m}(u(t_k)) c^{i,j}_{n,m}(t_k)}{2 + h_k\,\kappa}\Big]\\ &+ \frac{h_k}{2}\Big[\frac{h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_{k+1}) \,+\, 2\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{2 + h_k\,\kappa}\Big],\end{aligned}$$

thus

$$\begin{aligned}\Big(4 + 2\, h_k\,\kappa\Big)\, c^{i,j}_{n,m}(t_{k+1}) \;&=\; \Big(4 + 2\, h_k\,\kappa\Big)\, c^{i,j}_{n,m}(t_k) \,+\, h_k\,\Big(2 + h_k\,\kappa\Big)\, v^{i,j}_{n,m}(t_k)\\ &+\, h_k\Big(2 - h_k\kappa\Big) v^{i,j}_{n,m}(t_k) \,+\, h_k^2\, f_{n,m}(u(t_k)) c^{i,j}_{n,m}(t_k)\\ &+\, h_k^2\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_{k+1}) \,+\, 2\, h_k\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}.\end{aligned}$$

Hence

$$\begin{aligned}&\Big(4 + 2\, h_k\,\kappa \,-\, h_k^2\, f(u_{n,m}(t_k))\Big)\, c^{i,j}_{n,m}(t_{k+1})\\ &\quad = \Big(4 + 2\, h_k\,\kappa \,+\, h_k^2\, f_{n,m}(u(t_k))\Big)\, c^{i,j}_{n,m}(t_k) + 4\, h_k\, v^{i,j}_{n,m}(t_k) \,+\, 2\, h_k\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}.\end{aligned}$$

Therefore

$$c^{i,j}_{n,m}(t_{k+1}) \;=\; \frac{\Big(1 \,+\, \dfrac{h_k\,\kappa}{2} \,+\, \dfrac{h_k^2\, f(u_{n,m}(t_k))}{4}\Big) c^{i,j}_{n,m}(t_k) \,+\, h_k\, v^{i,j}_{n,m}(t_k)}{1 \,+\, \dfrac{h_k\,\kappa}{2} \,-\, \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}} \,+\, \frac{\dfrac{h_k}{2}\, g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{1 \,+\, \dfrac{h_k\,\kappa}{2} \,-\, \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}}$$

but we know that

$$f_{n,m}(u(t_k)) \;=\; -\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c^{i,j}_{q,l}(t_k)]^2,$$

and so by the assumptions that $a_2 \,\geq\, 0,\ \kappa \,\geq\, 0$ and

$$\forall\, n,m \in \mathbb{N} : \frac{\Big[-\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1\Big]\, h_k^2}{4} \;<\; 1 \,+\, \frac{h_k\,\kappa}{2},$$

we have

$$\begin{aligned} & 1 \,+\, \frac{h_k\,\kappa}{2} \,-\, \frac{h_k^2}{4}\Big[-\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c^{i,j}_{q,l}(t_k)]^2\Big] \\ & = 1 \,+\, \frac{h_k\,\kappa}{2} \,+\, \frac{h_k^2}{4}\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c^{i,j}_{q,l}(t_k)]^2\Big] \\ & > \; 0 \;\; (\textit{by assumption}). \end{aligned}$$

Which implies that

$$\Big|c^{i,j}_{n,m}(t_{k+1})\Big| \;<\; \infty$$

iff $\Big|c^{i,j}_{n,m}(t_k)\Big| \;<\; \infty$. 2) Now we want to prove the explicit representation of equation (5.16) and also it is finite. To prove that, we will plug $c^{i,j}_{n,m}(t_{k+1})$ from equation (5.10) into equation (5.11), and then we have

$$\begin{aligned} v^{i,j}_{n,m}(t_{k+1}) \;=\; & v^{i,j}_{n,m}(t_k) \,+\, \frac{h_k\, f_{n,m}(u(t_k))}{2} c^{i,j}_{n,m}(t_k) \,+\, \frac{h_k\, f_{n,m}(u(t_k))}{2}\Big[c^{i,j}_{n,m}(t_k) \\ & +\, \frac{h_k}{2}\, v^{i,j}_{n,m}(t_k) \,+\, \frac{h_k}{2}\, v^{i,j}_{n,m}(t_{k+1})\Big] \,-\, \frac{h_k\,\kappa}{2}\, v^{i,j}_{n,m}(t_k) \\ & -\, \frac{h_k\,\kappa}{2}\, v^{i,j}_{n,m}(t_{k+1}) \,+\, g_{n,m}(u(t_k))\, \triangle_k\, W^{i,j}_{n,m} \end{aligned}$$

i,e,

$$
\begin{aligned}
v^{i,j}_{n,m}(t_{k+1}) \;=\; & v^{i,j}_{n,m}(t_k) \;+\; \frac{h_k\, f_{n,m}(u(t_k))}{2} c^{i,j}_{n,m}(t_k) \;+\; \frac{h_k\, f_{n,m}(u(t_k))}{2} c^{i,j}_{n,m}(t_k) \\
& \frac{h_k^2\, f_{n,m}(u(t_k))}{4} v^{i,j}_{n,m}(t_k) \;+\; \frac{h_k^2\, f_{n,m}(u(t_k))}{4} v^{i,j}_{n,m}(t_{k+1}) \;-\; \frac{h_k\, \kappa}{2}\, v^{i,j}_{n,m}(t_k) \\
& -\; \frac{h_k\, \kappa}{2}\, v^{i,j}_{n,m}(t_{k+1}) \;+\; g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\Big[1 \;+\; \frac{h_k\, \kappa}{2} \;-\; \frac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big] v^{i,j}_{n,m}(t_{k+1}) \;=\; & \Big(1 \;-\; \frac{h_k\, \kappa}{2} \;+\; \frac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v^{i,j}_{n,m}(t_k) \\
& +\; h_k\, f_{n,m}(u(t_k)) c^{i,j}_{n,m}(t_k) \\
& +\; g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
v^{i,j}_{n,m}(t_{k+1}) \;=\; & \frac{\Big(1 \;-\; \frac{h_k\, \kappa}{2} \;+\; \frac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v^{i,j}_{n,m}(t_k) \;+\; h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_k)}{1 \;+\; \frac{h_k\, \kappa}{2} \;-\; \frac{h_k^2\, f(u_{n,m}(t_k))}{4}} \\
& +\; \frac{g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m}}{1 \;+\; \frac{h_k\, \kappa}{2} \;-\; \frac{h_k^2\, f_{n,m}(u(t_k))}{4}}
\end{aligned}
$$

And the same argument as above for our assumptions, we have

$$
\Big| v^{i,j}_{n,m}(t_{k+1}) \Big| \;<\; \infty,
$$

that is, the explicit representation of $v^{i,j}_{n,m}(t_{k+1})$ in equation (5.16) is finite. □

5.2 STABILITY OF METHODS (LIEM) AND (LIMM)

Lemma 5.2.1. *(Tower Property of Conditional Expectation). Let* $(\Omega,\, \mathcal{F},\, \mathbb{P})$ *be probability space.* $\forall\; \mathcal{F}_k \subseteq \mathcal{F}$*, then for any* $Z \in (\Omega,\, \mathcal{F},\, \mathbb{P})$*, we have*

$$
\mathbb{E}[\mathbb{E}(Z|\mathcal{F}_k)] \;=\; \mathbb{E}[Z].
$$

Corollary 5.2.2. *For (LIEM), Theorem 5.1.1, if* $\sigma^2(\lambda_n + \beta_m) \geq a_1$, $a_2 \geq 0$, *then* $\forall\, n, m = 1, ..., N \in \mathbb{N}$, *and* $k \in \mathbb{N}$,

$$1)\, \mathbb{E}[c^{i,j}_{n,m}(k+1)]^2 \leq \mathbb{E}\Big[\Big(1 + h_k\,\kappa\Big) c^{i,j}_{n,m}(k) + h_k\, v^{i,j}_{n,m}(k)\Big]^2 + h_k^3\, \mathbb{E}\Big(g_{n,m}(u(k))\Big)^2 \quad (5.17)$$

and

$$2)\, \mathbb{E}[v^{i,j}_{n,m}(k+1)]^2 \leq \mathbb{E}[v^{i,j}_{n,m}(k) + h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(k)]^2 + h_k\, \mathbb{E}\Big(g_{n,m}(u(k))\Big)^2 \quad (5.18)$$

Proof. 1) Let $a_2 \geq 0$ and $\sigma^2(\lambda_n + \beta_m) \geq a_1$, then $\forall\, n, m = 1, ..., N \in \mathbb{N}$, and $k \in \mathbb{N}$,and using (5.12) we have

$$\mathbb{E}[c^{i,j}_{n,m}(k+1)]^2 = \mathbb{E}\Big[\frac{\Big(1 + h_k\,\kappa\Big) c^{i,j}_{n,m}(k) + h_k\, v^{i,j}_{n,m}(k) + h_k\, g_{n,m}(u(k))\, \triangle_k W^{i,j}_{n,m}}{1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))}\Big]^2$$

$$= \mathbb{E}\Big\{\frac{\Big[\Big(1 + h_k\,\kappa\Big) c^{i,j}_{n,m}(k) + h_k\, v^{i,j}_{n,m}(k)\Big]^2}{[1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))]^2}$$

$$+ \frac{2\, h_k \Big[\Big(1 + h_k\,\kappa\Big) c^{i,j}_{n,m}(k) + h_k\, v^{i,j}_{n,m}(k)\Big]\, g_{n,m}(u(k))\, \triangle_k W^{i,j}_{n,m}}{[1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))]^2}$$

$$+ \frac{h_k^2 \Big(g_{n,m}(u)\Big)^2 \Big(\triangle_k W^{i,j}_{n,m}\Big)^2}{[1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))]^2}\Big\}$$

But $\triangle_k W^{i,j}_{n,m} = W^{i,j}_{n,m}(t_{k+1}) - W^{i,j}_{n,m}(t_k) \in \mathcal{N}(0,\, h_k)$ are independent and by using the tower property Lemma 5.1.3, so $\mathbb{E}[\triangle_k W^{i,j}_{n,m} | \mathcal{F}_k] = \mathbb{E}[\triangle_k W^{i,j}_{n,m}] = 0$ and $\mathbb{E}[(\triangle_k W^{i,j}_{n,m})^2] = h_k$. Thus

$$\mathbb{E}[c^{i,j}_{n,m}(k+1)]^2 = \mathbb{E}\Big[\frac{\Big[\Big(1 + h_k\,\kappa\Big) c^{i,j}_{n,m}(k) + h_k\, v^{i,j}_{n,m}(k)\Big]^2 + h_k^3 \Big(g_{n,m}(u)\Big)^2}{[1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))]^2}\Big]$$

Now, using the assumption that $a_2 \geq 0$ and $\sigma^2(\lambda_n + \beta_m) \geq a_1$, that is, $f(u_{n,m}) < 0$, then

$$1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k)) > 0,$$

thus

$$\mathbb{E}[c^{i,j}_{n,m}(k+1)]^2 = \mathbb{E}\Big[\Big(1 + h_k\,\kappa\Big) c^{i,j}_{n,m}(k) + h_k\, v^{i,j}_{n,m}(k)\Big]^2 + h_k^3\, \mathbb{E}\Big(g_{n,m}(u)\Big)^2$$

2) Let $a_2 \geq 0$ and $\sigma^2(\lambda_n + \beta_m) \geq a_1$, then $\forall\, n, m = 1, ..., N \in \mathbb{N}$, and $k \in \mathbb{N}$,and using (5.13) we have

$$\mathbb{E}[v^{i,j}_{n,m}(k+1)]^2 = \mathbb{E}\Big[\frac{v^{i,j}_{n,m}(k) + h_k\, f_{n,m}(u(k))c^{i,j}_{n,m}(k) + g_{n,m}(u(k)) \triangle_k W^{i,j}_{n,m}}{1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))}\Big]^2$$

$$= \mathbb{E}\Big\{\frac{\Big[v^{i,j}_{n,m}(k) + h_k\, f_{n,m}(u(k))c^{i,j}_{n,m}(k)\Big]^2}{[1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))]^2}$$

$$+ \frac{2\Big[v^{i,j}_{n,m}(k) + h_k\, f_{n,m}(u(k))c^{i,j}_{n,m}(k)\Big]\, g_{n,m}(u(k)) \triangle_k W^{i,j}_{n,m}}{[1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))]^2}$$

$$+ \frac{\Big(g_{n,m}(u)\Big)^2 \Big(\triangle_k W^{i,j}_{n,m}\Big)^2}{[1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))]^2}\Big\}$$

But $\triangle_k W^{i,j}_{n,m} = W^{i,j}_{n,m}(t_{k+1}) - W^{i,j}_{n,m}(t_k) \in \mathcal{N}(0,\, h_k)$ are independent and by using the tower property Lemma 5.1.3, so $\mathbb{E}[\triangle_k W^{i,j}_{n,m}|\mathcal{F}_k] = \mathbb{E}[\triangle_k W^{i,j}_{n,m}] = 0$ and $\mathbb{E}[(\triangle_k W^{i,j}_{n,m})^2] = h_k$. Thus

$$\mathbb{E}[v^{i,j}_{n,m}(k+1)]^2 = \mathbb{E}\Big[\frac{\Big[v^{i,j}_{n,m}(k) + h_k\, f_{n,m}(u(k))c^{i,j}_{n,m}(k)\Big]^2 + h_k\Big(g_{n,m}(u)\Big)^2}{[1 + h_k\,\kappa - h_k^2\, f_{n,m}(u(k))]^2}\Big]$$

Now, using the assumption that $a_2 \geq 0$ and $\sigma^2(\lambda_n + \beta_m) \geq a_1$, that is, $f_{n,m}(u) < 0$, then

$$1 + h_k\,\kappa - h_k^2\, f(u_{n,m}(k)) > 0,$$

thus

$$\mathbb{E}[v^{i,j}_{n,m}(k+1)]^2 = \mathbb{E}\Big[v^{i,j}_{n,m}(k) + h_k\, f_{n,m}(u(k))c^{i,j}_{n,m}(k)\Big]^2 + h_k\,\mathbb{E}\Big(g_{n,m}(u)\Big)^2$$

□

Corollary 5.2.3. *For (LIMM), Theorem 5.1.2, if* $\sigma^2(\lambda_n + \beta_m) \geq a_1$, $a_2 \geq 0$, *then* $\forall\, n, m = 1, ..., N \in \mathbb{N}$, *and* $k \in \mathbb{N}$,

$$1)\, \mathbb{E}[c^{i,j}_{n,m}(k+1)]^2 \leq 2\,\mathbb{E}\Big[\Big(1 + \frac{h_k\,\kappa}{2} + \frac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big)\, c^{i,j}_{n,m}(k)\Big]^2 + 2\, h_k^2\, \mathbb{E}[v^{i,j}_{n,m}(k)]^2 + h_k^3\, \mathbb{E}\Big(g_{n,m}(u)\Big)^2 \tag{5.19}$$

and

$$2)\,\mathbb{E}[v^{i,j}_{n,m}(k+1)]^2 \;\leq\; 2\,\mathbb{E}\Big[\Big(1 \,-\, \frac{h_k\,\kappa}{2} \,+\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}\Big)v^{i,j}_{n,m}(k)\Big]^2 \qquad (5.20)$$
$$+\; 2\,h_k^2\,\mathbb{E}\Big[f_{n,m}(u(k))c^{i,j}_{n,m}(k)\Big]^2 \,+\, h_k\,\mathbb{E}\Big(g_{n,m}(u)\Big)^2.$$

Proof. 1) Let $a_2 \geq 0$ and $\sigma^2(\lambda_n + \beta_m) \geq a_1$, then $\forall\, n, m = 1, ..., N \in \mathbb{N}$, and $k \in \mathbb{N}$, and using (5.15) we have

$$\begin{aligned}
\mathbb{E}[c^{i,j}_{n,m}(k+1)]^2 \;=\; &\mathbb{E}\Big[\frac{\Big(1 \,+\, \frac{h_k\,\kappa}{2} \,+\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}\Big)c^{i,j}_{n,m}(k) \,+\, h_k\,v^{i,j}_{n,m}(k)}{1 \,+\, \frac{h_k\,\kappa}{2} \,-\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}} \\
&+\, \frac{h_k\,g_{n,m}(u(k))\,\triangle_k\,W^{i,j}_{n,m}}{1 \,+\, \frac{h_k\,\kappa}{2} \,-\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}}\Big]^2 \\
=\mathbb{E}\Big\{&\frac{\Big[\Big(1 \,+\, \frac{h_k\,\kappa}{2} \,+\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}\Big)c^{i,j}_{n,m}(k) \,+\, h_k\,v^{i,j}_{n,m}(k)\Big]^2}{[1 \,+\, \frac{h_k\,\kappa}{2} \,-\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}]^2} \\
&+\, \frac{2\,h_k\,\Big[\Big(1 \,+\, \frac{h_k\,\kappa}{2} \,+\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}\Big)c^{i,j}_{n,m}(k)}{[1 \,+\, \frac{h_k\,\kappa}{2} \,-\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}]^2} \\
&+\, \frac{h_k\,v^{i,j}_{n,m}(k)\Big]\,g_{n,m}(u(k))\,\triangle_k\,W^{i,j}_{n,m}}{[1 \,+\, \frac{h_k\,\kappa}{2} \,-\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}]^2} \\
&+\, \frac{h_k^2\,\Big(g_{n,m}(u)\Big)^2\,\Big(\triangle_k\,W^{i,j}_{n,m}\Big)^2}{[1 \,+\, \frac{h_k\,\kappa}{2} \,-\, \frac{h_k^2\,f_{n,m}(u(t_k))}{4}]^2}\Big\}
\end{aligned}$$

But $\triangle_k\,W^{i,j}_{n,m} \;=\; W^{i,j}_{n,m}(t_{k+1}) \,-\, W^{i,j}_{n,m}(t_k) \,\in\, \mathcal{N}(0,\,h_k)$ are independent and by using the tower property Lemma 5.1.3, so $\mathbb{E}[\triangle_k\,W^{i,j}_{n,m}|\mathcal{F}_k] \;=\; \mathbb{E}[\triangle_k\,W^{i,j}_{n,m}] = 0$ and

$\mathbb{E}[(\triangle_k W_{n,m}^{i,j})^2] = h_k$. Thus

$$\mathbb{E}[c_{n,m}^{i,j}(k+1)]^2 \quad = \mathbb{E}\Big[\frac{\Big[\Big(1 + \dfrac{h_k\,\kappa}{2} + \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) c_{n,m}^{i,j}(k) + h_k\, v_{n,m}^{i,j}(k)\Big]^2}{[1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}]^2} + \frac{h_k^3\,\Big(g_{n,m}(u)\Big)^2}{[1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}]^2}\Big]$$

Now, using the assumption that $a_2 \geq 0$ and $\sigma^2(\lambda_n + \beta_m) \geq a_1$, that is, $f_{n,m}(u) < 0$, then

$$1 + \frac{h_k\,\kappa}{2} - \frac{h_k^2\, f_{n,m}(u(t_k))}{4} > 1,$$

thus

$$\begin{aligned}
&\mathbb{E}[c_{n,m}^{i,j}(k+1)]^2 \\
&\leq \mathbb{E}\Big[\Big(1 + \frac{h_k\,\kappa}{2} + \frac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) c_{n,m}^{i,j}(k) + h_k\, v_{n,m}^{i,j}(k)\Big]^2 + h_k^3\,\mathbb{E}\Big(g_{n,m}(u(t_k))\Big)^2 \\
&\leq 2\,\mathbb{E}\Big[\Big(1 + \frac{h_k\,\kappa}{2} + \frac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big)\, c_{n,m}^{i,j}(k)\Big]^2 + 2\,h_k^2\,\mathbb{E}[v_{n,m}^{i,j}(k)]^2 + h_k^3\,\mathbb{E}\Big(g_{n,m}(u(t_k))\Big)^2.
\end{aligned}$$

2) Let $a_2 \geq 0$ and $\sigma^2(\lambda_n + \beta_m) \geq a_1$, then $\forall\ n, m = 1, ..., N \in \mathbb{N}$, and $k \in \mathbb{N}$,and using (5.16) we have

$$\mathbb{E}[v_{n,m}^{i,j}(k+1)]^2 = \mathbb{E}\Big[\frac{\Big(1 - \dfrac{h_k\,\kappa}{2} + \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v_{n,m}^{i,j}(k)}{1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}} + \frac{h_k\, f_{n,m}(u(k)) c_{n,m}^{i,j}(k) + g_{n,m}(u(k))\,\triangle_k W_{n,m}^{i,j}}{1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}}\Big]^2$$

thus

$$
\begin{aligned}
&\mathbb{E}[v^{i,j}_{n,m}(k+1)]^2\\
&= \mathbb{E}\Big\{ \frac{\Big[\Big(1 - \dfrac{h_k\,\kappa}{2} + \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v^{i,j}_{n,m}(k) + h_k\, f_{n,m}(u(k)) c^{i,j}_{n,m}(k)\Big]^2}{[1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}]^2}\\
&+ \frac{2\Big(1 - \dfrac{h_k\,\kappa}{2} + \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v^{i,j}_{n,m}(k) g_{n,m}(u(k)) \triangle_k W^{i,j}_{n,m}}{[1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}]^2}\\
&+ \frac{2\, h_k\, f_{n,m}(u(k))\, c^{i,j}_{n,m}(k)\, g_{n,m}(u(k)) \triangle_k W^{i,j}_{n,m} + \Big(g_{n,m}(u)\Big)^2 \Big(\triangle_k W^{i,j}_{n,m}\Big)^2}{[1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}]^2}
\end{aligned}
$$

But $\triangle_k W^{i,j}_{n,m} = W^{i,j}_{n,m}(t_{k+1}) - W^{i,j}_{n,m}(t_k) \in \mathcal{N}(0,\, h_k)$ are independent and by using the tower property Lemma 5.1.3, we find that $\mathbb{E}[\triangle_k W^{i,j}_{n,m} | \mathcal{F}_k] = \mathbb{E}[\triangle_k W^{i,j}_{n,m}] = 0$ and $\mathbb{E}[(\triangle_k W^{i,j}_{n,m})^2] = h_k$. Thus

$$
\begin{aligned}
\mathbb{E}[v^{i,j}_{n,m}(k+1)]^2 &= \mathbb{E}\Big[\frac{\Big[\Big(1 - \dfrac{h_k\,\kappa}{2} + \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v^{i,j}_{n,m}(k) + h_k\, f_{n,m}(u(k)) c^{i,j}_{n,m}(k)\Big]^2}{[1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}]^2}\\
&\qquad + \frac{h_k \Big(g_{n,m}(u)\Big)^2}{[1 + \dfrac{h_k\,\kappa}{2} - \dfrac{h_k^2\, f_{n,m}(u(t_k))}{4}]^2}\Big]
\end{aligned}
$$

Now, using the assumption that $a_2 \geq 0$ and $\sigma^2(\lambda_n + \beta_m) \geq a_1$, that is, $f_{n,m}(u) < 0$, then

$$
1 + \frac{h_k\,\kappa}{2} - \frac{h_k^2\, f_{n,m}(u(t_k))}{4} > 1,
$$

thus

$$
\begin{aligned}
\mathbb{E}[v^{i,j}_{n,m}(k+1)]^2 &\leq \mathbb{E}\Big[\Big(1 - \frac{h_k\,\kappa}{2} + \frac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v^{i,j}_{n,m}(k)\\
&\qquad + h_k\, f_{n,m}(u(k)) c^{i,j}_{n,m}(k)\Big]^2 + h_k\, \mathbb{E}\Big(g_{n,m}(u)\Big)^2\\
&\leq 2\,\mathbb{E}\Big[\Big(1 - \frac{h_k\,\kappa}{2} + \frac{h_k^2\, f_{n,m}(u(t_k))}{4}\Big) v^{i,j}_{n,m}(k)\Big]^2\\
&\qquad + 2\, h_k^2\, \mathbb{E}\Big[f_{n,m}(u(k)) c^{i,j}_{n,m}(k)\Big]^2 + h_k\, \mathbb{E}\Big(g_{n,m}(u)\Big)^2.
\end{aligned}
$$

$\square$

5.3 LOCAL MEAN CONSISTENCY OF NUMERICAL METHODS

In this section, for convenience we refer to the following abbreviation: $(\forall\, i,j)$, for $u = u_{n,m} \in \mathbb{R}^{N\times N}$ *and* $f = f_{n,m},\ g = g_{n,m}$,

$$c^{i,j}_{n,m}(t_{k+1}) = c^{i,j}_{n,m}(t_k) + h_k\, v^{i,j}_{n,m}(t_k) \tag{5.21}$$

and

$$\begin{aligned} v^{i,j}_{n,m}(t_{k+1}) &= v^{i,j}_{n,m}(t_k) + h_k\, f_{n,m}(u(k))c^{i,j}_{n,m}(k) \\ &\quad - \kappa\, v^{i,j}_{n,m}(t_k) + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m} \end{aligned} \tag{5.22}$$

where $f_{n,m}(u) = \Big(-\sigma^2(\lambda_n + \beta_m) + a_1 - a_2\, ||u||^2_{\mathbb{L}^2(\mathbb{D})} \Big)\, c_{n,m}(t_k)$ and $g_{n,m}(u) = \Big(b_0 + b_1\, ||u||_{\mathbb{L}^2(\mathbb{D})} + b_2\, ||v||_{\mathbb{L}^2(\mathbb{D})} \Big)\, \alpha_{n,m}$

Definition. A numerical method with the scheme

$$\hat{u}(t+h) = \hat{u}(t) + h\, f(\hat{u}(t)) + g(\hat{u}(t)) \triangle W(t) \tag{5.23}$$

where $h = t_{k+1} - t_k = \int_{t_k}^{t_{k+1}} ds$,
$\triangle W^k = W(t_{k+1}) - W(t_k) = \int_{t_k}^{t_{k+1}} dW(s) \in \mathcal{N}(0,h)$ and $f(u(t))$ and $g(u(t))$ as above applied to SDE (5.23) is said to be **mean consistent** with rate $r_0 > 0$ on $\mathbb{D}$ $iff\ \exists K_0^c =$ consistency constant $\forall\, 0 \le t \le t+h \le T$, where h is sufficiently small, i.e., $0 < h \le \delta \le 1$, $\forall\, u(t) \in H$, where $H := \{u \in \mathbb{L}^2(\mathbb{D}) \big| \dot{u} \in \mathbb{L}^2(\mathbb{D})\}$, and $||u||_H = \sqrt{||u||^2_{\mathbb{L}^2(\mathbb{D})} + ||\dot{u}||^2_{\mathbb{L}^2(\mathbb{D})}} : (\mathcal{F}_t,\, \mathcal{B}(\mathbb{L}^2))$ measurable, then

$$\Big|\Big| \mathbb{E}\Big(u_{n,m}(t+h|t,\mathcal{F}_t) - \hat{u}_{n,m}(t+h|t,\mathcal{F}_t) \Big) \Big|\Big|_d \le K_0^c \Big(V(u(t)) \Big)\, h^{r_0}$$

where $\hat{u}(t+h|t,u(t)) = \hat{u}(t) + \int_t^{t+h} f(u(s))\, ds + \int_t^{t+h} g(u(s))\, dW(s)\, ds$.

Lemma 5.3.1. *Let* $a_2 = 0$, $\forall\, u, \hat{u} \in \mathbb{R}^N$, $\forall\, 0 \le s, t \le T$, *we have* $||f(u) - f(\hat{u})||_N \le L_f\, ||u - \hat{u}||_N$, *where*
$L_f(N) = \max_{n,m=1,\ldots,N} \sigma^2 (\lambda_n + \beta_m) - a_1 + O(N^2) = N^2\pi^2\Big\{\frac{1}{l_x^2} + \frac{1}{l_y^2}\Big\} - a_1 \ge 0$

Proof. Let $a_2 = 0$, $\sigma^2 (\lambda_n + \beta_m) - a_1 \ge 0$, then we have

$$f(u) = \Big(\Big(-\sigma^2 (\lambda_n + \beta_m) + a_1\Big)\, u_{n,m}(t)\Big)_{n,m \in I_N}.$$

To simplify, let $l_f(n,m) = -\sigma^2 (\lambda_n + \beta_m) + a_1$. Thus

$$\begin{aligned} ||f(u(t)) - f(\hat{u}(s))||_N &= ||l_f(n,m)\, u(t) - l_f(n,m)\, \hat{u}(s)||_N \\ &\le l_f\, ||u(t) - \hat{u}(s)||_N \\ &\le L_f(N)\, ||u(t) - \hat{u}(s)||_N \end{aligned}$$

□

Lemma 5.3.2. $\forall\, u, \hat{u} \in \mathbb{R}^N$, $\forall\, 0 \le s, t \le T$, *we have* $||g(u) - g(\hat{u})||_N \le L_g\, ||u - \hat{u}||_N$, *where* $L_g = |b_1|\, ||\alpha||_N \le |b_1|\, ||\alpha||_\infty$.

Proof. We know that $g(u(t)) = \Big(b_0 + b_1\, ||u(t)||_N\Big)\,\alpha$, thus

$$\begin{aligned} ||g(u) - g(\hat{u})||_N &= \Big|\Big|\Big(b_0 + b_1\, ||u||_N\Big)\,\alpha - \Big(b_0 + b_1\, ||\hat{u}||_N\Big)\,\alpha\Big|\Big| \\ &\le |b_1|\, ||\alpha||\, ||u - \hat{u}||_N \\ &\le L_g\, ||u - \hat{u}||_N\,, \end{aligned}$$

where $L_g = |b_1|\, ||\alpha||_N$ □

Theorem 5.3.3. *The method (LIEM) given by*

$$c^{i,j}_{n,m}(t_{k+1}) = c^{i,j}_{n,m}(t_k) + h_k\, v^{i,j}_{n,m}(t_{k+1}) \tag{5.24}$$

$$\begin{aligned} v^{i,j}_{n,m}(t_{k+1}) &= v^{i,j}_{n,m}(t_k) + h_k\Big[f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_{k+1}) - \kappa\, v^{i,j}_{n,m}(t_{k+1})\Big] \\ &\quad + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m} \end{aligned} \tag{5.25}$$

is mean consistent with rate $r_0 \geq 1.5$, *where*

$$f_{n,m}(u(t_k)) = -\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2,$$

and

$$g_{n,m}(u(t_k)) = \Big(b_0 + b_1 \sqrt{\sum_{i,j=1}^{2} \sum_{q,l=1}^{N} (c_{q,l}^{i,j}(t_k))^2} + b_2 \, ||v|| \Big) \alpha_{n,m},$$

and

$$\triangle_k W_{n,m}^{i,j} = W_{n,m}^{i,j}(t_{k+1}) - W_{n,m}^{i,j}(t_k) \in \mathcal{N}(0, h_k), \; h_k = t_{k+1} - t_k.$$

But in the Fourier space $||u||^2_{\mathbb{L}^2(\mathbb{D})} = ||c||^2_{l^2}$, *so*

$$f_{n,m}(u(t_k)) = -\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \, ||c||^2_{l^2},$$

and

$$g_{n,m}(u(t_k)) = \Big(b_0 + b_1 \, ||c||_{l^2} + b_2 \, ||v||^2_{l^2} \Big) \alpha_{n,m},$$

Proof. We know that from Theorem 5.1.1 that

$$c_{n,m}^{i,j}(t_{k+1}) = \frac{\Big(1 + h_k \, \kappa\Big) c_{n,m}^{i,j}(t_k) + h_k \, v_{n,m}^{i,j}(t_k) + h_k \, g_{n,m}(u(t_k)) \triangle_k W_{n,m}^{i,j}}{1 + h_k \, \kappa - h_k^2 \, f_{n,m}(u(t_k))}$$

so

$$c_{n,m}^{i,j}(t_{k+1}) = \frac{\Big(1 + h_k \, \kappa\Big) c_{n,m}^{i,j}(t_k) + h_k \, v_{n,m}^{i,j}(t_k) + h_k \, g_{n,m}(u(t_k)) \triangle_k W_{n,m}^{i,j}}{1 + h_k \, \kappa - h_k^2 \Big(-\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \, ||c(k)||^2_{l^2} \Big)}$$

by adding and subtracting in the numerator

$$h_k^2 \Big(-\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \, ||c(k)||^2_{l^2},$$

we have

$$
\begin{aligned}
c^{i,j}_{n,m}(t_{k+1}) \,&=\, \frac{\Big\{1 + h_k\,\kappa + h_k^2\Big(\sigma^2(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\Big)\Big\}c^{i,j}_{n,m}(t_k)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\left[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\right]} \\
&- \frac{h_k^2\Big(\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\Big)c^{i,j}_{n,m}(t_k)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\left[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\right]} \\
&+ \frac{h_k\,v^{i,j}_{n,m}(t_k) \,+\, h_k\,g_{n,m}(u(t_k))\triangle_k\,W^{i,j}_{n,m}}{1 \,+\, h_k\,\kappa \,+\, h_k^2\left[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\right]} \\
&= c^{i,j}_{n,m}(t_k) + h_k\frac{v^{i,j}_{n,m}(t_k) - h_k\Big(\sigma^2(\lambda_n + \beta_m) - a_1 + a_2\,||c(k)||^2_{l^2}\Big)c^{i,j}_{n,m}(t_k)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\left[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\right]} \\
&+ \frac{h_k\,g_{n,m}(u(t_k))\triangle_k\,W^{i,j}_{n,m}}{1 \,+\, h_k\,\kappa \,+\, h_k^2\left[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\right]}
\end{aligned}
$$

let

$$
\hat{f}_c(u(t)) \,=\, \frac{v^{i,j}_{n,m}(t_k) - h_k\Big(\sigma^2(\lambda_n + \beta_m) - a_1 + a_2\,||c(k)||^2_{l^2}\Big)c^{i,j}_{n,m}(t_k)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\left[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\right]}
$$

and

$$
\hat{g}_c(u(t)) \,=\, \frac{h_k\,g_{n,m}(u(t_k))}{1 \,+\, h_k\,\kappa \,+\, h_k^2\left[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||^2_{l^2}\right]}
$$

so we can define

$$
\hat{c}(t \,+\, h_k) \,=\, \hat{c}(t) \,+\, h_k\,\hat{f}_c(u(t)) \,+\, \hat{g}_c(u(t))\,\triangle_k\,W \tag{5.26}
$$

but we know that

$$
d\,c(t) \,=\, v(t)\,d\,t,
$$

then

$$
c(t \,+\, h_k) \,=\, c(t) \,+\, \int_t^{t+h} v(s)d\,.s \tag{5.27}
$$

Now, from equation (5.26) and equation (5.27) and local Lipschitz which means that

$\hat{c}(t) = c(t)\, and\, \hat{v}(t) = v(t)$, then

$$\begin{aligned} c(t + h_k) - \hat{c}(t + h_k) &= c(t) + \int_t^{t+h} v(s)d\,s - \hat{c}(t) - h_k\, \hat{f}_c(u(t)) - \hat{g}_c(u(t))\, \triangle_k W \\ &= \int_t^{t+h} v(s)d\,s - \int_t^{t+h} \hat{f}_c(u(t))d\,s - \int_t^{t+h} \hat{g}_c(u(t))\, d\,W(s) \\ &= \int_t^{t+h} \Big(v(s) - \hat{f}_c(u(t))\Big)ds - \int_t^{t+h} \hat{g}_c(u(t))\, d\,W(s) \end{aligned}$$

pulling the expectation and using the fact that

$$\mathbb{E} \int_t^{t+h} \hat{g}_c(u(t))\, d\,W(s) = 0 \ \ (martingale),$$

then we have

$$\begin{aligned} \Big|\Big|\mathbb{E}\Big[c(t + h_k) - \hat{c}(t + h_k)\Big]\Big|\Big|_N &= \Big|\Big|\mathbb{E} \int_t^{t+h} \Big(v(s) - \hat{f}_c(u(t))\Big)d\,s\Big|\Big|_N \\ &\le \int_t^{t+h} \Big|\Big|\mathbb{E}\Big(v(s) - \hat{f}_c(u(t))\Big)\Big|\Big|_N \, d\,s \end{aligned}$$

we have the last inequality because of non random partition of $(t_n)_{n \in \mathbb{N}}$. If we adding and subtracting $v(t)$, so

$$\begin{aligned} \Big|\Big|\mathbb{E}\Big[c(t + h_k) - \hat{c}(t + h_k)\Big]\Big|\Big|_N &\le \int_t^{t+h} \Big|\Big|\mathbb{E}\Big(v(s) - v(t)\Big)\Big|\Big|_N \, d\,s \\ &+ \int_t^{t+h} \Big|\Big|\mathbb{E}\Big(v(t) - \hat{f}_c(u(t))\Big)\Big|\Big|_N \, d\,s \qquad (5.28) \end{aligned}$$

but

$$
\begin{aligned}
v(t) \,-\, \hat{f}_c(u(t)) \,&=\, v(t) \,-\, \frac{v(t) - h_k\Big(\sigma^2(\lambda_n + \beta_m) - a_1 + a_2\,||c(k)||_{l^2}^2\Big)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big]} \\
&= \frac{v(t) \,+\, h_k\,\kappa\,v(t) \,+\, h_k^2\Big(\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big)v(t)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big]} \\
&\quad + \frac{-v(t) \,+\, h_k\Big(\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big)v(t)c(t)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big]} \\
&= h_k\Bigg\{\frac{\Big(\kappa \,+\, h_k\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]\Big)v(t)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big]} \\
&\quad + \frac{h_k\,a_2\,||c(k)||_{l^2}^2\,v(t) \,+\, \Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]c(t)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big]} \\
&\quad + \frac{a_2\,||c(k)||_{l^2}^2\,c(t)}{1 \,+\, h_k\,\kappa \,+\, h_k^2\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big]}\Bigg\}
\end{aligned}
$$

and by using that

$$
1 \,+\, h_k\,\kappa \,+\, h_k^2\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1 \,+\, a_2\,||c(k)||_{l^2}^2\Big] \,>\, 1,
$$

thus

$$
\begin{aligned}
&\Big|\Big|\mathbb{E}\Big[v(t) \,-\, \hat{f}_c(u(t))\Big]\Big|\Big|_N \\
&\leq\, h_k\Big\{\Big(\kappa \,+\, h_k\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]\Big)\,||\mathbb{E}[v(t)]||_N \,+\, h_k\,a_2\,\mathbb{E}\,||||c(t)||_{l^2}\;v(t)||_N \\
&\qquad + \,\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]\,||\mathbb{E}[c(t)]||_N + a_2\,\mathbb{E}\,||||c(t)||_{l^2}\;c(t)||_N\Big\} \\
&\leq\, h_k\Big\{\Big(\kappa \,+\, h_k\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]\Big)\,||\mathbb{E}[v(t)]||_N \,+\, h_k\,a_2\,\Big|\Big|\mathbb{E}\,||c(t)||_{l^2}^4\Big|\Big|_N^{\frac{1}{2}}\;\Big|\Big|\mathbb{E}[v^2(t)]\Big|\Big|_N^{\frac{1}{2}} \\
&\qquad + \,\Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]\,||\mathbb{E}[c(t)]||_N + \,a_2\,\Big|\Big|\mathbb{E}\,||c(t)||_{l^2}^4\Big|\Big|_N^{\frac{1}{2}}\;\Big|\Big|\mathbb{E}[c^2(t)]\Big|\Big|_N^{\frac{1}{2}}\Big\}
\end{aligned}
$$

therefore

$$\Big|\Big|\mathbb{E}\Big[v(t)\,-\,\hat{f}_c(u(t))\Big]\Big|\Big|_N$$

$$\leq\, h_k\,\Big\{\Big(\kappa\,+\,h_k\Big[\sigma^2\,(\lambda_n\,+\,\beta_m)\,-\,a_1\Big]\Big)\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}$$

$$+\,2\,h_k\,\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}$$

$$+\,\Big[\sigma^2\,(\lambda_n\,+\,\beta_m)\,-\,a_1\Big]\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}+\,2\,\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big\}$$

$$\leq\, h_k\,\Big\{\Big(\kappa\,+\,h_k\Big[\sigma^2\,(\lambda_n\,+\,\beta_m)\,-\,a_1\Big]+\,\sigma^2\,(\lambda_n\,+\,\beta_m)-\,a_1\Big)\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}$$

$$+\,2\,(1\,+\,h_k)\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big\}$$

$$\leq\, h_k\Big\{\kappa\,+\,(h_k+1)\Big[\sigma^2\,(\lambda_n\,+\,\beta_m)\,-\,a_1+\,2\,\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big]\Big\}\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}$$

then when we take the conditional expectation we get

$$\Big|\Big|\mathbb{E}\Big[\,\Big(v(t)\,-\,\hat{f}_c(u(t))\Big)\,\big|u(t)\Big]\Big|\Big|_N$$

$$\leq\, h_k\Big\{\kappa\,+\,(h_k+1)\Big[\sigma^2\,(\lambda_n\,+\,\beta_m)\,-\,a_1+\,2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big]\Big\}\Big(V(u(t))\Big)^{\frac{1}{2}}$$

hence

$$\int_t^{t+h}\Big|\Big|\mathbb{E}\Big[\,\Big(v(t)\,-\,\hat{f}_c(u(t))\Big)\,\big|u(t)\Big]\Big|\Big|_N\,d\,s$$

$$\leq\, h_k^2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big\{\kappa\,+\,(h_k\,+\,1)\Big[\sigma^2\,(\lambda_n\,+\,\beta_m)\,-\,a_1\,+\,2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big]\Big\}$$

$$\leq\, h_k^2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big\{\kappa\,+\,2\,\Big[\sigma^2\max_{1\leq n,m\leq N}(\lambda_n\,+\,\beta_m)\,-\,a_1\,+\,2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big]\Big\}$$

and for large N, we have

$$\int_t^{t+h}\Big|\Big|\mathbb{E}\Big[\,\Big(v(t)\,-\,\hat{f}_c(u(t))\Big)\,\big|u(t)\Big]\Big|\Big|_N\,d\,s$$

$$\leq\, h_k^2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big\{\kappa\,+\,2\,\Big[\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})\,+\,2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big]\Big\}$$

$$\leq\,\Big\{\kappa\,+\,2\,\Big[\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})\,+\,2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big]\Big\}\,(1+V(u(t)))\,h_k^2$$

which implies that

$$\int_t^{t+h} \Big|\Big|\mathbb{E}\Big[\Big(v(t) \,-\, \hat{f}_c(u(t))\Big)\,|u(t)\Big]\Big|\Big|_N \, d\,s \;\le\; K_1\,\tilde{V}(u(t))\,h_k^2 \tag{5.29}$$

where $\tilde{V}(u(t)) = 1 + V(u(t))$ and

$$K_1 \;=\; \kappa \,+\, 2\Big[\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2} + \frac{1}{l_y^2}) \,+\, 2\Big(V(u(t))\Big)^{\frac{1}{2}}\Big].$$

And we know that

$$\begin{aligned} d\,v_{n,m}(r) \quad &= \Big[-\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2\sum_{n,m=1}^{N}[c_{n,m}(r)]^2\Big]c_{n,m}(r) \\ &\qquad -\,\kappa\,v_{n,m}(r) \,+\, g(u(r))\,d\,W(r) \end{aligned}$$

if $s \,>\, t$, then

$$\begin{aligned} &\Big|\Big|\mathbb{E}\Big[v_{n,m}(s) \,-\, v_{n,m}(t)\Big]\Big|\Big|_N \\ &\le \Big|\Big|\Big[-\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1\Big]\mathbb{E}\int_t^s c_{n,m}(r)\,d\,r\Big|\Big|_N \\ &\quad +\, a_2\,\Big|\Big|\mathbb{E}\int_t^s \Big[\sum_{n,m=1}^{N}[c_{n,m}(r)]^2\Big]c_{n,m}(r)\,d\,r\Big|\Big|_N \\ &\quad +\,\kappa\,\Big|\Big|\mathbb{E}\int_t^s v_{n,m}(r)d\,r\Big|\Big|_N \,+\, \Big|\Big|\mathbb{E}\int_t^s g(u(r))\,d\,W(r)\Big|\Big|_N \\ &= \Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]\,\Big|\Big|\int_t^s \mathbb{E}c_{n,m}(r)\,d\,r\Big|\Big|_N \\ &\quad +\, a_2]\,\Big|\Big|\int_t^s \mathbb{E}\,||u(r)||^2\;c_{n,m}(r)\,d\,r\Big|\Big|_N \,+\,\kappa\,\Big|\Big|\int_t^s \mathbb{E}v_{n,m}(r)\,d\,r\Big|\Big|_N \\ &\le \Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]\int_t^s \Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\,d\,r \\ &\quad +\, a_2\int_t^s \Big(\mathbb{E}\,||u(r)||_N^4\Big)^{\frac{1}{2}}\Big(\mathbb{E}c^2(r)\Big)^{\frac{1}{2}}\,d\,r \,+\,\kappa\int_t^s \Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}} \\ &\le \Big[\sigma^2\,(\lambda_n \,+\, \beta_m) \,-\, a_1\Big]\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big|s \,-\, t\Big| \\ &\quad +\, 2\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big|s \,-\, t\Big| \\ &\quad +\,\kappa\Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big|s \,-\, t\Big|. \end{aligned}$$

Pulling the conditional expectation over the last inequality leads to

$$\begin{aligned}
&\Big|\Big|\mathbb{E}\Big[\,(v_{n,m}(s)\,-\,v_{n,m}(t))\,|u(t)\Big]\Big|\Big|_N \\
&\le\,\Big[\sigma^2\,(\lambda_n\,+\,\beta_m)\,-\,a_1\Big]\Big(V(u(t))\Big)^{\frac{1}{2}}\Big|s\,-\,t\Big| + 2\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big(V(u(t))\Big)^{\frac{1}{2}}\Big|s\,-\,t\Big| \\
&\quad+\,\kappa\,\Big(V(u(t))\Big)^{\frac{1}{2}}\Big|s\,-\,t\Big|
\end{aligned}$$

hence

$$\begin{aligned}
&\Big|\Big|\mathbb{E}\Big[\,(v_{n,m}(s)\,-\,v_{n,m}(t))\,|u(t)\Big]\Big|\Big|_N \\
&\le\,\Big[\kappa+\,\sigma^2\,\max_{1\le\,n,m\le\,N}(\lambda_n\,+\,\beta_m)\,-\,a_1\,+\,2\Big(V(u(t))\Big)^{\frac{1}{2}}\Big]\Big(V(u(t))\Big)^{\frac{1}{2}}\Big|s\,-\,t\Big|
\end{aligned}$$

and for large N, we find that

$$\begin{aligned}
&\Big|\Big|\mathbb{E}\Big[\,(v_{n,m}(s)\,-\,v_{n,m}(t))\,|u(t)\Big]\Big|\Big|_N \\
&\le\,\Big[\kappa+\,\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_x^2})\,+\,2\Big(V(u(t))\Big)^{\frac{1}{2}}\Big]\Big(V(u(t))\Big)^{\frac{1}{2}}\Big|s\,-\,t\Big| \\
&\le\,K_2\,(1+V(u(t)))\Big|s\,-\,t\Big| \\
&\le\,K_2\,\tilde{V}(u(t))\Big|s\,-\,t\Big|
\end{aligned}$$

where $\tilde{V}(u(t)) = 1\,+\,V(u(t))$ and

$$K_2\,=\,\kappa+\,\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_x^2})\,+\,2\Big(V(u(t))\Big)^{\frac{1}{2}},$$

thus

$$\begin{aligned}
\int_t^{t+h}\Big|\Big|\mathbb{E}\Big[\,(v_{n,m}(s)\,-\,v_{n,m}(t))\,|u(t)\Big]\Big|\Big|_N\,d\,s\,&\le\,K_2\int_t^{t+h}\tilde{V}(u(t))\Big|s\,-\,t\Big|\,d\,s \\
&=\,K_2\,\tilde{V}(u(t))\,\frac{h^2}{2}
\end{aligned}$$

which implies that

$$\int_t^{t+h}\Big|\Big|\mathbb{E}\Big[\,(v_{n,m}(s)\,-\,v_{n,m}(t))\,|u(t)\Big]\Big|\Big|_N\,d\,s\quad\le\,K_2\,\tilde{V}(u(t))\,h_k^2. \tag{5.30}$$

therefore,

$$\int_t^{t+h} \Big|\Big|\mathbb{E}\Big(v(s)\,-\,\hat{f}_c(u(t))\Big)\Big|\Big|_N\, d\,s \le\, K_1\,\tilde{V}(u(t))\,h_k^2\,+\,K_2\,\tilde{V}(u(t))\,h_k^2$$

that is

$$\int_t^{t+h} \Big|\Big|\mathbb{E}\Big(v(s)\,-\,\hat{f}_c(u(t))\Big)\Big|\Big|_N\, d\,s \le\, K_0^c\,\tilde{V}(u(t))\,h_k^2. \tag{5.31}$$

where

$$\begin{aligned}
K_0^c \,&=\, K_1\,+\,K_2\\
&=\kappa\,+\,2\Big[\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})\,+\,2\Big(V(u(t))\Big)^{\frac{1}{2}}\Big]\\
&\quad+\,\kappa+\,\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_x^2})\,+\,2\Big(V(u(t))\Big)^{\frac{1}{2}}\\
&=2\kappa+\,3\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_x^2})\,+\,6\Big(V(u(t))\Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus by using inequalities (5.29) and (5.30), we have

$$\Big|\Big|\mathbb{E}\Big[c(t+h)\,-\,\hat{c}(t+h)\Big]\Big|\Big|_N\,\le\,K_0^c\,\tilde{V}(u(t))\,h_k^2. \tag{5.32}$$

For $v_{n,m}^{i,j}(t_k)$, to simplify the inequalities let

$$F(u(t))=\sigma^2\,(\lambda_n\,+\,\beta_m)\,-\,a_1\,+\,a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,,$$

and we know from Theorem 5.1.1 that

$$v_{n,m}^{i,j}(t_{k+1})\,=\,\frac{v_{n,m}(t_k)\,+\,h_k\Big[-\,F(u(t))\Big]c_{n,m}(t_k)+g(u(t_k))\,\triangle_k\,W_{n,m}^{i,j}}{1\,+\,h_k\,\kappa\,+\,h_k^2\,\Big[F(u(t))\Big]}$$

thus

$$\begin{aligned}
&v_{n,m}^{i,j}(t_{k+1})\\
&=\frac{\Big(1\,+\,h_k\,\kappa\,+\,h_k^2\,\Big[F(u(t))\Big]\Big)v_{n,m}(t_k)+\Big(-\,h_k\,\kappa\,-\,h_k^2\Big[F(u(t))\Big]\Big)v_{n,m}(t_k)}{1\,+\,h_k\,\kappa\,+\,h_k^2\,\Big[F(u(t))\Big]}\\
&\quad+\frac{h_k\Big[-\,F(u(t))\Big]c_{n,m}(t_k)+g(u(t_k))\,\triangle_k\,W_{n,m}^{i,j}}{1\,+\,h_k\,\kappa\,+\,h_k^2\,\Big[F(u(t))\Big]}
\end{aligned}$$

hence

$$
\begin{aligned}
v^{i,j}_{n,m}(t_{k+1}) \;&=\; v_{n,m}(t_k) -\; h_k \Bigg\{ \frac{\Big(\kappa + h_k \Big[F(u(t))\Big]\Big) v_{n,m}(t_k)}{1 \,+\, h_k\, \kappa \,+\, h_k^2 \,\Big[F(u(t))\Big]} \\
&\qquad - \;\frac{\Big(-\;F(u(t))\Big) c_{n,m}(t_k)}{1 \,+\, h_k\, \kappa \,+\, h_k^2 \,\Big[F(u(t))\Big]} \Bigg\} + \;\frac{g(u(t_k))\, \triangle_k\, W^{i,j}_{n,m}}{1 \,+\, h_k\, \kappa \,+\, h_k^2 \,\Big[F(u(t))\Big]} \\
&= \; v_{n,m}(t_k) \,+\, h_k\, \hat{f}_v(u(t)) \,+\, \hat{g}_v(u(t)) \triangle_k\, W
\end{aligned}
$$

where

$$
\hat{f}_v(u(t)) \;=\; \frac{\Big(-\;F(u(t))\Big) c_{n,m}(t_k) \;-\; \Big(\kappa \,+\, h_k \Big[F(u(t))\Big]\Big) v_{n,m}(t_k)}{1 \,+\, h_k\, \kappa \,+\, h_k^2 \,\Big[F(u(t))\Big]}
$$

and

$$
\hat{g}_v(u(t)) \;=\; \frac{g(u(t_k))}{1 \,+\, h_k\, \kappa \,+\, h_k^2 \,\Big[F(u(t))\Big]}.
$$

Thus we can write $v_{,nm}(t_k)$ as

$$
v(t_{k+1}) \;=\; v(t_k) \;+\; h_k\, \hat{f}_v(u(t)) \;+\; \hat{g}_v(u(t))\, \triangle_k\, W \tag{5.33}
$$

and hence we can define

$$
\hat{v}(t+h) \;=\; \hat{v}(t) \;+\; h_k\, \hat{f}_v(\hat{u}(t)) \;+\; \hat{g}_v(\hat{u}(t))\, \triangle_k\, W \tag{5.34}
$$

and we know that

$$
\begin{aligned}
v_{n,m}(t+h) \quad &= \; v_{n,m}(t) + \textstyle\int_t^{t+h} \Big[-\;\sigma^2\,(\lambda_n \;+\; \beta_m) \;+\; a_1 \;-\; a_2\; ||c(s)||^2_{l^2}\Big] c_{n,m}(s)\, d\,s \\
&\qquad - \kappa\, \textstyle\int_t^{t+h}\; v_{n,m}(s)\, d\,s \;+\; \int_t^{t+h}\; g(u(s))\, d\,W(s)
\end{aligned}
$$

let

$$
f_v(u_{n,m}(s)) \;=\; \Big[-\;\sigma^2\,(\lambda_n \;+\; \beta_m) \;+\; a_1 \;-\; a_2\; ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\Big] c_{n,m}(s) \;-\; \kappa\, v_{n,m}(s)
$$

then

$$
v_{n,m}(t+h) \;=\; v_{n,m}(t) \;+\; \int_t^{t+h}\; f_v(u(s))\, d\,s \;+\; \int_t^{t+h}\; g(u(s))\, d\,W(s) \tag{5.35}
$$

from equations (5.34) and (5.35) and local Lipschitz that $\hat{v}(t) = v(t)$, we find

$$\begin{aligned} v(t+h) \,-\, \hat{v}(t+h) \;&=\; v(t) \,+\, \int_t^{t+h} f_v(u(s))\,d\,s \,+\, \int_t^{t+h} g(u(s))\,d\,W(s) \\ &\quad -\, \hat{v}(t) \,-\, h_k\,\hat{f}_v(\hat{u}(t)) \,-\, \hat{g}_v(\hat{u}(t))\,\triangle_k W \\ &= \int_t^{t+h} f_v(u(s))\,d\,s \,+\, \int_t^{t+h} g(u(s))\,d\,W(s) \\ &\quad -\int_t^{t+h} \hat{f}_v(\hat{u}(t))\,d\,s \,-\, \int_t^{t+h} \hat{g}_v(\hat{u}(t))\,d\,W(s) \\ &= \int_t^{t+h} \Big[f_v(u(s)) \,-\, \hat{f}_v(\hat{u}(t))\Big]\,d\,s \\ &\quad +\int_t^{t+h} \Big[g(u(s)) \,-\, \hat{g}_v(\hat{u}(t))\Big]\,d\,W(s). \end{aligned}$$

Pulling the expectation over the last identity and using the fact that

$$\mathbb{E}\int_t^{t+h} \Big[g(u(s)) \,-\, \hat{g}_v(\hat{u}(t))\Big]\,d\,W(s) \;=\; 0 \;\; (\textit{martingale}),$$

then we have

$$\begin{aligned} \Big|\Big|\mathbb{E}\Big[v(t\,+\,h_k) \,-\, \hat{v}(t\,+\,h_k)\Big]\Big|\Big|_N \;&=\; \Big|\Big|\mathbb{E}\int_t^{t+h} \Big[f_v(u(s)) \,-\, \hat{f}_v(\hat{u}(t))\Big]\,d\,s\Big|\Big|_N \\ &= \Big|\Big|\int_t^{t+h} \mathbb{E}\Big[f_v(u(s)) \,-\, \hat{f}_v(\hat{u}(t))\Big]\,d\,s\Big|\Big|_N \end{aligned}$$

we had last equation because of non random partition of $(t_n)_{n\in\mathbb{N}}$. By using the triangle inequality, we have

$$\begin{aligned} \Big|\Big|\mathbb{E}\Big[v(t\,+\,h_k) \,-\, \hat{v}(t\,+\,h_k)\Big]\Big|\Big|_N \;&\leq\; \mathbb{E}\int_t^{t+h} \Big|\Big|f_v(u(s)) \,-\, \hat{f}_v(\hat{u}(t))\Big|\Big|_N\;d\,s \\ &\leq\; \mathbb{E}\int_t^{t+h} ||f_v(u(s)) \,-\, f_v(u(t))||_N\;d\,s \\ &\quad +\, \mathbb{E}\int_t^{t+h} ||f_v(u(t)) \,-\, f_v(\hat{u}(t))||_N\;d\,s \\ &\quad +\, \mathbb{E}\int_t^{t+h} \Big|\Big|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\Big|\Big|_N\;d\,s \end{aligned}$$

the middle integration of the last inequality vanished because of local Lipschitz

$\Big(u(t) \,=\, \hat{u}(t)\Big)$, which means that

$$\begin{aligned}\Big|\Big|\mathbb{E}\Big[v(t\,+\,h_k)\,-\,\hat{v}(t\,+\,h_k)\Big]\Big|\Big|_N \quad &\leq \mathbb{E}\int_t^{t+h} ||f_v(u(s))\,-\,f_v(u(t))||_N\,d\,s \\ &+\mathbb{E}\int_t^{t+h} \Big|\Big|f_v(\hat{u}(t))\,-\,\hat{f}_v(\hat{u}(t))\Big|\Big|_N\,d\,s. \quad (5.36)\end{aligned}$$

But we know that

$$d\,c(r) \,=\, v(r)\,d\,r,$$

if $s \,<\, t$, thus

$$c(t)\,-\,c(s) \,=\, \int_s^t v(r)\,d\,r,$$

hence

$$||c(t)\,-\,c(s)||_N \,\leq\, \int_s^t ||v(r)||_N\,d\,r,$$

therefore

$$\begin{aligned}\mathbb{E}\,||c(t)\,-\,c(s)||_N \quad &\leq \mathbb{E}\int_s^t \Big(V(u(r))\Big)^{\frac{1}{2}}\,d\,r \\ &\leq \int_s^t \Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\,d\,r \qquad (5.37)\end{aligned}$$

which implies that

$$\mathbb{E}\,||c(t)\,-\,c(s)||_N \quad \leq \Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}}\Big|s\,-\,t\Big| \qquad (5.38)$$

hence

$$\mathbb{E}\,[(||c(t)\,-\,c(s)||_N)\,|u(t)] \quad \leq \Big(1+V(u(t))\Big)\,|s\,-\,t| \qquad (5.39)$$

and therefore

$$\begin{aligned}\int_t^{t+h}\mathbb{E}\,[(||c(t)\,-\,c(s)||_N)\,|u(t)]\,d\,s \quad &\leq \tilde{V}(u(t))\,\frac{h_k^2}{2} \\ &\leq \tilde{V}(u(t))\,h_k^2, \qquad (5.40)\end{aligned}$$

where $\tilde{V}(u(t)) = 1 + V(u(t))$, and

$$\begin{aligned} f_v(u(s)) \, - \, f_v(u(t)) \, &= \, \Big[- \, \sigma^2 \, (\lambda_n \, + \, \beta_m) \, + \, a_1 \, - \, a_2 \, ||u(s)||^2 \Big] c(s) \, - \, \kappa \, v(s) \\ &\quad - \, \Big[- \, \sigma^2 \, (\lambda_n \, + \, \beta_m) \, + \, a_1 \, - \, a_2 \, ||u(t)||^2 \Big] c(s) \, + \, \kappa \, v(t) \\ &= \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1 \, + \, a_2 \, ||u(t)||^2 \Big] c(t) \, + \, \kappa \, v(t) \\ &\quad - \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1 \, + \, a_2 \, ||u(s)||^2 \Big] c(s) \, - \, \kappa \, v(s) \\ &= \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1 \Big] \Big(c(t) \, - \, c(s) \Big) \\ &\quad + a_2 \Big(\, ||u(t)||_N^2 \, c(t) \, - \, ||u(s)||_N^2 \, c(s) \Big) \, + \, \kappa \Big(v(t) \, - \, v(s) \Big). \end{aligned}$$

Pulling the expectation over the last identity leads to

$$\begin{aligned} &\mathbb{E} \int_t^{t+h} ||f_v(u(s)) - f_v(u(t))||_N \, ds \\ &\leq \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1 \Big] \int_t^{t+h} \mathbb{E} \, ||c(t) \, - \, c(s)||_N \, d\,s + \kappa \, \int_t^{t+h} \mathbb{E} \, ||v(t) \, - \, v(s)||_N \, d\,s \\ &\qquad + a_2 \int_t^{t+h} \mathbb{E} \Big(\, ||u(t)||_N^2 \, c(t) - ||u(s)||_N^2 \, c(s) \Big) \, ds. \end{aligned} \tag{5.41}$$

Now, recall the following definition

Definition. (Frobenius Matrix Norm). The **Frobenius norm** of $A \in \mathcal{C}^{m \times n}$ is defined by the equations

$$||A||_F^2 \, = \, \sum_{i,j} \left| a_{ij} \right|^2 \, = \, \sum_i \, ||A_{i*}||_2^2 \, = \, \sum_j \, ||A_{*j}||_2^2 \, = \, trace(A^* \, A)$$

But we know from [] that,if A as above and $x \in \mathcal{C}^{n \times 1}$, then

$$||A \, x||_2 \, \leq \, ||A||_F \, ||x||_2$$

Lemma 5.3.4.

$$\left|\left| A \left(c(t) \, - \, c(s) \right) \right|\right|_2 \, \leq \, ||A||_F \, ||c(t) \, - \, c(s)||_2 \, ,$$

such that

$$A := \begin{bmatrix} v_1(\gamma)\Big(2c_1^2 + \sum_i^N c_i^2(\gamma)\Big) & \cdots & 2v_N(\gamma)c_1(\gamma)c_N(\gamma) \\ 2v_1(\gamma)c_1(\gamma)c_2(\gamma) & \cdots & 2v_N(\gamma)c_2(\gamma)c_N(\gamma) \\ \vdots & \ddots & \vdots \\ 2v_1(\gamma)c_1(\gamma)c_N(\gamma) & \cdots & v_N(\gamma)\Big(2c_N^2 + \sum_i^N c_i^2(\gamma)\Big) \end{bmatrix}$$

where $A \in \mathcal{C}^{N\times N}$

Proof.

$$\begin{aligned} &\bigtriangledown_c \left(||u(\gamma)||^2 \; c(\gamma)\right) \\ &= \bigtriangledown_c \left(||c(\gamma)||^2 \; c(\gamma)\right) \\ &= \bigtriangledown_c \Big[\Big(\sum_i^N c_i^2(\gamma)\Big)c(\gamma)\Big] \end{aligned}$$

$$= \begin{bmatrix} \frac{\partial}{\partial c_1}\Big(\sum_i^N c_i^2(\gamma)\Big)c_1 & \frac{\partial}{\partial c_2}\Big(\sum_i^N c_i^2(\gamma)\Big)c_1 & \cdots & \frac{\partial}{\partial c_N}\Big(\sum_i^N c_i^2(\gamma)\Big)c_1 \\ \frac{\partial}{\partial c_1}\Big(\sum_i^N c_i^2(\gamma)\Big)c_2 & \frac{\partial}{\partial c_2}\Big(\sum_i^N c_i^2(\gamma)\Big)c_2 & \cdots & \frac{\partial}{\partial c_N}\Big(\sum_i^N c_i^2(\gamma)\Big)c_2 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial c_1}\Big(\sum_i^N c_i^2(\gamma)\Big)c_N & \frac{\partial}{\partial c_2}\Big(\sum_i^N c_i^2(\gamma)\Big)c_N & \cdots & \frac{\partial}{\partial c_N}\Big(\sum_i^N c_i^2(\gamma)\Big)c_N \end{bmatrix}$$

which is equal to

$$A := \begin{bmatrix} v_1(\gamma)\Big(2c_1^2 + \sum_i^N c_i^2(\gamma)\Big) & \cdots & 2v_N(\gamma)c_1(\gamma)c_N(\gamma) \\ 2v_1(\gamma)c_1(\gamma)c_2(\gamma) & \cdots & 2v_N(\gamma)c_2(\gamma)c_N(\gamma) \\ \vdots & \ddots & \vdots \\ 2v_1(\gamma)c_1(\gamma)c_N(\gamma) & \cdots & v_N(\gamma)\Big(2c_N^2 + \sum_i^N c_i^2(\gamma)\Big). \end{bmatrix}$$

Now

$$
\begin{aligned}
||A||_F^2 \;&=\; \sum_i ||A_{i*}||_2^2 \;=\; ||A_1||_2^2 + ||A_2||_2^2 + ... + ||A_N||_2^2 \\
&=\; v_1^2\Big(2\,c_1^2 \;+\; ||c(\gamma)||^2\Big)^2 \;+\; 4\,v_2^2\,c_1^2\,c_2^2 \;+ ... +\; 4\,v_N^2\,c_1^2\,c_N^2 \\
&+\; 4\,v_1^2\,c_1^2\,c_2^2 \;+\; v_2^2\Big(2\,c_2^2 \;+\; ||c(\gamma)||^2\Big)^2 \;+ ... +\; 4\,v_N^2\,c_2^2\,c_N^2 \\
&+ \vdots \cdots\cdots\cdots \vdots \cdots\cdots \vdots \\
&+\; 4\,v_1^2\,c_1^2\,c_N^2 \;+\; 4\,v_2^2\,c_2^2\,c_N^2 \;+ ... +\; v_N^2\Big(2\,c_N^2 \;+\; ||c(\gamma)||^2\Big)^2
\end{aligned}
$$

By adding and subtracting $4\,v_1^2\,c_1^2\,c_1^2$, $4\,v_2^2\,c_2^2\,c_2^2, ...,$ and $4\,v_N^2\,c_N^2\,c_N^2$, we have

$$
\begin{aligned}
||A||_F^2 \;=\;& 4\,v_1^2(\gamma)c_1^2(\gamma)\,||c(\gamma)||^2 \;-\; 4\,v_1^2(\gamma)c_1^4 \;+\; 4\,v_2^2(\gamma)c_2^2(\gamma)\,||c(\gamma)||^2 \;-\; 4\,v_2^2(\gamma)c_2^4 \\
&+\; ... \;+\; 4\,v_N^2(\gamma)c_N^2(\gamma)\,||c(\gamma)||^2 \;-\; 4\,v_N^2(\gamma)c_N^4 \;+\; 2\,v_1^2(\gamma)c_1^4 \\
&+\; 4\,v_1^2(\gamma)c_1^2(\gamma)\,||c(\gamma)||^2 \;+\; v_1^2(\gamma)\,||c(\gamma)||^4 \;+\; 2\,v_2^2(\gamma)c_2^4(\gamma) \\
&+\; 4\,v_2^2(\gamma)c_2^2(\gamma)\,||c(\gamma)||^2 \;+\; v_2^2(\gamma)\,||c(\gamma)||^4 \;+\; ... \\
&+\; 2\,v_N^2(\gamma)c_N^4(\gamma) \;+\; 4\,v_N^2(\gamma)c_N^2(\gamma)\,||c(\gamma)||^2 \;+\; v_N^2(\gamma)\,||c(\gamma)||^4 \\
=\;& 8\,v_1^2(\gamma)c_1^2(\gamma)\,||c(\gamma)||^2 \;+\; 8\,v_2^2(\gamma)c_2^2(\gamma)\,||c(\gamma)||^2 \\
&+ ... +\; 8\,v_N^2(\gamma)c_N^2(\gamma)\,||c(\gamma)||^2 \;-\; 2\,v_1^2(\gamma)c_1^4 \;-\; 2\,v_2^2(\gamma)c_2^4 \\
&-\; ... -\; 2\,v_N^2(\gamma)c_N^4 \;+\; v_1^2(\gamma)\,||c(\gamma)||^4 + ... +\; v_N^2(\gamma)\,||c(\gamma)||^4
\end{aligned}
$$

hence

$$
\begin{aligned}
||A||_F^2 \,=\, & 8\, v_1^2(\gamma)c_1^2(\gamma)\, ||c(\gamma)||^2 \,+\, 8\, v_2^2(\gamma)c_2^2(\gamma)\, ||c(\gamma)||^2 \\
& + ... + \, 8\, v_N^2(\gamma)c_N^2(\gamma)\, ||c(\gamma)||^2 \,-\, 2\, v_1^2(\gamma)c_1^4 \,-\, 2\, v_2^2(\gamma)c_2^4 \\
& - \, ... - \, 2\, v_N^2(\gamma)c_N^4 \,+\, v_1^2(\gamma)\, ||c(\gamma)||^4 + ... + \, v_N^2(\gamma)\, ||c(\gamma)||^4 \\
= \, & 8\, ||c(\gamma)||^2 \Big(v_1^2(\gamma)c_1^2(\gamma) \,+\, v_2^2(\gamma)c_2^2(\gamma) \,+\, ... + \, v_N^2(\gamma)c_N^2(\gamma)\Big) \\
& - \, 2\, \Big(v_1^2(\gamma)c_1^4(\gamma) \,+\, v_2^2(\gamma)c_2^4(\gamma) \,+\, ... + \, v_N^2(\gamma)c_N^4(\gamma)\Big) \\
& + \, ||c(\gamma)||^4 \Big(v_1^2 + v_2^2 + ... + v_N^2\Big) \\
= \, & 8\, ||c(\gamma)||^2 \sum_i^N \Big(v_i^2(\gamma)c_i^2(\gamma)\Big) \,-\, 2\, \sum_i^N \Big(v_i^2(\gamma)c_i^4(\gamma)\Big) \,+\, ||c(\gamma)||^4\, ||v(\gamma)||^2 .
\end{aligned}
$$

Thus

$$
||A||_F^2 \,\leq\, 8\, ||c(\gamma)||^2 \sum_i^N \Big(v_i^2(\gamma)c_i^2(\gamma)\Big) \,+\, ||c(\gamma)||^4\, ||v(\gamma)||^2 . \tag{5.42}
$$

But we know that

$$
\sum_i^N v_i^2(\gamma)c_i^2(\gamma) \,\leq\, ||v||^2 \sum_i^N c_i^2(\gamma) \,\leq\, ||v(\gamma)||^2\, ||c(\gamma)||^2 .
$$

Thus inequality (5.42) becomes

$$
\begin{aligned}
||A||_F^2 \;\; & \leq\, 8\, ||c(\gamma)||^2\, ||v(\gamma)||^2\, ||c(\gamma)||^2 \,+\, ||c(\gamma)||^4 \\
& = \, 9\, ||c(\gamma)||^4\, ||v(\gamma)||^2 .
\end{aligned}
$$

Therefore,

$$
||A||_F \,\leq\, 3\, ||v(\gamma)||\, ||c(\gamma)||^2 \tag{5.43}
$$

which gives us

$$
\Big|\Big|A\Big(c(t) \,-\, c(s)\Big)\Big|\Big| \,\leq\, 3\, ||c(\gamma)||^2\, ||v(\gamma)||\, ||c(t) \,-\, c(s)|| .
$$

And therefore,

$$\begin{aligned}
\mathbb{E}\left|\left|A\Big(c(t)\,-\,c(s)\Big)\right|\right| &\leq 3\,\mathbb{E}\Big[\,||c(\gamma)||^2\;||v(\gamma)||\;||c(t)\,-\,c(s)||\,\Big]\\
&\leq 3\Big(\mathbb{E}\Big[\,||c(\gamma)||^4\;||v(\gamma)||^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\,||c(t)\,-\,c(s)||\,\Big)^{\frac{1}{2}}\\
&\leq 3\Big[\Big(\mathbb{E}\,||c(\gamma)||^8\,\Big)^{\frac{1}{2}}\Big(\mathbb{E}\,||v(\gamma)||^4\,\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}\Big(\mathbb{E}\,||c(t)\,-\,c(s)||^2\,\Big)^{\frac{1}{2}}\\
&\leq 3\Big(\mathbb{E}\,||c(\gamma)||^8\,\Big)^{\frac{1}{4}}\Big(\mathbb{E}\,||v(\gamma)||^4\,\Big)^{\frac{1}{4}}\Big(\mathbb{E}\,||c(t)\,-\,c(s)||^2\,\Big)^{\frac{1}{2}}\\
&\leq 3\Big(\mathbb{E}V^2(u(\gamma))\Big)^{\frac{1}{4}}\Big(\mathbb{E}V^2(u(\gamma))\Big)^{\frac{1}{4}}\Big(\mathbb{E}\,||c(t)\,-\,c(s)||^2\,\Big)^{\frac{1}{2}}
\end{aligned}$$

thus

$$\mathbb{E}\left|\left|A\Big(c(t)\,-\,c(s)\Big)\right|\right| \leq 3\Big(\mathbb{E}V^2(u(\gamma))\Big)^{\frac{1}{2}}\Big(\mathbb{E}\,||c(t)\,-\,c(s)||^2\,\Big)^{\frac{1}{2}} \tag{5.44}$$

but from inequality (5.37), we know that

$$\mathbb{E}\,||c(t)\,-\,c(s)||_N \leq \mathbb{E}\int_s^t \Big(V(u(r))\Big)^{\frac{1}{2}}\,d\,r\,,$$

hence

$$\begin{aligned}
\mathbb{E}\,||c(t)\,-\,c(s)||_N^2 &\leq \mathbb{E}\Big(\int_s^t \Big(V(u(r))\Big)^{\frac{1}{2}}\,d\,r\Big)^2\\
&\leq \int_s^t \Big(\mathbb{E}V(u(r))\Big)\,d\,r\,(t\,-\,s) \,=\, \mathbb{E}V(u(t))(t\,-\,s)^2
\end{aligned}$$

take the conditional expectation, we get

$$\mathbb{E}\left[||c(t)\,-\,c(s)||_N^2\,|u(t)\right] \quad \leq V(u(t))(t\,-\,s)^2 \tag{5.45}$$

Thus

$$\begin{aligned}
\mathbb{E}\Big[\Big|\Big|A\Big(c(t)\,-\,c(s)\Big)\Big|\Big|_N\,|u(t)\Big] &\leq 3\Big(V^2(u(\gamma))\Big)^{\frac{1}{2}}\Big(V(u(t))(t\,-\,s)^2\Big)^{\frac{1}{2}}\\
&\leq 3\;(\max\{V(u(s)),\,V(u(t))\})^{\frac{1}{2}}\,\Big(V(u(t))\Big)^{\frac{1}{2}}|t\,-\,s|
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathbb{E}\Big[\Big|\Big|A\Big(c(t)-c(s)\Big)\Big|\Big|_N\,|u(t)\Big] &\leq 3\;(\max\{V(u(s)),V(u(t))\})^{\frac{1}{2}}\,\Big(1+V(u(t))\Big)\,|t-s|\\
&\leq 3\;(\max\{V(u(s)),V(u(t))\})^{\frac{1}{2}}\;\tilde{V}(u(t))\,|t-s| \qquad (5.46)
\end{aligned}$$

□

by using inequalities (5.30) and (5.40) and Lemma[5.2.4] , inequality (5.41) becomes

$$
\begin{aligned}
&\mathbb{E}\int_t^{t+h} [||f_v(u(s)) - f_v(u(t))||_N \,|u(t)]\; ds \\
&\leq \Big[\sigma^2(\lambda_n + \beta_m) - a_1\Big]\tilde{V}(u(t))\, h_k^2 + \; K_2\,\kappa\,\tilde{V}(u(t))\, h_k^2 \\
&\quad + a_2 \int_t^{t+h} \mathbb{E}\Big[\nabla_c\Big(\,||u(\gamma)||_N^2\, c(\gamma)\Big)\; ||c(t) - c(s)||_N\,\Big]d\,s \\
&\leq \left(\sigma^2\, N^2\, \pi^2(\frac{1}{l_x^2} + \frac{1}{l_x^2}) + \; K_2\kappa\right)\tilde{V}(u(t))\, h_k^2 \\
&\quad + \; 3\, a_2 \int_t^{t+h} (\max\{V(u(s)), V(u(t))\})^{\frac{1}{2}}\;\; \tilde{V}(u(t))\, |t - s| \\
&\leq \left(\sigma^2\, N^2\, \pi^2(\frac{1}{l_x^2} + \frac{1}{l_x^2}) + \; K_2\,\kappa\right)\tilde{V}(u(t))\, h_k^2 \\
&\quad + \frac{3}{2}\,(\max\{V(u(s)), V(u(t))\})^{\frac{1}{2}}\;\; \tilde{V}(u(t))\, h_k^2 \\
&\leq \left[\sigma^2\, N^2\, \pi^2(\frac{1}{l_x^2} + \frac{1}{l_x^2}) + \; K_2\,\kappa\; + 3\; (\max\{V(u(s)), V(u(t))\})^{\frac{1}{2}}\right]\tilde{V}(u(t))\, h_k^2
\end{aligned}
$$

hence

$$
\mathbb{E}\int_t^{t+h} [||f_v(u(s)) - f_v(u(t))||_N \,|u(t)]\; ds \leq \; K_3\,\tilde{V}(u(t))\, h_k^2 \tag{5.47}
$$

where

$$
k_2 = \; \kappa + \; \sigma^2\, N^2\, \pi^2(\frac{1}{l_x^2} + \frac{1}{l_x^2}) \; + \; 2\Big(V(u(t))\Big)^{\frac{1}{2}},
$$

and

$$
k_3 = \sigma^2\, N^2\, \pi^2(\frac{1}{l_x^2} + \frac{1}{l_x^2}) + \; K_2\,\kappa\; + 3\; (\max\{V(u(s)), V(u(t))\})^{\frac{1}{2}}\,.
$$

Finally, to simplify, let

$$
F(u(t)) = \sigma^2\,(\lambda_n \; + \; \beta_m) \; - \; a_1 \; + \; a_2 \sum_{q,l=1}^{N} [c_{q,l}(t_k)]^2,
$$

then

$$
\begin{aligned}
f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t)) = & \Big[-\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \, ||\hat{u}(t)||^2 \Big] \hat{c}(t) \\
& - \kappa \, \hat{v}(t) - \frac{\Big[-\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \, ||\hat{u}(t)||^2 \Big] \hat{c}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \\
& + \frac{\Big(\kappa + h \Big[-\sigma^2 (\lambda_n + \beta_m) + a_1 - a_2 \, ||\hat{u}(t)||^2 \Big] \Big) \hat{v}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \\
= & \frac{- \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \, ||\hat{u}(t)||^2 \Big] \hat{c}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \\
& + \frac{- h_k \, \kappa \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \, ||\hat{u}(t)||^2 \Big] \hat{c}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \\
& + \frac{- h_k^2 \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \, ||\hat{u}(t)||^2 \Big]^2 \hat{c}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \\
& + \frac{-\kappa \, \hat{v}(t) - h_k \kappa^2 \hat{v}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \\
& + \frac{-\kappa \, h_k^2 \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \, ||\hat{u}(t)||^2 \Big] \hat{v}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \\
& + \frac{- \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \, ||\hat{u}(t)||^2 \Big] \hat{c}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \\
& + \frac{\kappa \hat{v}(t) + h_k \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \, ||\hat{u}(t)||^2 \Big] \hat{v}(t)}{1 + h_k \, \kappa + h_k^2 \Big[F(u(t)) \Big]} \Big\}
\end{aligned}
$$

thus

$$f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t)) = -h_k \Big\{ \frac{\kappa \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \,||\hat{u}(t)||^2 \Big] \hat{c}(t)}{1 + h_k\,\kappa + h_k^2 \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \,||\hat{u}(t)||^2 \Big]}$$

$$+ \frac{h_k \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \,||\hat{u}(t)||^2 \Big]^2 \hat{c}(t)}{1 + h_k\,\kappa + h_k^2 \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \,||\hat{u}(t)||^2 \Big]}$$

$$+ \frac{\kappa^2\,\hat{v}(t) + h_k\,\kappa \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \,||\hat{u}(t)||^2 \Big] \hat{v}(t)}{1 + h_k\,\kappa + h_k^2 \Big[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \,||\hat{u}(t)||^2 \Big]}.$$

Pulling the expectation over the last equation leads to

$$\mathbb{E}\Big|\Big|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\Big|\Big|_N$$

$$\leq h_k \Big\{ \mathbb{E}\left|\left| \frac{\kappa \Big[F_v(\hat{u}(t))\Big] \hat{c}(t)}{1 + h_k\,\kappa + h_k^2 \Big[F_v(\hat{u}(t))\Big]} \right|\right|_N + \mathbb{E}\left|\left| \frac{h_k \Big[F_v(\hat{u}(t))\Big]^2 \hat{c}(t)}{1 + h_k\,\kappa + h_k^2 \Big[F_v(\hat{u}(t))\Big]} \right|\right|_N$$

$$+ \mathbb{E}\left|\left| \frac{\kappa^2\,\hat{v}(t)}{1 + h_k\,\kappa + h_k^2 \Big[F_v(\hat{u}(t))\Big]} \right|\right|_N + \mathbb{E}\left|\left| \frac{h_k\,\kappa\,\hat{v}(t)}{1 + h_k\,\kappa + h_k^2 \Big[F_v(\hat{u}(t))\Big]} \right|\right|_N \Big\}$$

$$\leq h_k \Big\{ \kappa \Big[\sigma^2(\lambda_n + \beta_m) - a_1\Big] \mathbb{E}\,||\hat{c}(t)|| + a_2 \mathbb{E}\,\Big|\Big| ||\hat{u}(t)||^2\, \hat{c}(t) \Big|\Big|_N$$

$$+ h_k^2 \Big[\sigma^2(\lambda_n + \beta_m) - a_1\Big]^2 \mathbb{E}\,||\hat{c}(t)||$$

$$+ a_2^2\, h_k^2 \mathbb{E}\,\Big|\Big| ||\hat{u}(t)||^4\, \hat{c}(t) \Big|\Big|_N + \kappa^2 \mathbb{E}\,||\hat{v}(t)|| + h_k \kappa \mathbb{E}\,||\hat{c}(t)||$$

$$+ 2\,a_2\,h_k \Big[\sigma^2(\lambda_n + \beta_m) - a_1\Big] \mathbb{E}\,\Big|\Big| ||\hat{u}(t)||^2\, \hat{c}(t) \Big|\Big|_N \Big\}$$

for large N, we have

$$\mathbb{E}\Big|\Big|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\Big|\Big|_N$$

$$\leq h_k \Big\{ \kappa \sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2}) \mathbb{E}\,||\hat{c}(t)||_N + a_2 \mathbb{E}\,\Big|\Big| ||\hat{u}(t)||_{l^2}^2\, \hat{c}(t) \Big|\Big|_N$$

$$+ h_k^2 \sigma^4\, N^4\, \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 \mathbb{E}\,||\hat{c}(t)||_N + 2\,a_2\,h_k \sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2}) \mathbb{E}\,\Big|\Big| ||\hat{u}(t)||_{l^2}^2\, \hat{c}(t) \Big|\Big|_N$$

$$+ a_2^2\, h_k^2 \mathbb{E}\,\Big|\Big| ||\hat{u}(t)||_{l^2}^4\, \hat{c}(t) \Big|\Big|_N + \kappa^2 \mathbb{E}\,||\hat{v}(t)||_N + h_k \kappa\, \mathbb{E}\,||\hat{c}(t)||_N \Big\}$$

hence

$$
\begin{aligned}
&\mathbb{E}\left|\left|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\right|\right|_N \\
&\leq\ h_k\Big\{\kappa\,\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}} +\ a_2\left(\mathbb{E}\,||\hat{u}(t)||^4_{l^2}\right)^{\frac{1}{2}}(\mathbb{E}\,||\hat{c}(t)||_N)^{\frac{1}{2}} \\
&\quad + h_k^2\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}} \\
&\quad + 2\,a_2\,h_k\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})\left(\mathbb{E}\,||\hat{u}(t)||^4_{l^2}\right)^{\frac{1}{2}}(\mathbb{E}\,||\hat{c}(t)||_N)^{\frac{1}{2}} \\
&\quad + a_2^2\,h_k^2\left(\mathbb{E}\,||\hat{u}(t)||^8_{l^2}\right)^{\frac{1}{2}}(\mathbb{E}\,||\hat{c}(t)||_N)^{\frac{1}{2}} + \kappa^2\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}} + h_k\kappa\,\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}}\Big\} \\
&\leq\ h_k\Big\{\kappa\,\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}} +\ 2\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}}\left(\mathbb{E}V^{\frac{1}{2}}(\hat{u}(t))\right)^{\frac{1}{2}} \\
&\quad + h_k^2\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}} \\
&\quad + 4\,h_k\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}}\left(\mathbb{E}V^{\frac{1}{2}}(\hat{u}(t))\right)^{\frac{1}{2}} \\
&\quad + 4\,h_k^2\,(\mathbb{E}V(\hat{u}(t)))\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}} + \kappa^2\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}} + h_k\kappa\,\,(\mathbb{E}V(\hat{u}(t)))^{\frac{1}{2}}\Big\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathbb{E}\left[\left|\left|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\right|\right|_N\Big|\hat{u}(t)\right] \\
&\leq\ h_k\Big\{\kappa\,\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2}) +\ 2\left(V^{\frac{1}{2}}(\hat{u}(t))\right)^{\frac{1}{2}} + h_k^2\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2 \\
&\quad + 4\,h_k\sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})\left(V^{\frac{1}{2}}(\hat{u}(t))\right)^{\frac{1}{2}} + 4\,h_k^2\,(V(\hat{u}(t))) + \kappa^2 + h_k\kappa\Big\}\,(V(\hat{u}(t)))^{\frac{1}{2}}
\end{aligned}
$$

thus

$$
\begin{aligned}
\mathbb{E}\left[\left|\left|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\right|\right|_N\Big|\hat{u}(t)\right] \ &\leq\ K_4\,(1+V(\hat{u}(t)))\,h_k \\
&\leq\ K_4\,\tilde{V}(\hat{u}(t))\,h_k
\end{aligned}
\tag{5.48}
$$

where

$$
\begin{aligned}
K_4 \quad &=\ \kappa\,\sigma^2\,N^2\,\pi^2(\tfrac{1}{l_x^2}+\tfrac{1}{l_y^2}) +\ 2\left(V^{\frac{1}{2}}(\hat{u}(t))\right)^{\frac{1}{2}} + h_k^2\sigma^4\,N^4\,\pi^4(\tfrac{1}{l_x^2}+\tfrac{1}{l_y^2})^2 \\
&\quad +4\,h_k\sigma^2\,N^2\,\pi^2(\tfrac{1}{l_x^2}+\tfrac{1}{l_y^2})\left(V^{\frac{1}{2}}(\hat{u}(t))\right)^{\frac{1}{2}} + 4\,h_k^2\,(V(\hat{u}(t))) + \kappa^2 + h_k\kappa
\end{aligned}
$$

thus

$$\int_t^{t+h} \mathbb{E}\left[\left|\left|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\right|\right|_N \left|\hat{u}(t)\right]\right. \leq\ K_4\, \tilde{V}(\hat{u}(t))\, h_k^2. \tag{5.49}$$

Using inequalities (5.47) and (5.49), inequality (5.36) becomes

$$\begin{aligned} \left|\left|\mathbb{E}\Big[\,(v(t\,+\,h_k)\,-\,\hat{v}(t\,+\,h_k))\,|\hat{u}(t)\Big]\right|\right|_N \ &\leq\ K_3\,\tilde{V}(\hat{u}(t))\,h_k^2\,+\,K_4\,\tilde{V}(\hat{u}(t))\,h_k^2 \\ &\leq\ K_{v_0}^c \tilde{V}(\hat{u}(t))\,h_k^2 \end{aligned} \tag{5.50}$$

where $K_{v_0}^c\ =\ K_3\,+\,K_4$,

$$K_3\ = \sigma^2\,N^2\,\pi^2(\frac{1}{l_x^2} + \frac{1}{l_x^2}) + \,K_2\,\kappa\, + 3\,\left(\max\{V(u(s)), V(u(t))\}\right)^{\frac{1}{2}},$$

and

$$\begin{aligned} K_4 \quad &=\ \kappa\,\sigma^2\,N^2\,\pi^2(\tfrac{1}{l_x^2} + \tfrac{1}{l_y^2}) + \,2\left(V^{\frac{1}{2}}(\hat{u}(t))\right)^{\frac{1}{2}} + h_k^2\sigma^4\,N^4\,\pi^4(\tfrac{1}{l_x^2} + \tfrac{1}{l_y^2})^2 \\ &+4\,h_k\sigma^2\,N^2\,\pi^2(\tfrac{1}{l_x^2} + \tfrac{1}{l_y^2})\left(V^{\frac{1}{2}}(\hat{u}(t))\right)^{\frac{1}{2}} + 4\,h_k^2\,(V(\hat{u}(t))) + \kappa^2 + h_k\kappa. \end{aligned}$$

□

5.4 LOCAL MEAN SQUARE CONSISTENCY OF NUMERICAL METHODS

In this section, for convenience we refer to the following abbreviation: $(\forall\, i, j)$, for $u\ =\ u_{n,m} \in\ \mathbb{R}^{N\times N}$ *and* $f\ =\ f_{n,m},\ g\ =\ g_{n,m}$,

$$c_{n,m}^{i,j}(t_{k+1})\ =\ c_{n,m}^{i,j}(t_k)\ + h_k\, v_{n,m}^{i,j}(t_k) \tag{5.51}$$

and

$$\begin{aligned} v_{n,m}^{i,j}(t_{k+1}) \qquad &=\ v_{n,m}^{i,j}(t_k)\ + h_k\, f_{n,m}(u(k))c_{n,m}^{i,j}(k) \\ &-\kappa\, v_{n,m}^{i,j}(t_k)\ +\ g_{n,m}(u(t_k))\,\triangle_k\, W_{n,m}^{i,j} \end{aligned} \tag{5.52}$$

where $f_{n,m}(u)\ =\ \Big(-\ \sigma^2(\lambda_n\ +\ \beta_m) + a_1\ -\ a_2\ ||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big)\, c_{n,m}(t_k)$ and
$g_{n,m}(u)\ =\ \Big(b_0\ +\ b_1\,||u||_{\mathbb{L}^2(\mathbb{D})}\ +\ b_2\,||v||_{\mathbb{L}^2(\mathbb{D})}\Big)\,\alpha_{n,m}$

Definition. A numerical method with the scheme

$$\hat{u}(t+h) \;=\; \hat{u}(t) \;+\; h\, f(\hat{u}(t)) \;+\; g(\hat{u}(t)) \,\triangle\, W(t) \tag{5.53}$$

where $h \;=\; t_{k+1} - t_k \;=\; \int_{t_k}^{t_{k+1}} \, ds$, $\triangle\, W^k \;=\; W(t_{k+1}) \;-\; W(t_k) \;=\; \int_{t_k}^{t_{k+1}} \, dW(s) \;\in\; \mathcal{N}(0, h)$ and $f(u_{n,m}(t))$ and $g_{n,m}(u(t))$ as above applied to SDE (5.53) is said to be **mean square consistent** with rate $r_2 \;>\; 0$ on $\mathbb{D}iff$

$$\exists\, K_2^c = \text{ consistency constant} \,\forall\, 0 \;\leq\; t \;\leq\; t+h \;\leq\; T$$

where h is sufficiently small, i.e., $0 < \; h \leq \; \delta \leq\; 1$, $\forall\, u(t) \;\in\; H$, where $H := \{u \in\; \mathbb{L}^2(\mathbb{D}) \big| \dot{u} \in\; \mathbb{L}^2(\mathbb{D})\}$, and $||u||_H \;=\; \sqrt{||u||^2_{\mathbb{L}^2(\mathbb{D})} + ||\dot{u}||^2_{\mathbb{L}^2(\mathbb{D})}} : (\mathcal{F}_t,\, \mathcal{B}(\mathbb{L}^2))-$ measurable, then

$$\Big(\mathbb{E}\Big[\, ||u(t+h) \;-\; \hat{u}(t+h)||^2_{N\times N} \,\Big|\mathcal{F}_t\Big]\Big)^{\frac{1}{2}} \;\leq\; K_2^c \,\Big(V(u(t))\Big)\, h^{r_2}$$

where $\hat{u}(t+h|t, u(t)) \;=\; \hat{u}(t) \;+\; \int_t^{t+h} \, f(u(s))\, ds \;+\; \int_t^{t+h} g(u(s))\, dW(s)\, ds.$

Lemma 5.4.1. *(The Burkholder-Davis-Gundy Inequality)*

$$\mathbb{E}\bigg|\bigg|\int_0^t \; g(s)\, dW(s)\bigg|\bigg|^p \;\leq\; C_B\, \mathbb{E}\Big(\int_0^t \; ||g(s)||^2 \; ds\Big)^{\frac{p}{2}} \tag{5.54}$$

Proof. See Karatzas and Shreve [10]. □

Lemma 5.4.2. *Let* $X \;\sim\; \mathcal{N}(0,\, h_k)$*,* $\forall\, n \;\in\; \mathbb{N},\; n \;\geq\; 1$*, we have*

$$\mathbb{E}(X)^{2\,n} \;=\; (2\,n \;-\; 1)!!\, h_k^n$$

Proof. See appendix □

Theorem 5.4.3. *The method (LIEM) given by*

$$c^{i,j}_{n,m}(t_{k+1}) \quad = c^{i,j}_{n,m}(t_k) \;+\; h_k\, v^{i,j}_{n,m}(t_{k+1}) \tag{5.55}$$

$$v^{i,j}_{n,m}(t_{k+1}) \;=\; v^{i,j}_{n,m}(t_k) \,+\, h_k \Big[f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_{k+1}) \,-\, \kappa\, v^{i,j}_{n,m}(t_{k+1}) \Big] + g_{n,m}(u(t_k)) \triangle_k W^{i,j}_{n,m} \qquad (5.56)$$

is locally mean square consistent with rate $r_2 \geq 1$, *where*

$$f_{n,m}(u(t_k)) \,=\, -\,\sigma^2\,(\lambda_n \,+\, \beta_m) \,+\, a_1 \,-\, a_2 \sum_{i,j=1}^{2} \sum_{q,l=1}^{N} [c^{i,j}_{q,l}(t_k)]^2,$$

and

$$g_{n,m}(u(t_k)) \,=\, \Big(b_0 \,+\, b_1\, \|u^N\|_{\mathbb{L}^2(\mathbb{D})} \,+\, b_2\, \|v^N\|_{\mathbb{L}^2(\mathbb{D})} \Big)\, \alpha^{i,j}_{n,m},$$

and

$$\triangle_k W^{i,j}_{n,m} \,=\, W^{i,j}_{n,m}(t_{k+1}) \,-\, W^{i,j}_{n,m}(t_k) \in \mathcal{N}(0, h_k),\; h_k \,=\, t_{k+1} - t_k$$

Proof. We want to prove that

$$1) \Big(\mathbb{E}\left[\|c(t+h) \,-\, \hat{c}(t+h)\|^2_N \,|\hat{u}(t)\right] \Big)^{\frac{1}{2}} \,\leq\, K^c_{c_2}\, V(u(t))\, h^{r_2} \qquad (5.57)$$

$$2) \Big(\mathbb{E}\left[\|v_{n,m}(t+h) \,-\, \hat{v}_{n,m}(t+h)\|^2_N \,|\hat{u}(t)\right] \Big)^{\frac{1}{2}} \,\leq\, K^c_{v_2}\, V(u(t))\, h^{r_2} \qquad (5.58)$$

Provided that $c(t) \,=\, \hat{c}(t)$ and $v(t) \,=\, \hat{v}(t)$. To prove inequality (5.57), we know from Theorem 5.2.4 that

$$c(t+h) \,-\, \hat{c}(t+h) \;=\; \int_t^{t+h} \Big(v(s) \,-\, \hat{f}_c(u(t)) \Big)\, d\,s \,-\, \int_t^{t+h} \hat{g}_c(u(t))\, d\,W(s).$$

Thus,

$$\begin{aligned} &\|c_{n,m}(t+h) \,-\, \hat{c}_{n,m}(t+h)\|^2_N \\ &= \left\| \int_t^{t+h} \Big(v(s) \,-\, \hat{f}_c(u(t)) \Big)\, d\,s - \int_t^{t+h} \hat{g}_c(u(t))\, d\,W(s) \right\|^2_N \\ &\leq\, 2 \left\| \int_t^{t+h} \Big(v(s) \,-\, \hat{f}_c(u(t)) \Big)\, d\,s \right\|^2_N \,+\, 2 \left\| \int_t^{t+h} \hat{g}_c(u(t)) d\,W(s) \right\|^2_N, \end{aligned}$$

by using the fact that

$$(a + b)^2 \le 2a^2 + 2b^2.$$

Hence

$$\mathbb{E}\,||c(t+h) - \hat{c}(t+h)||_N^2 \le 2\,\mathbb{E}\left|\left|\int_t^{t+h}\Big(v(s) - \hat{f}_c(u(t))\Big)\,d\,s\right|\right|_N^2 + 2\,\mathbb{E}\left|\left|\int_t^{t+h}\hat{g}_c(u(t))d\,W(s)\right|\right|_N^2$$

which implies that

$$\mathbb{E}\,||c(t+h) - \hat{c}(t+h)||_N^2 \le 2\int_t^{t+h}\mathbb{E}\left|\left|v(s) - \hat{f}_c(u(t))\right|\right|_N^2 d\,s + 2\,\mathbb{E}\left|\left|\int_t^{t+h}\hat{g}_c(u(t))d\,W(s)\right|\right|_N^2 \tag{5.59}$$

first, we will find the first part of the right hand side of inequality (5.59),

$$\left|\left|v(s) - \hat{f}_c(u(t))\right|\right|_N^2 = \left|\left|v(s) - v(t) + v(t) - \hat{f}_c(u(t))\right|\right|_N^2 \le 2\,||v(s) - v(t)||_N^2 + 2\left|\left|v(t) - \hat{f}_c(u(t))\right|\right|_N^2 \tag{5.60}$$

but we know that

$$dv_{n,m}(r) = \Big(\Big[-\sigma^2(\lambda_n+\beta_m) - a_1 + a_2\,||u(r)||_{\mathbb{L}^2(\mathbb{D})}^2\Big]\,c_{n,m}(r)d\,r - \kappa\,v_{n,m}(r)\Big)d\,r + g(u(r))\,d\,W_{n,m}(r)$$

thus

$$v_{n,m}(s) - v_{n,m}(t) = \int_t^s\Big[-\sigma^2(\lambda_n+\beta_m) - a_1 + a_2\,||u(r)||_{\mathbb{L}^2(\mathbb{D})}^2\Big]\,c_{n,m}(r) - \kappa\int_t^s v_{n,m}(r)d\,r + \int_t^s g(u(r))\,d\,W_{n,m}(r)$$

so

$$||v_{n,m}(s) - v_{n,m}(t)||_N^2 \le 4\Big[\sigma^2(\lambda_n+\beta_m) - a_1\Big]^2\left|\left|\int_t^s c_{n,m}(r)\,dr\right|\right|_N^2 + 4\,a_2^2\left|\left|\int_t^s ||u(r)||_{\mathbb{L}^2(\mathbb{D})}^2\,c_{n,m}(r)\,dr\right|\right|_N^2 + 4\,\kappa^2\left|\left|\int_t^s v_{n,m}(r)\,dr\right|\right|_N^2 + 4\left|\left|\int_t^s g(u(r))\,d\,W_{n,m}(r)\right|\right|_N^2$$

hence

$$\mathbb{E}\,||v_{n,m}(s)\,-\,v_{n,m}(t)||_N^2\,\leq\,4\left[\sigma^2(\lambda_n+\beta_m)-a_1\right]^2\mathbb{E}\left|\left|\int_t^s\,c_{n,m}(r)\,dr\right|\right|_N^2$$

$$+\,4\,a_2^2\,\mathbb{E}\left|\left|\int_t^s\,||u(r)||^2_{\mathbb{L}^2(\mathbb{D})}\,c_{n,m}(r)\,dr\right|\right|_N^2$$

$$+\,4\,\kappa^2\mathbb{E}\left|\left|\int_t^s\,v_{n,m}(r)\,dr\right|\right|_N^2\,+\,4\,\mathbb{E}\left|\left|\int_t^s\,g(u(r))\,dW_{n,m}(r)\right|\right|_N^2$$

therefore,

$$\mathbb{E}\,||v_{n,m}(s)\,-\,v_{n,m}(t)||_N^2\,\leq\,4\left[\sigma^2(\lambda_n+\beta_m)-a_1\right]^2\,\int_s^t\,\mathbb{E}\,||c(r)||_N^2\,dr$$

$$+\,4\,a_2^2\,\int_s^t\,\Big(\mathbb{E}\,||u(r)||^8_{\mathbb{L}^2(\mathbb{D})}\Big)^{\frac{1}{2}}\Big(\mathbb{E}\,||c(r)||_N^2\Big)^{\frac{1}{2}}\,dr$$

$$+\,4\,\kappa^2\int_s^t\,\mathbb{E}\,||v(r)||_N^2\,dr\,+\,4\,C_B\,\mathbb{E}\int_s^t\,||g(u(r))||_N^2\,dW(r)$$

by using Hölder and Burkerholder-Davis-Gundy inequalities, Lemma 5.3.1, then substituting $g(u(t))\,=\,\Big(b_0+b_1\,||u(t)||_{l^2}\,+\,||v(t)||_{l^2}\Big)\alpha^2$, we get

$$\mathbb{E}\,||v_{n,m}(s)\,-\,v_{n,m}(t)||_N^2\,\leq\,4\left(\left[\sigma^2(\lambda_n+\beta_m)-a_1\right]^2\,+\,\kappa^2\right)\mathbb{E}V(u(s))|s\,-\,t|$$

$$+\,12\,C_B\,\alpha^2\left[b_0^2+(b_1^2\,+\,b_2^2)\mathbb{E}V(u(t))\right]|s\,-\,t|$$

$$+\,4\,a_2^2\,\mathbb{E}V^2(u(t))\mathbb{E}V(u(s))|s\,-\,t|,$$

which implies that

$$\mathbb{E}\,||v_{n,m}(s)\,-\,v_{n,m}(t)||_N^2\,\leq\,\max_{1\leq\,n,m\leq\,N}4\left(\left[\sigma^2(\lambda_n+\beta_m)-a_1\right]^2\,+\,\kappa^2\right)\mathbb{E}V(u(s))|s\,-\,t|$$

$$+\,12\,C_B\,\alpha^2\left[b_0^2+(b_1^2\,+\,b_2^2)\mathbb{E}V(u(t))\right]|s\,-\,t|$$

$$+\,8\,a_2\,\mathbb{E}V^2(u(t))\mathbb{E}V(u(s))|s\,-\,t|,$$

and, if N is sufficiently large, we have

$$\mathbb{E}\,||v_{n,m}(s)\,-\,v_{n,m}(t)||_N^2\,\leq\,4\left(\left[\sigma^2\,\pi^2\,N^2(\frac{1}{l_x^2}\,+\,\frac{1}{l_y^2})\right]^2\,+\,\kappa^2\right)\mathbb{E}V(u(s))|s\,-\,t|$$

$$+\,12\,C_B\,\alpha^2\left[b_0^2+(b_1^2\,+\,b_2^2)\mathbb{E}V(u(t))\right]|s\,-\,t|$$

$$+\,8\,a_2\,\mathbb{E}V^2(u(t))\mathbb{E}V(u(s))|s\,-\,t|,$$

taking the conditional expectation gives us

$$\begin{aligned}
&\mathbb{E}\left[||v_{n,m}(s) - v_{n,m}(t)||_N^2 \, |u(t)\right] \\
&\leq 4\left(\sigma^4 \pi^4 N^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + \kappa^2 + 3\, C_B\, \alpha^2 (b_1^2 + b_2^2)\right) V(u(t))|s - t| \\
&\quad + 12\, C_B\, \alpha^2\, b_0^2 |s - t| + 8\, a_2\, V^2(u(t)) V(u(t))|s - t| \\
&\leq K_5\,(1 + V(u(t)))|s - t| \\
&\leq K_5\, \tilde{V}(u(t))|s - t|
\end{aligned}$$

where

$$\begin{aligned}
K_5 &= 4\left(\sigma^4 \pi^4 N^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + \kappa^2 + 3\, C_B\, \alpha^2 (b_1^2 + b_2^2)\right) \\
&\qquad + 12\, C_B\, \alpha^2\, b_0^2 + 8\, a_2\, V^2(u(t))
\end{aligned}$$

therefore,

$$\begin{aligned}
\int_t^{t+h} \mathbb{E}\Big[\, ||v_{n,m}(s) - v_{n,m}(t)||_N^2 \,\Big|\, u(t)\Big]\, d\, s &\leq k_5\, \tilde{V}(u(t))\, \frac{h_k^2}{2} \\
&\leq k_5\, \tilde{V}(u(t))\, h_k^2. \qquad (5.61)
\end{aligned}$$

from Theorem 5.2.3, we know that,

$$
\begin{aligned}
& v_{n,m}(t) \, - \, \hat{f}_c(u(t)) \\
& = \, h_k \Bigg\{ \frac{\Big(\kappa \, + \, h_k \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1\Big]\Big) v_{n,m}(t)}{1 \, + \, h_k \, \kappa \, + \, h_k^2 \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1 \, + \, a_2 \, \sum\limits_{i,j=1}^{2} \, \sum\limits_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2\Big]} \\
& \quad + \, \frac{h_k \, a_2 \Big(\sum\limits_{i,j=1}^{2} \, \sum\limits_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2\Big) v_{n,m}(t) \, + \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1\Big] c_{n,m}(t)}{1 \, + \, h_k \, \kappa \, + \, h_k^2 \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1 \, + \, a_2 \, \sum\limits_{i,j=1}^{2} \, \sum\limits_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2\Big]} \\
& \quad + \, \frac{a_2 \Big(\sum\limits_{i,j=1}^{2} \, \sum\limits_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2\Big) c_{n,m}(t)}{1 \, + \, h_k \, \kappa \, + \, h_k^2 \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1 \, + \, a_2 \, \sum\limits_{i,j=1}^{2} \, \sum\limits_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2\Big]} \Bigg\}
\end{aligned}
$$

to simplify, let

$$
F_c(u(t)) \, = \, 1 \, + \, h_k \, \kappa \, + \, h_k^2 \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1 \, + \, a_2 \, \sum_{i,j=1}^{2} \, \sum_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2,
$$

which gives $F(u(t)) \, \geq \, 1$, thus

$$
\begin{aligned}
& v_{n,m}(t) \, - \, \hat{f}_c(u(t)) \\
& = \, h_k \Bigg\{ \frac{\Big(\kappa \, + \, h_k \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1\Big]\Big) v_{n,m}(t) + a_2 \Big(\sum\limits_{i,j=1}^{2} \, \sum\limits_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2\Big) c_{n,m}(t)}{F_c(u(t))} \\
& \quad + \, \frac{h_k \, a_2 \Big(\sum\limits_{i,j=1}^{2} \, \sum\limits_{q,l=1}^{N} [c_{q,l}^{i,j}(t_k)]^2\Big) v_{n,m}(t) \, + \, \Big[\sigma^2 \, (\lambda_n \, + \, \beta_m) \, - \, a_1\Big] c_{n,m}(t)}{F_c(u(t))} \Bigg\}.
\end{aligned}
$$

By pulling the expectation over the squared norm of latter identity, we find that

$$
\begin{aligned}
&\mathbb{E}\left|\left|v_{n,m}(t) - \hat{f}_c(u(t))\right|\right|_N^2 \\
&\leq h_k^2 \Big\{ 4\left(\kappa + h_k\Big[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big]\right)^2 \mathbb{E}\left|\left|\frac{v(t)}{F_c(u(t))}\right|\right|_N^2 \\
&\quad + 4\, h_k^2\, a_2^2\, \mathbb{E}\left|\left|\frac{||u(t)||_N^2\; v(t)}{F_c(u(t))}\right|\right|_N^2 + 4\left[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\right]^2 \mathbb{E}\left|\left|\frac{c(t)}{F_c(u(t))}\right|\right|_N^2 \\
&\quad + 4\, a_2^2\, \mathbb{E}\left|\left|\frac{||u(t)||_N^2\; c(t)}{F_c(u(t))}\right|\right|_N^2 \Big\}
\end{aligned}
$$

but we know that $F_c(u(t)) \geq 1$, hence

$$
\begin{aligned}
&\mathbb{E}\left|\left|v(t) - \hat{f}_c(u(t))\right|\right|_N^2 \\
&\leq h_k^2 \Big[4\left(\kappa + h_k\Big[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big]\right)^2 \mathbb{E}\,||v(t)||_N^2 + 4\, h_k^2\, a_2^2\, \mathbb{E}\left|\left|\,||u(t)||_N^2\; v(t)\right|\right|_N^2 \\
&\quad + 4\left[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\right]^2 \mathbb{E}\,||c(t)||_N^2 + 4\, a_2^2\, \mathbb{E}\left|\left|\,||u(t)||_N^2\; c(t)\right|\right|_N^2 \Big] \\
&\leq h_k^2 \Big[4\left(\kappa + h_k\Big[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big]\right)^2 \mathbb{E}V(u(t)) \\
&\quad + 4\, h_k^2\, a_2^2 \left(\mathbb{E}\,||u(t)||_N^4\right)^{\frac{1}{2}} \left(\mathbb{E}\,||v(t)||_N^2\right)^{\frac{1}{2}} \\
&\quad + 4\left[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\right]^2 \mathbb{E}V(u(t)) + 4\, a_2^2 \left(\mathbb{E}\,||u(t)||_N^4\right)^{\frac{1}{2}} \left(\mathbb{E}\,||c(t)||_N^2\right)^{\frac{1}{2}} \Big] \\
&\leq h_k^2 \Big[4\left[\left(\kappa + h_k\Big[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big]\right)^2 + \Big[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big]^2\right] \mathbb{E}V(u(t)) \\
&\quad + 4\, h_k^2\, a_2^2 \Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}} \Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}} + 4\, a_2^2 \Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}} \Big(\mathbb{E}V(u(t))\Big)^{\frac{1}{2}} \Big]
\end{aligned}
$$

thus

$$
\begin{aligned}
&\mathbb{E}\left|\left|v(t) - \hat{f}_c(u(t))\right|\right|_N^2 \\
&\leq h_k^2 \Big(4\left[\left(\kappa + h_k\Big[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big]\right)^2 + \Big[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big]^2\right] \mathbb{E}V(u(t)) \\
&\quad + 4\, a_2^2\,(1 + h_k^2)\mathbb{E}V(u(t)) \Big) \\
&= 4h_k^2 \Big[\left(\kappa + h_k\Big[\sigma^2(\lambda_n + \beta_m) - a_1\Big]\right)^2 + \Big[\sigma^2(\lambda_n + \beta_m) - a_1\Big]^2 + a_2^2(1 + h_k^2) \Big] \mathbb{E}V(u(t)) \\
&\leq 4\, h_k^2 \Big[\Big(\kappa + \sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big)^2 + \Big[\sigma^2\left(\lambda_n + \beta_m\right) - a_1\Big]^2 + 2\, a_2^2 \Big] \mathbb{E}V(u(t)),
\end{aligned}
$$

for large N and take the conditional expectation, then we get

$$\mathbb{E}\left[\left|\left|v(t)\;-\;\hat{f}_c(u(t))\right|\right|_N^2|u(t)\right]$$
$$\leq\;4\Big[\kappa+\sigma^2N^2\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2+\;\sigma^4N^4\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2+\;2\,a_2^2\Big]\,(1+V(u(t)))\,h_k^2$$

thus

$$\mathbb{E}\left[\left|\left|v(t)\;-\;\hat{f}_c(u(t))\right|\right|_N^2|u(t)\right]\;\;\leq\;K_6\,\tilde{V}(u(t))\,h_k^2$$

where

$$K_6\;=\;4\Big[\kappa+\sigma^2N^2\pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2+\;\sigma^4N^4\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2+\;2\,a_2^2\Big].$$

Therefore,

$$\int_t^{t+h}\mathbb{E}\left[\left|\left|v(t)\;-\;\hat{f}_c(u(t))\right|\right|_N^2|u(t)\right]\,d\,s\;\;\leq\;K_6\,\tilde{V}(u(t))\,h_k^2\,\int_t^{t+h}\,d\,s$$
$$\leq\;K_6\,\tilde{V}(u(t))\,h_k^3$$

and since $h_k\leq\;1$, then

$$\int_t^{t+h}\mathbb{E}\left[\left|\left|v(t)\;-\;\hat{f}_c(u(t))\right|\right|_N^2|u(t)\right]\,d\,s\;\;\leq\;K_6\,\tilde{V}(u(t))\,h_k^2. \tag{5.62}$$

Form inequalities (5.61) and (5.62), we have

$$\int_t^{t+h}\mathbb{E}\left[\left|\left|v(s)\;-\;\hat{f}_c(u(t))\right|\right|_N^2|u(t)\right]\,d\,s\;\;\leq\;K_5\,\tilde{V}(u(t))\,h_k^2\;+\;K_6\,\tilde{V}(u(t))\,h_k^2$$
$$\leq\;K_7\,\tilde{V}(u(t))\,h_k^2 \tag{5.63}$$

where $K_7=K_5+K_6$. Now, the second part of inequality (5.59) is

$$\mathbb{E}\left|\left|\int_t^{t+h}\hat{g}_c(u(t))\,d\,W(s)\right|\right|_N^2$$
$$=\;\mathbb{E}\left|\left|\int_t^{t+h}\frac{h_k\Big(b_0+b_1\,||u(t)||_{l^2}\;+\;b_2\,||v(t)||_{l^2}\Big)\alpha^2}{F_c(u(t))}d\,W(s)\right|\right|_N^2$$
$$=\;\mathbb{E}\left|\left|\frac{h_k\Big(b_0+b_1\,||u(t)||_{l^2}\;+\;b_2\,||v(t)||_{l^2}\Big)\alpha^2}{F_c(u(t))}\int_t^{t+h}\,d\,W(s)\right|\right|_N^2.$$

Then

$$\mathbb{E}\left|\left|\int_t^{t+h} \hat{g}_c(u(t))\, dW(s)\right|\right|_N^2$$
$$\leq \mathbb{E}\left[\left|\left|\frac{h_k\Big(b_0 + b_1\,||u(t)|| \,+\, b_2\,||v(t)||\Big)\alpha^2}{F_c(u(t))}\right|\right|_N^2 \left|\left|\int_t^{t+h} dW(s)\right|\right|_N^2\right]$$
$$\leq \left(\mathbb{E}\left|\left|\frac{h_k\Big(b_1\,||u(t)|| \,+\, b_2\,||v(t)||\Big)\alpha^2}{F_c(u(t))}\right|\right|_N^4\right)^{\frac{1}{2}} \left(\mathbb{E}\left|\left|\int_t^{t+h} dW(s)\right|\right|_N^4\right)^{\frac{1}{2}}$$

using Lemma 5.3.1, $F_c(u(t)) \geq 1$, and

$$(a+b+c)^4 = ((a+b+c)^2)^2 \leq (3(a^2+b^2+c^2))^2 \leq 27(a^4+b^4+c^4),$$

then we find

$$\mathbb{E}\left|\left|\int_t^{t+h} \hat{g}_c(u(t))\, dW(s)\right|\right|_N^2$$
$$\leq h_k^2\,\alpha^4\Big(27\Big[b_0^4 + b_1^4\mathbb{E}\,||u(t)||^4 + b_2^4\mathbb{E}\,||v(t)||^4\Big]\Big)^{\frac{1}{2}}(3\,h_k^2)^{\frac{1}{2}}$$
$$\leq 9\,\alpha^4\Big(b_0^4 + b_1^4\,\mathbb{E}V(u(t)) \,+\, b_2^4\,\mathbb{E}V^2(u(t))\Big)^{\frac{1}{2}}\, h_k^3.$$

Hence, by redoing above steps for the conditional expectation, we arrive at

$$\begin{aligned}\mathbb{E}\Big[\left|\left|\int_t^{t+h} \hat{g}_c(u(t))\, dW(s)\right|\right|_N^2 \Big|\mathcal{F}_t\Big] &\leq 9\,\alpha^4\Big(b_0^4 + b_1^4\,V(u(t)) \,+\, b_2^4\,V^2(u(t))\Big)^{\frac{1}{2}}\, h_k^3\\ &\leq 9\,\alpha^4\Big(b_0^2 + b_1^2\,\sqrt{V(u(t))} \,+\, b_2^2\,V(u(t))\Big)\, h_k^3\\ &\leq 9\,\alpha^4\Big(b_0^2 + \frac{b_1^4}{2} + \frac{V(u(t))}{2} \,+\, b_2^2\,V(u(t))\Big)\, h_k^3,\end{aligned}$$

because we know from the Young's inequality that if $p = 2\,\text{and}\,q = 2,$ that $a\,\cdot\,b\,\leq\,\frac{a^2}{2}\,+\,\frac{b^2}{2}$, then

$$\mathbb{E}\Big[\left|\left|\int_t^{t+h} \hat{g}_c(u(t))\, dW(s)\right|\right|_N^2 \Big|\mathcal{F}_t\Big] \leq 9\,\alpha^4\Big(b_0^2 + \frac{b_1^4}{2} + (\frac{1}{2} + b_2^2)\,V(u(t))\Big)\, h_k^3$$

and therefore

$$\mathbb{E}\Big[\left|\left|\int_t^{t+h} \hat{g}_c(u(t))\, dW(s)\right|\right|_N^2 \Big|\mathcal{F}_t\Big] \leq 9\alpha^4 \max\Big\{b_0^2 + \frac{b_1^4}{2} + \frac{1}{2} + b_2^2,\, \frac{1}{2} + b_2^2\Big\} V(u(t))h_k^2.$$

Thus

$$\mathbb{E}\left|\left|\int_t^{t+h} \hat{g}_c(u(t))\,dW(s)\right|\right|_N^2 \quad\le\; K_8\,V(u(t))\,h_k^2$$

$$\le\; k_8\;(1+V(u(t)))\;h_k^2 \tag{5.64}$$

$$\le\; k_8\,\tilde{V}(u(t)))\,h_k^2. \tag{5.65}$$

where

$$K_8 \;=\; 9\alpha^4 \max\left\{b_0^2+\frac{b_1^4}{2}+\frac{1}{2}+\,b_2^2,\;\frac{1}{2}+b_2^2\right\}.$$

Thus inequality (5.59) equivelant to

$$\mathbb{E}\left[||c(t+h)\;-\;\hat{c}(t+h)||_N^2\,\middle|\mathcal{F}_t\right]$$

$$\le\;2\left(K_7\,\tilde{V}(u(t))\,h_k^2\;+\;2\,K_8\,\tilde{V}(u(t))\,h_k^2\right.$$

$$\le\;K_{c_2}^c\,\tilde{V}(u(t))\,h_k^2$$

where $K_{c_2}^c \;=\; 2\left(K_7\;+\;K_8\right)$, which is implies that

$$\left(\mathbb{E}\left[||c(t+h)\;-\;\hat{c}(t+h)||_N^2\,\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \le\;(K_{c_2}^c)^{\frac{1}{2}}\left(\tilde{V}(u(t))\right)^{\frac{1}{2}}\,h_k. \tag{5.66}$$

2) To prove inequality (5.58), we know from Theorem 5.2.4 that,

$$v(t+h)\;-\;\hat{v}(t+h)\;=\;\int_t^{t+h}\left[f_v(u(s))\;-\;\hat{f}_v(u(t))\right]d\,s$$

$$+\;\int_t^{t+h}\left[g_v(u(s))\;-\;\hat{g}_v(u(t))\right]d\,W(s)$$

thus

$$||v(t+h)\;-\;\hat{v}(t+h)||_N^2\;\le\;2\left|\left|\int_t^{t+h}\left[f_v(u(s))\;-\;\hat{f}_v(u(t))\right]d\,s\right|\right|_N^2 + 2\left|\left|\int_t^{t+h}\left[g_v(u(s))\;-\;\hat{g}_v(u(t))\right]d\,W(s)\right|\right|_N^2 \tag{5.67}$$

and we know that

$$f_v(u(s))\;-\;\hat{f}_v(u(t))\;=\;f_v(u(s))\;-\;f_v(u(t))\;+\;f_v(u(t))\;-\;f_v(\hat{u}(t))$$

$$+\;f_v(\hat{u}(t))-\;\hat{f}_v(\hat{u}(t))$$

and, by using the local property that $u(t) = \hat{u}(t)$, hence

$$\left|\left|f_v(u(s)) - \hat{f}_v(u(t))\right|\right|_N^2 \leq 2\,||f_v(u(s)) - f_v(u(t))||_N^2 + 2\left|\left|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\right|\right|_N^2 \tag{5.68}$$

where

$$f_v(u(s)) - f_v(u(t)) = \left[\sigma^2(\lambda_n + \beta_m) - a_1\right](c(t) - c(s)) + a_2\left(||u(t)||_N^2\, c(t) - ||u(s)||_N^2\, c(s)\right) + \kappa\,(v(t) - v(s))$$

thus

$$||f_v(u(s)) - f_v(u(t))||_N^2 \leq 3\left(\sigma^2(\lambda_n + \beta_m) - a_1\right)^2 ||c(t) - c(s)||_N^2 + 3\,a_2^2\left|\left|\,||u(t)||_N^2\, c(t) - ||u(s)||_N^2\, c(s)\right|\right|_N^2 + 3\,\kappa^2\,||v(t) - v(s)||_N^2.$$

From Lemma 5.2.5, we have

$$\left|\left|\,||u(t)||_N^2\, c(t) - ||u(s)||_N^2\, c(s)\right|\right|_N \leq 3\,||c(\gamma)||_N^2\,||v(\gamma)||_N\,||c(t) - c(s)||_N$$

hence

$$\begin{aligned}
&\mathbb{E}\left|\left|\,||u(t)||_N^2\, c(t) - ||u(s)||_N^2\, c(s)\right|\right|_N \\
&\leq 9\,\mathbb{E}\Big(\,||c(\gamma)||_N^4\,||v(\gamma)||_N^2\,||c(t) - c(s)||_N^2\Big) \\
&\leq 9\Big[\mathbb{E}\Big(\,||c(\gamma)||_N^8\,||v(\gamma)||_N^4\Big)\Big]^{\frac{1}{2}}\Big[\mathbb{E}\,||c(t) - c(s)||_N^4\Big]^{\frac{1}{2}} \\
&\leq 9\Big[\mathbb{E}\,||c(\gamma)||_N^{16}\Big]^{\frac{1}{2}}\Big[\mathbb{E}\,||v(\gamma)||_N^8\Big]^{\frac{1}{2}}\Big[\mathbb{E}\,||c(t) - c(s)||_N^4\Big]^{\frac{1}{2}} \\
&\leq \frac{36}{a_2^2}\Big[\mathbb{E}V^4(u(t))\Big]^{\frac{1}{2}}\Big[\mathbb{E}V^4(u(t))\Big]^{\frac{1}{2}}\Big[\mathbb{E}\,||c(t) - c(s)||_N^4\Big]^{\frac{1}{2}}
\end{aligned}$$

therefore

$$\mathbb{E}\left|\left|\,||u(t)||_N^2\, c(t) - ||u(s)||_N^2\, c(s)\right|\right|_N \leq \frac{36}{a_2^2}\mathbb{E}V^4(u(t))\Big[\mathbb{E}\,||c(t) - c(s)||_N^4\Big]^{\frac{1}{2}} \tag{5.69}$$

but we can find that

$$\begin{aligned}\mathbb{E}\,||c(t)-c(s)||_N^4 &\le \mathbb{E}\Big(\int_t^s \big[\mathbb{E}V(u(r))\big]^{\frac{1}{2}}\,d\,r\Big)^4\\ &\le \int_t^s \big[\mathbb{E}V(u(r))\big]^2\,d\,r\,(s-t)^2\\ &\le \big[\mathbb{E}V(u(t))\big]^2\,(s-t)^2\int_t^s d\,r\end{aligned}$$

$$\mathbb{E}\,||c(t)-c(s)||_N^4 \le \big[\mathbb{E}V(u(t))\big]^2\,(s-t)^3. \tag{5.70}$$

Substituting inequality (5.70) in inequality (5.69), we find that

$$\mathbb{E}\,\big|\big|\,||u(t)||_N^2\;c(t)-||u(s)||_N^2\;c(s)\big|\big|_N \le \frac{36}{a_2^2}\mathbb{E}V^4(u(t))\big[\mathbb{E}V(u(t))\big]^2\,(s-t)^3 \tag{5.71}$$

and we know that

$$\mathbb{E}\,\big|\big|\,||c(t)-c(s)||_N^2\;c(s)\big|\big|_N \le \mathbb{E}V(u(t))\,(s-t)^2. \tag{5.72}$$

Also from Theorem 5.2.3. part 1), we know that

$$\int_t^{t+h}\mathbb{E}\,||v(t)-v(s)||_N^2\;d\,s \le K_5\,\tilde{V}(u(t))\,h_k^2. \tag{5.73}$$

Therefore, by using inequalities (5.71), (5.72), and (5.73), we find that

$$\begin{aligned}&\int_t^{t+h}\mathbb{E}\,||f_v(u(s))-f_v(u(t))||_N^2\;ds\\ &\le 3\Big[\sigma^2(\lambda_n+\beta_m)-a_1\Big]^2\int_t^{t+h}\mathbb{E}\,||c(t)-c(s)||_N^2\;ds\\ &\quad+3\,a_2^2\int_t^{t+h}\mathbb{E}\,\big|\big|\,||u(t)||_N^2\;c(t)-||u(s)||_N^2\;c(s)\big|\big|_N^2\;ds\\ &\quad+\;3\,\kappa^2\int_t^{t+h}\mathbb{E}\,||v(t)\;-\;v(s)||_N^2\;d\,s,\end{aligned}$$

hence

$$
\begin{aligned}
&\int_t^{t+h} \mathbb{E}\left[||f_v(u(s)) - f_v(u(t))||_N^2 \,|u(t)\right] \, ds \\
&\leq 3\Big[\sigma^2(\lambda_n + \beta_m) - a_1\Big]^2 \int_t^{t+h} V(u(t))(s-t)^2 \, ds \\
&\quad + 108 \int_t^{t+h} V^4(u(t))\mathbb{E}V(u(t))|s-t|^3 \, ds + \, 3\, K_5\, \kappa^2\, \tilde{V}(u(t))\, h_k^2 \\
&\leq \Big[\sigma^2\,(\lambda_n + \beta_m) \, - \, a_1\Big]^2 \, V(u(t))\, h_k^3 + \frac{108}{4}\, V^4(u(t))V(u(t))\, h_k^4 \, + \, 3\, K_5\, \kappa\, \tilde{V}(u(t))\, h_k^2 \\
&\leq \Big[\sigma^2\,(\lambda_n + \beta_m) \, - \, a_1\Big]^2 \, V(u(t))\, h_k^2 + \, 27\, V^4(u(t))V(u(t))\, h_k^2 \, + \, 3\, K_5\, \kappa\, \tilde{V}(u(t))\, h_k^2.
\end{aligned}
$$

Thus, for large N, we have

$$
\begin{aligned}
&\int_t^{t+h} \mathbb{E}\left[||f_v(u(s)) - f_v(u(t))||_N^2 \,|u(t)\right] \, ds \\
&\leq \sigma^4\, N^4\, \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2\, V(u(t))\, h_k^2 + \, 27\, V^4(u(t))V(u(t))\, h_k^2 \, + \, 3\, K_5\, \kappa\, \tilde{V}(u(t))\, h_k^2 \\
&\leq \sigma^4\, N^4\, \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 \, \left(1 + V(u(t))\right) \, h_k^2 \\
&\quad + \, 27\, V^4(u(t))\,(1 + V(u(t))) \, h_k^2 \, + \, 3\, K_5\, \kappa\, \tilde{V}(u(t))\, h_k^2
\end{aligned}
$$

therefore

$$
\int_t^{t+h} \mathbb{E}\left[||f_v(u(s)) - f_v(u(t))||_N^2 \,|u(t)\right] \, ds \;\; \leq \; K_9\, \tilde{V}(u(t))\, h_k^2 \tag{5.74}
$$

where

$$
K_9 \, = \sigma^4\, N^4\, \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 \, + \, 27\, V^4(u(t)) + \, 3\, K_5\, \kappa.
$$

The second part of inequality (5.68) is,

$$
\begin{aligned}
&\Big|\Big| f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t)) \Big|\Big|_N^2 \\
&\leq\, h_k^2 \,\Big[\left|\left| \frac{\kappa \left[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||_N^2 \right] \hat{c}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \\
&\quad +\, \left|\left| \frac{h_k \Big[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||_N^2 \Big]^2 \hat{c}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \,+\, \left|\left| \frac{\kappa^2\,\hat{v}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \\
&\quad +\, \left|\left| \frac{h_k\,\kappa \Big[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||_N^2 \Big] \hat{v}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \Big] \\
&\leq\, 4\,h_k^2 \,\Big[\kappa^2 \Big[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||_N^2 \Big]^2 \left|\left| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \\
&\quad +\, h_k^2 \Big[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||_N^2 \Big]^4 \left|\left| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \\
&\quad +\, \kappa^4 \left|\left| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \\
&\quad +\, h_k^2 \kappa^2 \Big[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||_N^2 \Big]^2 \left|\left| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \Big] \\
&\leq\, 4\,h_k^2 \,\Big[\kappa^2 \Big(\Big[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||_N^2 \Big]^2 \left|\left| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \\
&\quad +\, \kappa^2 \left|\left| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \Big) \\
&\quad +\, h_k^2 \Big[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||_N^2 \Big]^2 \Big(\Big[\sigma^2\,(\lambda_n + \beta_m) \,-\, a_1 \\
&\quad +\, a_2\,||\hat{u}(t)||_N^2 \Big]^2 \left|\left| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \,+\, \kappa^2 \left|\left| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right|\right|_N^2 \Big) \Big]
\end{aligned}
$$

hence

$$\Big|\Big|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\Big|\Big|^2_N$$
$$\leq 4\,h_k^2 \Big[\Big(\kappa^2 \,+\, h_k^2\Big[\sigma^2\,(\lambda_n+\beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||^2_N\Big]^2\Big)$$
$$\Big(\Big[\sigma^2\,(\lambda_n+\beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||^2_N\Big]^2 \Big|\Big|\frac{\hat{c}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N$$
$$+\,\kappa^2 \Big|\Big|\frac{\hat{v}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N\Big)\Big]$$

using $F(\hat{u}(t)) \geq 1$, and pulling the expectation over the last identity, then we find

$$\mathbb{E}\Big|\Big|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\Big|\Big|^2_N$$
$$\leq 4\,h_k^2\,\mathbb{E}\Big[\Big(\kappa^2 \,+\, h_k^2\Big[\sigma^2\,(\lambda_n+\beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||^2_N\Big]^2\Big)\Big(\Big[\sigma^2\,(\lambda_n+\beta_m) \,-\, a_1$$
$$+\, a_2\,||\hat{u}(t)||^2_N\Big]^2 \Big|\Big|\frac{\hat{c}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N +\,\kappa^2 \Big|\Big|\frac{\hat{v}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N\Big)\Big]$$

therefore,

$$\mathbb{E}\Big|\Big|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\Big|\Big|^2_N$$
$$\leq 4\,h_k^2\,\mathbb{E}\Big[\Big(\kappa^2 \,+\, \Big[\sigma^2\,(\lambda_n+\beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||^2_N\Big]^2\Big)\Big(\Big[\sigma^2\,(\lambda_n+\beta_m) \,-\, a_1$$
$$+\, a_2\,||\hat{u}(t)||^2_N\Big]^2 \Big|\Big|\frac{\hat{c}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N +\,\kappa^2 \Big|\Big|\frac{\hat{v}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N\Big)\Big]$$

hence

$$\mathbb{E}\Big|\Big|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\Big|\Big|^2_N$$
$$\leq 4\,h_k^2\,\Big(\mathbb{E}\Big[\kappa^2 \,+\, \Big[\sigma^2\,(\lambda_n+\beta_m) \,-\, a_1 \,+\, a_2\,||\hat{u}(t)||^2_N\Big]^2\Big]^2\Big)^{\frac{1}{2}}\Big[\mathbb{E}\Big(\Big[\sigma^2\,(\lambda_n+\beta_m) \,-\, a_1$$
$$+\, a_2\,||\hat{u}(t)||^2_N\Big]^2 \Big|\Big|\frac{\hat{c}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N +\,\kappa^2 \Big|\Big|\frac{\hat{v}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N\Big)^2\Big]^{\frac{1}{2}}$$

for large N, we have

$$\mathbb{E}\Big|\Big|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\Big|\Big|^2_N$$
$$\leq 4\,h_k^2\,\Big(\mathbb{E}\Big[\kappa^2 \,+\, \Big[\sigma^2\,N^2\,\pi^2(\tfrac{1}{l_x^2}+\tfrac{1}{l_y^2})^2 \,+\, a_2\,||\hat{u}(t)||^2_N\Big]^2\Big]^2\Big)^{\frac{1}{2}}\Big[\mathbb{E}\Big(\Big[\sigma^2\,N^2\,\pi^2(\tfrac{1}{l_x^2}+\tfrac{1}{l_y^2})^2$$
$$+\, a_2\,||\hat{u}(t)||^2_N\Big]^2 \Big|\Big|\frac{\hat{c}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N +\,\kappa^2 \Big|\Big|\frac{\hat{v}(t)}{F(\hat{u}(t))}\Big|\Big|^2_N\Big)^2\Big]^{\frac{1}{2}}. \tag{5.75}$$

First, I will calculate the first part of the right side of inequality (5.75) which is

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\kappa^2 + h_k^2\Big[\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + a_2\, ||\hat{u}(t)||_N^2\Big]^2\Big]^2\Big)^{\frac{1}{2}} \\
&\le \Big(\mathbb{E}\Big[2\kappa^4 + 2\,\Big[\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 \,+\, a_2\, ||\hat{u}(t)||_N^2\Big]^4\Big]\Big)^{\frac{1}{2}} \\
&\le \Big(\mathbb{E}\Big[2\kappa^4 + 2\,\Big[4\,\left(\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2\right)^4 \,+\, 4\, a_2^4\, ||\hat{u}(t)||_N^8\Big]\Big]\Big)^{\frac{1}{2}}
\end{aligned}$$

hence

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\kappa^2 + h_k^2\Big[\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + a_2\, ||\hat{u}(t)||_N^2\Big]^2\Big]^2\Big)^{\frac{1}{2}} \\
&\le \Big(2\kappa^4 + 2\,\Big[4\,\left(\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2\right)^4 \,+\, 4\, a_2^4 \mathbb{E}\, ||\hat{u}(t)||_N^8\Big]\Big)^{\frac{1}{2}} \\
&\le \Big(2\kappa^4 + \,8\,\Big[\,\left(\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2\right)^4 \,+\, a_2^4 \mathbb{E} V^2(u(t))\Big]\Big)^{\frac{1}{2}} \\
&\quad = \Big(2\,\kappa^4 + \,8\,\left(\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2\right)^4 \,+\, 8\, a_2^4 \mathbb{E} V^2(u(t))\Big)^{\frac{1}{2}} \\
&\le \sqrt{2}\left(\kappa^2 + \,2\,\left(\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2\right)^2 + 2a_2^2\big[\mathbb{E} V^2(u(t))\big]^{\frac{1}{2}}\right) \\
&\le \sqrt{2}\left(\kappa^2 + \,2\sigma^4\, N^4\, \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + 2a_2^2\big[\mathbb{E} V^2(u(t))\big]^{\frac{1}{2}}\right).
\end{aligned}$$

Now, the second part of the right side of inequality (5.75) is

$$\begin{aligned}
&\mathbb{E}\Big(\Big[\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 +\, a_2\, ||\hat{u}(t)||_N^2\Big]^2 \left|\left|\frac{\hat{c}(t)}{F(\hat{u}(t))}\right|\right|_N^2 +\, \kappa^2\, \left|\left|\frac{\hat{v}(t)}{F(\hat{u}(t))}\right|\right|_N^2\Big)^2 \\
&\le\, 2\mathbb{E}\left(\Big[\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 +\, a_2\, ||\hat{u}(t)||_N^2\Big]^4 \left|\left|\frac{\hat{c}(t)}{F(\hat{u}(t))}\right|\right|_N^4\right) +\, 2\,\kappa^2\, \mathbb{E}\left|\left|\frac{\hat{v}(t)}{F(\hat{u}(t))}\right|\right|_N^2 \\
&\le\, 2\left(\mathbb{E}\Big[\sigma^2\, N^2\, \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 +\, a_2\, ||\hat{u}(t)||_N^2\Big]^8\right)^{\frac{1}{2}} \left(\mathbb{E}\left|\left|\frac{\hat{c}(t)}{F(\hat{u}(t))}\right|\right|_N^8\right)^{\frac{1}{2}} \\
&\quad +\, 2\,\kappa^2\, \mathbb{E}\left|\left|\frac{\hat{v}(t)}{F(\hat{u}(t))}\right|\right|_N^2 \\
&\le\, 2\left(16\,\sigma^{16}\, N^{16}\, \pi^{16} (\frac{1}{l_x^2} + \frac{1}{l_y^2})^{16} +\, a_2^8 \mathbb{E}\, ||\hat{u}(t)||_N^{16}\right)^{\frac{1}{2}} \left(\mathbb{E}\left|\left|\frac{\hat{c}(t)}{F(\hat{u}(t))}\right|\right|_N^8\right)^{\frac{1}{2}} \\
&\quad +\, 2\,\kappa^2\, \mathbb{E}\left|\left|\frac{\hat{v}(t)}{F(\hat{u}(t))}\right|\right|_N^2
\end{aligned}$$

but $F(\hat{u}(t)) \geq 1$, then

$$\mathbb{E}\Big(\Big[\sigma^2 N^2 \pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2 + a_2\,||\hat{u}(t)||_N^2\Big]^2 ||\hat{c}(t)||_N^2 + \kappa^2\,||\hat{v}(t)||_N^2\Big)^2$$

$$\leq 2\left(16\,\sigma^{16} N^{16}\pi^{16}(\frac{1}{l_x^2}+\frac{1}{l_y^2})^{16} + a_2^8\mathbb{E}\,||\hat{u}(t)||_N^{16}\right)^{\frac{1}{2}}\left(\mathbb{E}\,||\hat{c}(t)||_N^8\right)^{\frac{1}{2}} + 2\,\kappa^2\,\mathbb{E}\,||\hat{v}(t)||_N^2$$

$$\leq 2\left(4\,\sigma^8 N^8\pi^8(\frac{1}{l_x^2}+\frac{1}{l_y^2})^8 + a_2^4\mathbb{E}\,||\hat{u}(t)||_N^8\right)\left(\mathbb{E}\,||\hat{c}(t)||_N^8\right)^{\frac{1}{2}} + 2\,\kappa^2\,\mathbb{E}\,||\hat{v}(t)||_N^2\,,$$

hence

$$\left[\mathbb{E}\Big(\Big[\sigma^2 N^2 \pi^2(\frac{1}{l_x^2}+\frac{1}{l_y^2})^2 + a_2\,||\hat{u}(t)||_N^2\Big]^2 ||\hat{c}(t)||_N^2 + \kappa^2\,||\hat{v}(t)||_N^2\Big)^2\right]^{\frac{1}{2}}$$

$$\leq \left[2\left(4\,\sigma^8 N^8\pi^8(\frac{1}{l_x^2}+\frac{1}{l_y^2})^8 + a_2^4\mathbb{E}\,||\hat{u}(t)||_N^8\right)\left(\mathbb{E}\,||\hat{c}(t)||_N^8\right)^{\frac{1}{2}} + 2\,\kappa^2\,\mathbb{E}\,||\hat{v}(t)||_N^2\right]^{\frac{1}{2}}$$

$$\leq \sqrt{2}\left(4\,\sigma^8 N^8\pi^8(\frac{1}{l_x^2}+\frac{1}{l_y^2})^8 + a_2^4\mathbb{E}\,||\hat{u}(t)||_N^8\right)^{\frac{1}{2}}\left(\mathbb{E}\,||\hat{c}(t)||_N^8\right)^{\frac{1}{4}}$$

$$+\sqrt{2}\,\kappa\,\left(\mathbb{E}\,||\hat{v}(t)||_N^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{2}\left(2\,\sigma^4 N^4\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 + a_2^2\left(\mathbb{E}\,||\hat{u}(t)||_N^8\right)^{\frac{1}{2}}\right)\left(\mathbb{E}\,||\hat{c}(t)||_N^8\right)^{\frac{1}{4}}$$

$$+\sqrt{2}\,\kappa\,\left(\mathbb{E}\,||\hat{v}(t)||_N^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{2}\left(2\,\sigma^4 N^4\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 + 2\,a_2\left(\mathbb{E}V^2(u(t))\right)^{\frac{1}{2}}\right)\left(\frac{2}{a_2}\mathbb{E}V^8(u(t))\right)^{\frac{1}{4}}$$

$$+\sqrt{2}\,\kappa\,(\mathbb{E}V(u(t)))^{\frac{1}{2}}\,.$$

Thus inequality (5.75) equivelant to

$$\mathbb{E}\left|\left|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\right|\right|_N^2$$

$$\leq 4\,h_k^2\cdot\sqrt{2}\left(\kappa^2 + 2\sigma^4 N^4\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 + 2a_2^2\big[\mathbb{E}V^2(u(t))\big]^{\frac{1}{2}}\right)$$

$$\Big[\sqrt{2}\left(2\,\sigma^4 N^4\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 + 2\,a_2\left(\mathbb{E}V^2(u(t))\right)^{\frac{1}{2}}\right)\left(\frac{2}{a_2}\mathbb{E}V^8(u(t))\right)^{\frac{1}{4}}$$

$$+\sqrt{2}\,\kappa\,(\mathbb{E}V(u(t)))^{\frac{1}{2}}\Big].$$

Pulling the conditional expectation over the last identity, we find that

$$\mathbb{E}\left[\left|\left|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\right|\right|_N^2 |u(t)\right]$$

$$\leq\, 8\,h_k^2\,\cdot\,\left(\kappa^2 +\, 2\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 + 2a_2^2\big[V^2(u(t))\big]^{\frac{1}{2}}\right)$$

$$\Big[\left(2\,\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 +\, 2\,a_2\left(V^2(u(t))\right)^{\frac{1}{2}}\right)\left(\frac{2}{a_2}V^8(u(t))\right)^{\frac{1}{4}}$$

$$+\,\kappa\,\left(V(u(t))\right)^{\frac{1}{2}}\Big].$$

Thus

$$\mathbb{E}\left[\left|\left|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\right|\right|_N^2 |u(t)\right]$$

$$\leq\, 8\,h_k^2\,\cdot\,\left(\kappa^2 +\, 2\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 + 2a_2^2\,V(u(t))\right)$$

$$\Big[\left(2\,\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 +\, 2\,a_2\,V(u(t))\right)\left(\frac{2}{a_2}\right)^{\frac{1}{4}}V^2(u(t)) +\,\kappa\,V^{\frac{1}{2}}(u(t))\Big]$$

$$\leq\, 8\,h_k^2\,\cdot\,\left(\kappa^2 +\, 2\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 + 2a_2^2\,V(u(t))\right)$$

$$\Big[\left(2\,\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 +\, 2\,a_2\,V(u(t))\right)\left(\frac{2}{a_2}\right)^{\frac{1}{4}}V^{\frac{3}{2}}(u(t)) +\,\kappa\Big]V^{\frac{1}{2}}(u(t))$$

hence

$$\mathbb{E}\left[\left|\left|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\right|\right|_N^2 |u(t)\right] \leq\, K_{10}\,(1+V(u(t)))\,h_k^2$$

$$\leq\, K_{10}\tilde{V}(u(t))\,h_k^2. \tag{5.76}$$

where

$$K_{10} = 8\,\cdot\,\left(\kappa^2 +\, 2\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 + 2a_2^2\,V(u(t))\right)$$

$$\Big[\left(2\,\sigma^4\,N^4\,\pi^4(\frac{1}{l_x^2}+\frac{1}{l_y^2})^4 +\, 2\,a_2\,V(u(t))\right)\left(\frac{2}{a_2}\right)^{\frac{1}{4}}V^{\frac{3}{2}}(u(t)) +\,\kappa\Big],$$

therefore

$$\int_t^{t+h}\mathbb{E}\left[\left|\left|f_v(\hat{u}(t)) \,-\, \hat{f}_v(\hat{u}(t))\right|\right|_N^2 |u(t)\right]\,d\,s \;\leq\; K_{10}\tilde{V}(u(t))\,h_k^3$$

$$\leq\; K_{10}\tilde{V}(u(t))\,h_k^2. \tag{5.77}$$

To complete the proof, we have to simplify the second part of the right side of inequality (5.67) which is

$$\left|\left|\int_t^{t+h} \Big[g_v(u(s)) \, - \, \hat{g}_v(u(t))\Big] d\,W(s)\right|\right|_N^2$$
$$\leq C_B \int_t^{t+h} ||g_v(u(s)) \, - \, \hat{g}_v(u(t))||_N^2 \, d\,s$$

by using Lemma 5.3.4, we get

$$\left|\left|\int_t^{t+h} \Big[g_v(u(s)) \, - \, \hat{g}_v(u(t))\Big] d\,W(s)\right|\right|_N^2$$
$$\leq C_B \int_t^{t+h} ||g_v(u(s)) \, - \, g_v(u(t))||_N^2 \, d\,s \, + \, C_B \int_t^{t+h} ||g_v(u(t)) \, - \, \hat{g}_v(u(t))||_N^2 \, d\,s. \tag{5.78}$$

We start with the first part of the right side of inequality (5.78) which is

$$\int_t^{t+h} ||g_v(u(s)) \, - \, g_v(u(t))||_N^2 \, d\,s$$
$$\leq \int_t^{t+h} ||b_1 \left(||u(s)||_{l^2} - ||u(t)||_{l^2}\right) + b_2 \left(||v(s)||_{l^2} - ||v(t)||_{l^2}\right)||_N^2 \, d\,s$$
$$\leq \int_t^{t+h} \left[2\,b_1^2 \left(||u(s)||_{l^2} - ||u(t)||_{l^2}\right)^2 + 2\,b_2^2 \left(||v(s)||_{l^2} - ||v(t)||_{l^2}\right)^2\right] \, d\,s$$
$$\leq \int_t^{t+h} \max\{2\,b_1^2, 2\,b_2^2\} \left[\left(||u(s)||_{l^2} - ||u(t)||_{l^2}\right)^2 + \left(||v(s)||_{l^2} - ||v(t)||_{l^2}\right)^2\right] \, d\,s$$
$$\leq \int_t^{t+h} 2\max\{2\,b_1^2, 2\,b_2^2\} \left[||u(\gamma)||_{l^2}^2 \, |s-t| + ||v(\gamma)||_{l^2}^2 \, |s-t|\right] \, d\,s$$
$$\leq \max\{2\,b_1^2, 2\,b_2^2\} V(u(t))\, h_k^2$$

thus

$$\int_t^{t+h} \mathbb{E}\, ||g_v(u(s)) \, - \, g_v(u(t))||_N^2 \, d\,s \leq \, K_{11}\, V(u(t))\, h_k^2 \leq \, K_{11}\, \tilde{V}(u(t))\, h_k^2 \tag{5.79}$$

where $K_{11} = \max\{2\,b_1^2, 2\,b_2^2\}$. And the second part of the right side of inequality (5.78), we start with

$$
\begin{aligned}
&\mathbb{E}\left[||g_v(u(t)) \,-\, \hat{g}_v(u(t))||_N^2\right] \\
&= h_k \mathbb{E}\left[\left|\left|\frac{[\kappa + h_k\, F(u(t))]\,[b_0 + b_1\,||u(t)||_{l^2} + b_2\,||v(t)||_{l^2}]}{1 + h_k\,\kappa + h_k^2\,F(u(t))}\right|\right|_N^2\right] \\
&\leq h_k \left(\mathbb{E}\,||\kappa + h_k\,F(u(t))||_N^4\right)^{\frac{1}{2}} \left(\mathbb{E}\,||b_0 + b_1\,||u(t)||_{l^2} + b_2\,||v(t)||_{l^2}||_N^4\right)^{\frac{1}{2}} \\
&\leq 81\, h_k \left(\mathbb{E}\left[\kappa^4 + h_k^4\,\sigma^4 N^4 \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + a_2^4\,||u(t)||_{l^2}^8\right]\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\mathbb{E}\left[b_0^4 + b_1^4\,||u(t)||_{l^2}^4 + b_2^4\,||v(t)||_{l^2}^4\right]\right)^{\frac{1}{2}},
\end{aligned}
$$

hence

$$
\begin{aligned}
&\mathbb{E}\left[||g_v(u(t)) \,-\, \hat{g}_v(u(t))||_N^2 \,|u(t)\right] \\
&\leq\ 81\, h_k \left(\kappa^4 + h_k^4\, \sigma^4 N^4 \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + a_2^4 \mathbb{E}\, ||u(t)||_{l^2}^8\right)^{\frac{1}{2}} \\
&\quad \cdot\ \left(b_0^4 + b_1^4 \mathbb{E}\, ||u(t)||_{l^2}^4 + b_2^4 \mathbb{E}\, ||v(t)||_{l^2}^4\right)^{\frac{1}{2}} \\
&\leq\ 81\, h_k \left(\kappa^4 + h_k^4\, \sigma^4 N^4 \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + 4\, a_2^2 \mathbb{E} V^2(u(t))|u(t)\right)^{\frac{1}{2}} \\
&\quad \cdot\ \left(b_0^4 + \frac{2\, b_1^4}{a_2} \mathbb{E} V(u(t))|u(t) + b_2^4 \mathbb{E} V^2(u(t))|u(t)\right)^{\frac{1}{2}} \\
&\leq\ 81\, h_k \left(\kappa^4 + h_k^4\, \sigma^4 N^4 \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + 4\, a_2^2\, V^2(u(t))\right)^{\frac{1}{2}} \\
&\quad \cdot\ \left(b_0^4 + \frac{2\, b_1^4}{a_2}\, V(u(t)) + b_2^4\, V^2(u(t))\right)^{\frac{1}{2}} \\
&\leq\ 81\, h_k \left(\kappa^2 + h_k^2\, \sigma^2 N^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + 2\, a_2\, V(u(t))\right) \\
&\quad \cdot\ \left(b_0^2 + \frac{\sqrt{2}\, b_1^2}{\sqrt{a_2}}\, (V(u(t)))^{\frac{1}{2}} + b_2^2\, V(u(t))\right) \\
&\leq\ 81\ \left(\kappa^2 + h_k^2\, \sigma^2 N^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + 2\, a_2\, (V(u(t)))^{\frac{1}{2}}\right) \\
&\quad \cdot\ \left(b_0^2 + \frac{\sqrt{2}\, b_1^2}{\sqrt{a_2}}\ + b_2^2\, (V(u(t)))^{\frac{1}{2}}\right) (V(u(t)))^{\frac{1}{2}}\, h_k \\
&\leq\ 81\ \left(\kappa^2 + h_k^2\, \sigma^2 N^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + 2\, a_2\, (V(u(t)))^{\frac{1}{2}}\right) \\
&\quad \cdot\ \left(b_0^2 + \frac{\sqrt{2}\, b_1^2}{\sqrt{a_2}}\ + b_2^2\, (V(u(t)))^{\frac{1}{2}}\right) (1 + V(u(t)))\, h_k
\end{aligned}
$$

thus

$$
\mathbb{E}\left[||g_v(u(t)) \,-\, \hat{g}_v(u(t))||_N^2 \,|u(t)\right] \leq\ K_{12} \tilde{V}(u(t))\, h_k \tag{5.80}
$$

where

$$K_{12} = 81 \left(\kappa^2 + h_k^2\, \sigma^2 N^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + 2\, a_2\, (V(u(t)))^{\frac{1}{2}} \right) \cdot \left(b_0^2 + \frac{\sqrt{2}\, b_1^2}{\sqrt{a_2}} + b_2^2\, (V(u(t)))^{\frac{1}{2}} \right)$$

therefore

$$\int_t^{t+h} \mathbb{E}\left[||g_v(u(t)) \,-\, \hat{g}_v(u(t))||_N^2 \,|u(t) \right] \, d\,s \leq\, K_{12} \tilde{V}(u(t))\, h_k^2. \tag{5.81}$$

Thus by using inequalities (5.74), (5.77), (5.79), and (5.81) we can write inequality (5.67) as

$$\begin{aligned} \mathbb{E}\left[||v_{n,m}(t+h) \,-\, \hat{v}_{n,m}(t+h)||_N^2 \,|\hat{u}(t) \right] \;&\leq\, [4(K_9 + K_{10}) + 2\, C_B (K_{11} + K_{12})] \tilde{V}(u(t))\, h_k^2 \\ &\leq\, K_{v_2}^c \tilde{V}(u(t))\, h_k^2 \end{aligned} \tag{5.82}$$

where $K_{v_2}^c = 4(K_9 + K_{10}) + 2\, C_B (K_{11} + K_{12})$. Thus

$$\begin{aligned} \left(\mathbb{E}\left[||v_{n,m}(t+h) \,-\, \hat{v}_{n,m}(t+h)||_N^2 \,|\hat{u}(t) \right] \right)^{\frac{1}{2}} \;&\leq\, K_{v_2}^c \left(\tilde{V}(u(t)) \right)^{\frac{1}{2}} \, h_k \\ &\leq\, K_{v_2}^c \tilde{V}(u(t))\, h_k. \end{aligned} \tag{5.83}$$

□

5.5 LOCAL MEAN SQUARE CONTRACTIVITY

Let $\hat{u}(t+h|t, u(t))$ be the continuous one step representation of numerical method (5.3).

Definition. The numerical sequence $\hat{u} \,=\, (\hat{u}_{n,m})_{n,m=1,\ldots,N}$ is said to be **numerically mean square contractive** on $[0,\, T]$ iff $\exists\, K_c$ constant such that $\forall\, h$ sufficiently small $\forall\, 0 \leq\, t \leq\, t+h \leq\, T$ it satisfies the inequality

$$\begin{aligned} &\mathbb{E}\left[\left|\left| \hat{u}_{u(t),t}(t+h) - \hat{u}_{\tilde{u}(t),t}(t+h) \right|\right|_H^2 |\hat{u}_{u(t),t}(t), \hat{u}_{\tilde{u}(t),t}(t) \right] \\ &\leq\, \exp\left(2 K_2^c\, h\right) \left(||u(t) - \tilde{u}(t)||_H^2 \right) \end{aligned} \tag{5.84}$$

where $u(t),\, \tilde{u}(t) \,\in\, H := \{u \in\, \mathbb{L}^2(\mathbb{D}) |\, \dot{u} \in\, \mathbb{L}^2(\mathbb{D})\}$ and $||u||_H = \sqrt{||u||_{\mathbb{L}^2(\mathbb{D})}^2 + ||\dot{u}||_{\mathbb{L}^2(\mathbb{D})}^2}$.

The aim of the following is to investigate mean square contractivity of numerical method (5.3).

Lemma 5.5.1. *Let $\hat{u}(t)$ be the numerical solution of the system (5.3) and let $u(t),\ \tilde{u}(t) \in H$. Then*

$$\left(\left|\left|\hat{u}_{u(t),t}(t)\right|\right|_H^2 - \left|\left|\hat{u}_{\tilde{u}(t),t}(t)\right|\right|_H^2\right)^2 \leq 8\,||\hat{u}(\gamma_1)||_H^2\,||u(t)-\tilde{u}(t)||_H^2 \tag{5.85}$$

where $\hat{u}(\gamma_1)$ on the line segment between $u(t)$ and $\tilde{u}(t)$.

Proof. By using the mean value theorem, we have

$$\left|\left|\left|\hat{u}_{u(t),t}(t)\right|\right|_H^2 - \left|\left|\hat{u}_{\tilde{u}(t),t}(t)\right|\right|_H^2\right| = \left|\left|\nabla_{(\hat{c},\hat{v})}\,||\hat{u}(\gamma_1)||_H^2\right|\right|_H\,||u(t)-\tilde{u}(t)||_H \tag{5.86}$$

where

$$\begin{aligned}
\nabla_{(\hat{c},\hat{v})}\,||\hat{u}(\gamma_1)||_H^2 &= \nabla_{(\hat{c},\hat{v})}\left(\sum_{i=1}^{N}\hat{c^2}_i(\gamma_1) + \sum_{i=1}^{N}\hat{v^2}_i(\gamma_1)\right)\\
&= (2\,\hat{c}_1 + 2\,\hat{v}_1,\ 2\,\hat{c}_2 + 2\,\hat{v}_2,\ ...,\ 2\,\hat{c}_N + 2\,\hat{v}_N)\\
&= 2\,(\hat{c}_1,\ \hat{c}_2,\ ...,\ \hat{c}_N) + 2\,(\hat{v}_1,\ \hat{v}_2,\ ...,\ \hat{v}_N)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left(\left|\left|\hat{u}_{u(t),t}(t)\right|\right|_H^2 - \left|\left|\hat{u}_{\tilde{u}(t),t}(t)\right|\right|_H^2\right)^2\\
&= 4\,||(\hat{c}_1,\ \hat{c}_2,\ ...,\ \hat{c}_N) + (\hat{v}_1,\ \hat{v}_2,\ ...,\ \hat{v}_N)||_{l^2}^2\,||u(t)-\tilde{u}(t)||_H^2\\
&\leq 8\left(||\hat{c}_1,\ \hat{c}_2,\ ...,\ \hat{c}_N||_{l^2}^2 + ||\hat{v}_1,\ \hat{v}_2,\ ...,\ \hat{v}_N||_{l^2}^2\right)||u(t)-\tilde{u}(t)||_H^2\\
&= 8\left(||\hat{c}(\gamma_1)||_{l^2}^2 + ||\hat{v}(\gamma_1)||_{l^2}^2\right)\left(||u(t)-\tilde{u}(t)||_H^2\right).
\end{aligned}$$

Hence,

$$\left(\left|\left|\hat{u}_{u(t),t}(t)\right|\right|_H^2 - \left|\left|\hat{u}_{\tilde{u}(t),t}(t)\right|\right|_H^2\right)^2 \leq 8\,||\hat{u}(\gamma_1)||_H^2\,||u(t)-\tilde{u}(t)||_H^2\,.$$

The proof is complete. □

Theorem 5.5.2. *Assume that $u(t)$, $\tilde{u}(t) \in H$. Then $\forall\, t : 0 \le t \le t+h \le T$, h small enough, the linear-implicit Euler method (5.3) is locally mean square Lipschitz continuous with Lipschitz constant $\hat{K}_2^c$ on $\mathbb{D}$, such that*

$$\mathbb{E}\left[||\hat{u}(t+h|u(t)) - \hat{u}(t+h|\tilde{u}(t))||_H^2 \,|u(t), \tilde{u}(t)\right] \\ \le \left(3 + \hat{K}_2^c(N, u(t), \tilde{u}(t))\, h\right) ||u(t) - \tilde{u}(t)||_H^2 .$$

where

$$\hat{K}_2^c = \hat{K}_2^c(N, u(t), \tilde{u}(t)) = \hat{K}_6(N, u(t), \tilde{u}(t)) + \hat{K}_{11}(N, u(t), \tilde{u}(t)).$$

Note: During the proof, we will explain the derivation and estimation of $\hat{K}_2^c$ because we have many constants and we do not like to repeat them.

Proof. We know that

$$\mathbb{E}\left[||\hat{u}(t+h|u(t)) - \hat{u}(t+h|\tilde{u}(t))||_H^2 \,|u(t), \tilde{u}(t)\right] \\ = \mathbb{E}\left[||\hat{c}(t+h|u(t)) - \hat{c}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t), \tilde{u}(t)\right] \\ +\mathbb{E}\left[||\hat{v}(t+h|u(t)) - \hat{v}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t), \tilde{u}(t)\right] .$$

So, we will separate the right side of last equation to

$$1)\, \hat{c}_{n,m}(t+h) = \hat{c}_{n,m}(t+h) + h_k\, (\hat{f}_c(\hat{u}(t)))_{n,m} + (\hat{g}_c(\hat{u}(t)))_{n,m} \triangle_k W_{n,m}(t) \qquad (5.87)$$

where, and to simplify let $F_1(n, m) = \sigma^2(\lambda_n + \beta_m) - a_1$,

$$(\hat{f}_c(\hat{u}(t)))_{n,m} = \frac{\hat{v}_{n,m}(t) - h_k(F_1(n,m) + s_2\, ||\hat{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2)\hat{c}_{n,m}(t)}{1 + h_k\, \kappa + h_k^2(F_1(n,m) + a_2\, ||\hat{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2)}$$

and

$$(\hat{g}_c(\hat{u}(t)))_{n,m} = \frac{h_k\, g(\hat{u}(t))}{1 + h_k\, \kappa + h_k^2(F_1(n,m) + a_2\, ||\hat{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2)}$$

$$2)\, \hat{v}_{n,m}(t+h) = \hat{v}_{n,m}(t+h) + h_k\, (\hat{f}_v(\hat{u}(t)))_{n,m} + (\hat{g}_v(\hat{u}(t)))_{n,m} \triangle_k W_{n,m}(t) \qquad (5.88)$$

where

$$(\hat{f}_v(\hat{u}(t)))_{n,m}$$
$$= \frac{-(F_1(n,m) + a_2\,||\hat{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\hat{c}_{n,m}(t) - (\kappa + h_k[F_1(n,m) + a_2\,||\hat{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}])\hat{v}_{n,m}(t)}{1 + h_k\,\kappa + \; h_k^2(F_1(n,m) + s_2\,||\hat{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})}$$

and

$$(\hat{g}_v(\hat{u}(t)))_{n,m} = \frac{g(\hat{u}(t))}{1 + h_k\,\kappa + \; h_k^2(F_1(n,m) + s_2\,||\hat{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})}.$$

First, we will start with equation (5.87).$\forall\; u(t),\, \tilde{u}(t) \in H$, we have

$$\begin{aligned}
&\mathbb{E}\left[||\hat{c}(t+h|u(t)) - \hat{c}(t+h|\tilde{u}(t))||^2_{l^2}\,|u(t), \tilde{u}(t))\right] \\
&\le\; 3\mathbb{E}\left[||\hat{c}(t|u(t)) - \hat{c}(t|\tilde{u}(t))||^2_{l^2}\,\Big| u(t), \hat{u}(t)\right] \\
&\quad +\; 3\,h_k^2\mathbb{E}\left[\left|\left|\hat{f}_c(\hat{u}(t)|u(t)) - \hat{f}_c(\hat{u}(t)|\tilde{u}(t))\right|\right|^2_{l^2}\Big| u(t), \tilde{u}(t)\right] \\
&\quad +\; 3\,\mathbb{E}\left[||\hat{g}_c(\hat{u}(t)|u(t)) - \hat{g}_c(\hat{u}(t)|\tilde{u}(t))||^2_{l^2}\,||(\triangle_k\,W_{n,m}(t))_{n,m=1,..,N}||^2_{l^2}\,\Big| u(t), \tilde{u}(t)\right] \\
&\le\; 3\mathbb{E}\left[||\hat{c}(t|u(t)) - \hat{c}(t|\tilde{u}(t))||^2_{l^2}\right] \\
&\quad +\; 3\,h_k^2\mathbb{E}\left[\left|\left|\hat{f}_c(\hat{u}(t)|u(t)) - \hat{f}_c(\hat{u}(t)|\tilde{u}(t))\right|\right|^2_{l^2}\right] \\
&\quad +\; 3\,\left(\mathbb{E}\,||\hat{g}_c(\hat{u}(t)|u(t)) - \hat{g}_c(\hat{u}(t)|\tilde{u}(t))||^4_{l^2}\right)^{\frac{1}{2}}\left(\mathbb{E}\,||\triangle_k\,W(t)||^4_{l^2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\hat{g}_c(\hat{u}(t)|u(t)) - \hat{g}_c(\hat{u}(t)|\tilde{u}(t)) = \hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t)) \\
&= \frac{h_k\,g(u(t))}{1 + h_k\,\kappa + h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})} - \frac{h_k\,g(\tilde{u}(t))}{1 + h_k\,\kappa + h_k^2(F_1 + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})} \\
&= \frac{h_k(1 + h_k\,\kappa + h_k^2\,F_1)[g(u(t)) - g(\tilde{u}(t))] + a_2\,h_k\left[||\tilde{u}(t)||^2\;g(u(t)) - ||u(t)||^2\;g(\tilde{u}(t))\right]}{\left[1 + h_k\,\kappa + h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + h_k^2(F_1 + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]}.
\end{aligned}$$

Thus

$$\begin{aligned}
&||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||^4_{l^2} \\
&\qquad \le\; 8\,h_k^4(1 + h_k\,\kappa + h_k^2\,F_1)^4\,||g(u(t)) - g(\tilde{u}(t))||^4_{l^2} \\
&\quad +8\,a_2^4\,h_k^4\left|\left|\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\;g(u(t)) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\;g(\tilde{u}(t))\right|\right|^4_{l^2}.
\end{aligned} \tag{5.89}$$

But

$$||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\ g(u(t)) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\ g(\tilde{u}(t))$$

$$= ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\ g(u(t)) - ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\ g(\tilde{u}(t)) + ||\tilde{u}(t)||^2\ g(\tilde{u}(t)) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\ g(\tilde{u}(t))$$

$$= ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,[g(u(t)) - g(\tilde{u}(t))] + \Big[||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})} - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\Big]\, g(\tilde{u}(t)),$$

therefore,

$$\Big|\Big|\, ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\ g(u(t)) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\ g(\tilde{u}(t))\Big|\Big|^4_{l^2}$$
$$\leq\ 8\,||\tilde{u}(t)||^8_{\mathbb{L}^2(\mathbb{D})}\,||g(u(t)) - g(\tilde{u}(t))||^4_{l^2}$$
$$+\,8\left(||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})} - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right)^4 ||g(\tilde{u}(t))||^4_{l^2}\,.$$

Thus inequality (5.89) equivelant to

$$||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||^4_{l^2}$$
$$\leq\ 8\,h_k^4(1 + h_k\,\kappa + h_k^2\,F_1)^4\,||g(u(t)) - g(\tilde{u}(t))||^4_{l^2}$$
$$+\,8\,a_2^4\,h_k^4\Big[8\,||\tilde{u}(t)||^8_{\mathbb{L}^2(\mathbb{D})}\,||g(u(t)) - g(\tilde{u}(t))||^4_{l^2}$$
$$+\,8\left(||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})} - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right)^4 ||g(\tilde{u}(t))||^4_{l^2}\,\Big]$$

hence

$$||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||^4_{l^2}$$
$$\leq\ 8\,h_k^4(1 + h_k\,\kappa + h_k^2\,F_1)^4\,||g(u(t)) - g(\tilde{u}(t))||^4_{l^2}$$
$$+\,64a_2^4h_k^4\,||\tilde{u}(t)||^8\,||g(u(t)) - g(\tilde{u}(t))||^4_{l^2}$$
$$+\,64a_2^4h_k^4\left(||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})} - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right)^4 ||g(\tilde{u}(t))||^4_{l^2}$$

using Lemma 5.4.1 and inequality (5.90), we have

$$
\begin{aligned}
&||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||_{l^2}^4 \\
&\leq \; 8\, h_k^4 (1 + h_k\, \kappa + h_k^2\, F_1)^4\, ||g(u(t)) - g(\tilde{u}(t))||_{l^2}^4 + 64 a_2^4 h_k^4\, ||\tilde{u}(t)||^8 \\
&\quad \cdot\, ||g(u(t)) - g(\tilde{u}(t))||_{l^2}^4 + \; 2^{12} a_2^4\, h_k^4\, ||\hat{u}(\gamma_1)||_H^2\, ||u(t) - \tilde{u}(t)||_H^4\, ||g(\tilde{u}(t))||_{l^2}^4
\end{aligned}
$$

and we can estimate $||g(u(t)) - g(\tilde{u}(t))||_{l^2}^4$ as the following

$$
\begin{aligned}
&||g(u(t)) - g(\tilde{u}(t))||_{l^2}^4 = [g(u(t)) - g(\tilde{u}(t))]^4 \\
&= \Big(b_1(||u(t)||_{\mathbb{L}^2(\mathbb{D})} - ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}) + b_2(||v(t)||_{\mathbb{L}^2(\mathbb{D})} - ||\tilde{v}(t)||_{\mathbb{L}^2(\mathbb{D})}) \Big)^4 \\
&\leq \; 8\, b_1^4 \Big(||u(t)||_{\mathbb{L}^2(\mathbb{D})} - ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})} \Big)^4 + \; 8\, b_2^4 \Big(||v(t)||_{\mathbb{L}^2(\mathbb{D})} - ||\tilde{v}(t)||_{\mathbb{L}^2(\mathbb{D})} \Big)^4 \\
&\leq \; 8\, b_1^4\, ||u(t) - \tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^4 + 8\, b_2^4\, ||v(t) - \tilde{v}(t)||_{\mathbb{L}^2(\mathbb{D})}^4
\end{aligned}
$$

hence

$$
||g(u(t)) - g(\tilde{u}(t))||_{l^2}^4 \; \leq \; 8(b_1^4 + b_2^4)\, ||u(t) - \tilde{u}(t)||_H^4\,, \tag{5.90}
$$

thus

$$
\begin{aligned}
&||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||_{l^2}^4 \\
&\leq \; 64\, h_k^4 (b_1^4 + b_2^4)(1 + h_k\, \kappa + h_k^2\, F_1)^4\, ||u(t) - \tilde{u}(t)||_H^4 \\
&\quad + \; 2^9\, a_2^4\, h_k^4 (b_1^4 + b_2^4)\, ||\tilde{u}(t)||^8\, ||u(t) - \tilde{u}(t)||_H^4 \\
&\quad + \; 2^{12} a_2^2\, h_k^4\, ||\hat{u}(\gamma_1)||_H^2\, ||u(t) - \tilde{u}(t)||_H^4\, ||g(\tilde{u}(t))||_{l^2}^4 \\
&\leq \; 64\, h_k^4 \Big[(b_1^4 + b_2^4)(1 + h_k\, \kappa + h_k^2\, F_1)^4 \\
&\quad + 8\, a_2^4\, (b_1^4 + b_2^4)\, ||\tilde{u}(t)||^8 + 64\, a_2^2\, ||\hat{u}(\gamma_1)||_H^2\, ||g(\tilde{u}(t))||_{l^2}^4 \Big] \\
&\quad \cdot\, ||u(t) - \tilde{u}(t)||_H^4\,.
\end{aligned}
$$

Pulling the conditional expectation over the last identity leads to

$$\mathbb{E}\left[||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||_{l^2}^4 \,\Big|\, u(t), \tilde{u}(t)\right]$$

$$\leq \; 64\, h_k^4 \Big\{ (b_1^4 + b_2^4) \left[(1 + h_k\,\kappa + h_k^2\, F_1)^4 + 8\, a_2^4\, ||\tilde{u}(t)||^8 \right] + 64\, a_2^2\, ||\hat{u}(\gamma_1)||_H^2\, ||g(\tilde{u}(t))||_{l^2}^4 \Big\}$$

$$\cdot\, ||u(t) - \tilde{u}(t)||_H^4$$

$$\leq \; 64\, h_k^4 \Big\{ (b_1^4 + b_2^4) \left[(1 + h_k\,\kappa + h_k^2\, F_1)^4 + 32\, a_2^2 V^2(\tilde{u}(t)) \right]$$

$$+\, 64\, a_2^2\, ||\hat{u}(\gamma_1)||_H^2\, ||g(\tilde{u}(t))||_{l^2}^4 \Big\} \cdot ||u(t) - \tilde{u}(t)||_H^4$$

for large N,

$$\mathbb{E}\left[||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||_{l^2}^4 \,\Big|\, u(t), \tilde{u}(t)\right]$$

$$\leq \; 64\, h_k^4 \Big\{ (b_1^4 + b_2^4) \left[\Big(1 + h_k\,\kappa + h_k^2\, \sigma^2\, N^2(\lambda_1 + \beta_1)\Big)^4 + 32\, a_2^2 V^2(\tilde{u}(t)) \right]$$

$$+\, 64\, a_2^2\, ||\hat{u}(\gamma_1)||_H^2\, ||g(\tilde{u}(t))||_{l^2}^4 \Big\} \cdot ||u(t) - \tilde{u}(t)||_H^4 \,.$$

Let

$$[\hat{K}_1(N, \tilde{u}(t))]^2 = 64 \Big\{ (b_1^4 + b_2^4) \left[\Big(1 + h_k\,\kappa + h_k^2\, \sigma^2\, N^2(\lambda_1 + \beta_1)\Big)^4 + 32\, a_2^2 V^2(\tilde{u}(t)) \right]$$

$$+\, 64\, a_2^2\, ||\hat{u}(\gamma_1)||_H^2\, ||g(\tilde{u}(t))||_{l^2}^4 \Big\},$$

then

$$\mathbb{E}\left[||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||_{l^2}^4 \,\Big|\, u(t), \tilde{u}(t)\right] \;\leq\; [\hat{K}_1(N, \tilde{u}(t))]^2\, h_k^4\; ||u(t) - \tilde{u}(t)||_H^4 \,.$$

Thus

$$\Big(\mathbb{E}\left[||\hat{g}_c(u(t)) - \hat{g}_c(\tilde{u}(t))||_{l^2}^4 \,\Big|\, u(t), \tilde{u}(t)\right]\Big)^{\frac{1}{2}} \;\leq\; \hat{K}_1(N, \tilde{u}(t))\, h_k^2\; ||u(t) - \tilde{u}(t)||_H^2 \,. \quad (5.91)$$

Now

$$\left|\left| \hat{f}_c(\hat{u}(t)|u(t)) - \hat{f}_c(\hat{u}(t)|\tilde{u}(t)) \right|\right|_{l^2}^2 = \left|\left| \hat{f}_c(u(t)) - \hat{f}_c(\tilde{u}(t)) \right|\right|_{l^2}^2$$

$$= \left|\left| \frac{v(t) - h_k \Big(F_1 + a_2\, ||u(t)||_{\mathbb{L}^2(\mathbb{D})}^2\Big)\, c(t)}{1 + h_k\kappa + h_k^2 \Big[F_1 + a_2\, ||u(t)||_{\mathbb{L}^2(\mathbb{D})}^2\Big]} - \frac{\tilde{v}(t) - h_k \Big(F_1 + a_2\, ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2\Big)\, \tilde{c}(t)}{1 + h_k\kappa + h_k^2 \Big[F_1 + a_2\, ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2\Big]} \right|\right|_{l^2}^2,$$

after we doing some algebra, we find that

$$\begin{aligned}
&\left|\left|\hat{f}_c(\hat{u}(t)|u(t)) - \hat{f}_c(\hat{u}(t)|\tilde{u}(t))\right|\right|^2_{l^2} = \left|\left|\hat{f}_c(u(t)) - \hat{f}_c(\tilde{u}(t))\right|\right|^2_{l^2} \\
&\leq \frac{6\,(1+h_k\kappa+h_k^2\,F_1)^2\,||v(t)-\tilde{v}(t)||^2_{l^2}}{\left[1+h_k\kappa+h_k^2[F_1+a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]\left[1+h_k\kappa+h_k^2[F_1+a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]} \\
&\quad + \frac{6\,a_2^2\left|\left|\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,v(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}(t)\right|\right|^2_{l^2}}{\left[1+h_k\kappa+h_k^2[F_1+a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]\left[1+h_k\kappa+h_k^2[F_1+a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]} \\
&\quad + \frac{6\,h_k^2\,F_1^2(1+h_k\kappa+h_k^2\,F_1)^2\,||c(t)-\tilde{c}(t)||^2_{l^2}}{\left[1+h_k\kappa+h_k^2[F_1+a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]\left[1+h_k\kappa+h_k^2[F_1+a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]} \\
&\quad + \frac{6\,h^6\,F_1^2\,a_2^2\left|\left|\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,c(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{c}(t)\right|\right|^2_{l^2}}{\left[1+h_k\kappa+h_k^2[F_1+a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]\left[1+h_k\kappa+h_k^2[F_1+a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]} \\
&\quad + \frac{6\,h_k^2\,a_2^2(1+h_k\kappa+h_k^2\,F_1)^2\left|\left|\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,c(t) - ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{c}(t)\right|\right|^2_{l^2}}{\left[1+h_k\kappa+h_k^2[F_1+a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]\left[1+h_k\kappa+h_k^2[F_1+a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]} \\
&\quad + \frac{6\,h_k^6\,a_2^4\,||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\,||u(t)||^4_{\mathbb{L}^2(\mathbb{D})}\,||c(t)-\tilde{c}(t)||^2_{l^2}}{\left[1+h_k\kappa+h_k^2[F_1+a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]\left[1+h_k\kappa+h_k^2[F_1+a_2\,||\tilde{c}(t)||^2_{\mathbb{L}^2(\mathbb{D})}]\right]}.
\end{aligned}$$

to simplify last inequality, we will find

$$\begin{aligned}
&||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,v(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}(t) \\
&= ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,v(t) - ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}(t) + ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}(t) \\
&= ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,(v(t)-\tilde{v}(t)) + \left(||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})} - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right)\tilde{v}(t),
\end{aligned}$$

thus

$$\begin{aligned}
&\left|\left|\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,v(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}(t)\right|\right|^2 \\
&\leq\; 2\,||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\,||v(t)-\tilde{v}(t)||^2_{l^2} + 2\left(||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})} - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right)^2||\tilde{v}(t)||^2_{l^2}.
\end{aligned}$$

Now by using Lemma 5.4.1, we get

$$\begin{aligned}
&\left|\left|\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,v(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}(t)\right|\right|^2 \\
&\leq\; 2\,||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\,||v(t)-\tilde{v}(t)||^2 + 16\;||\hat{u}(\gamma_1)||_H\,||u(t)-\tilde{u}(t)||^2_H\,||\tilde{v}(t)||^2 \qquad (5.92)
\end{aligned}$$

if we replace v by c in inequality (5.92), we get

$$\begin{aligned} &\Big|\Big| ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\, c(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\, \tilde{c}(t) \Big|\Big|^2_{l^2} \\ &\quad \leq\ 2\, ||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\, ||c(t) - \tilde{c}(t)||^2_{l^2} + 16\ ||\hat{u}(\gamma_1)||_H\, ||u(t) - \tilde{u}(t)||^2_H\, ||\tilde{c}(t)||^2_{l^2} \end{aligned} \tag{5.93}$$

and by using Lemma 5.2.5, we get

$$\Big|\Big| ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\, c(t) - ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\, \tilde{c}(t) \Big|\Big|^2_{l^2} \leq\ 9\, ||c(\gamma_1)||^4_{l^2}\, ||v(\gamma_1)||^2_{l^2}\, ||c(t) - \tilde{c}(t)||^2_{l^2}\,. \tag{5.94}$$

Now, using inequalities (5.92), (5.93), and (5.94) and pulling the conditional expectation we have

$$\begin{aligned} &\mathbb{E}\left[\Big|\Big| \hat{f}_c(\hat{u}(t)|u(t)) - \hat{f}_c(\hat{u}(t)|\tilde{u}(t)) \Big|\Big|^2_{l^2} \Big| u(t), \tilde{u}(t) \right] = \mathbb{E}\left[\Big|\Big| \hat{f}_c(u(t)) - \hat{f}_c(\tilde{u}(t)) \Big|\Big|^2_{l^2} \Big| u(t), \tilde{u}(t) \right] \\ &\leq\ 6\, \big(1 + h_k\kappa + h_k^2\, F_1\big)^2\, ||v(t) - \tilde{v}(t)||^2_{l^2} \\ &\quad +\ 6\, a_2^2\, \Big(2\, ||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\, ||v(t) - \tilde{v}(t)||^2_{l^2} + 16\ ||\hat{u}(\gamma_1)||_H\, ||u(t) - \tilde{u}(t)||^2_H\, ||\tilde{v}(t)||^2_{l^2}\Big) \\ &\quad +\ 6\, h_k^2\, F_1^2 (1 + h_k\kappa + h_k^2\, F_1)^2\, ||c(t) - \tilde{c}(t)||^2_{l^2} \\ &\quad +\ 6\, h^6\, F_1^2\, a_2^2\, \Big(2\, ||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\, ||c(t) - \tilde{c}(t)||^2_{l^2} + 16\ ||\hat{u}(\gamma_1)||_H\, ||u(t) - \tilde{u}(t)||^2_H\, ||\tilde{c}(t)||^2_{l^2}\Big) \\ &\quad +\ 6\, h_k^2\, a_2^2 (1 + h_k\kappa + h_k^2\, F_1)^2\, \big(9\, ||c(\gamma_1)||^4_{l^2}\, ||v(\gamma_1)||^2_{l^2}\big)\, ||c(t) - \tilde{c}(t)||^2_{l^2} \\ &\quad +\ 6\, h_k^6\, a_2^4\, ||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\, ||u(t)||^4_{\mathbb{L}^2(\mathbb{D})}\, ||c(t) - \tilde{c}(t)||^2_{l^2} \\ &\leq\ 6\, \Big[\big(1 + h_k\kappa + h_k^2\, F_1\big)^2 + 2\, a_2^2\, ||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\Big]\, ||v(t) - \tilde{v}(t)||^2_{l^2} \\ &\quad + 6\, h_k^2 \Big[F_1^2 (1 + h_k\kappa + h_k^2\, F_1)^2 + 2\, h_k^4\, F_1^2\, a_2^2\, ||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})} \\ &\quad + 9\, a_2^2 (1 + h_k\kappa + h_k^2\, F_1)^2\, ||c(\gamma_1)||^4_{l^2}\, ||v(\gamma_1)||^2_{l^2} + h_k^4\, a_2^4\, ||\tilde{u}(t)||^4_{\mathbb{L}^2(\mathbb{D})}\, ||u(t)||^4_{\mathbb{L}^2(\mathbb{D})}\Big] \\ &\quad \cdot\, ||c(t) - \tilde{c}(t)||^2_{l^2} \\ &\quad + 96\, a_2^2\, ||\hat{u}(\gamma_1)||_H\, \Big[\, ||\tilde{v}(t)||_{l^2} + h_k^6\, F_1^2\ ||\tilde{c}(t)||_{l^2}\Big]\, ||u(t) - \tilde{u}(t)||^2_H\,, \end{aligned}$$

hence

$$\mathbb{E}\left[\left|\left|\hat{f}_c(\hat{u}(t)|u(t)) - \hat{f}_c(\hat{u}(t)|\tilde{u}(t))\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right]$$

$$\leq \hat{K}_2(N,u,\tilde{u})\,||v(t)-\tilde{v}(t)||_{l^2}^2 + \hat{K}_3(N,u,\tilde{u})\,||c(t)-\tilde{c}(t)||_{l^2}^2$$

$$+ \hat{K}_4(N,u,\tilde{u})\,||u(t)-\tilde{u}(t)||_H^2$$

$$\leq \max\{\hat{K}_2(N,u,\tilde{u}), \hat{K}_3(N,u,\tilde{u})\}\left(||v(t)-\tilde{v}(t)||_{l^2}^2 + ||c(t)-\tilde{c}(t)||_{l^2}^2\right)$$

$$+ \hat{K}_4(N,u,\tilde{u})\,||u(t)-\tilde{u}(t)||_H^2$$

where, for N large enough,

$$\hat{K}_2(N,u,\tilde{u}) = 6\left[\left(1 + h_k\kappa + h_k^2\,\sigma^2\,N^2(\lambda_1+\beta_1)\right)^2 + 2\,a_2^2\,||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^4\right]$$

and

$$\begin{aligned}\hat{K}_3(N,u,\tilde{u}) = 6\,h_k^2\Big[&(\sigma^2\,N^2(\lambda_1+\beta_1))^2(1+h_k\kappa+h_k^2\,\sigma^2\,N^2(\lambda_1+\beta_1))^2\\ &+ 2\,h_k^4\,F_1^2\,a_2^2\,||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^4 + 9\,a_2^2(1+h_k\kappa+h_k^2\,\sigma^2\,N^2(\lambda_1+\beta_1))^2\\ &\cdot\,||c(\gamma_1)||_{l^2}^4\,||v(\gamma_1)||_{l^2}^2 + h_k^4\,a_2^4\,||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^4\,||u(t)||_{\mathbb{L}^2(\mathbb{D})}^4\Big]\end{aligned}$$

and

$$\hat{K}_3(N,u,\tilde{u}) = 96\,a_2^2\,\max\{||\hat{u}(t)||_H, ||u(t)||_H\}\left[||\tilde{v}(t)||_{l^2} + h_k^6\,(\sigma^2\,N^2(\lambda_1+\beta_1))^2\,||\tilde{c}(t)||_{l^2}\right].$$

Thus

$$\begin{aligned}&\mathbb{E}\left[\left|\left|\hat{f}_c(\hat{u}(t)|u(t)) - \hat{f}_c(\hat{u}(t)|\tilde{u}(t))\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right]\\ &\leq \max\{\hat{K}_2(N,u,\tilde{u}), \hat{K}_3(N,u,\tilde{u})\}\,||u(t)-\tilde{u}(t)||_H^2 + \hat{K}_4(N,u,\tilde{u})\,||u(t)-\tilde{u}(t)||_H^2\\ &\leq \hat{K}_5(N,u,\tilde{u})\,||u(t)-\tilde{u}(t)||_H^2 \qquad (5.95)\end{aligned}$$

where

$$\hat{K}_5(N,u,\tilde{u}) = \max\{\hat{K}_2(N,u,\tilde{u}), \hat{K}_3(N,u,\tilde{u})\} + \hat{K}_4(N,u,\tilde{u}).$$

Using the inequalities (5.91) and (5.95), we get

$$\mathbb{E}\left[||\hat{c}(t+h|u(t)) - \hat{c}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t), \tilde{u}(t))\right]$$

$$\leq 3\mathbb{E}\left[||\hat{c}(t|u(t)) - \hat{c}(t|\tilde{u}(t))||_{l^2}^2\right] + 3\,h_k^2 \hat{K}_5(N,u,\tilde{u})\,||u(t)-\tilde{u}(t)||_H^2$$

$$+\ 3\,\hat{K}_1(N,\tilde{u}(t))\,h_k^2\ ||u(t)-\tilde{u}(t)||_H^2 \left(\mathbb{E}\,||\triangle_k\,W(t)||_{l^2}^4\right)^{\frac{1}{2}},$$

but we know from Lemma 5.3.1 that

$$\mathbb{E}\,||\triangle_k\,W(t)||_{l^2}^4 = 3\,h_k^2,$$

hence

$$\mathbb{E}\left[||\hat{c}(t+h|u(t)) - \hat{c}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t), \tilde{u}(t))\right]$$

$$\leq 3\mathbb{E}\left[||\hat{c}(t|u(t)) - \hat{c}(t|\tilde{u}(t))||_{l^2}^2\right] + 3\,h_k^2 \hat{K}_5(N,u,\tilde{u})\,||u(t)-\tilde{u}(t)||_H^2$$

$$+\ 3\,\hat{K}_1(N,\tilde{u}(t))\,h_k^2\ ||u(t)-\tilde{u}(t)||_H^2\,(3\,h_k^2)^{\frac{1}{2}}$$

$$\leq 3\mathbb{E}\left[||\hat{c}(t|u(t)) - \hat{c}(t|\tilde{u}(t))||_{l^2}^2\right]$$

$$+\,2\ \max\{3\hat{K}_5(N,u,\tilde{u}), 3\sqrt{3}\hat{K}_1(N,u,\tilde{u})h_k\}\,||u(t)-\tilde{u}(t)||_H^2\ h_k^2.$$

Thus

$$\begin{aligned}&\mathbb{E}\left[||\hat{c}(t+h|u(t)) - \hat{c}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t), \tilde{u}(t))\right]\\ &\quad\leq 3\mathbb{E}\left[||\hat{c}(t|u(t)) - \hat{c}(t|\tilde{u}(t))||_{l^2}^2\right] + \hat{K}_6(N,u,\tilde{u})\,||u(t)-\tilde{u}(t)||_H^2\ h_k^2\end{aligned} \tag{5.96}$$

where $\hat{K}_6(N,u,\tilde{u}) = 2\ \max\{3\hat{K}_5(N,u,\tilde{u}), 3\sqrt{3}\hat{K}_1(N,u,\tilde{u})h_k\}$.

Now, we work on equation (5.88). $\forall\, u(t),\ \tilde{u}(t) \in H$, we have

$$
\begin{aligned}
&\mathbb{E}\left[||\hat{v}(t+h|u(t)) - \hat{v}(t+h|\tilde{u}(t))||^2_{l^2}\,|u(t),\tilde{u}(t))\right]\\
&\le\ 3\mathbb{E}\left[||\hat{v}(t|u(t)) - \hat{v}(t|\tilde{u}(t))||^2_{l^2}\,\Big|u(t),\hat{u}(t)\right]\\
&\quad +\ 3\,h_k^2\mathbb{E}\left[\left|\left|\hat{f}_v(\hat{u}(t)|u(t)) - \hat{f}_v(\hat{u}(t)|\tilde{u}(t))\right|\right|^2_{l^2}\,\Big|u(t),\tilde{u}(t)\right]\\
&\quad +\ 3\,\mathbb{E}\left[||\hat{g}_v(\hat{u}(t)|u(t)) - \hat{g}_v(\hat{u}(t)|\tilde{u}(t))||^2_{l^2}\,||(\triangle_k\, W_{n,m}(t))_{n,m=1,..,N}||^2_{l^2}\,\Big|u(t),\tilde{u}(t)\right]\\
&\le\ 3\mathbb{E}\left[||\hat{v}(t|u(t)) - \hat{v}(t|\tilde{u}(t))||^2_{l^2}\right]\\
&\quad +\ 3\,h_k^2\mathbb{E}\left[\left|\left|\hat{f}_v(\hat{u}(t)|u(t)) - \hat{f}_v(\hat{u}(t)|\tilde{u}(t))\right|\right|^2_{l^2}\right]\\
&\quad +\ 3\ \left(\mathbb{E}\,||\hat{g}_v\hat{u}(t)|u(t)) - \hat{g}_v(\hat{u}(t)|\tilde{u}(t))||^4_{l^2}\right)^{\frac{1}{2}}\left(\mathbb{E}\,||\triangle_k\, W(t)||^4_{l^2}\right)^{\frac{1}{2}}.
\end{aligned}
$$

If we compare between $\hat{g}_v$ and $\hat{g}_c$, we note that $\hat{g}_c = h_k\,\hat{g}_v$. Therefore, by using inequality (5.91), we get

$$
\left(\mathbb{E}\left[||\hat{g}_v(u(t)) - \hat{g}_v(\tilde{u}(t))||^4_{l^2}\,\Big|u(t),\tilde{u}(t)\right]\right)^{\frac{1}{2}} \le\ \hat{K}_1(N,\tilde{u}(t))\ ||u(t)-\tilde{u}(t)||^2_H\,. \qquad (5.97)
$$

and

$$
\begin{aligned}
&(\hat{f}_v(u(t)))_{n,m} - (\hat{f}_v(\tilde{u}(t)))_{n,m} \\
&= \frac{-(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})c_{n,m}(t) - (\kappa + h_k[F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}])v_{n,m}(t)}{1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})} \\
&\quad + \frac{-(F_1 + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\tilde{c}_{n,m}(t) - (\kappa + h_k[F_1 + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}])\tilde{v}_{n,m}(t)}{1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})} \\
&= \frac{-[1 + h_k\kappa + h_k^2 F_1]\left[F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right]c_{n,m}(t)}{\left[1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]} \\
&\quad - \frac{a_2 h_k^2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\left[F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right]c_{n,m}(t)}{\left[1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]} \\
&\quad - \frac{[1 + h_k\kappa + h_k^2 F_1]\left[F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right]v_{n,m}(t)}{\left[1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]} \\
&\quad - \frac{a_2 h_k^2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\left[F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right]v_{n,m}(t)}{\left[1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]} \\
&\quad + \frac{[1 + h_k\kappa + h_k^2 F_1]\left[F_1 + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right]\tilde{c}_{n,m}(t)}{\left[1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]} \\
&\quad + \frac{a_2 h_k^2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\left[F_1 + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right]\tilde{c}_{n,m}(t)}{\left[1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]} \\
&\quad + \frac{[1 + h_k\kappa + h_k^2 F_1]\left[F_1 + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right]\tilde{v}_{n,m}(t)}{\left[1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]} \\
&\quad + \frac{a_2 h_k^2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\left[F_1 + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\right]\tilde{v}_{n,m}(t)}{\left[1 + h_k\,\kappa + \, h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]\left[1 + h_k\,\kappa + \, h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\right]}
\end{aligned}
$$

hence

$$
\begin{aligned}
&(\hat{f}_v(u(t)))_{n,m} - (\hat{f}_v(\tilde{u}(t)))_{n,m} \\
&= \frac{F_1\,[1 + h_k\kappa + h_k^2\,F_1]\,(\tilde{c}_{n,m}(t) - c_{n,m}(t))}{\Big[1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]\,\Big[1 + h_k\,\kappa + \,h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]} \\
&\quad + \frac{a_2\,[1 + h_k\kappa + h_k^2\,F_1]\,\Big(||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{c}_{n,m}(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,c_{n,m}(t)\Big)}{\Big[1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]\,\Big[1 + h_k\,\kappa + \,h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]} \\
&\quad + \frac{a_2 h_k^2\,F_1\,\Big(||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{c}_{n,m}(t) - ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,c_{n,m}(t)\Big)}{\Big[1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]\,\Big[1 + h_k\,\kappa + \,h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]} \\
&\quad + \frac{a_2^2 h_k^2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,(\tilde{c}_{n,m}(t) - c_{n,m}(t))}{\Big[1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]\,\Big[1 + h_k\,\kappa + \,h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]} \\
&\quad + \frac{(\kappa + h_k F_1)\,[1 + h_k\kappa + h_k^2 F_1]\,(\tilde{v}_{n,m}(t) - v_{n,m}(t))}{\Big[1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]\,\Big[1 + h_k\,\kappa + \,h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]} \\
&\quad + \frac{a_2\,[1 + h_k\kappa + h_k^2 F_1]\,\Big(||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}_{n,m}(t) - ||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,v_{n,m}(t)\Big)}{\Big[1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]\,\Big[1 + h_k\,\kappa + \,h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]} \\
&\quad + \frac{a_2 h_k^2\,(\kappa + h_k F_1)\,\Big(||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,\tilde{v}_{n,m}(t) - ||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,v_{n,m}(t)\Big)}{\Big[1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]\,\Big[1 + h_k\,\kappa + \,h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]} \\
&\quad + \frac{a_2^2 h_k^3\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}\,(\tilde{v}_{n,m}(t) - v_{n,m}(t))}{\Big[1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]\,\Big[1 + h_k\,\kappa + \,h_k^2(F_1(n,m) + a_2\,||\tilde{u}(t)||^2_{\mathbb{L}^2(\mathbb{D})})\Big]}
\end{aligned}
$$

and since

$$1 + h_k\,\kappa + \,h_k^2(F_1 + a_2\,||u(t)||^2_{\mathbb{L}^2(\mathbb{D})}) \geq\, 1.$$

Taking the norm squared and the conditional expectation yield that

$$\mathbb{E}\left[\left|\left|(\hat{f}_v(u(t)))_{n,m} - (\hat{f}_v(\tilde{u}(t)))_{n,m}\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right]$$

$$\begin{aligned}
&\leq 8F_1^2 \left[1 + h_k\kappa + h_k^2 F_1\right]^2 \mathbb{E}\left[||\tilde{c}_{n,m}(t) - c_{n,m}(t)||_{l^2}^2 \Big| u(t), \tilde{u}(t)\right] \\
&\quad + 8a_2^2 \left[1 + h_k\kappa + h_k^2 F_1\right]^2 \mathbb{E}\left[\left|\left|\,||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, \tilde{c}_{n,m}(t) - ||u(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, c_{n,m}(t)\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right] \\
&\quad + 8a_2^2 h_k^4 \, F_1^2 \mathbb{E}\left[\left|\left|\,||u(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, \tilde{c}_{n,m}(t) - ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, c_{n,m}(t)\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right] \\
&\quad + 8a_2^4 h_k^4 \mathbb{E}\left[\left|\left|\,||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, ||u(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, (\tilde{c}_{n,m}(t) - c_{n,m}(t))\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right] \\
&\quad + 8(\kappa + h_k F_1)^2 \left[1 + h_k\kappa + h_k^2 F_1\right]^2 \mathbb{E}\left[||\tilde{v}_{n,m}(t) - v_{n,m}(t)||_{l^2}^2 \Big| u(t), \tilde{u}(t)\right] \\
&\quad + 8a_2^2 h_k^2 \left[1 + h_k\kappa + h_k^2 F_1\right]^2 \mathbb{E}\left[\left|\left|\,||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, \tilde{v}_{n,m}(t) - ||u(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, v_{n,m}(t)\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right] \\
&\quad + 8a_2^2 h_k^4 \, (\kappa + h_k F_1)^2 \mathbb{E}\left[\left|\left|\,||u(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, \tilde{v}_{n,m}(t) - ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, v_{n,m}(t)\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right] \\
&\quad + 8a_2^4 h_k^6 \mathbb{E}\left[\left|\left|\,||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, ||u(t)||_{\mathbb{L}^2(\mathbb{D})}^2 \, (\tilde{v}_{n,m}(t) - v_{n,m}(t))\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right].
\end{aligned}$$

Using the inequalities (5.92), (5.93), and (5.94) we find that

$$\mathbb{E}\left[\left|\left|(\hat{f}_v(u(t)))_{n,m} - (\hat{f}_v(\tilde{u}(t)))_{n,m}\right|\right|_{l^2}^2 \Big| u(t), \tilde{u}(t)\right]$$

$$\leq 8F_1^2\left[1 + h_k\kappa + h_k^2 F_1\right]^2 ||\tilde{c}_{n,m}(t) - c_{n,m}(t)||_{l^2}^2$$

$$+ 72a_2^2\left[1 + h_k\kappa + h_k^2 F_1\right]^2 ||c(\gamma_1)||_{l^2}^4\, ||v(\gamma_1)||_{l^2}^2\, ||c_{n,m}(t) - \tilde{c}_{n,m}(t)||_{l^2}^2$$

$$+ 16a_2^2 h_k^4\, F_1^2\, ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^4\, ||c_{n,m}(t) - \tilde{c}_{n,m}(t)||_{l^2}^2$$

$$+ 128a_2^2 h_k^4\, F_1^2\, ||\hat{u}(\gamma_1)||_H\, ||u(t) - \tilde{u}(t)||_H^2\, ||\tilde{c}(t)||_{l^2}^2$$

$$+ 8a_2^4 h_k^4\, ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^4\, ||u(t)||_{\mathbb{L}^2(\mathbb{D})}^4\, ||\tilde{c}_{n,m}(t) - c_{n,m}(t)||_{l^2}^2$$

$$+ 8(\kappa + h_k F_1)^2\left[1 + h_k\kappa + h_k^2 F_1\right]^2 ||\tilde{v}_{n,m}(t) - v_{n,m}(t)||_{l^2}^2$$

$$+ 72a_2^2 h_k^2\left[1 + h_k\kappa + h_k^2 F_1\right]^2 ||v(\gamma_1)||_{l^2}^4\, ||c(\gamma_1)||_{l^2}^2\, ||v_{n,m}(t) - \tilde{v}_{n,m}(t)||_{l^2}^2$$

$$+ 16a_2^2 h_k^4\,(\kappa + h_k F_1)^2\, ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^4\, ||v_{n,m}(t) - \tilde{v}_{n,m}(t)||_{l^2}^2$$

$$+ 128a_2^2 h_k^4\,(\kappa + h_k F_1)^2\, ||\hat{u}(\gamma_1)||_H\, ||u(t) - \tilde{u}(t)||_H^2\, ||\tilde{v}(t)||_{l^2}^2$$

$$+ 8a_2^4 h_k^6\, ||\tilde{u}(t)||_{\mathbb{L}^2(\mathbb{D})}^4\, ||u(t)||_{\mathbb{L}^2(\mathbb{D})}^4\, ||v_{n,m}(t) - \tilde{v}_{n,m}(t)||_{l^2}^2$$

$$\leq \Big[4\left[1 + h_k\kappa + h_k^2\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2\left(2\sigma^4 N^4(\lambda_1 + \beta_1)^2 + 9a_2 V^2(u(t))\right)$$

$$+ 2a_2 h_k^4 V(\tilde{u}(t))\left(4\sigma^4 N^4(\lambda_1 + \beta_1)^2 + a_2 V(u(t))\right)\Big]\, ||c_{n,m}(t) - \tilde{c}_{n,m}(t)||_{l^2}^2$$

$$+ \Big[8\left[1 + h_k\kappa + h_k^2\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2\Big(\left[\kappa + h_k\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2 + 9\, a_2^2\, h_k^2\, V^3(u(t))\Big)$$

$$+ 2\, a_2\, h_k^2\, V(\tilde{u}(t))\Big(4\left[\kappa + h_k\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2 + a_2\, h_k^4\, V(u(t))\Big)\Big]\, ||v_{n,m}(t) - \tilde{v}_{n,m}(t)||_{l^2}^2$$

$$+ \Big[128\, a_2^2\, h_k^4\, \max\{||u(t)||_H\,, ||\tilde{u}(t)||_H\} V(\tilde{u}(t))$$

$$\cdot\Big(\sigma^4 N^4(\lambda_1 + \beta_1)^2 + \left[\kappa + h_k\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2\Big)\Big]\, ||u(t) - \tilde{u}(t)||_H^2$$

$$\leq \hat{K}_7(N, u(t), \tilde{u}(t))\, ||c_{n,m}(t) - \tilde{c}_{n,m}(t)||_{l^2}^2 + \hat{K}_8(N, u(t), \tilde{u}(t))\, ||v_{n,m}(t) - \tilde{v}_{n,m}(t)||_{l^2}^2$$

$$+ \hat{K}_9(N, u(t), \tilde{u}(t))\, ||u(t) - \tilde{u}(t)||_H^2$$

where

$$\hat{K}_7(N, u(t), \tilde{u}(t)) = 4\left[1 + h_k\kappa + h_k^2\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2 \left(2\sigma^4 N^4(\lambda_1 + \beta_1)^2 + 9a_2V^2(u(t))\right)$$

$$+ 2a_2h_k^4V(\tilde{u}(t))\left(4\sigma^4 N^4(\lambda_1 + \beta_1)^2 + a_2V(u(t))\right)$$

and

$$\hat{K}_8(N, u(t), \tilde{u}(t))$$

$$= 8\left[1 + h_k\kappa + h_k^2\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2 \Big(\left[\kappa + h_k\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2 + 9\, a_2^2\, h_k^2\, V^3(u(t))\Big)$$

$$+ 2\, a_2\, h_k^2\, V(\tilde{u}(t))\Big(4\left[\kappa + h_k\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2 + a_2\, h_k^4\, V(u(t))\Big)$$

and

$$\hat{K}_9(N, u(t), \tilde{u}(t)) = 128\, a_2^2\, h_k^4\, \max\{||u(t)||_H\,, ||\tilde{u}(t)||_H\}V(\tilde{u}(t))$$

$$\cdot\;\Big(\sigma^4 N^4(\lambda_1 + \beta_1)^2 + \left[\kappa + h_k\sigma^2 N^2(\lambda_1 + \beta_1)\right]^2\Big)\,.$$

Thus

$$\mathbb{E}\left[\Big|\Big|(\hat{f}_v(u(t)))_{n,m} - (\hat{f}_v(\tilde{u}(t)))_{n,m}\Big|\Big|_{l^2}^2 \,\Big|\, u(t), \tilde{u}(t)\right]$$

$$\leq\; \max\{\hat{K}_7(N, u(t), \tilde{u}(t)), \hat{K}_8(N, u(t), \tilde{u}(t))\}\Big(\,||c_{n,m}(t) - \tilde{c}_{n,m}(t)||_{l^2}^2$$

$$+ ||v_{n,m}(t) - \tilde{v}_{n,m}(t)||_{l^2}^2\Big) + \hat{K}_9(N, u(t), \tilde{u}(t))\, ||u(t) - \tilde{u}(t)||_H^2$$

$$\leq\; \max\{\hat{K}_7(N, u(t), \tilde{u}(t)), \hat{K}_8(N, u(t), \tilde{u}(t))\}\, ||u(t) - \tilde{u}(t)||_H^2$$

$$+ \hat{K}_9(N, u(t), \tilde{u}(t))\, ||u(t) - \tilde{u}(t)||_H^2\,.$$

Hence

$$\mathbb{E}\left[\Big|\Big|(\hat{f}_v(u(t)))_{n,m} - (\hat{f}_v(\tilde{u}(t)))_{n,m}\Big|\Big|_{l^2}^2 \,\Big|\, u(t), \tilde{u}(t)\right]$$
$$\leq\; \hat{K}_{10}(N, u(t), \tilde{u}(t))\, ||u(t) - \tilde{u}(t)||_H^2 \tag{5.98}$$

where $\hat{K}_{10} = \max\{\hat{K}_7(N, u(t), \tilde{u}(t)), \hat{K}_8(N, u(t), \tilde{u}(t))\} + \hat{K}_9(N, u(t), \tilde{u}(t))$.

Thus, by using inequalities (5.83) and (5.98) we get

$$\mathbb{E}\left[||\hat{v}(t+h|u(t)) - \hat{v}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t),\tilde{u}(t))\right]$$

$$\leq 3\mathbb{E}\left[||\hat{v}(t|u(t)) - \hat{v}(t|\tilde{u}(t))||_{l^2}^2 \,\Big|u(t),\hat{u}(t)\right] + 3\,h_k^2\hat{K}_{10}(N,u(t),\tilde{u}(t))\,||u(t)-\tilde{u}(t)||_H^2$$

$$+\ 3\,\hat{K}_1(N,\tilde{u}(t))\ ||u(t)-\tilde{u}(t)||_H^2 \left(\mathbb{E}\,||\triangle_k W(t)||_{l^2}^4\right)^{\frac{1}{2}},$$

but we know from Lemma 5.3.1 that

$$\mathbb{E}\,||\triangle_k W(t)||_{l^2}^4 = 3\,h_k^2,$$

thus

$$\mathbb{E}\left[||\hat{v}(t+h|u(t)) - \hat{v}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t),\tilde{u}(t))\right]$$

$$\leq 3\mathbb{E}\left[||\hat{v}(t|u(t)) - \hat{v}(t|\tilde{u}(t))||_{l^2}^2 \,\Big|u(t),\hat{u}(t)\right] + 3\,h_k^2\hat{K}_{10}(N,u(t),\tilde{u}(t))\,||u(t)-\tilde{u}(t)||_H^2$$

$$+\ 3\sqrt{3}\,\hat{K}_1(N,\tilde{u}(t))\ ||u(t)-\tilde{u}(t)||_H^2\ h_k$$

therefore,

$$\begin{aligned}\mathbb{E}&\left[||\hat{v}(t+h|u(t)) - \hat{v}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t),\tilde{u}(t))\right]\\ &\leq 3\mathbb{E}\left[||\hat{v}(t|u(t)) - \hat{v}(t|\tilde{u}(t))||_{l^2}^2 \,\Big|u(t),\hat{u}(t)\right]\\ &\quad+\hat{K}_{11}(N,u(t),\tilde{u}(t))\,||u(t)-\tilde{u}(t)||_H^2\ h_k \end{aligned} \tag{5.99}$$

where $\hat{K}_{11}(N,u(t),\tilde{u}(t)) = 3\,h_k\hat{K}_{10}(N,u(t),\tilde{u}(t)) + 3\sqrt{3}\,\hat{K}_1(N,\tilde{u}(t))$.

Now, combine inequalities (5.96) and (5.99), we get

$$\mathbb{E}\left[||\hat{c}(t+h|u(t)) - \hat{c}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t),\tilde{u}(t)\right]$$

$$+\,\mathbb{E}\left[||\hat{v}(t+h|u(t)) - \hat{v}(t+h|\tilde{u}(t))||_{l^2}^2 \,|u(t),\tilde{u}(t)\right]$$

$$\leq\; 3\mathbb{E}\left[||\hat{c}(t|u(t)) - \hat{c}(t|\tilde{u}(t))||_{l^2}^2 \,\Big|u(t),\hat{u}(t)\right] + \hat{K}_6(N,u(t),\tilde{u}(t))\,||u(t)-\tilde{u}(t)||_H^2\; h_k^2$$

$$+\,3\mathbb{E}\left[||\hat{v}(t|u(t)) - \hat{v}(t|\tilde{u}(t))||_{l^2}^2 \,\Big|u(t),\hat{u}(t)\right] + \hat{K}_{11}(N,u(t),\tilde{u}(t))\,||u(t)-\tilde{u}(t)||_H^2\; h_k$$

$$\leq\; 3\mathbb{E}\Big(\left[||\hat{c}(t|u(t)) - \hat{c}(t|\tilde{u}(t))||_{l^2}^2 \,|u(t),\tilde{u}(t)\right] + \left[||\hat{v}(t|u(t)) - \hat{v}(t|\tilde{u}(t))||_{l^2}^2 \,|u(t),\tilde{u}(t)\right]\Big)$$

$$+\,\hat{K}_{11}(N,u(t),\tilde{u}(t))\,||u(t)-\tilde{u}(t)||_H^2\; h_k$$

where $\hat{K}_2^c(N,u(t),\tilde{u}(t)) = \hat{K}_6(N,u(t),\tilde{u}(t)) + \hat{K}_{11}(N,u(t),\tilde{u}(t))$.

Thus

$$\mathbb{E}\left[||\hat{u}(t+h|u(t)) - \hat{u}(t+h|\tilde{u}(t))||_H^2 \,|u(t),\tilde{u}(t)\right]$$

$$\leq\; 3\mathbb{E}\left[||\hat{u}(t+h|u(t)) - \hat{u}(t+h|\tilde{u}(t))||_H^2 \,|u(t),\tilde{u}(t)\right]$$

$$+\,\hat{K}_2^c(N,u(t),\tilde{u}(t))\,||u(t)-\tilde{u}(t)||_H^2\; h_k$$

$$\leq\; 3\,||u(t)-\tilde{u}(t)||_H^2 + \hat{K}_2^c(N,u(t),\tilde{u}(t))\,||u(t)-\tilde{u}(t)||_H^2\; h_k.$$

Therefore

$$\begin{aligned}&\mathbb{E}\left[||\hat{u}(t+h|u(t)) - \hat{u}(t+h|\tilde{u}(t))||_H^2 \,|u(t),\tilde{u}(t)\right]\\ &\quad\leq\; \Big(3 + \hat{K}_2^c(N,u(t),\tilde{u}(t))\,h\Big)\,||u(t)-\tilde{u}(t)||_H^2\,. \end{aligned} \tag{5.100}$$

The proof is completed. □

CHAPTER 6

SIMULATION

We simulate the following equation

$$u_{tt} = 4\,(u_{xx} + u_{yy}) - u_t + \Big(1 - ||u||^2_{\mathbb{L}^2(\mathbb{D})}\Big)\, u + \Big(2 + ||u||_{\mathbb{L}^2(\mathbb{D})}\Big)\, \dot{W} \tag{6.1}$$

where $0 \le\, t \le\, 1$ and $l_x = 5$, and $l_y = 5$ on $\mathbb{D} = [0,\, 5] \times\, [0,\, 5]$. We apply the two explicit representations of linear-implicit Euler-type numerical approximations that we've found in equations (5.12) and (5.13) at $\sigma\, =\, 2$, $a_1 = 1$, $a_2 = 1$, $b_0 = 2$, $b_1 = 1$, and $\kappa = 1$, which are

$$1)\; c^{i,j}_{n,m}(t_{k+1}) \;=\; \frac{\Big(1\,+\,h_k\Big) c^{i,j}_{n,m}(t_k) \;+\; h_k\, v^{i,j}_{n,m}(t_k) \;+\; h_k\, g_{n,m}(u(t_k)) \triangle_k\, W^{i,j}_{n,m}}{1\,+\,h_k\,-\,h_k^2\, f_{n,m}(u(t_k))} \tag{6.2}$$

and

$$2)\; v^{i,j}_{n,m}(t_{k+1}) \;=\; \frac{v^{i,j}_{n,m}(t_k) \;+\; h_k\, f_{n,m}(u(t_k))\, c^{i,j}_{n,m}(t_k) \;+\; g_{n,m}(u(t_k)) \triangle_k\, W^{i,j}_{n,m}}{1\,+\,h_k\,-\,h_k^2\, f_{n,m}(u(t_k))} \tag{6.3}$$

where

$$f_{n,m}(u(t_k)) \;=\; -\,4(\frac{n^2\,\pi^2}{25} + \frac{m^2\,\pi^2}{25}) \,+\, 1 \,-\, ||u||^2_{\mathbb{L}^2(\mathbb{D})}\,,$$

and

$$g_{n,m}(u(t_k)) \;=\; \frac{2\,+||u||_{\mathbb{L}^2(\mathbb{D})}}{n\,\cdot\,m}.$$

We take the step size $h_k = 10^{-4}$. Also, we simulate the energy identity, which is

$$e(t) = \frac{1}{2}\mathbb{E}\Big[\,||u_t||^2 + 4\,||\nabla u||^2 \,-\, ||u||^2 + \frac{1}{2}\,||u||^4\,\Big]. \tag{6.4}$$

We use the following:

1) $C++$ compiler, built in uniform generator.

2) Architecture: 64 bites/ Samsung/ 4 GB ram.

3) Polar Marsaglia method.

4) Gnuplot for data.

5) Sample size $M = 10^4$.

We simulate the above equations at $N = 5$, 25, 50, and 100. We start with fairly smooth initial data, as we can see from all plots of truncated displacements u_N in Figure 1. Now, as the integration time advances and the dimension N of finite-dimensional truncation increases, the $3D$ plots are getting rougher, hence they seem to "loose smoothness". This observation is natural since our solutions of related infinite-dimensional system and its derivatives are in $\mathbb{L}^2(\mathbb{D})$. Interestingly, the simulated averaged energy displays monotone increasing convexity as time t increases - a fact which has not been proven yet. This is due to the presence of multiplicative noise which may show exponentially increasing energy.

Figure 1. Initial displacement $u_N(x, y, 0)$ for dimensions $N = 5,\ 25,\ 50,$ and 100

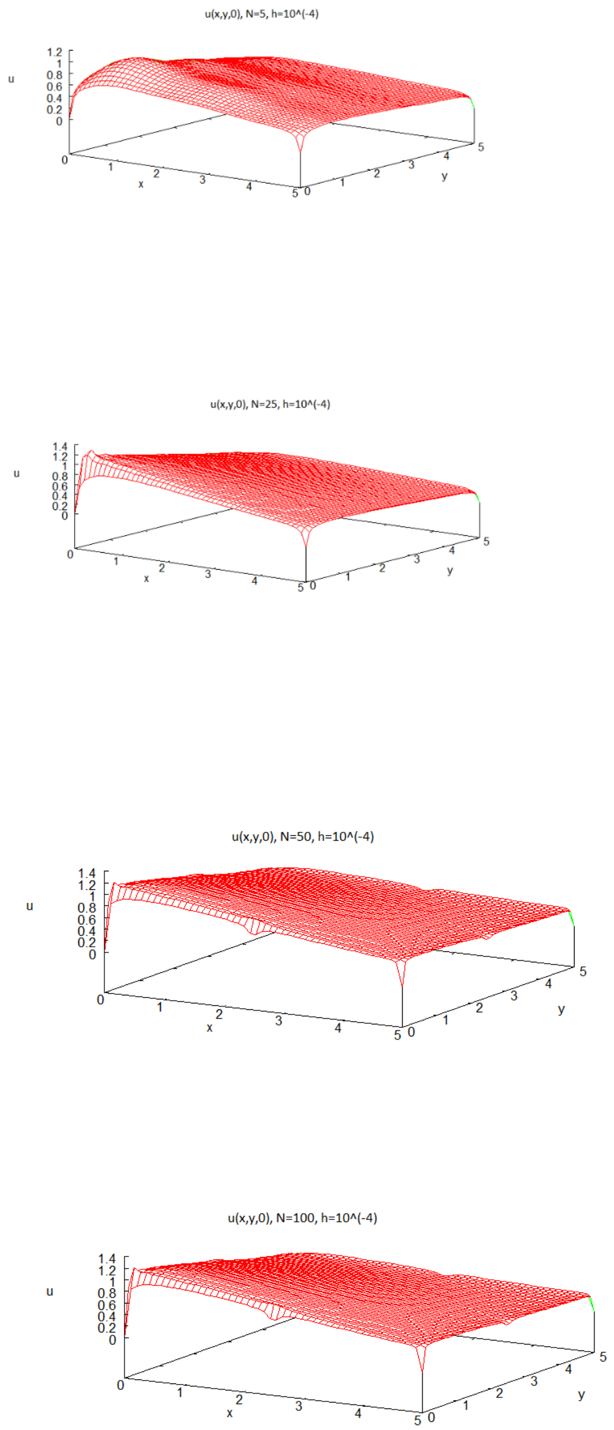
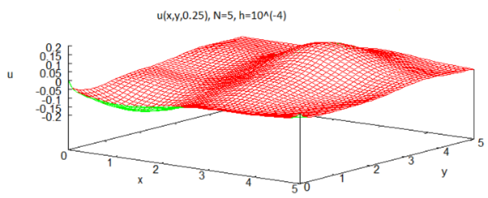
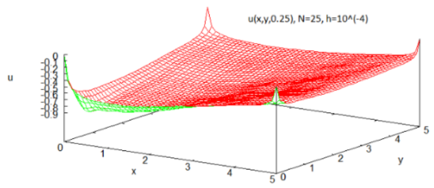
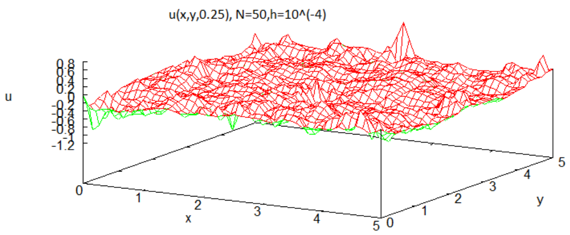


Figure 2. The displacement $u_N(x, y, 0.25)$ for dimensions $N = 5$, 25, 50, and 100







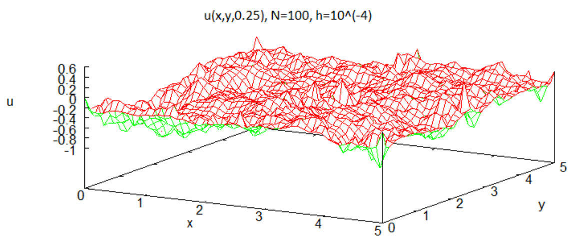


Figure 3. The displacement $u_N(x, y, 0.5)$ for dimensions $N = 5,\ 25,\ 50,$ and 100

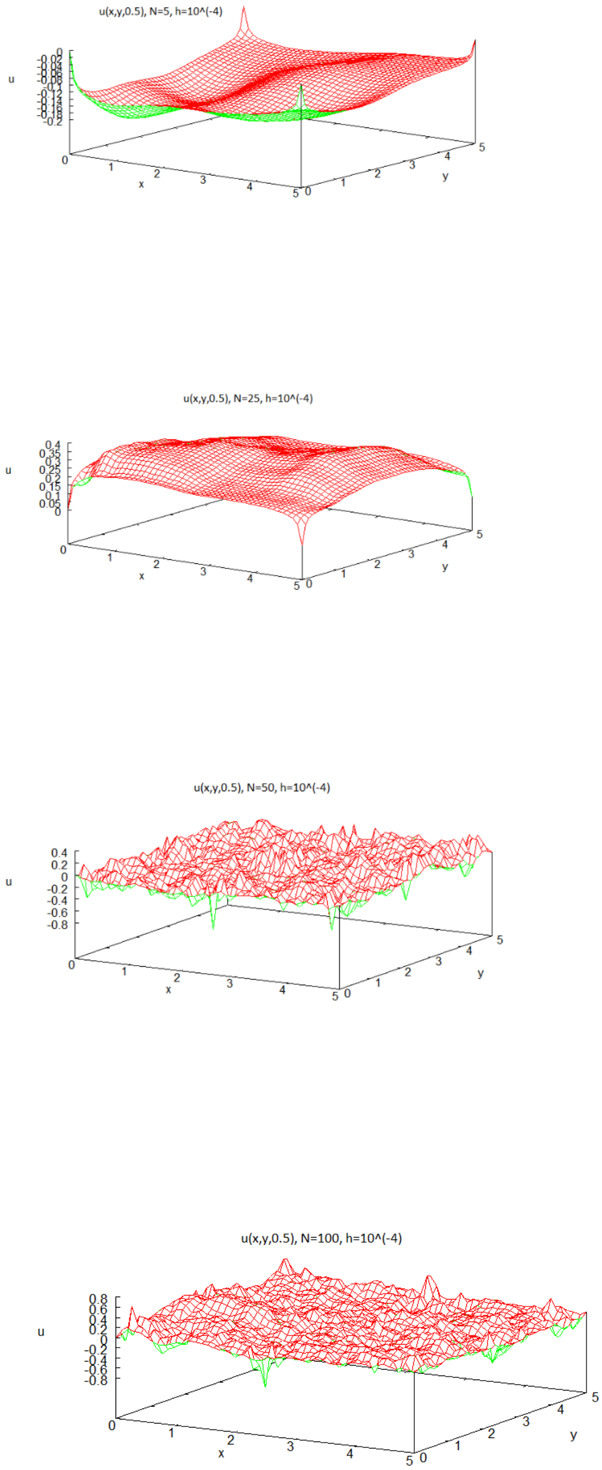


Figure 4. The displacement $u_N(x, y, 0.75)$ for dimensions $N = 5$, 25, 50, and 100

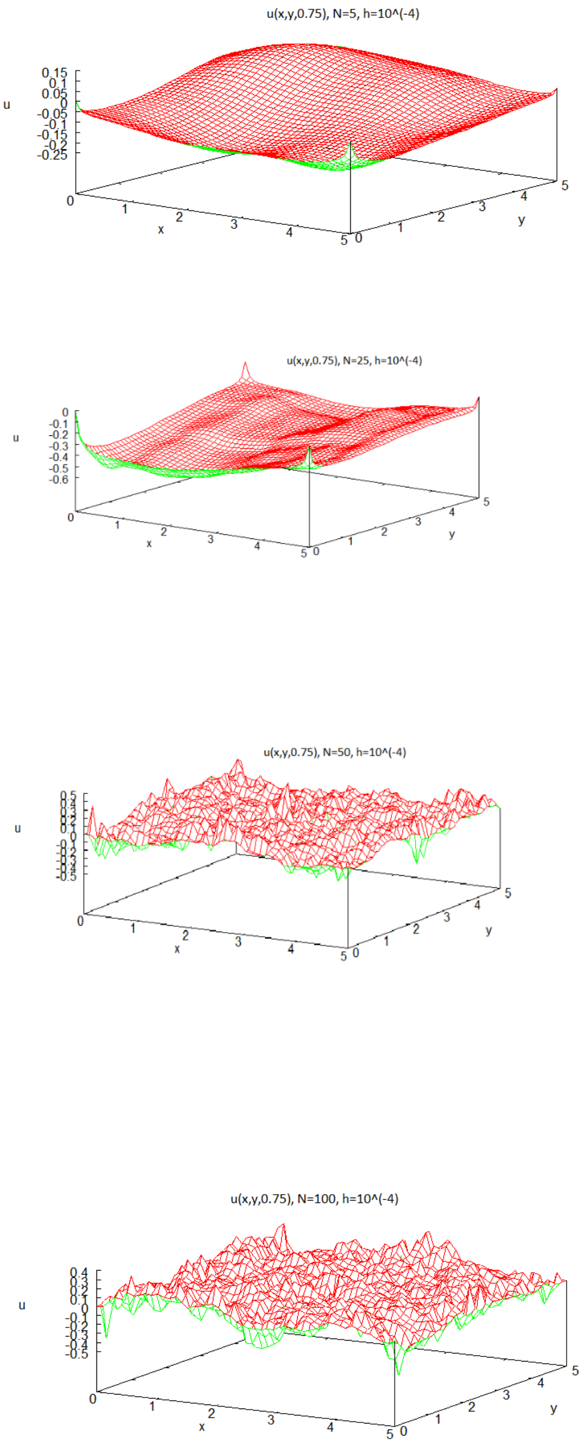


Figure 5. The displacement $u_N(x, y, 1)$ for dimensions $N = 5$, 25, 50, and 100

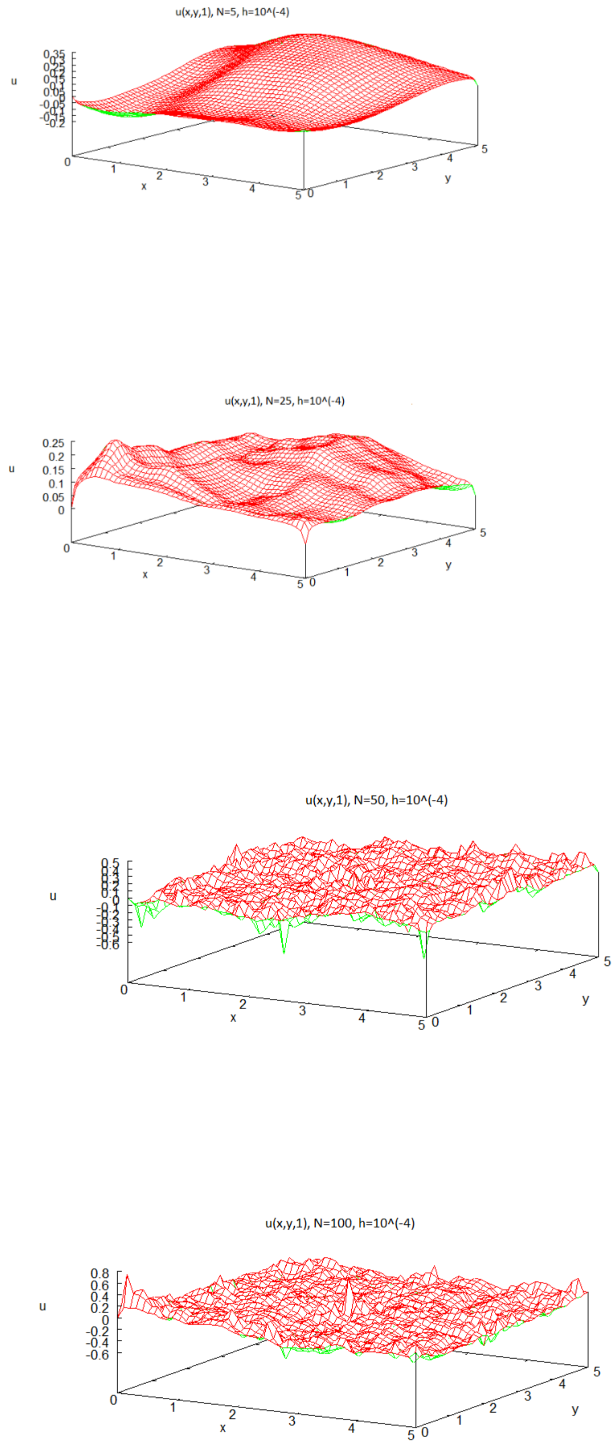


Figure 6. The expected energy e_N for dimensions $N = 5$, 25, 50, and 100

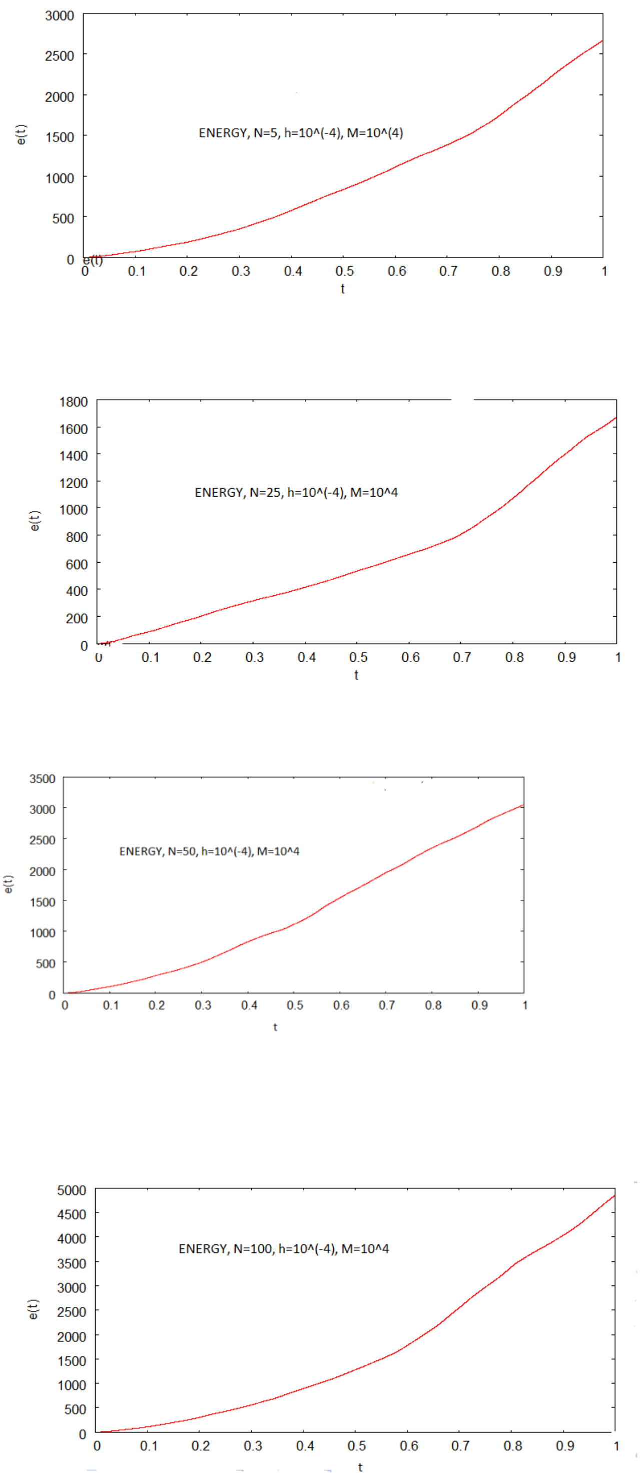


Figure 6

shows the expected energy evolving with time t. The sample size for all plots in Figure 6 is left constant with $M = 10^4$. Simulation with larger sample sizes was impossible due to large computational times such as 13 hours for one plot. All 4 plots in Figure 6 use different sets of random variables. All the plots show increasing energy which is feeded by the interaction with external noise source. The computational complexity of nonlinear equations (i.e. high dimensions N, large sample sizes M, small step sizes h, long term integration with large T) restricted our possibilities to simulate. That is why we had to work with the parameter ranges as given in figures above.

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APPENDIX I

Theorem 6.0.3. *(The Borel-Cantelli Lemma) If* $\{A_n\} \subset \mathcal{F}$, *where* $\mathcal{F}$ *is a* $\sigma-$*algebra of all subsets of* Ω *and* $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, *then*

$$\mathbb{P}\Big(\limsup_{n\to\infty} A_n\Big) = 0.$$

That is, $\exists$ *a set* $\Omega_0 \in \mathcal{F}$ *with* $\mathbb{P}(\Omega_0) = 1$ *and an integer-valued random variable* n_0 *such that for every* $\omega \in \Omega_0$ *we have* $\omega \notin A_n$ *whenever* $n \geq n_0(\omega)$.

Proof. $\mathbb{P}(\{A_n\, i.o.\}) = \mathbb{P}(\limsup_{n\to\infty} A_n)$

$= \lim_{n\to\infty} \mathbb{P}(\bigcup_{k=n}^{\infty} A_k)$

$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k)$ and by the assumption $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then the right hand side of last inequality approaches to 0 as $n \to \infty$. Thus

$$\mathbb{P}\Big(\limsup_{n\to\infty} A_n\Big) = 0.$$

□

Lemma 6.0.4. *(MVT Estimates) Assume* $f \in \mathcal{C}^1(\mathbb{D})$, $f : \mathbb{D} \to \mathbb{R}$, ∇f *is convex on* $\mathbb{D}$, $\mathbb{D}$ *convex, too. Then* $\forall\, x,\, y \in \mathbb{D}$, *we have*

$$1)\; f(x) - f(y) - \nabla f(\eta)\,(x - y)\; (MVT)$$

$$where\, \nu \in [0,1] : \eta = y + \nu\,(x - y)$$

$$2)\; ||\nabla f(\eta)|| \leq \max\Big\{ ||\nabla f(x)||,\, ||\nabla f(y)|| \Big\}$$

$$3)\; ||\nabla f(\eta)|| \leq \sup_{z\in\partial\mathbb{D}} ||\nabla f(z)||$$

Lemma 6.0.5. *Let* $X \sim \mathcal{N}(0, h_k)$, $\forall\, n \in \mathbb{N}$, $n \geq 1$, *we have*

$$\mathbb{E}(X)^{2n} = (2n - 1)!!\, h_k$$

where $(2n - 1)!! = (2n-1)(2n-3)(2n-5) \times \ldots \times 5 \times 3 \times 1.$

Proof. Since $x \sim \mathcal{N}(0,\, h_k)$, then the density function is

$$f(x) \;=\; \frac{1}{\sqrt{2\,\pi\,h_k}}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x$$

and therefore

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi\,h_k}}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x \;=\; 1.$$

Thus

$$\mathbb{E}x^{2n} \;=\; \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi\,h_k}}\,x^{2n}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x$$

but $x^{2n}\;\exp(\frac{-\,x^2}{2\,h_k})$ is an even function, so

$$\begin{aligned}
\mathbb{E}x^{2n} \;&=\; \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi\,h_k}}\,x^{2n}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x \;=\; 2\,\int_{0}^{\infty} \frac{1}{\sqrt{2\,\pi\,h_k}}\,x^{2n}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x \\
&=\; \frac{2}{\sqrt{2\,\pi\,h_k}}(2n-1)\,h_k \int_{0}^{\infty} x^{2n-2}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x \;\;\text{(by parts)} \\
&=\; \frac{2}{\sqrt{2\,\pi\,h_k}}(2n-1)(2n-3)\,h_k^2 \int_{0}^{\infty} x^{2n-4}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x \\
&=\; \frac{2}{\sqrt{2\,\pi\,h_k}}(2n-1)(2n-3)(2n-5)\,h_k^3 \int_{0}^{\infty} x^{2n-4}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x \\
&=\; (2n-1)(2n-3)\times\,...\,\times\,3\,\times\,1\,h_k^n \int_{-\,\infty}^{\infty} \frac{2}{\sqrt{2\,\pi\,h_k}}\,x^{2n-4}\;\exp(\frac{-\,x^2}{2\,h_k})\,d\,x \\
&=\; (2n-1)(2n-3)\times\,...\,\times\,3\,\times\,1\,h_k^n \;=\; (2n\,-\,1)!!\,h_k^n.
\end{aligned}$$

□

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