Notes on Complete Affine Flows without Closed Orbits

on 3-Manifolds

by

Atsushi SATO\textsuperscript{1,2}

Department of Mathematics
School of Science and Technology, Meiji University
1-1-1, Higashi-mita, Tama Kawasaki 214 JAPAN

Received December 18, 1992; Accepted January 19, 1993

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

Synopsis: In this note we consider affine flows with some conditions (called the completeness) on closed 3-manifolds. We will determine all the 3-manifolds which bear the complete flows, and will exhibit all the distinct types of examples of minimal flows in this category.

§1. Introduction and the statement of results

It is an interesting as well as a very difficult problem to find a condition for a given non-singular flow to admit a transverse codimension one foliation. There we find a partial answer on 3-manifolds for $S^1$-bundles([M], [W]), Seifert manifolds([E-H-N]), Morse-Smale flows([G1], [S], [Y]) and Smale flows([G2]).

\textsuperscript{1} Partially supported by Chief Research in 1989 of the Institute of Science and Technology in Meiji University

\textsuperscript{2} Partially supported by Grant-in-Aid for Encouragement of Young Scientists in 1991
On the other hand flows without closed orbits or especially minimal flows (flows with all orbits dense) are also fascinating objects in considering the above problem.

There we know, however, few examples of flows of this type. One of this reason is that in general flows are difficult to describe because of their vast variety, which, in contrast to the codimension one foliation case, comes mainly from the abundance of diffeomorphisms on manifolds of dimension $> 1$. Among them affine flows or more generally transversely geometric flows are objects fairly easy to treat.

In these context the main interest in this paper is to find minimal examples in affine flows. In this note we will investigate 3-manifolds admitting some affine flows with restricted conditions (the assumptions (a) (b) in §2), and on these 3-manifolds we will exhibit distinct types of examples of minimal flows.

In §2, we state our assumptions (a) (b) and prepare some tools necessary to advance our arguments. By these assumptions we will see that the 3-manifolds which we must treat are only $T^2$ or $K^2$-bundles over $S^1$. In §3, we treat the area preserving case, in which we find "non-linear" minimal flows on $T^3$ and minimal flows on parabolic $T^2$-bundles over $S^1$. In §4, we treat the general case. We determine 3-manifolds which support our flow. In §5, we treat the $K^2$-bundle case. The notion of the uniform distribution is used as a criterion for the minimality of flows.

This paper is intended as a preparatory issue as well as a memorandum for further study of affine flows on closed 3-manifolds. Therefore the proofs and examples in this paper are described in a fairly precise manner and without sophistication.

The author would like to express his gratitude for the members of the Saturday seminar at the Tokyo Institute of Technology.
§2. Preliminaries

Let $M$ be a closed connected 3-manifold and $\varphi$ be a foliation with 1-dimensional leaves (we call such a foliation a flow). Each leaf of $\varphi$ is called an orbit and a compact leaf a closed orbit. Also a flow with all orbits dense is called a minimal flow.

Consider the group of the affine transformations

$$\text{Aff}(2) = \text{GL}(2; \mathbb{R}) \cdot \mathbb{R}^2$$

acting on the 2-plane $\mathbb{R}^2$. Each element $f \in \text{Aff}(2)$ has the following form: for $\forall x \in \mathbb{R}^2$,

$$f(x) = Ax + a,$$

where $A$ is a 2 by 2 matrix with $\det A \neq 0$ and $a \in \mathbb{R}^2$. In this case we denote $f = (A, a)$. One can easily verify that for $f = (A, a)$ and $g = (B, b)$, $fg = (AB, a + Ab)$ and $f^{-1} = (A^{-1}, -A^{-1}a)$. We call $A$ the linear part of $f$ and $a$ the translation part of $f$ respectively. An element $f \in \text{Aff}(2)$ is hyperbolic if all eigenvalues of its linear part are real and $\neq \pm 1$.

Let $G$ be a subgroup of all the real analytic diffeomorphisms of $\mathbb{R}^2$ and $\varphi$ be a (transversely) $(G, \mathbb{R}^2)$-flow on $M$. By definition there exists a family of triples $\{(U_i, \pi_i, g_{ij}) \mid i, j \in I\}$ such that $\{U_i\}$ be an open covering of $M$, $\pi_i : U_i \to \mathbb{R}^2$ a smooth submersion, and $g_{ij}$ the restriction of an element of $G$ on $\pi_j(U_i \cap U_j)$ satisfying $g_{ij} \circ \pi_j = \pi_i$ on $U_i \cap U_j$.

For any $(G, \mathbb{R}^2)$-flow $\varphi$, we can consider the developing map $D : \widetilde{M} \to \mathbb{R}^2$ and the holonomy homomorphism $h : \pi_1(M) \to G$ such that $D$ is a submersion from the universal covering space $\widetilde{M}$ of $M$, and for $\forall \gamma \in \pi_1(M)$ and for $\forall p \in \widetilde{M}$, $D(\gamma \cdot p) = h(\gamma)(D(p))$, where $\gamma$ acts on $\widetilde{M}$ by the deck transformation. The image $\Gamma = h(\pi_1(M))$ is called the holonomy group of $\varphi$. If $G = \text{Aff}(2)$, then a $(G, \mathbb{R}^2)$-flow is also called
an affine flow. Similarly we can formulate the notion of \((G, X)\)-flows for simply connected real analytic manifold \(X\) and for a subgroup \(G\) of real analytic diffeomorphisms on \(X\).

**Definition 2.1.**

An affine flow \(\varphi\) is complete if and only if \(D\) is surjective and is a fibre bundle map.

**Remark.** In this case each \(D\)-fiber is diffeomorphic to \(\mathbb{R}\) and the universal cover \(\widetilde{M}\) is diffeomorphic to \(\mathbb{R}^3\). In particular \(M\) is irreducible.

**Assumption:** In this paper we assume all the flows are
(a) complete affine flows,
(b) without closed orbits.

**Lemma 2.2.** Let \(X\) be a singular 2 by 2 matrix. Then by an element in \(\text{GL}(2; \mathbb{R})\), \(X\) is conjugate to \[
\begin{bmatrix}
u & v \\
0 & 0
\end{bmatrix}
\] for some \(u, v \in \mathbb{R}\).

**Proof.** If \(X = \begin{bmatrix} a & b \\ ta & tb \end{bmatrix}\) for some \(t \in \mathbb{R}\), then

\[
\begin{bmatrix}
1 & 0 \\
-t & 1
\end{bmatrix}
X
\begin{bmatrix}
1 & 0 \\
t & 1
\end{bmatrix} = \begin{bmatrix} a + tb & b \\ 0 & 0 \end{bmatrix}.
\]

Another case is similar. q.e.d.

**Proposition 2.3.** Up to conjugacy in \(\text{Aff}(2)\), a fixed point free element \(f \in \Gamma\) has the following form:

\[f = \left( \begin{bmatrix} p & q \\ 0 & 1 \end{bmatrix}, a \right),\]

with either
(i) \(p = 1\) and \(q = 0\) and \(a \neq 0\) (nontrivial translation)
or
(ii) not (i) and \( a_2 \neq 0 \) where \( a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \).
In particular a fixed point free \( f \) is of infinite order.

**Proof.** The equation \( Ax + a = x \) does not have a solution only if \( A - I \) is singular. In this case by Lemma 2.2 we can assume \( A = \begin{bmatrix} p & q \\ 0 & 1 \end{bmatrix} \) for some \( p, q \in \mathbb{R} \). Then \( (A - I)x = a \) does not have a solution only if (i) \( A - I = O \) and \( a \neq 0 \) or \( A - I \neq O \) and \( a_2 \neq 0 \). The sufficiency of the condition is obvious. If \( f \) is of type (ii) above, then since the 2nd component of the translation part of \( f^n \) is equal to \( na_2 \), \( f \) is of infinite order (see 4.3 (2) below). q.e.d.

**Proposition 2.4.** We have the following:

(1) \( h : \pi_1(M) \to \text{Aff}(2) \) is injective.

(2) Every non-trivial \( g \in \Gamma \) is periodic point free (especially fixed point free). In particular \( \Gamma \) is a torsion free group.

**Proof.** (1) If \( h \) is not injective, then \( K = \ker h \) is non-trivial and since \( K \) is a subgroup of \( \pi_1(M) \), \( K \) acts on each \( D \)-fiber(\( \cong \mathbb{R} \)) freely and properly discontinuously. Hence \( K \cong \mathbb{Z} \), and this means the flow \( \varphi \) is a Seifert fibration, which contradicts the assumption (b).

(2) If \( g \) is of finite order, then by Proposition 2.3, \( g \) has a fixed point \( p \). Since \( g \) acts on \( D^{-1}(p) \) freely, \( D^{-1}(p) \) corresponds to a closed orbit of \( \varphi \), which contradicts the assumption (b). Thus every non-trivial \( g \) is of infinite order. If \( g \) has a periodic point of period \( n \), then \( g^n \) is a nontrivial element with a fixed point which also corresponds to a closed orbit of \( \varphi \), a contradiction. q.e.d.

**Theorem 2.5.** ([E-M]) (a version used in Plante [P])

Let \( M \) be an irreducible compact 3-manifold such that \( \pi_1(M) \) is almost solvable and \( H^1(M; \mathbb{Z}) \neq 0 \). Then either

(1) \( \pi_1(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \) or
(2) $M$ is a bundle over $S^1$ with the fiber a disc, annulus, Möbius band, Klein bottle $K^2$, or torus $T^2$.

For the proof, see Plante [P].

§3. Area preserving case

In this section we treat some restricted case and we observe that in this case the fundamental group of $M$ is nilpotent, from which we determine 3-manifolds supporting our flows. Also we will obtain all typical examples of minimal flows.

Let $G_0 = SL(2; \mathbb{R}) \cdot \mathbb{R}^2$ (the area preserving affine transformations) and consider the $(G_0, \mathbb{R}^2)$-flows.

**Proposition 3.1.** An element $f = (A, a) \in G_0$ is fixed point free if and only if

- either (1) $f = (I, a)$ and $a \neq 0$ (non-trivial translation)
- or (2) trace $A = 2$ and $A \neq I$ (we call such an $f$ parabolic), and by a suitable conjugacy in $SL(2; \mathbb{R})$, $f$ is of the following form:

$$f = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{with } a \neq 0 \text{ and } a_2 \neq 0.$$ 

**Proof.** This is a corollary of Proposition 2.3.

Put $G_1 = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \mid a \in \mathbb{R}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 \right\}.$

**Proposition 3.2.** By a suitable conjugacy in $SL(2; \mathbb{R})$, we have $\Gamma \subset G_1$, and every non-trivial $f \in \Gamma$ satisfies:

$$\text{if } a \neq 0 \text{, then } a_2 \neq 0.$$
Proof. If $\forall f \in \Gamma$ is a translation, then we are done. Otherwise by 3.1, up to conjugacy, there exists an element $f = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ with $a \neq 0$ and $a_2 \neq 0$. By 3.1, for each element in $\Gamma$, the trace of the linear part is equal to 2. For each $g = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \Gamma$ with $p + s = 2$ and $ps - qr = 1$, the trace of the linear part of $fg$ is $p + s + ar$, which must be equal to 2, and thus $r = 0$, then we have $p = s = 1$. q.e.d.

Definition 3.3. Let $G_2 = \{ f = (I, \begin{bmatrix} a_1 \\ 0 \end{bmatrix}) \mid a_1 \in \mathbb{R} \}$, and call an element of $G_2$ a special translation.

One can easily verify the following lemma.

Lemma 3.4. Let $f = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $g = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be two elements of $G_1$. Then the following formulae hold:

1. $fg = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1+ab_2 \\ b_2 \end{bmatrix}$.
2. $gf^{-1}g^{-1} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} ba_2-ab_2 \\ 0 \end{bmatrix}$.
3. $gfg^{-1}f^{-1} = (I, \begin{bmatrix} ba_2-ab_2 \\ 0 \end{bmatrix})$.

Lemma 3.4 shows the following lemma.

Lemma 3.5. The commutator subgroup $[G_1,G_1] = G_2$, and $G_2$ is the center of $G_1$.

Corollary 3.6. $\Gamma$ is nilpotent.

Proposition 3.7. $H^1(M;\mathbb{R}) \neq 0$ and $\pi_1(M)$ is torsion free.

Proof. By 3.4 (1), we can construct a non-trivial homomorphism $\rho : \pi_1(M) \cong \Gamma \to \mathbb{R}$ by assigning $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \mapsto a_2$ (If $\Gamma \subset G_2$ then
consider \((I, \begin{bmatrix} a_{11} \\ 0 \end{bmatrix}) \mapsto a_{11}\). Thus \(H^1(M; R) \neq 0\). By 3.2, every non-trivial element of \(\Gamma\) is of infinite order. Hence \(\pi_1(M)\) is torsion free. q.e.d.

**Proposition 3.8.** \(M\) is the total space of a \(T^2\)-bundle over \(S^1\).

**Proof.** Since \(\partial M = \emptyset\), by 2.5 and 3.7, \(M\) is the total space of either a \(T^2\) or \(K^2\)-bundle over \(S^1\). In the Klein bottle case, \(\pi_1(K^2) \cong \langle s, t \mid sts^{-1} = t^{-1}\rangle\) must inject to \(\Gamma\) by \(h\). Then from the relation \(sts^{-1}t^{-1} = t^{-2}\), \(h(t^2)\) and also \(h(t)\) must be a special translation. By 3.4 (2) and the group relation, we have \(h(t) = 1\), which contradicts the injectivity of \(h\). Thus \(M\) is a \(T^2\)-bundle over \(S^1\). q.e.d.

Next consider the monodromy \(A \in GL(2; \mathbb{Z})\) of \(M\).

**Proposition 3.9.** The possible monodromy \(A\) of \(M\) is one of the followings.

1. \(
\det A = 1 \text{ and } |\text{trace} A| \leq 2
\)
2. \(
\det A = -1 \text{ and } \text{trace} A = 0.
\)

**Proof.** If the monodromy \(A\) is not a one as above, then \(A\) is a hyperbolic matrix and this implies that \(\pi_1(M)\) is not nilpotent, which contradicts 3.6. q.e.d.

Now we determine the monodromy. The following two propositions are well known.

**Proposition 3.10.** The conjugacy classes in \(GL(2; \mathbb{Z})\) with \(\det = -1\) and \(\text{trace} = 0\) are as follows:

\[
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]
Proposition 3.11. The conjugacy classes in $\text{GL}(2;\mathbb{Z})$ of elements with $\det = 1$ and $|\text{trace}| \leq 2$ are as follows:

1. trace $= 0$ case: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
2. trace $= 1$ case: $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.
3. trace $= -1$ case: $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$.
4. trace $= 2$ case: $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} (n \in \mathbb{Z})$.
5. trace $= -2$ case: $\begin{bmatrix} -1 & n \\ 0 & -1 \end{bmatrix} (n \in \mathbb{Z})$.

Theorem 3.12.

1. $M$ admits a $(G_0, \mathbb{R}^2)$-flow with the assumptions (a) and (b) if and only if $M$ is the $T^2$-bundle over $S^1$ with monodromy

   $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} (n \in \mathbb{Z})$.

2. For each $M$ arising in (1), minimal flows are realizable.
3. In $T^3$ case ($n = 0$), there exists an affine minimal flow other than the linear ones.

Lemma 3.13. If $\beta \neq 1$, then there cannot exist a relation $\gamma \beta \gamma^{-1} = \beta^{-1}$ in $\pi_1(M)$.

Proof. Denote the image $h(\beta)$ by $\hat{\beta}$. By 3.2 we can set $\hat{\beta} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and $\hat{\gamma} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. By 3.4, the relation $\gamma \hat{\beta} \gamma^{-1} = \hat{\beta}^{-1}$ implies $\hat{\beta} = 1$, a contradiction. q.e.d.

Proof of Theorem 3.12.

We examine the existence of the structure for each monodromy $A$.

case 1. $\det A = -1$ and trace $A = 0$. 

21
We will see that this case does not occur.

(1) \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

In this case the fundamental group of \( M \) has the following presentation:
\[
\pi_1(M) = \langle \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta, \gamma \beta \gamma^{-1} = \alpha \rangle.
\]

Set
\[
\tilde{\alpha} = ( \begin{bmatrix} 1 \\ a \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}) , \tilde{\beta} = ( \begin{bmatrix} 1 \\ b \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} b_2 \\ \cdot \end{bmatrix}) , \tilde{\gamma} = ( \begin{bmatrix} 1 \\ c \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} c_2 \end{bmatrix}).
\]

From the relation \( \tilde{\gamma} \tilde{\alpha} \tilde{\gamma}^{-1} = \tilde{\beta} \), we obtain \( a_1 + ca_2 - ac_2 = b_1, a = c = b \) and \( a_2 = b_2 \).

From the relation \( \tilde{\gamma} \tilde{\beta} \tilde{\gamma}^{-1} = \tilde{\alpha} \), we obtain \( b_1 + cb_2 - bc_2 = a_1 \). Then we have \( a_1 = b_1 \). This shows \( \tilde{\alpha} = \tilde{\beta} \), and that \( \text{Ker} \ h \neq 1 \), a contradiction.

(2) \( A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \).

In this case we have
\[
\pi_1(M) = \langle \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha \beta, \gamma \beta \gamma^{-1} = \beta^{-1} \rangle.
\]
The last relation contradicts Lemma 3.13.

(3) \( A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \).

In this case we have
\[
\pi_1(M) = \langle \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha, \gamma \beta \gamma^{-1} = \beta^{-1} \rangle.
\]
The last relation contradicts Lemma 3.13.

**case 2.** \( \det A = 1 \) and \( |\text{trace } A| \leq 2 \).

(1) \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

In this case we have
\[
\pi_1(M) = \langle \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta, \gamma \beta \gamma^{-1} = \alpha^{-1} \rangle.
\]

Then we have \( \tilde{\gamma}^2 \tilde{\alpha} \tilde{\gamma}^{-2} = \tilde{\alpha}^{-1} \) and by 3.13, we have a contradiction.

(2) \( A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \).

In this case we have
Notes on Complete Affine Flows without Closed Orbits on 3-Manifolds

\[ \pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha \beta, \gamma \beta \gamma^{-1} = \alpha^{-1} \rangle. \]

Then by the 2nd relation \( \bar{\beta} \) is a special translation, and then from the 3rd relation we have \( \bar{\beta} = \bar{\alpha}^{-1} \), which contradicts the injectivity of \( h \).

\[ (3) \ A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}. \]

In this case we have
\[ \pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha^{-1} \beta^{-1}, \gamma \beta \gamma^{-1} = \alpha \rangle. \]

Comparing the linear part and 2nd component of the translation part in the last two relations, we have \( a = b = 0 \) and \( a_2 = b_2 = 0 \). Then both \( \bar{\alpha} \) and \( \bar{\beta} \) commute with \( \bar{\gamma} \), and we have \( \bar{\alpha}^3 = 1 \), which contradicts the injectivity of \( h \).

\[ (4) \ A = \begin{bmatrix} -1 & n \\ 0 & -1 \end{bmatrix} \quad (n \in \mathbb{Z}). \]

In this case we have
\[ \pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha^{-1} \beta^n, \gamma \beta \gamma^{-1} = \beta^{-1} \rangle. \]

The last relation contradicts Lemma 3.13.

\[ (5) \ A = I. \]

There exists a minimal representation \( h \) which does not correspond to the usual linear flows. See examples (ii) and (iii) below.

Let \( T_0 \) denote the subgroup consisting of all the translations in \( \text{Aff}(2) \).

Example (i)
By considering the representation into \( T_0 \), we obviously obtain usual linear flows on \( T^3 \).

Definition 3.14. A sequence \( \omega = \{x_n\} \) in \( \mathbb{R} \) is uniformly distributed mod 1 (u.d.mod 1 for short) if and only if the following holds: for \( \forall a, b \) satisfying \( 0 \leq a < b \leq 1 \),

\[ \lim_{N \to \infty} \frac{T(N)}{N} = b - a \]
where \( T(N) = \#\{x_n \mid a < \tilde{x}_n < b \ (1 \leq n \leq N)\} \) and \( \tilde{x}_n = x_n - [x_n] \) is
the fractional part of \( x_n \).

For \( \mathbb{R}^k \)-valued sequences \( (k > 1) \) the notion \( u.d. \mod 1 \) is similarly
defined.

**Theorem 3.15.** ([K-N]) If \( f(x) = \theta_k x^k + \cdots + \theta_1 x \) is a polynomial
and some \( \theta_i \) is irrational, then the sequence \( \{f(n)\} (n \in \mathbb{N}) \) is \( u.d. \mod 1 \).

**Theorem 3.16.** ([K-N]) An \( \mathbb{R}^k \)-valued sequence \( \{x_n\} \) is \( u.d. \mod 1 \)
if and only if for \( \forall a \in \mathbb{Z}^k \setminus \{0\} \), the real valued sequence \( \{a \cdot x_n\} \) is
\( u.d. \mod 1 \).

From the above two theorems one can easily verify the following

**Lemma 3.17.** If \( a \) and \( b \) are irrational and \( 1, a, b \) is linearly in-
dependent over \( \mathbb{Q} \), then the subset \( D = [\begin{bmatrix} a \\ b \end{bmatrix}] \mathbb{Z} + \mathbb{Z}^2 \) is dense in \( \mathbb{R}^2 \)
(Kronecker). Moreover \( \{x_n\} \), where \( x_n = [\begin{bmatrix} an \\ bn \end{bmatrix}] \), is \( u.d. \mod 1 \).

**Example (ii)** There are flows such that \( \Gamma \cap T_0 \cong \mathbb{Z} \).

Let \( \hat{\alpha} = \left( \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right), \hat{\beta} = (I, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}), \hat{\gamma} = (\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}) \) with
\( a, c \neq 0 \) be the generators of \( \Gamma \). By the commuting relation we see that
\( b_2 = 0 \) and that there is a constant \( \kappa \neq 0 \) satisfying \( a_2/a = c_2/c = \kappa \).

Furthermore conjugating by \( \begin{bmatrix} 1/\kappa a^2 & 0 \\ 0 & 1/\kappa a \end{bmatrix} \) we may assume \( \kappa = 1 \) and
\[ \hat{\alpha} = \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right), \hat{\beta} = (I, \begin{bmatrix} b_1 \\ 0 \end{bmatrix}), \hat{\gamma} = (\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c \end{bmatrix}) \).

One can verify that \( \hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \left( \begin{bmatrix} 1 & p + cr \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda(p, q, r) \end{bmatrix} \right) \) where
\[ \lambda(p, q, r) = \frac{1}{2} p(p - 1) + a_1 p + \frac{1}{2} c^2 r(r - 1) + c_1 r + cpr + b_1 q \]
\[ = \frac{1}{2} (p + cr)^2 + (a_1 - \frac{1}{2}) p + (c_1 - \frac{1}{2} c^2) r + b_1 q. \]

Since we are assuming that \( \Gamma \cap T_0 \cong \mathbb{Z}, 1, c \) must be linearly inde-
dependent over \( \mathbb{Q} \).
Since the subgroup $H$ generated by $\hat{\alpha}$ and $\hat{\beta}$ is discrete in $\text{Aff}(2)$ and $\mathbb{R}^2/H \cong T^2$, the following action of $\pi_1(T^3)$ on $\mathbb{R}^3$ (whose coordinates are $(x, z)$ with $x \in \mathbb{R}^2$ and $z \in \mathbb{R}$)
\[
\alpha(x, z) = (\hat{\alpha}(x), z), \quad \beta(x, z) = (\hat{\beta}(x), z), \quad \gamma(x, z) = (\hat{\gamma}(x), z + 1),
\]
defines deck transformations on the universal cover $\tilde{T}^3$, and the projection $D : (x, z) \mapsto x$ gives our developing map.

Minimal flows are given as follows:
Choose $c, c'$ such that $1, c, c'/b_1$ is linearly independent over $\mathbb{Q}$. Set $a_1 = \frac{1}{2}$ and $c_1 = \frac{1}{2}c^2 + c'$. Then
\[
\lambda(p, q, r) = \frac{1}{2}(p + cr)^2 + c'r + b_1q.
\]
For any $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and $u = \begin{bmatrix} u \\ v \end{bmatrix}$, by 3.17, we can choose a sequence $\{(p_n, q_n, r_n)\}$ such that
\[
p_n + cr_n \to v - y
\]
and
\[
b_1q_n + c'r_n \to u - (x + (v - y)y + \frac{1}{2}(v - y)^2)
\]
as $n \to \infty$. Then we have $\hat{\alpha}^{p_n}\hat{\beta}^{q_n}\hat{\gamma}^{r_n}(x) \to u$. Thus this representation is a minimal one.

For non-minimal flows we see that each minimal set is a saturated $T^2$ in $T^3$ as follows.

By the diffeomorphism $T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x - \frac{1}{2}y^2 \\ y \end{bmatrix}$, the action of $H$ on $\mathbb{R}^2$ is analytically diffeomorphic to the usual $\mathbb{Z}^2$-action, and on the torus $\mathbb{R}^2/H$ the diffeomorphism induced by $\hat{\gamma}$ is differentiably conjugate to a rotation. Since $c$ is irrational, each minimal set is a saturated $T^2$. 
Example (iii) There are flows such that every non-trivial element in \( \Gamma \) is parabolic (we call such \( \Gamma \) a \textit{purely parabolic} subgroup).

As above, up to conjugacy we may assume

\[
\hat{\alpha} = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} a_1 \\ 1 \end{array} \right), \quad \hat{\beta} = \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} b_1 \\ b \end{array} \right) \quad \text{and} \quad \hat{\gamma} = \left( \begin{array}{cc} 1 & c \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} c_1 \\ c \end{array} \right).
\]

Then one can verify that

\[
\hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \left( \begin{array}{cc} 1 & p + bq + cr \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} \lambda(p, q, r) \\ p + bq + cr \end{array} \right)
\]

where

\[
\lambda(p, q, r) = a_1 p + \frac{1}{2} p(p - 1) + b_1 q + \frac{1}{2} b^2 q(q - 1) + c_1 r + c^2 \frac{1}{2} r(r - 1) + bpq + cpr + bcqr = \frac{1}{2} (p + bq + cr)^2 + (a_1 - \frac{1}{2}) p + (b_1 - \frac{1}{2} b^2) q + (c_1 - \frac{1}{2} c^2) r.
\]

Since we are assuming that \( \Gamma \) is purely parabolic, \( 1, b, c \) must be linearly independent over \( \mathbb{Q} \).

The action of \( \pi_1(T^3) \) on \( \hat{T}^3 \) is given by

\[
\alpha(x, z) = (\hat{\alpha}(x), z), \beta(x, z) = (\hat{\beta}(x), z), \gamma(x, z) = (\hat{\gamma}(x), z + 1),
\]

and the projection \( D : (x, z) \mapsto x \) gives our developing map as before.

Minimal flows are given as follows:

Choose irrational \( b, c, d \) such that both \( 1, b, c, d \) and \( 1, c - bd, d \) are linearly independent over \( \mathbb{Q} \). Set \( a_1 = \frac{1}{2}, b_1 = 1 + \frac{1}{2} b^2, c_1 = \frac{1}{2} c^2 + d \).

Then

\[
\lambda(p, q, r) = \frac{1}{2} (p + bq + cr)^2 + q + dr.
\]

**Lemma 3.18.** The set

\[
D = \left\{ \left[ \begin{array}{c} q + dr \\ p + bq + cr \end{array} \right] \mid p, q, r \in \mathbb{Z} \right\}
\]

is dense in \( \mathbb{R}^2 \).
Proof. Consider the linear isomorphism $T = \begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. $T$ satisfies that

$$T\left( \begin{bmatrix} \frac{-bd + c}{dr} \\ p \end{bmatrix} \right) = \begin{bmatrix} q + dr \\ p + bq + cr \end{bmatrix}.$$ 

Since the sequence $\{x_r\}$, where $x_r = \begin{bmatrix} \frac{-bd + c}{dr} \end{bmatrix}$, is u.d. mod 1, we have the desired result. q.e.d.

By 3.18, for any $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and $u = \begin{bmatrix} u \\ v \end{bmatrix}$, we can choose a sequence $\{(p_n, q_n, r_n)\}$ such that

$$p_n + bq_n + cr_n \rightarrow v - y$$

and

$$q_n + dr_n \rightarrow u - (x + (v - y)y + \frac{1}{2}(v - y)^2)$$
as $n \rightarrow \infty$. Then we have $\hat{\alpha}^{p_n} \hat{\beta}^{q_n} \hat{\gamma}^{r_n}(x) \rightarrow u$. Thus this representation is a minimal one.

For non-minimal flows, each minimal set is a saturated $T^2$ as before.

(6) $A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \ (0 \neq n \in \mathbb{Z})$.

In this case, there also exist minimal flows.

Since $M_n$ with the monodromy $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is an $n$-fold cover over $M_1$, we only construct flows on $M = M_1$.

Consider the fundamental group of $M$:

$$\pi_1(M) = \langle \alpha, \beta, \gamma \mid \alpha \beta = \beta \alpha, \gamma \alpha \gamma^{-1} = \alpha \beta, \gamma \beta \gamma^{-1} = \beta \rangle.$$ 

Here $\alpha$ and $\beta$ are the generators of the fundamental group of the fiber $T^2$. 

27
In this case note that every element of \( \pi_1(M) \) can be uniquely expressed as \( \alpha^p \beta^q \gamma^r \) for some \( p, q, r \in \mathbb{Z} \).

Example (i) We construct an example such that \( \Gamma \cap T_0 \cong \mathbb{Z}^2 \).

Since the roles of \( \alpha \) and \( \gamma \) in \( \pi_1(M) \) are symmetric and \( \alpha, \gamma \) do not commute, we may assume

\[
\hat{\alpha} = (I, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}), \hat{\beta} = (I, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}), \text{ and } \hat{\gamma} = (\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}).
\]

From the 2nd and the 3rd relations of \( \pi_1(M) \) we have \( b_2 = 0 \) and \( a_2 \neq 0 \). Conjugating by \( \begin{bmatrix} 1/b_1 & -a_1/a_2b_1 \\ 0 & 1/a_2 \end{bmatrix} \), and considering the 2nd relation again, we have

\[
\hat{\alpha} = (I, \begin{bmatrix} 0 \\ 1 \end{bmatrix}), \hat{\beta} = (I, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), \text{ and } \hat{\gamma} = (\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c \end{bmatrix}).
\]

Then one can verify that \( \hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = (\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda(p, q, r) \\ p + cr \end{bmatrix}) \) where

\[
\lambda(p, q, r) = q + \frac{1}{2}cr^2 + (c_1 - \frac{1}{2}c)r.
\]

Since every non-trivial element is fixed point free, \( c \) must be irrational.

Since the subgroup \( H = \langle \hat{\alpha}, \hat{\beta} \rangle \) is discrete and \( \mathbb{R}^2/H \cong T^2 \), the action of \( \pi_1(M) \) on \( \widetilde{M} \) given by

\[
\alpha(x, z) = (\hat{\alpha}(x), z), \beta(x, z) = (\hat{\beta}(x), z), \gamma(x, z) = (\hat{\gamma}(x), z + 1) \)
\]

and the projection \( \pi : (x, z) \mapsto x \) give our developing map.

In this case we see any flow is minimal as follows:

From 3.15 and 3.16 we have

**Lemma 3.19.** For an arbitrary real \( t \), the sequence \( \{x_r\}(r \in \mathbb{N}) \), where \( x_r = \left[ \begin{array}{c} \frac{1}{2}cr^2 + tr \\ cr \end{array} \right] \), is u.d. mod. 1.

For any \( x = \left[ \begin{array}{c} x \\ y \end{array} \right] \) and \( u = \left[ \begin{array}{c} u \\ v \end{array} \right] \), choose a sequence \( \{(p_n, q_n, r_n)\} \) such that

\[
p_n + cr_n \rightarrow v - y.
\]
and
\[ q_n + \frac{1}{2}cr_n^2 + (c_1 - \frac{1}{2}c + y)r_n \rightarrow u - x \]
as \( n \rightarrow \infty \). Then we have \( \hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r(x) \rightarrow u \). Thus an arbitrary representation is a minimal one.

Example (ii) Flows such that \( \Gamma \cap T_0 \cong \mathbb{Z} \).

In this case \( \hat{\alpha} \) cannot be a translation. For otherwise from the 1st relation, \( \hat{\alpha} \) must be a special translation, and then \( \hat{\alpha} \) must commute with \( \hat{\gamma} \), which is a contradiction. Thus we may assume
\[ \hat{\alpha} = \left[ \begin{array}{c} 1 \\ a \\ \end{array} \right], \hat{\beta} = (I, \left[ \begin{array}{c} b_1 \\ 0 \\ \end{array} \right]), \hat{\gamma} = \left[ \begin{array}{c} 1 \\ c \\ \end{array} \right], \left[ \begin{array}{c} c_1 \\ \end{array} \right] \].

Conjugating by
\[ \left[ \begin{array}{cc} 1/aa_2 & (aa_2/2 - a_1)/aa_2^2 \\ 0 & 1/aa_2 \\ \end{array} \right], \]
we may assume
\[ \hat{\alpha} = \left[ \begin{array}{c} 1 \\ 1 \\ \end{array} \right], \hat{\beta} = (I, \left[ \begin{array}{c} b_1 \\ 0 \\ \end{array} \right]), \hat{\gamma} = \left[ \begin{array}{c} 1 \\ c \\ \end{array} \right], \left[ \begin{array}{c} c_1 \\ \end{array} \right] \].

One can verify that \( \hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \left[ \begin{array}{c} 1 \\ p + cr \\ \end{array} \right], \left[ \begin{array}{c} \lambda(p, q, r) \\ p + (c - b_1)r \\ \end{array} \right] \)
where
\[ \lambda(p, q, r) = \frac{1}{2}p^2 + b_1q + c_1r + \frac{1}{2}c(c - b_1)r(r - 1) + (c - b_1)pr \]

\[ = \frac{1}{2}(p + (c - b_1)r)^2 + \frac{1}{2}b_1(c - b_1)r^2 + (c_1 - \frac{1}{2}c(c - b_1))r + b_1q. \]

Then we see that \( c - b_1 \) must be irrational as follows.

Suppose the contrary. Then \( p + (c - b_1)r = 0 \) for some \( p, r \in \mathbb{Z} \).
Since every non-trivial element is fixed point free we have \( p + cr = 0 \),
which implies that \( b_1 = 0 \) or \( r = 0 \), a contradiction.

Then we can verify that every non-trivial \( g \in \Gamma \) is fixed point free.

As before
\[ \alpha(x, z) = (\hat{\alpha}(x), z), \beta(x, z) = (\hat{\beta}(x), z), \gamma(x, z) = (\hat{\gamma}(x), z + 1) \]
and the projection $D : (x, z) \mapsto x$ give our developing map.

Minimal flows arise as follows:

Set $c_1 = \frac{1}{2} c (c - b_1)$. Then

$$\lambda(p, q, r) = \frac{1}{2} (p + (c - b_1) r)^2 + \frac{1}{2} b_1 (c - b_1) r^2 + b_1 q.$$ 

Choose $c, b_1$ such that $c - b_1$ is irrational. Then the sequence $\{x_r\} (r \in \mathbb{N})$ is u.d. mod 1, where $x_r = \left\lfloor \frac{1}{2} (c - b_1) r^2 \right\rfloor \right\rfloor.

For any $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and $u = \begin{bmatrix} u \\ v \end{bmatrix}$, choose a sequence $\{(p_n, q_n, r_n)\}$ such that

$$p_n + (c - b_1) r_n \to v - y$$

and

$$b_1 q_n + \frac{1}{2} b_1 (c - b_1) r_n^2 + b_1 y r_n \to u - (x + (v - y) y + \frac{1}{2} (v - y)^2)$$

as $n \to \infty$. Then we have $\alpha^p \beta^q \gamma^r (x) \to u$. Thus this representation is a minimal one.

For non-minimal flows, we see any minimal set is a saturated $T^2$ as follows.

Conjugating by the diffeomorphism $T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x - \frac{1}{2} y^2 \\ y \end{bmatrix}$ on $\mathbb{R}^2$, we have

$$T \alpha T^{-1} = (I, \begin{bmatrix} 0 \\ 1 \end{bmatrix}), T \beta T^{-1} = \tilde{\beta},$$

$$T \gamma T^{-1} = \left( \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 - \frac{1}{2} (c - b_1)^2 \\ c - b_1 \end{bmatrix} \right).$$

In contrast to example (i) above, there are non-minimal flows. Since $c - b_1$ is irrational, every minimal set for a non-minimal flow is a saturated $T^2$. 

**Proposition 3.20.** On $M_1$ above, there are no representations with purely parabolic $\Gamma$.

**Proof.** Let

$$\hat{\alpha} = \left( \begin{array}{c} 1 \\ a \\ 1 \end{array} \right), \quad \hat{\beta} = \left( \begin{array}{c} 1 \\ b \\ 1 \end{array} \right), \quad \hat{\gamma} = \left( \begin{array}{c} 1 \\ c \\ 1 \end{array} \right).$$

Since $[\alpha, \beta] = 1$ and $[\beta, \gamma] = 1$, we have $a/a_2 = b/b_2$ and $b/b_2 = c/c_2$, which implies that $\alpha$ and $\gamma$ commute, a contradiction.

**Proposition 3.21.** On $T^3$ there is no minimal flow such that $\Gamma \cap T_0 \cong \mathbb{Z}^2$.

**Proof.** Set

$$\hat{\alpha} = (I, \left[ \begin{array}{c} a_1 \\ 0 \\ 1 \end{array} \right]), \quad \hat{\beta} = (I, \left[ \begin{array}{c} b_1 \\ 0 \\ 1 \end{array} \right]), \quad \hat{\gamma} = (I, \left[ \begin{array}{c} c_1 \\ 0 \\ 1 \end{array} \right]).$$

Then we have for $p, q, r \in \mathbb{Z}$

$$\hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \left( \begin{array}{c} 1 \\ cr \\ 1 \end{array} \right), \quad \left[ \begin{array}{c} \lambda(p, q, r) \\ cr \end{array} \right],$$

where $\lambda(p, q, r) = a_1 p + b_1 q + c_1 r + \frac{1}{2} c c_2 r (r - 1)$. We see $a_1$ and $b_1$ must be linearly independent over $\mathbb{Q}$. Since the 2nd component of the translation part is discrete, this representation is not minimal.

The following action

$$\alpha(x, z) = (\hat{\alpha}(x), z), \beta(x, z) = (\hat{\beta}(x), z), \gamma(x, z) = (\hat{\gamma}(x), z + 1)$$

of $\pi_1(T^3)$ on $\tilde{T}^3$ with the projection $D : (x, z) \mapsto x$ gives us a developing map. Thus we can realize this representation as a flow on $T^3$. 

31
§4. General case

In this and the next sections we classify 3-manifolds which admit \((G, \mathbb{R}^2)\)-flows satisfying the assumptions (a) (b), where \(G = \text{Aff}(2)\).

**Proposition 4.1.** If \(\Gamma = \pi_1(M)\) is not nilpotent, then we have the following:

1. The linear part \(L\Gamma\) of \(\Gamma\) is abelian.
2. Up to conjugacy in \(\text{Aff}(2)\), each \(g \in \Gamma\) is of the following form simultaneously: \(g = \left(\begin{array}{cc} p & q \\ 0 & 1 \end{array}\right), \left(\begin{array}{c} b_1 \\ b_2 \end{array}\right)\) with

   - either (i) \(p = 1\) and \(q = 0\) or (ii) \(p \neq 1\) or \(q \neq 0\), and \(b_2 \neq 0\).

**Proof.** First assume that the linear part of \(\Gamma\) is abelian. Since \(\Gamma\) is not nilpotent, the linear part of \(\Gamma\) is non-trivial. Therefore we can find an element \(f \in \Gamma\) whose linear part is equal to \(A = \left(\begin{array}{cc} p & q \\ 0 & 1 \end{array}\right)\) with \(A \neq I\) after a suitable conjugation. This implies that if \(p = 1\) then \(q \neq 0\). Then for any \(g \in \Gamma\) with its linear part \(B = \left(\begin{array}{cc} s & t \\ u & v \end{array}\right)\), the condition that \(B\) commute with \(A\) implies that \(u = 0\). Since \(B - I\) is singular, we obtain \(s = 1\) or \(v = 1\). Since \(\Gamma\) is not nilpotent, there exists an element \(g \in \Gamma\) whose linear part \(B\) satisfies that either \(s\) or \(v\) is not equal to 1. Since every non-trivial element \(g \in \Gamma\) is fixed point free, we can assume the linear part \(B = \left(\begin{array}{cc} s & t \\ 0 & 1 \end{array}\right)\) simultaneously.

Next consider the case that the linear part is non-abelian. Since \(\Gamma\) is not nilpotent, there exists a non-trivial commutator element \([f, g]\) which is not a translation. Then by 2.3, up to conjugacy, we have \([f, g] = \left(\begin{array}{c} 1 \\ a \\ 0 \\ 1 \end{array}\right), \left(\begin{array}{c} a_1 \\ a_2 \end{array}\right)\) with \(a \neq 0\). Since \([f, g]\) is fixed point free, we have \(a_2 \neq 0\). Then by 4.2 below, up to conjugacy in \(G\), the linear part of each \(g\) is of the form \(B = \left(\begin{array}{cc} s & t \\ 0 & 1 \end{array}\right)\) simultaneously. Then by 4.3 (3) below, we see that \([\Gamma, \Gamma]\) is contained in the special translations, which contradicts
the assumption that $L\Gamma$ is non-abelian. Thus this case does not occur and we have the desired results. q.e.d.

**Lemma 4.2.** Assume that there exists an element
\[
f = \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) \in \Gamma \text{ with } a \neq 0 \text{ and } a_2 \neq 0.
\]
Then up to conjugacy in $G$, each $g$ has the following form simultaneously: $g = \left( \begin{array}{cc} p & q \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right)$

with either (1) $p = 1$ and $q = 0$ or (2) $p \neq 1$ or $q \neq 0$, and $b_2 \neq 0$.

**Proof.** Set $g = \left( \begin{array}{cc} p & q \\ r & s \end{array} \right), \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right)$. The linear part of $f^n g$ for $n \in \mathbb{N}$ is $C = \left[ \begin{array}{cc} p + arn & q + asn \\ r & s \end{array} \right]$. By 2.3 this matrix has an eigenvalue 1. Therefore $\det (C - I) = (ps - qr + 1 - s - p) - arn = 0$ for all $n \in \mathbb{N}$, which implies $r = 0$ and either $p = 1$ or $s = 1$. Since the product of two elements must also satisfy this condition, we may assume $s = 1$ simultaneously. q.e.d.

**Lemma 4.3.** Let
\[ f = \left( \begin{array}{cc} \alpha_1 & \alpha_2 \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) \text{ and } g = \left( \begin{array}{cc} \beta_1 & \beta_2 \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right) \]
be two elements of $G$. Then the followings hold:

(0) $f^{-1} = \left( \begin{array}{cc} 1/\alpha_1 & -\alpha_2/\alpha_1 \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} -a_1 + \alpha_2 a_2 \\ -a_2 \end{array} \right)$.

(1) $fg = \left( \begin{array}{cc} \alpha_1 \beta_1 & \alpha_1 \beta_2 + \alpha_2 \\ 0 & 1 \end{array} \right), \left( \begin{array}{c} a_1 + \alpha_1 b_1 + \alpha_2 b_2 \\ a_2 \end{array} \right)$.

In particular the 2nd component of the translation part is additive.

(2) $gfg^{-1} = \left( \begin{array}{cc} \alpha_1 (1 - \alpha_1) \beta_2 + \beta_1 \alpha_2 \\ 0 & 1 \end{array} \right), \left[ \beta_1 a_1 + \beta_2 a_2 + (1 - \alpha_1) b_1 + (\alpha_2 \beta_2 - \alpha_1 \beta_2) b_2 \right]$.

(3) If $f, g \in \Gamma$ then $fgf^{-1}g^{-1} = (I, \left[ \begin{array}{c} k \\ 0 \end{array} \right])$ where
\[ k = (1 - \beta_1) a_1 + (\alpha_1 - 1) b_1 + (\alpha_2 \beta_1 - \alpha_1 \beta_2 - \alpha_2) a_2 + (\alpha_2 \beta_1 - \alpha_1 \beta_2 + \beta_2) b_2. \]
Proof. (0),(1),(2) are easy. We prove (3). By (1) above, the 2nd component of the translation part of \( f g f^{-1} g^{-1} \) is equal to 0. Then by 2.2 this element must be a translation. q.e.d.

Remark. After we have established 4.1, since \( L \Gamma \) is abelian, we have \( k = (1 - \beta_1)a_1 + (\alpha_1 - 1)b_1 - \beta_2a_2 + \alpha_2b_2 \).

**Proposition 4.4.** \( \pi_1(M) \) is solvable.

Proof. For non-nilpotent \( \Gamma \), by 4.1 (1), \( [\Gamma, \Gamma] \) is abelian, which shows this proposition.

**Proposition 4.5.** \( \pi_1(M) \) is torsion free and \( H^1(M;R) \neq 0 \).

Proof. The first part is a conclusion of 2.4 (2). By 4.3 (1), we can easily construct a non-trivial homomorphism \( \rho : \pi_1(M) \to R \). This shows that \( H^1(M;R) \neq 0 \).

One can easily verify the following

**Lemma 4.6.** For \( p \neq 1 \) and \( k = -q/(p - 1) \), we have

\[
\begin{bmatrix}
1 & -k \\
0 & 1
\end{bmatrix} \cdot \left( \begin{bmatrix}
p & q \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}, \begin{bmatrix}
1 & k \\
0 & 1
\end{bmatrix} \right) = \left( \begin{bmatrix}
p & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
c_1 - kc_2 \\
c_2
\end{bmatrix} \right).
\]

**Lemma 4.7.** Let \( f = \left( \begin{bmatrix}
\alpha_1 & \alpha_2 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
a_1 \\
a_2
\end{bmatrix} \right) \in G \).

If \( f^k \) is a special translation for some integer \( k \neq 0 \), then \( f \) itself is a special translation.

Proof We may assume that \( k > 0 \).

Case 1. \( k = 2 \).

\[
f^2 = \left( \begin{bmatrix}
\alpha_1^2 & (\alpha_1 + 1) \alpha_2 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
a_1 \\
a_2
\end{bmatrix} + \begin{bmatrix}
\alpha_1 & \alpha_2 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
a_1 \\
a_2
\end{bmatrix} \right).
\]
If $f^2$ is a special translation and $\alpha_1 = 1$, then we obtain $\alpha_2 = 0$ and $f$ itself is a special translation. If $\alpha_1 = -1$, then we obtain $f^2 = 1$, a contradiction.

**case 2.** $k$ is an odd integer.

$$f^k = \left[ \begin{array}{cc} \alpha_1^k & \left( \sum_{j=0}^{k-1} \alpha_1^j \right) \alpha_2 \\ 0 & 1 \end{array} \right], \left[ \begin{array}{cc} \left( \sum_{j=0}^{k-1} \alpha_1^j \right) & * \\ 0 & k \end{array} \right] \left[ \begin{array}{c} a_1 \\ a_2 \end{array} \right]$$  for some $*$.  

If $f^k$ is a special translation, then $\alpha_1 = 1$ and $a_2 = 0$. From this we have $\alpha_2 = 0$.

**case 3.** general $k$.

Express $k = 2^m l$ where $l$ is odd. Use case 1 $m$ times and then use case 2.

**Lemma 4.8.** For non-trivial $f, g \in \Gamma$, if $gfg^{-1} = f^{-1}$, then by conjugating by an element $\left[ \begin{array}{cc} 1 & k \\ 0 & 1 \end{array} \right]$ for some $k$ we have that $f = (I, \left[ \begin{array}{c} a_1 \\ 0 \end{array} \right])$ (a special translation) and that the linear part of $g$ is equal to $\left[ \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right]$.

**Proof.** 4.3 (3) shows that $f^{-2}$ is a special translation, and by lemma 4.7 $f$ is also a special translation. Setting $g = \left( \begin{array}{cc} p & q \\ 0 & 1 \end{array} \right), \left[ \begin{array}{c} b_1 \\ b_2 \end{array} \right]$ and comparing the translation part of $gf$ and $f^{-1}g$, we have $p = -1$. Conjugating $\Gamma$ by $\left[ \begin{array}{cc} 1 & k \\ 0 & 1 \end{array} \right]$ where $k = \frac{1}{2}q$, we have the desired forms. q.e.d.

**Lemma 4.9.** Let $M$ be a $T^2$-bundle over $S^1$ with hyperbolic monodromy $A$. Then the generators $\alpha, \beta \in \pi_1(T^2) \subset \pi_1(M)$ are homologically torsion (some power of them are represented by products of commutators).

**Proof.** Abelianize the group relation of $\pi_1(M)$. Then we see some power (equal to $1 + \det A - \text{trace} A$) of $\alpha$ and $\beta$ are homologous to zero.
Corollary 4.10. $\hat{\alpha}$ and $\hat{\beta}$ are special translations.

Proof. This concludes from 4.7 and 4.9.

By 2.5 (Evans-Moser), we only consider $T^2$ or $K^2$-bundles over $S^1$. In the following of this section we treat $T^2$-bundles, and the $K^2$-bundle case in the next section.

We consider all cases with respect to their monodromy $A \in \text{GL}(2; \mathbb{Z})$.

I. $\det A = 1$ case.

(i) Elliptic case. $|\text{trace } A| < 2$

Proposition 4.11. The case $\det A = 1$ and $|\text{trace } A| < 2$ does not occur.

Proof. By proposition 3.11, we have 3 cases to deal with.

(1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

In this case we have

$\pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta, \gamma \beta \gamma^{-1} = \alpha^{-1} \rangle$.

In this case $\alpha^2$ and $\beta^2$ are homologous to zero.

By lemma 4.7, $\hat{\alpha}$ and $\hat{\beta}$ are special translations.

Set $\hat{\gamma} = (\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix})$. From the translation part of the relation $\hat{\gamma} \hat{\alpha} = \hat{\beta} \hat{\gamma}$, we have $ca_1 = b_1$. From $\hat{\gamma} \hat{\beta} = \hat{\alpha}^{-1} \hat{\gamma}$, we have $cb_1 = -a_1$. Then we have $c^2 = -1$, which is a contradiction. Thus this case does not occur.

(2) $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$.

In this case we have

$\pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha^{-1} \beta^{-1}, \gamma \beta \gamma^{-1} = \alpha \rangle$.
Since $\alpha^3$ is homologous to zero, we have $\hat{\alpha} = (I, \begin{bmatrix} a_1 \\ 0 \end{bmatrix})$ and $\hat{\beta} = (I, \begin{bmatrix} b_1 \\ 0 \end{bmatrix})$ for some $a_1$ and $b_1$. Then calculating the translation parts of the 2nd and the 3rd relation, we have $ca_1 = -a_1 - b_1$ and $cb_1 = a_1$, from which we deduce $a_1 = 0$, a contradiction. Thus this case does not occur.

(3) $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.

In this case we have

$$\pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha \beta, \gamma \beta \gamma^{-1} = \alpha^{-1} \rangle.$$ 

In this case trace $A^2 = -1$, and taking the double cover with respect to the base $S^1$ and reducing to the observation in case (2), we see this case does not occur.

(ii) **Hyperbolic case ($|\text{trace} A| > 2$)**

Let $A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$. Then

$$\pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha^k \beta^l, \gamma \beta \gamma^{-1} = \alpha^m \beta^n \rangle.$$ 

In this case by 4.10 the generators $\hat{\alpha}$ and $\hat{\beta}$ are special translations. Then by 3.5, the linear part of $\hat{\gamma}$ has an eigenvalue $\neq 1$. Conjugating by a suitable element, we have

$$\hat{\alpha} = (I, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), \quad \hat{\beta} = (I, \begin{bmatrix} b_1 \\ 0 \end{bmatrix}), \quad \hat{\gamma} = (\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}),$$

with $c \neq 1$. Then we see that $c$ is an eigenvalue of $A$ and that $\begin{bmatrix} 1 \\ b_1 \end{bmatrix}$ is an eigenvector corresponding to $c$. Then $c$ and also $b_1$ must be irrational. Furthermore we have

$$\hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \begin{bmatrix} c^r & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda(p, q, r) \\ c_2 r \end{bmatrix})$$

where $\lambda(p, q, r) = p + b_1 q + c_1 \varepsilon_r$, $\varepsilon_r = \sum_{j=0}^{r-1} c^j$.

37
Since $b_1$ is irrational, every non-trivial $g \in \Gamma$ is fixed point free.

The action of $\pi_1(M)$ on $\widetilde{M}$ is given as follows.

Let \[ \begin{bmatrix} 1 \\ d \end{bmatrix} \] be the eigenvector corresponding to the other eigenvalue $c^{-1}$. Then
\[
\alpha(x, z) = (\alpha(x), z + 1), \quad \beta(x, z) = (\beta(x), z + d), \quad \gamma(x, z) = (\gamma(x), c^{-1}z),
\]
is the desired action. On the $x$-$z$ plane in $\mathbb{R}^3$, the set $\mathbb{Z} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} b_1 \\ d \end{bmatrix}$
is identified to $\mathbb{Z} \oplus \mathbb{Z}$ by the linear map $\begin{bmatrix} 1 & b_1 \\ 1 & d \end{bmatrix}$, which shows that the action above is properly discontinuous and free.

Since the 2nd component of the translation part is $c_2r$ (discrete), this representation is not minimal.

Each minimal set of this flow is a saturated $T^2$.

(iii) \textbf{Parabolic case.} ($\left| \text{trace } A \right| = 2$)

(1) \[ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \]

In this case we have
\[ \pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha^{-1}, \gamma \beta \gamma^{-1} = \beta^{-1} \rangle. \]

From the 2nd and 3rd relations $\hat{\alpha}$ and $\hat{\beta}$ must be special translations. By a suitable conjugacy we may assume $\hat{\alpha} = (I, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $\hat{\beta} = (I, \begin{bmatrix} b_1 \\ 0 \end{bmatrix})$.

Then by 4.8 we may assume $\hat{\gamma} = (\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix})$.

For $p, q, r \in \mathbb{Z}$ we have
\[
\hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = (\begin{bmatrix} (-1)^r & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda(p, q, r) \\ c_2r \end{bmatrix})
\]
where $\lambda(p, q, r) = p + b_1q + c_1 \varepsilon_r$, $\varepsilon_r = \frac{1}{2}(1 + (-1)^{r-1})$. The injectivity of $h$ requires that $b_1$ must be irrational. Then we obtain a representation where every non-trivial $g \in \Gamma$ is fixed point free.

The action of $\pi_1(M)$ on $\widetilde{M}$ is given by
\[ \alpha(x, z) = (\widehat{\alpha}(x), z + 1), \beta(x, z) = (\widehat{\beta}(x), z), \gamma(x, z) = (\widehat{\gamma}(x), -z). \]

with the developing map \( D : (x, z) \mapsto x. \)

Since the 2nd component of the translation part is \( c_2 r \) (discrete), this representation is not minimal.

We see that every minimal set is a saturated \( T^2 \) corresponding to \( y = \text{const.} \)

\[ (2) \ A = \begin{bmatrix} -1 & n \\ 0 & -1 \end{bmatrix} \quad (0 \neq n \in \mathbb{Z}). \]

In this case we have \( \pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha^{-1} \beta^n, \gamma \beta \gamma^{-1} = \beta^{-1} \rangle. \)

From the 2nd and 3rd relations \( \widehat{\alpha} \) and \( \widehat{\beta} \) must be special translations. Set \( \widehat{\alpha} = (I, \begin{bmatrix} a_1 \\ 0 \end{bmatrix}) \) and \( \widehat{\beta} = (I, \begin{bmatrix} b_1 \\ 0 \end{bmatrix}) \) where \( a_1 \) and \( b_1 \) are non zero. By 4.8, set \( \widehat{\gamma} = (\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}) \). Then from the 2nd relation we have \(-a_1 = -a_1 + nb_1\), which implies \( n = 0 \), a contradiction. Thus this case does not occur.

\[ (3) \ A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \quad (n \in \mathbb{Z}). \]

The relations of \( \pi_1(M) \) are \([\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha \beta^n, \gamma \beta \gamma^{-1} = \beta \).

We can verify that if \( n \neq 0 \), then \( \Gamma \subset G_1 \) as follows.

By the 2nd relation \( \beta \) is homologous to zero, that is \( \widehat{\beta} = (I, \begin{bmatrix} b_1 \\ 0 \end{bmatrix}) \) is a special translation. Then from the 1st and 3rd relations we see \( \widehat{\alpha} \) and \( \widehat{\gamma} \) are contained in \( G_1 \). Thus no new examples arise from here.

When \( n = 0 \), we will observe that new examples do not arise as follows:

Let \( \widehat{\alpha} = (\begin{bmatrix} a & a' \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}) \) be a generator in \( \Gamma \) such that \( a \neq 1 \).

Since \( \widehat{\beta} = (\begin{bmatrix} b & b' \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}) \) and \( \widehat{\gamma} = (\begin{bmatrix} c & c' \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}) \) commute with \( \widehat{\alpha} \), comparing the linear part we have \( b' = \kappa(b - 1) \) and \( c' = \kappa(c - 1) \) where
\( \kappa = a'/ (a - 1) \). Conjugating by \( \begin{bmatrix} 1 & \kappa \\ 0 & 1 \end{bmatrix} \), we may assume \( a' = 0, b' = 0 \) and \( c' = 0 \). Then commuting condition again shows that \( b_1 = l(b - 1) \) and \( c_1 = l(c - 1) \) where \( l = a_1/(a - 1) \). If \( l \neq 0 \), then conjugating by \( \begin{bmatrix} 1/l & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), we may assume \( l = 0 \). Thus we have

\[
\hat{\alpha} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{\beta} = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{\alpha} = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}.
\]

Then for \( p, q, r \in \mathbb{Z} \)

\[
\hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \begin{bmatrix} a^p b^q c^r & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_2 p + b_2 q + c_2 r \end{bmatrix}.
\]

From this we have \( a_2, b_2, c_2 \) must be linearly independent over \( \mathbb{Q} \).

We will see that this representation is not realizable as a flow.

Suppose that there is a flow realizing this representation with the developing map \( D : \tilde{T}^3 \to \mathbb{R}^2 \).

First consider the case where \( a = -1 \) and \( |b| = |c| = 1 \). Then \( \Gamma \) is a subgroup of Euclidean isometries on \( \mathbb{R}^2 \). Let \( K \) be a relatively compact fundamental region in \( \tilde{T}^3 \). Then \( D(K) \) is also relatively compact and the \( \Gamma \) orbit of \( D(K) \) does not cover the whole \( \mathbb{R}^2 \), which contradicts that \( D \) is surjective.

Next consider the case where at least one of \( |a|, |b|, |c| \) is not equal to 1. Let \( \pi : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto x \) be the projection to the first factor. Then since the representation above leaves the line \( x = 0 \) fixed, the map \( \pi \circ D : \tilde{T}^3 \to \mathbb{R} \) defines a codimension one transversely affine foliation \( \mathcal{F} \) of \( T^3 \) with only one compact leaf \( L \) which corresponds to the point \( x = 0 \). In particular \( \mathcal{F} \) is almost without holonomy. Taking a finite cover we may assume that \( \mathcal{F} \) is transversely orientable and that \( L \) is \( T^2 \) with non-trivial linear holonomy corresponding to \( \pi \circ \hat{\alpha} \). Cutting out \( T^3 \) along \( L \), we have an affine foliation on \( T^2 \times [0, 1] \) which is almost without holonomy. Then by a theorem of Inaba [I], the transverse orientations on the boundary leaves \( T^2 \times \{0, 1\} \) are simultaneously inward or simultaneously outward,
which contradicts the transverse orientability of $\mathcal{F}$. Thus this case also does not occur.

**II.** $\det A = -1$ case.

**case 1.** $|\text{trace} A| > 0$.
In this case both of the eigenvalues of $A$ are real and $\neq 1$, and the treatment is the same as the hyperbolic case.

**case 2.** $\text{trace} A = 0$.
In this case up to conjugacy in $\text{GL}(2; \mathbb{Z})$, there are 3 cases as follows.

$$ (1) \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. $$

In this case we have

$$ \pi_1(M) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha, \gamma \beta \gamma^{-1} = \beta^{-1} \rangle. $$

From the 3rd relation necessarily we have

$$ \hat{\beta} = (I, \begin{bmatrix} b_1 \\ 0 \end{bmatrix}) \quad \text{and} \quad \hat{\gamma} = (\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}). $$

Since $\alpha$ and $\beta$ commute, we have $\hat{\alpha} = (\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix})$. Then by the 2nd relation we have $a = 0$ and $a_1 = 0$. Conjugating by $\begin{bmatrix} b_1^{-1} & 0 \\ 0 & a_2^{-1} \end{bmatrix}$ we may assume $a_2 = b_1 = 1$. Thus

$$ \hat{\alpha} = (I, \begin{bmatrix} 0 \\ 1 \end{bmatrix}), \quad \hat{\beta} = (I, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) \quad \text{and} \quad \hat{\gamma} = (\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}). $$

Then for $p, q, r \in \mathbb{Z}$ we have

$$ \hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \left( \begin{bmatrix} (-1)^r & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda(p, q, r) \\ p + c_2 r \end{bmatrix} \right) $$

where $\lambda(p, q, r) = q + c_1 \varepsilon_r, \varepsilon_r = \frac{1}{2}(1 + (-1)^{r-1})$. Since this representation must be injective, $c_2$ must be irrational. Then we see that every non-trivial $g \in \Gamma$ is fixed point free.

The action of $\pi_1(M)$ on $\tilde{M}$ is given by
\[\alpha(x, z) = (\hat{\alpha}(x), z), \quad \beta(x, z) = (\hat{\beta}(x), z), \quad \gamma(x, z) = (\hat{\gamma}(x), z + 1).\]

Since the 1st component of the translation part is discrete, this representation is not minimal. We see that each minimal set is a saturated \( T^2 \).

\[(2) \quad A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.\]

In this case we have

\[\pi_1(M) = \langle \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha \beta, \gamma \beta \gamma^{-1} = \beta^{-1} \rangle.\]

From the 3rd relation necessarily we have \( \hat{\beta} = (I, \begin{bmatrix} b_1 \\ 0 \end{bmatrix}) \) and \( \hat{\gamma} = (\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}) \).

Since \( \alpha \) and \( \beta \) commute, we have \( \hat{\alpha} = (\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}) \).

Then by the 2nd relation we have \( a = 0 \) and \( 2a_1 + b_1 = 0 \). Then for \( p, q, r \in \mathbb{Z} \) we have

\[\hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = (\begin{bmatrix} (-1)^r & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda(p, q, r) \\ a_2 p + c_2 r \end{bmatrix})\]

where \( \lambda(p, q, r) = a_1 p - 2a_1 q + c_1 \varepsilon_r \), \( \varepsilon_r = \frac{1}{2}(1 + (-1)^{r-1}) \). Since the representation is injective, \( a_2 \) and \( c_2 \) must be linearly independent. Then every non-trivial \( g \in \Gamma \) is fixed point free.

The action of \( \pi_1(M) \) on \( \hat{M} \) is given by

\[\alpha(x, z) = (\hat{\alpha}(x), z), \quad \beta(x, z) = (\hat{\beta}(x), z), \quad \gamma(x, z) = (\hat{\gamma}(x), z + 1).\]

Since \( \lambda(p, q, r) \) is discrete, this representation is not minimal. We see each minimal set is a saturated \( T^2 \).

\[(3) \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.\]

In this case

\[\pi_1(M) = \langle \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta, \gamma \beta \gamma^{-1} = \alpha \rangle.\]

Since \( \gamma \alpha \beta^{-1} \gamma^{-1} = (\alpha \beta^{-1})^{-1} \), by 4.8 we may assume that \( \hat{\alpha} \hat{\beta}^{-1} \) is a non-trivial special translation and that \( \hat{\gamma} = (\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}) \).

Since
both \( \hat{\alpha} \) and \( \hat{\beta} \) commute with \( \hat{\alpha}\hat{\beta}^{-1} \), we have 
\[
\hat{\alpha} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad \hat{\beta} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.
\]
Since \( \hat{\alpha}\hat{\beta}^{-1} \) is a special translation, we have \( a = b \).
Then from the 1st and the 2nd relations we have \( a = 0, b_2 = a_2 \) and 
\( b_1 = -a_1 \) with \( a_1 \neq 0 \) and \( a_2 \neq 0 \). Conjugating by 
\[
\begin{bmatrix} a_1^{-1} & 0 \\ 0 & a_2^{-1} \end{bmatrix}
\]
we may assume 
\[
\hat{\alpha} = (I, \begin{bmatrix} 1 \\ 1 \end{bmatrix}), \quad \hat{\beta} = (I, \begin{bmatrix} -1 \\ 1 \end{bmatrix}).
\]
Then we have 
\[
\hat{\alpha}^p \hat{\beta}^q \gamma^r = \begin{bmatrix} (-1)^r & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda(p, q, r) \\ p + q + c_2 r \end{bmatrix}
\]
where \( \lambda(p, q, r) = p - q + c_1 \varepsilon_r, \varepsilon_r = \frac{1}{2}(1 + (-1)^{r-1}) \). Since this representation must be injective, \( c_2 \) must be irrational. Then we see that every non-trivial \( g \in \Gamma \) is fixed point free.

The action of \( \pi_1(M) \) on \( \tilde{M} \) is given by 
\[
\alpha(x, z) = (\hat{\alpha}(x), z), \quad \beta(x, z) = (\hat{\beta}(x), z), \quad \gamma(x, z) = (\hat{\gamma}(x), z + 1).
\]
Since \( \lambda(p, q, r) \) is discrete, this representation is not minimal. We see each minimal set is a saturated \( T^2 \).

Summarizing these results we have

**Theorem 4.12.** Let \( M \) be a \( T^2 \)-bundle over \( S^1 \) with the monodromy map \( A \). Then the followings hold.

(1) \( M \) admits an affine flow with the assumptions (a) (b) if and only if \( A \) is one of the following:

(i) \( \det A = 1 \) case.

\[
|\text{trace } A| > 2 \text{ (hyperbolic)}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} (n \in \mathbb{Z}).
\]

(ii) \( \det A = -1 \) case.
\[ |\text{trace } A| > 0 \text{ (hyperbolic)}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}. \]

(2) \( M \) supports a complete affine minimal flow if and only if
\[ A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \quad (n \in \mathbb{Z}). \]

§5. Flows on \( K^2 \)-bundles

**Lemma 5.1.** If \( M \) is a \( K^2 \)-bundle over \( S^1 \), then the fiberwise double cover is a \( T^2 \)-bundle over \( S^1 \) with monodromy \( \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \).

**Proof.** Consider the free involution \( \sigma : T^2 \to T^2 \) satisfying \( T^2/\sigma = K^2 \). On \( H_1(T^2; \mathbb{R}) \cong \mathbb{R}^2 \) with the standard identification, the induced automorphism \( \sigma_* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). Since the monodromy map \( A \) on \( H_1(T^2; \mathbb{R}) \) must commute with \( \sigma_* \), we have \( A = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \). q.e.d.

One can easily verify the following

**Lemma 5.2.**

1. In \( \pi_1(K^2) = \langle \alpha, \beta \mid \beta \alpha \beta^{-1} = \alpha^{-1} \rangle \), \( \pi_1(T^2) \) is generated by \( \alpha \) and \( \beta^2 \).

2. Every element in \( \pi_1(K^2) \) is uniquely expressed as \( \alpha^k \beta^l \) for some \( k, l \in \mathbb{Z} \).

3. In \( \pi_1(K^2) \), \((\alpha^m \beta^n)^2 = \alpha^{m+(-1)^m} \beta^{2n} \).

4. The fundamental group of the total space \( M \) of a \( K^2 \)-bundle over \( S^1 \) has the following presentation:
\[ \pi_1(M) = \langle \alpha, \beta, \gamma \mid \beta \alpha \beta^{-1} = \alpha^{-1}, \gamma \alpha \gamma^{-1} = \alpha^k \beta^l, \gamma \beta \gamma^{-1} = \alpha^m \beta^n \rangle \]
for some \( k, l, m, n \in \mathbb{Z} \).

**Theorem 5.3.** Let \( A \) be the monodromy map of the fiberwise doubly covered \( T^2 \)-bundle over \( S^1 \).
(1) $M$ supports an affine flow with the assumptions (a) (b) if and only if $A = I$ or $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(2) Minimal flows do not arise in this case.

**Proof.** In the following we divide the cases to consider by the monodromy of the fiberwise doubly covered $T^2$-bundle over $S^1$.

**case 1.** the monodromy = $I$.

Set the relations of $\pi_1(M)$ as $\beta \alpha \beta^{-1} = \alpha^{-1}$, $\gamma \alpha \gamma^{-1} = \alpha^k \beta^l$, $\gamma \beta \gamma^{-1} = \alpha^m \beta^n$ where $k, l, m, n$ are integers. Since $\gamma \alpha \gamma^{-1} = \alpha$ and $\gamma \beta^2 \gamma^{-1} = \alpha^{m+(-1)^m} \beta^{2n} = \beta^2$, we have $k = 1, l = 0$ and $n = 1$.

Thus the relations of $\pi_1(M)$ are $\beta \alpha \beta^{-1} = \alpha^{-1}$, $\gamma \alpha \gamma^{-1} = \alpha$, $\gamma \beta \gamma^{-1} = \alpha^m \beta$ for some $m$.

Then the automorphism of $\pi_1(K^2)$ determined by the 2nd and the 3rd relations can be realized by a diffeomorphism of $K^2$. By the 1st relation and lemma 4.8, we may assume

$$\hat{\alpha} = (I, \begin{bmatrix} a_1 \\ 0 \end{bmatrix}), \quad \hat{\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$  

By the 2nd relation we have

$$\hat{\gamma} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$  

From the 3rd relation we have $c = 0$ and $c_1 = \frac{1}{2} ma_1$. Thus

$$\hat{\alpha} = (I, \begin{bmatrix} a_1 \\ 0 \end{bmatrix}), \quad \hat{\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}), \quad \hat{\gamma} = (I, \begin{bmatrix} \frac{1}{2} ma_1 \\ c_2 \end{bmatrix}).$$  

Then $\hat{\beta}^2 = (I, \begin{bmatrix} 0 \\ 2b_2 \end{bmatrix})$.

We see for $p, q, r \in \mathbb{Z}$

$$\hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \begin{bmatrix} (-1)^q & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda(p, q, r) \\ b_2 q + c_2 r \end{bmatrix}$$  

45
where \( \lambda(p, q, r) = a_1 p + b_1 \varepsilon_q + (-1)^q c_1 r, \varepsilon_q = \frac{1}{2}(1 + (-1)^q - 1) \). From this, \( b_2 \) and \( c_2 \) must be linearly independent over \( \mathbb{Q} \).

The action of \( \pi_1(M) \) on \( \tilde{M} \) is given by
\[
\alpha(x, z) = (\tilde{\alpha}(x), z), \beta(x, z) = (\tilde{\beta}(x), z), \gamma(x, z) = (\tilde{\gamma}(x), z + 1).
\]

Since the 1st components of the translation parts of \( \tilde{\alpha} \) and \( \tilde{\gamma} \) are rationally dependent, by considering on the fiberwise double cover \( T^3 \), we see there does not exist a minimal example. Each minimal set is a saturated \( K^2 \).

**Remark.** If \( m \neq 0 \), then \( M_m \) is the \( m \)-fold cover of \( M_1 \).

**case 2.** the monodromy = \[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\].

We will see this case does not occur.

In this case the relations of \( \pi_1(M) \) are \( \beta \alpha \beta^{-1} = \alpha^{-1}, \gamma \alpha \gamma^{-1} = \alpha, \gamma \beta \gamma^{-1} = \alpha^m \beta^{-1} \) for some \( m \).

The automorphism of \( \pi_1(K^2) \) determined by the 2nd and the 3rd relations can be realized by a diffeomorphism of \( K^2 \).

By the 1st relation we may assume
\[
\tilde{\alpha} = (I, \begin{bmatrix} a_1 \\ 0 \end{bmatrix}) \text{ and } \tilde{\beta} = (\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}).
\]

Then by the 2nd relation we may assume \( \tilde{\gamma} = (\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}) \). Since the 2nd component of the translation part is additive, from the 3rd relation we have \( b_2 = 0 \), which contradicts 2.3.

**case 3.** the monodromy = \[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\].

In this case the relations of \( \pi_1(M) \) are
\[
\beta \alpha \beta^{-1} = \alpha^{-1}, \gamma \alpha \gamma^{-1} = \alpha^{-1}, \gamma \beta \gamma^{-1} = \alpha^m \beta \text{ for some } m.
\]

Then the automorphism of \( \pi_1(K^2) \) determined by the 2nd and the 3rd relations can be realized by a diffeomorphism of \( K^2 \).

Then we may assume \( \tilde{\alpha} = (I, \begin{bmatrix} a_1 \\ 0 \end{bmatrix}) \) and \( \tilde{\beta} = (\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}) \). By
the 2nd relation we may assume \( \hat{\gamma} = \left( \begin{array}{c} -1 \\ 0 \\ \end{array} \right), \left[ \begin{array}{c} c_1 \\ c_2 \\ \end{array} \right] \). From the linear part of the 3rd relation we have \( c = 0 \). Then from the translation part of the same relation we have \( 2(c_1 - b_1) = ma_1 \). Thus

\[
\hat{\alpha} = (I, \left[ \begin{array}{c} a_1 \\ 0 \end{array} \right]), \quad \hat{\beta} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left[ \begin{array}{c} b_1 \\ b_2 \end{array} \right] \), \quad \hat{\gamma} = \left( \begin{array}{c} -1 \\ 0 \end{array} \right), \left[ \begin{array}{c} \frac{1}{2}ma_1 + b_1 \\ c_2 \end{array} \right].
\]

Then we have for \( p, q, r \in \mathbb{Z} \)

\[
\hat{\alpha}^p \hat{\beta}^q \hat{\gamma}^r = \left( \begin{array}{c} 0^{q+r} \\ 1 \end{array} \right), \left[ \begin{array}{c} \lambda(p, q, r) \\ b_2q + c_2r \end{array} \right]
\]

where \( \lambda(p, q, r) = a_1p + b_1\varepsilon_q + (-1)^q c_1\varepsilon_r, \varepsilon_q = \frac{1}{2}(1 + (-1)^{q-1}) \). Since the representation is injective, \( b_2 \) and \( c_2 \) must be linearly independent over \( \mathbb{Q} \). Then every non-trivial \( g \in \Gamma \) is fixed point free.

The developing map can be constructed similarly as before.

By the conclusion of the corresponding \( T^2 \)-bundle case, we see there are no minimal examples. Each minimal set is a saturated \( T^2 \).

**case 4.** the monodromy = \( \left[ \begin{array}{c} -1 \\ 0 \\ 0 \\ -1 \end{array} \right] \).

In this case the relations of \( \pi_1(M) \) are \( \beta\alpha\beta^{-1} = \alpha^{-1} \), \( \gamma\alpha\gamma^{-1} = \alpha^{-1} \), \( \gamma\beta\gamma^{-1} = \alpha^m\beta^{-1} \) for some \( m \).

The automorphism of \( \pi_1(K^2) \) determined by the 2nd and the 3rd relations can be realized by a diffeomorphism of \( K^2 \).

Then by the 1st relation we may assume

\[
\hat{\alpha} = (I, \left[ \begin{array}{c} a_1 \\ 0 \end{array} \right]), \quad \hat{\beta} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left[ \begin{array}{c} b_1 \\ b_2 \end{array} \right].
\]

By the 2nd relation we may assume

\[
\hat{\gamma} = \left( \begin{array}{c} -1 \\ 0 \end{array} \right), \left[ \begin{array}{c} c_1 \\ c_2 \end{array} \right].
\]

47
Then, as in case 2 above, from the 3rd relation we have $b_2 = 0$, which is a contradiction. Thus this case does not occur.

§6. Remarks and problems

The existence of transverse foliations.

We can see that all the flows arising above admit transverse codimension one foliations. In the hyperbolic case ($\det = 1$ and $|\text{trace}| > 2$, or $\det = -1$ and $|\text{trace}| > 0$) and in (1) of the parabolic case in §4, we must consider the foliations which are the suspensions by $A$ of suitable linear foliations on $T^2$.

Except for these two cases, we see that foliations by the fibers are transverse to flows.

Finally we pose some problems.

Problem 1. If we drop the condition that $D$ is surjective, then which types of domain do there arise?

Problem 2. For a given $T^2$ or $K^2$-bundle over $S^1$ admitting complete affine flows, determine the moduli space of transversely affine flows.
References


