Soliton propagation in an inhomogeneous plasma at critical density of negative ions: Effects of gyrotary and thermal motions of ions

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The effects of gyrotary and thermal motions of ions on soliton propagation in an inhomogeneous plasma that contains positive ions, negative ions, and electrons are studied at a critical density of negative ions. Since at this critical negative ion density the nonlinear term of the relevant Korteweg–deVries (KdV) equation vanishes, a higher order of nonlinearity is considered by retaining higher-order perturbation terms in the expansion of dependent quantities together with the appropriate set of stretched coordinates. Under this situation, time-dependent perturbation leads to the evolution of modified KdV solitons, which are governed by a modified form of the KdV equation that has an additional term due to the density gradient present in the plasma. On the basis of the solution of this equation and obliquely applied magnetic field, the effects of gyrotary and thermal motions of ions are analyzed on the soliton propagation for three cases, \( n_{i0} < n_{e0}, n_{i0} = n_{e0}, \) and \( n_{i0} > n_{e0} \), together with \( n_{i0} (n_{e0}) \) as the density of negative ions (electrons). The role of the negative ions in the evolution of the modes and the solitons is also discussed. Under the limiting cases, our calculations reduce to the ones obtained by other investigators in the past. This substantiates the generality of the present analysis. © 2007 American Institute of Physics.

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I. INTRODUCTION

It is well known that a solitary wave evolves as a soliton when it is a dynamical balance between the effects of nonlinearity and dispersion of the medium. These two effects are encompassed in a fascinating equation known as the Korteweg–deVries (KdV) equation. In a homogenous plasma, the behavior of one-dimensional solitons is governed by the usual form of the KdV equation. However, this equation gets modified with the variable coefficients and/or an additional term that appears due to the presence of density gradient in the case of inhomogeneous plasmas. On the other hand, relevant KdV equations have been reported in plasma under the effect of external static magnetic field, which show the modification in the soliton propagation characteristics.

In ordinary plasma with positive ions and electrons, usually compressive solitons are found to propagate. However, when negative ions are introduced, the response of the plasma to the perturbations gets drastically modified as the negative ions respond out of phase with the positive ions. Then, rarefactive solitons, which evolve due to the compression of the negative ions, can also propagate together with the compressive solitons. Another interesting point of negative ion containing plasmas is that the balance between nonlinearity and dispersion is lost at some critical density of negative ions, and then the coefficient of the nonlinear term of the KdV equation vanishes. For example, in a recent investigation of magnetized inhomogeneous plasma that has positive and negative ions of equal masses and temperatures, the nonlinear term of the relevant KdV equation vanishes at the density \( n_{i0} \) of the negative ions. This density \( n_{i0} \) shows the dependence on the positive ion density \( n_{p0} \) and temperatures of the positive and negative ions, as per the following equation:

\[
1 - 3 \left( \frac{n_{p0} - n_{i0}}{n_{p0} + n_{i0}} \right)^2 + \frac{2\sigma}{2\sigma + (n_{p0} + n_{i0}))(n_{p0} - n_{i0})} \times \frac{4n_{p0}n_{i0}}{(n_{p0} + n_{i0})^2} = 0, \tag{1}
\]

where \( \sigma = T_i/T_e \) together with \( T_i = T_e = T_0 \). If we calculate the critical negative to positive ion density ratio, i.e., \( n_{i0}/n_{p0} = \rho_0 \), for plasma having an ion to electron temperature ratio \( \sigma = 0.05 \), we obtain it as 0.25. This critical density ratio \( \rho_0 \) in the case of magnetized plasma is higher in comparison with the observed value of Ludwig et al. for an unmagnetized plasma.

In view of the above, the theory given in Ref. 12 fails to explain the behavior of soliton propagation at a critical density of negative ions, hence there is a need to investigate the same in a more realistic situation in which thermal motions of the ions are taken into account due to their finite temperatures. Furthermore, since the modification in the soliton characteristics is found in the presence of a magnetic field, it is of much interest to examine the effect of gyrotary motion of the ions in addition to their thermal motion effects on the soliton evolution. The present paper, therefore, focuses on the same problem.

II. SET OF BASIC FLUID EQUATIONS

We consider weakly inhomogeneous plasma that has positive ions, negative ions, and electrons. A static magnetic field \( B_0 \) is applied in the \( z \) direction and the wave is taken to propagate at an angle \( \theta \) with it in the \( x, z \) plane. Assuming...
both the positive and negative ions to be singly charged, the normalized basic fluid equations can be written as

\[ \partial n / \partial t + \nabla \cdot (n \mathbf{v}_n) = 0, \]

\[ \partial \mathbf{v}_n / \partial t + (\mathbf{u}_j \cdot \nabla) \mathbf{v}_j \pm \eta \nabla \phi \mp \eta B_0 \sqrt{(e_d n_m n_p)} [\mathbf{u}_j \times \mathbf{z}] + \eta \gamma \sigma \mathbf{n}_n / \mathbf{n}_j = 0, \]

\[ n_e - e^{\phi} = 0, \]

\[ \nabla^2 \phi - \nabla n_e - n_n + n_p = 0. \]

Here for the positive ions \( \mathbf{v}_j = \mathbf{u}_j \), \( \eta = 1 \), \( \sigma_j = \sigma_p = T_p / T_e, j \rightarrow p \), and the upper sign holds well, while for the negative ions \( \mathbf{v}_j = \mathbf{u}_j \), \( \eta = m_p / m_n, \sigma_j = \sigma_n = T_n / T_e, j \rightarrow n \), and the lower sign holds well. \( m_p(T_p) \) and \( m_n(T_n) \) are the masses (temperatures) of the positive and negative ions, respectively, and \( \gamma_j \) are their specific-heat ratios, which can be taken as 2 for the one-dimensional motions. Further, the densities are normalized by the zeroth-order plasma density \( n_0 \) at an arbitrary reference point, which we choose as \( x = z = 0 \), the velocities \( \mathbf{v}_p \) and \( \mathbf{u}_p \) of the positive and negative ions are normalized by the ion acoustic speed \( \sqrt{T_i / m_i} \), the electric potential \( \phi \) is normalized by \( T_e / e \), the space coordinates \( x \) and \( z \) are normalized by the electron Debye length \( \sqrt{e_0 T_e / n_0 e^2} \), and the time \( t \) is normalized by the plasma period \( \omega_0^{-1} = \sqrt{e_0 n_0 / n_0 e^2} \).

### III. REDUCTIVE PERTURBATION ANALYSIS

While using the usual reductive perturbation analysis of the above fluid equations to study the low-frequency ion acoustic waves in the present plasma model, the densities, fluid velocities, and the electric potential are expanded around their equilibrium position in terms of a smallness parameter \( \epsilon \), and the coordinates \( x, z \), and \( t \) are transformed to the wave frame of reference via stretched coordinates \( \xi \) and \( \tau \), given below,

\[ \xi = \epsilon [k \cdot \mathbf{r} / \lambda_0 - i] = \epsilon [(x \sin \theta + z \cos \theta / \lambda_0 - i], \]

\[ \tau = \epsilon^3 \hat{k} \cdot \mathbf{r} = \epsilon^3 (x \sin \theta + z \cos \theta), \]

\[ f = f_0(x, z) + \epsilon f_1(x, z, t) + \epsilon^2 f_2(x, z, t) + \epsilon^3 f_3(x, z, t) + \cdots, \quad f = n_p \rho_n \rho_e \phi \mathbf{u}_n \mathbf{u}_z \]

\[ g = g_0(x, z) + \epsilon^3 g_1(x, z, t) + \epsilon^2 g_2(x, z, t) + \epsilon g_3(x, z, t) + g = \mathbf{v}_n \mathbf{u}_n \mathbf{u}_z. \]

In these equations, \( \lambda_0 \) is the phase velocity of the wave and \( \mathbf{k} \) is the unit wave vector in the \( x, z \) plane at an angle \( \theta \) with the direction of magnetic field \( \mathbf{B}_0 = B_0 \mathbf{z} \). Here, it may be noted that the perturbations have been taken to third-order quantities in view of retaining a higher order of nonlinearity in the system.

In the forthcoming portion, we will concentrate only on the plasma for which positive and negative ions have the same masses. In nature, such a plasma model may occur in the \( D \) region of Earth’s ionosphere where relatively higher pressures permit the formation of negative ions by the attachment of the electrons to atoms or molecules under the effect of Earth’s magnetic field. Pair-ion plasmas consisting of positive and negative ions of equal masses have been generated in the laboratory using fullerene as an ion source. In these experiments, \( C_{60}^- \) and \( C_{60}^+ \) ions together with some fraction of electrons were produced. On the other hand, electron-positron plasmas are believed to be abundant in many astrophysical environments from pulsars to quasars. In laboratory also, electron-positron pair plasma has been generated with the help of ultraintense lasers and by trapping of positron in a magnetic mirror configuration by electron cyclotron resonance heating. Moreover, \( (H^+, H^+) \) compositions in a theoretical model were also taken for investigating the solitons in an unmagnetized plasma.

Under the situation of ions of equal masses \( (m_p / m_n = 1) \), we take their drift velocities to be the same as \( v_{x0} = u_{x0} = u_{z0}, v_{z0} = u_{z0} = u_{z0} \) together with the same temperatures as \( \sigma_j = \sigma_p \). Then as a part of the reductive perturbation technique, we use Eq. (3) in Eq. (2) and collect various coefficients at the different orders of \( \epsilon \) as follows:

**Zeroth-order equations**

\[ \sin \theta (n_p \mathbf{u}_0) + \cos \theta (n_p \mathbf{u}_0) = 0, \]

\[ n_{e0} = n_{p0} - n_{j0} = 0, \]

\[ (u_0 \sin \theta + v_0 \cos \theta) u_{0x} \pm \sin \theta \mathbf{u}_0 = (2 \epsilon \sin \theta n_{j0} \mathbf{n}_{j0} = 0, \]

**First-order equations**

\[ (\lambda_0 - \epsilon \sin \theta - \epsilon \cos \theta) n_{j1} - n_{j0} \mathbf{C}_{jz} = 0, \]

\[ n_{e1} + n_{e1} - n_{j1} = 0, \]

\[ (\lambda_0 - \epsilon \sin \theta - \epsilon \cos \theta) C_{jz} \mp \epsilon \cos \theta \sin \theta n_{j0} \mathbf{n}_{j0} = 0, \]

\[ n_{e1} - \epsilon \sin \theta n_{j1} = 0, \]

\[ \pm \sin \theta \mathbf{u}_1 \mp \lambda_0 B_0 \sqrt{(e_d m_p n_p)} C_{j1} + (2 \epsilon \sin \theta n_{j0} \mathbf{n}_{j1} = 0, \]

\[ v_{x1} = 0, \quad u_{z1} = 0. \]
Second-order equations

\[ n_m C_{z} + (n_{j1} C_{z1}) \xi - (\lambda_0 M \cos \theta) n_{j2} \xi = 0, \]

\[ C_{z1 \xi} = MB_0 \sqrt{\varepsilon_o \mu_m} C_{x2} = 0, \]

\[ (n_m \theta_0^2 + n_{j1} \phi_{\xi} + 2 \sigma n_{j2}) \sin \theta - \lambda_0 B_0 \sqrt{\varepsilon_o \mu_m} n_{j1} C_{z1} \]

\[ - \lambda_0 B_0 \sqrt{\varepsilon_o \mu_m} n_{j0} C_{x2} = 0, \quad (6) \]

\[ n_{e2} - e^\phi_0 (\phi_{\xi}^2 + \phi_z^2) = 0, \]

\[ (n_{j1} n_{j0}) \phi_{\xi} + \phi_{z} = (2 \sigma n_{j0}) n_{j2} \xi + (\lambda_0 M \cos \theta) C_{z2} \xi = 0, \]

\[ n_{e2} + n_{p2} = 0. \]

Third-order equations

\[ MB_0 \sqrt{\varepsilon_o \mu_m} C_{x3} \cos \theta \lambda_0 C_{z1} \xi + C_{y2} \xi = 0, \]

\[ (M \sin \theta \lambda_0)[n_m C_{z} + \lambda_0 C_{\theta} \phi_{\theta}] - M (u_0 \sin \theta + v_0 \cos \theta) \]

\[ \times (n_{j1} n_{j0}) n_{j0} - n_{j1} \xi + (M \cos \theta \lambda_0)[n_{j0} C_{z} \xi \]

\[ + (n_{j2} n_{j0}) \phi_{\xi} + (2 \sigma n_{j0}) n_{j2} \xi + (M \cos \theta \lambda_0)[n_{j2} C_{z} \xi \]

\[ + (n_{j1} n_{j0}) C_{z1} - (\lambda_0 M n_{j0} \cos \theta) n_{j2} C_{z1} \xi - C_{z3} \xi = 0, \]

\[ \pm \phi_{\xi} + (n_{j1} n_{j0}) \phi_{\theta} + (2 \sigma n_{j0}) n_{j2} \xi \]

\[ - (2 \sigma n_{j0} n_{j2}) n_{j1} \xi + M (u_0 \sin \theta \lambda_0) \]

\[ + v_0 \cos \theta \lambda_0 C_{z1} \xi \cos \theta - (1/M) C_{z3} \xi = 0, \quad (7) \]

\[ (1/M) C_{z3} - (u_0 \sin \theta + v_0 \cos \theta) C_{z1} \xi - (1/M) C_{z2} \xi \]

\[ + (1/M) C_{z2} \lambda_0 C_{y1} \xi \cos \theta = 0, \]

\[ \cos \theta \lambda_0 C_{z3} \cos \phi_{\xi} \xi \]

\[ \times [\xi \phi_{z} \pm (n_{j1} n_{j0}) \phi_{\xi} \pm (n_{j2} n_{j0}) \phi_{z} \xi + (2 \sigma n_{j0}) n_{j1} \xi \]

\[ + \lambda_0 \phi_{\xi} + (2 \sigma n_{j0} n_{j1}) \xi + B_0 \sqrt{\varepsilon_o \mu_m} [C_{\xi} \xi \]

\[ + (n_{j1} n_{j0}) C_{z2} \xi + (n_{j2} n_{j0}) C_{y1} \xi] - (2 \sigma \sin \theta \lambda_0 n_{j0} n_{j1} \xi \]

\[ - (1/M) C_{z2} \xi = 0, \]

\[ n_{e3} + n_{e3} - (1/\lambda_0^2) \phi_{\xi} \xi = 0, \]

\[ e^\phi_0 (\phi_{\xi}^2 + \phi_z^2 + \phi_{\theta}^2) \xi + n_{e3} - n_{p3} - (1/\lambda_0^2) \phi_{\xi} \xi = 0. \]

In the above equations, the subscripts \( \xi \) and \( \tau \) represent the respective differentiation and the symbol \( M \) has been used for \( \lambda_0/M \). \( \lambda_0 \) and \( \theta_0 \) are the ion to electron temperature ratio \( (T_e/T_i) \), and \( C \) for \( v \) or \( u \) as per the upper or lower sign, respectively, together with \( j \) or \( n \).

If we look at the physical aspect of the above equations, we realize that the set of zeroth-order equations (4) provides the conditions for the equilibrium of the plasma in the absence of perturbations, and these relations are called self-consistent relations between the zeroth-order quantities. The first-order equations (5) provide the information about the possible modes in the present plasma model, and also we get the relations between the oscillating quantities \( n_{p1}, n_{e1}, n_{n1}, \)

\( v_{a1}, v_{a1}, v_{c1}, u_{c1}, u_{c2}, \) and \( \phi_1 \) when these equations are integrated under the boundary conditions that \( n_{p1}, n_{e1}, n_{n1}, \)

\( v_{a1}, u_{c1}, v_{c1}, u_{c1}, u_{c2}, \) and \( \phi_1 \) go to zero as \( \xi \rightarrow \infty \). The various relations between the first-order quantities yield the following phase velocity relation for the ion acoustic wave:

\[ \lambda_0 = u_0 \sin \theta + v_0 \pm 2 \sigma \sqrt{\frac{n_{p0} + n_{n0}}{n_{p0} - n_{n0}}} \cos \theta. \quad (8) \]

Equation (8) shows that two types of modes are possible in the plasma. Since the velocity corresponding to the +ve sign is larger than the one corresponding to the –ve sign, we call them fast mode \( (\lambda_{0f}) \) and slow mode \( (\lambda_{0s}) \), respectively. This can be seen from the phase velocity relations that the velocity \( \lambda_{0f} \) is always positive, but the velocity \( \lambda_{0s} \) can become negative for some typical values of plasma parameters and the direction of magnetic field. Therefore, the wave propagation angle \( \theta \) requires some minimum value \( (= \theta_{m}) \) in order to keep \( \lambda_{0s} \) positive, which is given by

\[ \theta_m = \tan^{-1} \left\{ \frac{2 \sigma \pm \sqrt{[(n_{p0} + n_{n0})/(n_{p0} - n_{n0})] - u_0}}{u_0} \right\}. \quad (9) \]

The above condition reveals that the slow mode would propagate only if the magnetic field is applied at an angle larger than \( \theta_m \) for the given parameters of the plasma. The angle \( \theta_m \) is affected by the temperatures of the positive and negative ions in addition to their densities and drift velocities, and it gets larger in the presence of thermal motions of the ions (Fig. 1). When we compare the three cases of \( n_{p0} < n_{p0}, n_{n0} > n_{n0}, \) and \( n_{p0} > n_{n0} \) in the figure, we obtain that the effect of ion thermal motion is more significant when \( n_{p0} > n_{n0} \). It can also be seen from Eq. (9) that the angle \( \theta_m \) attains lower values for \( n_{p0} = 0 \), i.e., in the absence of negative ions. This means that the role of negative ions is to enlarge the minimum angle for the propagation of the slow modes.
Figure 2 shows that the effect of thermal motions of the ions is opposite in the phase velocities of the fast and slow modes: \( \lambda_{OF} \) gets increased whereas the velocity \( \lambda_{OS} \) gets decreased for the higher thermal energies of the ions. Here also, the phase velocities change faster with ion thermal energy when \( n_{n0} < n_{p0} \) (please see the slopes of the graphs). The increase in the velocity \( \lambda_{OF} \) with the ion thermal energy can be explained on the basis of restoring force acting on the ions during oscillations. Actually, the restoring force would be larger in the presence of ion thermal motions, because of the increased oscillations. Therefore, for fixed wavelength (or k) oscillations, the phase velocity \( \omega/k \) of the fast mode would be larger due to the increased \( \omega \), and the frequency \( \omega \) would be lower for the slow mode, as per the normal mode analysis. On the other hand, a comparison of \( \lambda_{OF} \) for the three cases of \( n_{n0} < n_{e0}, n_{n0} = n_{p0}, \) and \( n_{n0} > n_{e0} \) reveals that the fast mode attains larger velocity when more negative ions are present in the plasma. This behavior of the fast mode with negative ion density is similar to that observed by Nakamura and Tsukabayashi\(^6,27\) in an unmagnetized plasma. Finally, it can be seen from Eq. (8) that the velocity \( \lambda_{OF} \) (\( \lambda_{OS} \)) gets lower (larger) in the absence of negative ions \( (n_{n0} = 0) \). This means that the role of negative ions is to enhance (reduce) the phase velocity of the fast (slow) mode.

In the next section, we will derive the relevant modified KdV (mKdV) equation on the basis of which we will analyze whether both types of modes evolve as solitons, and rarefactive solitons also occur in such a plasma model at the critical density of negative ions.

**IV. mKdV EQUATION WITH AN ADDITIONAL TERM AND ITS SOLUTION**

With the help of Eqs. (4)–(7), we obtain an equation in the oscillating electrostatic potential \( \phi \) as

\[
A_1 \phi + A_2 \phi_{1+} + A_3 \phi_{1+} + A_4 \phi_{1+} + A_5 \phi_{1+} n_{p0} + A_6 (\phi_{1} + \phi_{2}) = 0.
\]

Here, various coefficients are given by

\[
A_1 = -\frac{2 T_0 \lambda_0^2 n_{p0}(1 + r_0)}{D_0^4 \cos \theta},
\]

\[
A_2 = -\frac{1}{\lambda_0^2} \left[ 1 + \frac{T_0^2 n_{p0}(1 + r_0) \sin^2 \theta}{4 \sigma D_0^4 B_{L0}^2 \eta_m \eta_p} \right],
\]

\[
A_3 = -\frac{n_{p0}(1 - r_0)}{D_0^4} \left[ \frac{2 \sigma (D_0^4 - 1)}{T_0^2} + \frac{4 \sigma}{D_0^4} + 12 \left( \frac{T_0}{D_0^4} \right)^4 \right],
\]

\[
A_4 = n_{p0} - n_{e0} - \frac{n_{p0}}{D_0^4} + \frac{n_{n0}}{D_0^4},
\]

\[
A_5 = \left( \frac{T_0}{D_0^4} \right)^2 \left[ \lambda_0 - M(u_0 \sin \theta + v_0 \cos \theta) \right]
\]

\[
- \lambda_0 \left[ \frac{2 \sigma \lambda_0}{T_0^2} \right],
\]

\[
A_6 = (n_{p0} - n_{e0}) \left[ 1 - \frac{3}{D_0^4} + \frac{2 \sigma}{T_0^2} \left( 1 - \frac{1}{D_0^4} \right) \right].
\]

Together with

\[
r_0 = \frac{n_{n0}}{n_{p0}}, \quad D_0^2 = \left( \frac{1 + r_0}{1 - r_0} \right), \quad T_0^2 = D_0^2 + 2 \sigma.
\]

It may be noted that the coefficient \( A_1 \) of \( \phi_{1+} \) becomes zero when we set the value of \( D_0^2 \). Further, in view of the critical density of negative ions, given by Eq. (1), the coefficient \( A_6 \) of the last term \( \phi_{1+} \) vanishes. With this, we obtain the following mKdV equation that has an additional term appearing due to the density gradient:

\[
\phi_{1+} + c_d \phi_{1+} + c_s \phi_{1+} + c_g \phi_{1+} n_{p0} = 0.
\]

**Coefficient of nonlinear term**

\[
c_n = \frac{A_1}{A_1} = \frac{\cos \theta}{4 T_0 \lambda_0^2 D_0^4} \left[ \frac{2 \sigma (D_0^4 - 1)}{T_0^2} + \frac{4 \sigma}{D_0^4} + 12 \left( \frac{T_0}{D_0^4} \right)^4 \right].
\]

**Coefficient of dispersive term**

\[
c_d = \frac{A_2}{A_1} = \frac{D_0^4 \cos \theta}{2 T_0 \lambda_0^2 n_{p0}(1 + r_0)} \left[ 1 + \frac{T_0^2 n_{p0}(1 + r_0) \eta_m \eta_p \sin^2 \theta}{D_0^4 B_{L0}^2 \eta_p} \right].
\]

**Coefficient of density gradient term**

\[
c_g = \frac{A_3}{A_1} = \frac{D_0^4 \cos \theta}{2 \lambda_0^2 n_{p0}} \left[ \lambda_0 - M(u_0 \sin \theta + v_0 \cos \theta) \right] - \frac{\lambda_0 \phi_{1+}}{T_0^2} \left[ \frac{2 \sigma \lambda_0}{T_0^2} \right].
\]

Now we analyze whether the coefficient \( c_n \) of the nonlinear term still vanishes at some negative ion density or is nonvanishing for all values of negative to positive ion density ratio \( r_0 \). Since \( r_0 < 1 \Rightarrow D_0 > 1 \Rightarrow (D_0^4 - 1) > 0 \) and other terms are always positive, the coefficient \( c_n \) is nonvanishing. Moreover, the above mKdV equation has been obtained by using the condition of the critical negative ion density. This means that this equation will govern the soliton behavior at the critical negative ion density in the plasma. In order to
study these solitons, we solve the mKdV equation (11) and use the transformation \( \phi_j = \Gamma(\tau) \phi^{(1)}(\xi, \tau) \) together with \( \Gamma(\tau) = e^{-\Gamma c_\eta \eta d \tau} \) for reducing it to the usual form of the mKdV equation,

\[
\phi^{(1)}(\tau) + \Gamma^2 c_n \phi^{(1)}^2 \phi^{(1)}_{\xi} + c_d \phi^{(1)}_{\xi \xi} = 0. \tag{12}
\]

This equation carries variable coefficients \( \Gamma, c_n, \) and \( c_d \). Therefore, it cannot be solved by the usual integration process. In view of this, we employ the method suggested by Yan\(^{29} \) and write the solution in the following form:

\[
\Gamma^2 c_n V C_1(C_0^2 + S_1^2) - C_1 - 2 C_2 V^3 C_1 W^2 + 2 \Gamma^2 c_n V C_0 C_1 S_1 \sin \psi + [2 \Gamma^2 c_n V C_0(C_1^2 - S_1^2)] \cos \psi + [S_1 + \Gamma^2 c_n V S_1(2 C_1^2 - C_0^2 - S_1^2) + 5 c_n V^3 S_1 W^2] \sin \psi \cos \phi - 4 \Gamma^2 c_n V C_0 C_1 S_1 \sin \psi \cos^2 \psi + [C_1 + \Gamma^2 c_n V C_1(C_0^2 - C_1^2) - 4 S_1^2] + 8 c_n V^3 C_1 W^2 \cos^2 \psi \\
- 2 \Gamma^2 c_n V C_0(C_1^2 - S_1^2) \cos^3 \psi + [\Gamma^2 c_n V S_1(S_1^2 - 3 C_1^2) - 6 c_n V^3 S_1 W^2] \sin \psi \cos^3 \psi + [\Gamma^2 c_n V C_1(3 S_1^2 - C_1^2)] \\
- 6 c_n V^3 C_1 W^2 \cos^3 \psi = 0. \tag{15}
\]

From the above equation, we separate out the terms containing \( \sin \psi \cos^m \psi, \cos^{m+1} \psi, \) etc., where \( m = 0, 1, 2, \) and \( 3 \). By setting the coefficients of these terms and the constant term separately equal to zero, we solve for obtaining the real values of \( C_0, C_1, S_1, \) and \( W \). The calculated values are given by

\[
C_0 = 0, \quad C_1 = 0, \quad S_1 = \pm \sqrt{ \frac{W^2 + 5 c_n V^3}{W^2 \Gamma^2 c_n V} }, \quad \text{and} \quad \frac{c_n V^3}{W^2} = 1.
\]

Along with these values,\(^{29} \) we arrive at the following soliton solution of the mKdV equation (12):

\[
\phi^{(1)}(\xi, \tau) = \pm \sqrt{ \frac{W^2 + 5 c_n V^3}{W^2 \Gamma^2 c_n V} } \sech \left[ \frac{(\tau - V \xi)}{\sqrt{c_n V^3}} \right]. \tag{16}
\]

From this solution, the peak soliton amplitude (say \( \phi_m \)) and the soliton width \( W \) are given as

\[
\phi_m = \sqrt{ \frac{W^2 + 5 c_n V^3}{W^2 \Gamma^2 c_n V} }, \quad W = \sqrt{c_n V^3}.
\]

In addition to the soliton amplitude and width, the energy is an important propagation characteristic of the soliton. For the weakly inhomogeneous plasma, we can calculate the soliton energy \( E_m \) by treating the zeroth-order quantities as the slowly varying functions and using the following integration\(^{30} \) along with \( \chi = (\tau - V \xi) \):

\[
E_m = \int_{-\infty}^{\infty} [\phi^{(1)}(\chi)]^2 d\chi = 2 \left( \frac{W^2 + 5 c_n V^3}{\Gamma^2 W c_n V} \right) = \frac{12 \sqrt{c_n V}}{\Gamma^2 c_n}.
\]

V. DISCUSSION AND RESULTS

The soliton solution (16) reveals that there are two types of solitons that carry positive and negative amplitudes of equal magnitudes, whose profiles in the oscillating density \( n_{p1} \) are recognized as the density hump and the density dip (Fig. 3). Therefore, corresponding to the \(+\)ve sign (density hump) and \(-\)ve sign (density dip) in Eq. (16), the compressive solitons and the rarefactive solitons of the same amplitude occur in the present plasma model. Since the solution (16) is valid for both the fast and slow modes, the compressive and rarefactive solitons are evolved for both types of the modes.

Figure 4 discusses the effect of ion thermal energy density on the peaks \( \phi_{mF} \) and \( \phi_{mS} \) of the fast and slow
solitons corresponding to the fast and slow modes, respectively. Since the compressive and rarefactive solitons carry the same amplitudes, this figure also represents the amplitudes of these two types of the solitons. It is obvious from the figure that both the amplitudes $\phi_{nf}$ and $\phi_{ms}$ go down for the larger ion thermal energies. A comparison of the cases $n_{i0} < n_{e0}$, $n_{i0} = n_{e0}$, and $n_{i0} > n_{e0}$ infers that the decrease of amplitude with the ion thermal energy is faster in the case of $n_{i0} > n_{e0}$. Also, the fast (slow) solitons evolve with higher (smaller) amplitudes for the higher density of negative ions in the plasma. This variation of the amplitudes with $n_{i0}$ is consistent with Fig. 2 and also is in agreement with the result of Nakamura and Tsukabayashi.\(^6\) On the other hand, the widths $W_F$ and $W_S$ of the fast and slow wave solitons in Fig. 5 show the opposite behavior for the cases of $n_{i0} < n_{e0}$ and $n_{i0} > n_{e0}$. The variation of the amplitudes and widths with the ion thermal energy can be explained on the basis of nonlinear and dispersive properties of the plasma. It seems plausible that the plasma would be comparatively weakly nonlinear when the plasma species, i.e., the ions, have finite thermal motions. This will cause the solitons to evolve with lower amplitude, as the nonlinearity tries to steepen the solitary waves. Furthermore, since the solitons occur when there is a balance between the nonlinear and dispersive effects, the lower dispersion would be required to balance the weaker nonlinearity in the presence of ion thermal motions (energies). Therefore, the soliton width gets decreased with the larger ion thermal energies in the presence of lower dispersion. Finally, a comparison between the fast and slow wave solitons in Figs. 4 and 5 yields that the fast wave solitons evolve with bigger size compared with the slow wave solitons.

Figure 6 shows the effects of gyromotion of the ions and the obliqueness of the magnetic field $B_0$ on the width $W_F$ and $W_S$ of the fast and slow wave solitons, respectively. Our analysis shows that the soliton amplitude does not show the dependence on the magnetic field or the gyromotion of the ions. A similar result was obtained by El-Labany and El-Shamy\(^18\) for solitary wave structures in a hot magnetized dusty plasma, and by Mishra \textit{et al.}\(^3\) in a negative ion containing homogeneous plasma. However, the effect of gyromotion of the ions is to broaden both types of solitons (Fig. 6), which is attributed to the modified dispersive properties of the plasma in the presence of magnetic field. Also, this effect is more pronounced on the fast compressive and fast rarefactive solitons. When we investigate the effect of magnetic field obliqueness $\theta$ on $W_F$ and $W_S$, the effect is found to be opposite: The width $W_F$ of the fast soliton gets decreased and the width $W_S$ of the slow wave soliton gets larger in the presence of larger angle $\theta$. In addition, the gaps or the slopes of the graphs in Fig. 6 show that the effect of obliqueness of the magnetic field on the solitons is more significant when the ions gyrate with larger frequencies, i.e., for the stronger magnetic field.

In Fig. 7, we show the dependence of energy $E_{mF}$ and $E_{ms}$ of the fast and slow wave solitons, respectively, on the ion gyration frequency, the obliqueness of magnetic field, and the ion thermal energy. Clearly, the energy of both types of solitons gets increased for the larger gyration frequency of the ions, and the fast wave solitons propagate with larger energy in comparison with the slow wave solitons. When we compare the graphs marked 75(0.3) and 75(0.6), we find that the effect of ion thermal motion is to reduce the energy of both solitons. This variation of the energy is consistent with
FIG. 7. Dependence of energy $E_{mf}$ and $E_{ms}$ of fast and slow wave solitons, respectively, on ion gyration frequency, obliqueness of magnetic field, and ion thermal energy, when $n_{e0}=0.1$, $T_{e0}=0.3$ eV, and other parameters are the same as in Fig. 5. Graphs marked with 75(0.3), 75(0.6), and 77(0.3) correspond to the cases of $\theta=75^\circ$ and $T_{e0}=0.3$ eV, $\theta=75^\circ$ and $T_{e0}=0.6$ eV, and $\theta=77^\circ$ and $T_{e0}=0.3$ eV.

Figs. 4 and 5, which show that both solitons evolve with smaller amplitudes and widths under the effect of ion thermal motions. Finally, a comparison of the graphs marked with 75(0.3) and 77(0.3) yields that the fast and slow wave solitons behave oppositely with the obliqueness of the magnetic field: The energy $E_{mf}$ is decreased for the larger obliqueness but the energy $E_{ms}$ gets increased. Also, it can be inferred from the figure that the energy $E_{ms}$ is more sensitive to the obliqueness of the magnetic field.

In addition to the above results, we can investigate the role of negative ions on the soliton propagation characteristics by setting $n_{n0}=0$. Under this situation, we find for the fast (slow) wave solitons that the coefficient of nonlinearity $c_{n0}$ increases (decreases) whereas the coefficient of the dispersive term $c_{d0}$ decreases (increases), which means the amplitude $\phi_{nf}$ ($\phi_{ms}$) gets lower (higher) and the width $W_{f}$ ($W_{s}$) gets broadened in the absence of negative ions. Therefore, it is concluded that the negative ions play an opposite role in the evolution of the fast and slow wave solitons. In the presence of negative ions, the fast wave solitons evolve with higher amplitude, lower width, and larger energy (see the expression of $E_{nf}$). On the other hand, it was observed during the calculations that the slow wave solitons do not evolve with noticeable amplitude and width when the magnetic field is applied at an angle close to $\theta_{nf}$. Also, when we consider the situation in which the plasma has only positive and negative ions of equal densities, i.e., when there are no electrons, the phase velocities and the amplitudes of the solitons attain infinitely large values. This means the nonlinearity and dispersion are not balanced in the plasma that has purely positive and negative ions of equal masses and temperatures. Further investigations show that the solitons do not occur in the plasma when the wave propagates in the direction perpendicular to the magnetic field as $c_{n0} \rightarrow 0$, $c_{d0} \rightarrow 0$ when $\theta \rightarrow 90^\circ$.

Finally, we check the authenticity of the calculations by examining the limiting cases of the present analysis. It is seen for $\sigma=0$ that the phase velocity relations and the mKdV equation reduce to the respective equations of Singh and Malik for a cold magnetized plasma. Furthermore, by setting $\theta=0^\circ$, we can reproduce the model of Chauhan and Dahiya for an unmagnetized cold plasma. It is fascinating here that the zeroth-, first-, second-, and third-order equations and, therefore, the mKdV equation reduce to those respective equations reported by them. Further limiting cases reproduce the results of earlier workers (see references given in Ref. 17). Moreover, when we calculate the critical negative ion density ratio $r_{0}$ from Eq. (1) for $\sigma=0$, it is obtained as 0.268, which is exactly the same as obtained by Yi et al. for $m_{p}/m_{n}=1$. Therefore, these comparisons and the limiting cases substantiate the generality of our calculations.

On the basis of more realistic problems, we have obtained some additional results in comparison with others’ investigations. For example, in Ref. 17, Chauhan and Dahiya could not calculate the width of the soliton and were unable to investigate the effect of negative ion density on it. Interestingly, we have obtained the complete solution of the mKdV equation and could investigate the behavior of compressive and rarefactive mKdV solitons corresponding to both the fast and slow modes. The occurrence of the slow wave solitons and a limit on the obliqueness of the magnetic field for their existence is also additional information, which could not be exposed in earlier works. Moreover, the evolution of rarefactive solitons for both types of modes, the nonvanishing coefficient of the nonlinear term of the mKdV equation, and the different behavior of the compressive and rarefactive solitons for the cases of $n_{e0}<n_{n0}$, $n_{e0}=n_{n0}$, and $n_{e0}>n_{n0}$ are additional investigations in comparison with those of Ref. 12.

VI. SUMMARY

The present analysis infers that two types of modes, namely fast and slow modes, evolve as mKdV solitons and that each mode corresponds to the compressive and the rarefactive solitons of the same amplitudes. However, there is a limit on the angle $\theta (=\theta_{nf})$ for the existence of the slow wave solitons. The three cases discussed in the present article, $n_{e0}<n_{n0}$, $n_{e0}=n_{n0}$, and $n_{e0}>n_{n0}$, could reveal interesting results, i.e., that the effect of ion thermal motion is more significant on the angle $\theta_{m}$ and the phase velocities $\lambda_{nf}$ and $\lambda_{ns}$ when the negative ions in smaller numbers are present in the plasma. On the other hand, the decrease of soliton amplitude and width with the ion thermal energy is faster in the case of $n_{e0}>n_{n0}$. The effect of the gyrotary motion of the ions was found to broaden both types of solitons, and this effect is more pronounced on the fast compressive and fast rarefactive solitons. In addition, the effect of obliqueness of the magnetic field on the solitons is more prominent when the ions gyrate with larger frequencies, i.e., for the stronger magnetic field. Both types of solitons evolve with larger energy under the effect of stronger magnetic field.

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