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Mode Selection Rules for Two-Delay Systems: Dynamical Explanation for the Function of the Register Hole on the Clarinet

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Generally, time-delay systems are regarded as multi-attractor systems. We investigate mode selection rules for two-delay systems considering which oscillation mode is first excited by the Hopf bifurcation with increasing a bifurcation parameter. In particular, we focus on the case that the strength of the short time delay $\alpha_1$ is lower than that of the long time delay $\alpha_2$. In a certain range of $\alpha_1/\alpha_2$ in which it is sufficiently small but still not negligible, the third-harmonic mode occupies a particular range of the ratio of the two delay times such that $2 < t_{R2}/t_{R1} < 4$, where $t_{R1}$ and $t_{R2}$ denote the short and long delay times, respectively. This is the key for understanding the function of the register hole on the clarinet, which is smaller in radius than the other tone holes, but works well to raise the pitch of first register notes in a wide range more than an octave by a twelfth (19 semitones), i.e., generating third harmonics, when opened. This is confirmed using a simple model of the clarinet with two delays: short and long time delays are regarded as delayed reflections caused by the register hole and the open end of the pipe, respectively. The working range of the register hole roughly corresponds to the particular range of the third-harmonic mode for two-delay systems ($2 < t_{R2}/t_{R1} < 4$).

1. Introduction

In the last quarter of a century, time-delay systems have attracted many authors’ attention, because they occur in a wide variety of physical, chemical, engineering, economic, and biological systems, and the delay plays a crucial role in understanding the underlying mechanics of those systems.\textsuperscript{1–4)} Even in the case of two-delay and multiple-delay systems, a large number of papers have been published in a wide range of research areas from pure mathematics to practical application in individual problems.\textsuperscript{5–15)} From viewpoints of mathematics and nonlinear physics, the stability of static solutions and the mechanism of Hopf bifurcation from a static solution to a vibrating mode have been extensively studied and there are many fruitful results.\textsuperscript{5,7,9–12,15)
Generally, time-delay systems, and of course two-delay systems, are considered as multi-attractor systems. There still exist open problems on what type of oscillation mode, periodic, quasi-periodic or chaotic, is excited with the change in a control parameter after the Hopf bifurcation and how many attractors coexist in the entire phase space for given parameters. As shown in Refs.5-7, even for one-dimensional differential equations with two delays, e.g., the model of a compound optical resonator with competing boundary conditions, selection rules for exciting modes are very complicated, sensitively depending on the ratio of two-delay times. To the best of the authors’ knowledge, the mode selection rules for multiple delay systems still have not been completely understood.

In the field of musical acoustics, it is well known that wind instruments, such as the clarinet, oboe, and trumpet, are modelized by differential and difference equations with a time delay.\textsuperscript{16–22)} Indeed, the reed valve of the instrument plays the role of a sound generator and drives a resonance pipe.\textsuperscript{16–19)} The acoustic properties of the resonance pipe are characterized by a response function, called the reflection function, which induces time delay operating on the reed valve. The reflection function changes with the shape of the bore.\textsuperscript{16)} For example, when one of the tone holes of a woodwind instrument is opened, it adds an additional delay peak to the reflection function, which has at least one delay peak caused by the open-end reflection. Therefore, the models for the woodwind instruments with tone holes are regarded as multiple-delay systems.

In this work, we focus on the particular tone hole on the clarinet called the register hole (key), which is used to play in the second register.\textsuperscript{18)} Namely, it raises the pitch of most first-register notes by a twelfth (19 semitones), i.e., generating third harmonics, when opened: for the B♭ clarinet, each note from D3 (147.5 Hz) to D4# (312.5 Hz) in the first register changes to a corresponding note from A4 (442 Hz) to A5# (936.6 Hz) in the second register. The register hole is placed at a distance of \( l_r \approx 0.14 \) m from the tip of the mouthpiece and is about 1.5 mm in radius, which is much smaller than those of the other tone holes (2.5-6.0 mm).

To consider the function of the register hole, we introduce a simple model formed by a closed pipe with only a tone hole (register hole), which is driven by a sound generator attached to the closed end (see Fig. 1). The pipe length \( l_p \) is adjusted to generate the pitch of a given note in the first register. Namely, it is calculated as \( l_p = c_0/(4 f_n) \), where \( c_0 \) is the sound speed and \( f_n \) is the frequency of the note. Since the sound speed \( c_0 \) is 340 m/s at a temperature of 288 K, the pipe length \( l_p \) is estimated as 0.576 and 0.272 m for the notes D3 and D4#, respectively. Therefore, the ratio of \( l_p \) to \( l_r \) has a range of 1.9 ≤ \( l_p/l_r \) ≤ 4.1.

The pipe with a tone hole is analogous to a two-delay system. The round trip between
the closed end and the tone hole, Loop 1, makes a short delay time \( t_{R1} = \text{Loop 1}/c_0 \), while the round trip between the closed and open ends, Loop 2, causes a long delay time \( t_{R2} = \text{Loop 2}/c_0 \). The sounding mechanism of single-reed instruments, such as a clarinet, is well explained by a simple model, which consists of a cylindrical pipe fitted with the reed valve (sound generator).\(^{17–19}\) Thus, delay models can be used to study the basic oscillation mechanism of single-reed instruments, and the two-delay model is regarded as the model of the clarinet when one of the tone holes is opened.

Let us consider a pipe with a small tone hole, i.e., the short time delay with weak intensity [see Fig. 1(a)]. If the tone hole is placed at 1/3 length of the pipe from the closed end [top picture of Fig. 1(a)], the third-harmonic mode satisfies the boundary condition of this pipe: one of the antinodes exists at the tone hole and the other exists at the open end. Even if the length of the pipe is changed so as to put the tone hole in the middle of the pipe (middle picture) and at the quarter (bottom picture), the third-harmonic mode can still be excited in the pipe, while the antinode shifts from the tone hole to the left and right, respectively. Indeed, when the tone hole is sufficiently small, it does not strictly require that one of the antinodes exists on it, but it still stimulates the excitation of the third-harmonic mode rather than the fundamental and other higher harmonic modes, even though the antinode strays off the tone hole to some extent. This really occurs for the clarinet as the function of the register hole. Indeed, the register hole works for a wide range of registers more than an octave. Since the change in fingering is regarded as the change in the effective length of the pipe, then the pipes with a small tone hole [Fig. 1(a)] likely explain the function of the register hole.
However, in the case of pipes with a large tone hole, i.e., the short time delay with a high intensity [see Fig. 1(b)], it makes a different choice of oscillation modes. If the tone hole is placed at 1/3 length of the pipe [top picture of Fig. 1(b)], the third-harmonic mode still satisfies the boundary condition of this pipe. However, the large tone hole strongly requires that oscillation modes satisfy the boundary condition caused by it: one of the antinodes is placed on it. In the cases of the tone hole in the middle of the pipe (middle picture) and at the quarter (bottom picture), it is necessary that even harmonic modes are excited, e.g., second-harmonic mode and fourth-harmonic mode, respectively. Since even-harmonic modes are prohibited owing to the boundary condition at the open end, the system falls into frustration owing to the conflicting requirements by the mismatched boundary conditions. Thus, it is anticipated that higher harmonic modes, which nearly satisfy both boundary conditions at the tone hole and at the open end, will be accepted reluctantly. The above discussion indicates that the register hole should be sufficiently small but not negligible. Actually, it is smaller in radius than the other tone holes. This intuitive explanation seems to describe well the characteristic and function of the register hole.

In this work, we give theoretical and numerical lines of evidence for this intuitive explanation by a two-delay model, which is appropriate for theoretical analyses, and a simple clarinet model with two delays, which is used for studying the sounding mechanism of single-reed instruments in the field of musical acoustic. The organization of this paper is as follows.

In Sect. 2, we introduce a two-delay model, which is essentially the same as the optical resonator model with two delays, although for convenience of analysis, our model is simplified in bifurcation compared with the optical resonator model. We briefly explain the selection rule of oscillation modes generated by the bifurcation based on the results given by Ikeda and Mizuno.

In Sect. 3, we provide a theoretical analysis of the two-delay model. We study the selection rules of oscillation modes as a function of the ratio of the two delay times, $t_{R1}/t_{R2}$ or $t_{R2}/t_{R1}$, where $t_{R1}$ and $t_{R2}$ denote the short and long delay times, respectively. In the case that the two delays have the same strength, our theoretical results are essentially the same as those given by Ikeda and Mizuno. We explore the selection rules of oscillation modes in the case that the strength of the short time delay becomes increasingly smaller, focusing on the excitation range in $t_{R1}/t_{R2}$ of the third-harmonic mode concerned with the function of the register hole.

In Sect. 4, we show the results of the numerical calculation of the two-delay model to
confirm the theoretical analysis provided in Sect. 3.

In Sect. 5, we introduce a delay difference model of the clarinet, which has a cylindrical pipe with only a register hole, namely, a simplified clarinet model with two time delays. In this simple model, changing notes is represented by changing the pipe length, although the distance of the register hole from the mouthpiece tip is fixed. The clarinet model has essentially the same characteristics as the two-delay model, and we obtain theoretical and numerical explanations for the function of the register hole. Sect. 6 is devoted to summary and discussion.

2. Model System

In this section and the following two sections, we treat the simplified model with two delays instead of the model of the clarinet. This is because this two-delay model is suitable for the theoretical analysis, i.e., the linear stability analysis, while the clarinet model is slightly complicated and difficult to treat analytically. However, both models suffer negative delay feedback, and their specific forms with one delay, more precisely, their reductions to 1D maps, undergo the period-doubling bifurcation. Therefore, it is expected that the two models have essentially the same properties.

2.1 Two-delay model system

The model system with two delays, which we treat in this paper, is

\[
\frac{1}{\gamma} \frac{dx}{dt} = -x + \mu[\alpha_1 f[x(t-t_{R1})] + \alpha_2 f[x(t-t_{R2})]],
\]

where \( \mu \) is a control parameter, \( \gamma \) is a decay constant, \( t_{R1} \) and \( t_{R2} \) denote two delay times set as \( t_{R1} \leq t_{R2} \); the parameters \( \alpha_1 \) and \( \alpha_2 \) (0 \( \leq \alpha_1 \leq \alpha_2 = 1 \)) determine the strength of individual delays, and the function \( f(x) \) is given by

\[
f(x) = \exp\left(-\frac{(x-x_0)^2}{\Delta x^2}\right). \tag{2}
\]

The formal solution of Eq. (1) is given by a delay integral equation:

\[
x(t) = \sum_{i=1}^{2} \int_{-\infty}^{t} \gamma \mu \alpha_i e^{-\gamma(t-s)} f[x(s-t_{Ri})] \, ds = \sum_{i=1}^{2} \int_{0}^{\infty} \mu \alpha_i G(t' - t_{Ri}) f[x(t - t')] \, dt' \tag{3}
\]

where the change in variables \( t' = t - (s - t_{Ri})(i = 1, 2) \) is utilized. The response function \( G \) is defined by

\[
G(t - t_{Ri}) = \gamma e^{-\gamma(t-t_{Ri})} H(t - t_{Ri}), \tag{4}
\]
where $H(t)$ is the Heaviside function.

In the case of $\alpha_1 = 0$, $\alpha_2 = 1$, and $\gamma \gg 1$, Eq. (3) is approximately reduced into a map, $x(t) \approx \mu f[x(t - t_{R2})]$. Hence, the map $x_{n+1} = \mu f(x_n)$ governs the basic properties of bifurcation of the delay system. If the parameters $x_0$ and $\Delta x$ are set at $x_0 = 0.2$ and $\Delta x = 0.5$, the solution of the map bifurcates only once with increasing $\mu$: the stable fixed point $x = x_c$ is bifurcated into period-2 points, where $x_c = \mu f(x_c)$ (see Fig. 2.1). In this work, we adopt this choice of parameter values. When the reduced map undergoes the period-doubling bifurcation, the slope of the tangent at the fixed point is $-1$, namely, the eigenvalue of the 1D Jacobi matrix is $-1$. This means that the original delay system effectively suffers negative feedback. As will be shown in the following sections, the two-delay system has a solution that turns from a stationary solution into an oscillating mode at a certain $\mu$ when it is increased.

In this work, we consider the following two cases. In Case I, $t_{R2}$ is fixed at 1 and $t_{R1}$ takes values in the range of $0 < t_{R1} \leq t_{R2}$. On the other hand, in Case II, $t_{R1}$ is fixed at $1/3$ and $t_{R2}$ takes values equal to or larger than $t_{R1}$, i.e., $t_{R2} \geq t_{R1}$. The decay constant $\gamma$ is taken at $\gamma = 210$ and 105 in both cases. As will be discussed in Sect. 5, Case II corresponds to the model of the clarinet, with which we will consider the function of the register hole.

### 2.2 Oscillation modes

We briefly explain the selection rules of oscillation modes generated by the bifurcation, which are observed for systems of two delays with the same strength, i.e., $\alpha_1 = \alpha_2$, based on the results given by Ikeda and Mizuno. Since the system suffers the negative delay feedback, odd harmonic components hold a dominant position over even harmonic components in an oscillation mode. In other words, the even harmonics are suppressed.

Let us consider the selection rule that specifies an oscillation mode for a given ratio of
the two delay times, $t_{R1}/t_{R2}$. There are two boundary conditions that should be satisfied by oscillation modes. That is, for each delay $t_{Ri}$ ($i = 1, 2$), the oscillation mode with a period $T_i$ such that $T_i \in P_i = \{ T_i | T_i = 2t_{Ri}/(2l + 1), l = 0, 1, 2, \cdots \}$ is allowed. Therefore, the oscillation mode with a period $T \in P_1 \cap P_2$ must be excited to satisfy the two boundary conditions. Normally, the mode of the largest period $T_{\text{max}} = \max \{ T | T \in P_1 \cap P_2 \}$ is the most stable oscillation and should be chosen to be the excited mode. When the ratio $t_{R1}/t_{R2}$ is an irreducible fraction of two odd numbers $m_1$ and $m_2$, i.e., $t_{R1}/t_{R2} = m_1/m_2$, $T_{\text{max}}$ is given as $T_{\text{max}} = 2t_{R1}/m_1 = 2t_{R2}/m_2$. Namely, the $m_2$-th-harmonic mode is excited. On the other hand, there is no periodic solution to satisfy the two boundary conditions for a ratio of even to odd (odd to even) numbers as well as for an irrational ratio, namely, a mismatch of the two delay times. Thus, the system falls into frustration owing to the contradictory requirements by the two boundary conditions, and it seems impossible to find a periodic oscillation mode.

However, the existence of the relaxation time ($\propto 1/\gamma$) relaxes the selection rule of oscillation modes. There is a cutoff frequency determined by $\gamma$, $t_{R1}$, and $t_{R2}$; generally, it is proportional to $\sqrt{\gamma(t_{R1} + t_{R2})}$. Hence, an odd-harmonic oscillation with a frequency $f_{\text{ap}}$ just below the cutoff can be excited for mismatched boundary conditions. Namely, there exists an odd-harmonic oscillation, which approximately satisfies both boundary conditions, $f_{\text{ap}} = m_2/2t_{R2} \approx m_1/2t_{R1}$, where $m_1$ and $m_2$ are odd integers. Even in the case that a higher odd-harmonic oscillation is obtained by the rigorous selection rule for an irreducible fraction of large odd numbers, $t_{R1}/t_{R2} = m_1/m_2$, it should be replaced with a lower odd harmonic oscillation with a frequency below the cutoff, which is obtained from two smaller odd numbers, $m_1'$ and $m_2'$, whose ratio still approximates $t_{R1}/t_{R2}$ well.

As a result, in the neighborhood of the fraction of small odd numbers, i.e., $t_{R1}/t_{R2} \approx \text{odd/odd}$, the odd harmonic determined by the fraction is observed, but the size of the neighborhood decreases with increasing $\gamma$, because the cutoff frequency increases with $\gamma$. On the other hand, in the neighborhood of the fraction of small odd and even numbers, i.e., $t_{R1}/t_{R2} \approx \text{odd/even or even/odd}$, odd harmonics of extremely higher orders but still below the cutoff are observed and they form a characteristic structure around it, called the tower structure: the tower becomes taller with increasing $\gamma$.

As shown in Fig. 1, the pipe with a tone hole driven by an acoustic generator at the closed end is regarded as a two-delay system, and the same selection rule of oscillation modes should be applied to the pipe with the large tone hole. As shown in Fig. 1 (b), when the ratio of the distance of the tone hole from the closed end to that of the open end is given by the ratio of small odd numbers, $m_1/m_2$, the $m_2$-th-harmonic mode is observed. On the other hand, if the...
ratio is given by even/odd or odd/even, a higher harmonic mode must be observed. However, for a pipe with a small tone hole, corresponding to the case of \( \alpha_1 \ll \alpha_2 \), the third harmonic should be excited in the wide range of the ratio, \( 1/4 < t_{R1}/t_{R2} < 1/2 \), as shown in Fig. 1 (a). This is the point that we will discuss in this work. Thus, we consider the selection rules of oscillation modes in the case of a small \( \alpha_1/\alpha_2 \).

3. Theoretical Analysis of Time-Delay Systems

In this section, we theoretically explore the selection rules of oscillation modes with the help of the stability analysis for time-delay systems.\(^{1-5,7,9-12}\)

3.1 Analysis of the system with a single delay

First, let us consider the system with a single delay. In this case, Eq. (1) is reduced to

\[
\frac{1}{\gamma} \frac{dx}{dt} = -x + \mu f[x(t - t_R)]. \tag{5}
\]

At the fixed point \( x = x_c \), the condition \( \dot{x} = 0 \) must be satisfied and it gives

\[
x(t) = \mu f[x(t - t_R)]. \tag{6}
\]

From \( x(t) = x(t - t_R) = x_c \), we obtain

\[
x_c = \mu f(x_c). \tag{7}
\]

As shown in Appendix, \( x_c \) is a monotonically increasing function of \( \mu \) when the parameters of the function \( f(x) \) in Eq. (2) are chosen as \( x_0 = 0.2 \) and \( \Delta x = 0.5 \).

Let us consider the linear stability of the fixed point \( x_c \). A solution starting at a point in the neighborhood of \( x_c \) can be written as

\[
x = x_c + \sum_{n=1}^{\infty} a_n \exp[\lambda_n t + in(\omega t + \theta)], \tag{8}
\]

where \( \omega \) should be taken as \( \omega \approx (2l + 1)\pi/\gamma \), because the Hopf bifurcation is induced by the period-doubling bifurcation on the map \( x_{n+1} = \mu f(x_n) \). In the lowest-order approximation, the solution is reduced to

\[
x = x_c + a_1 \exp[\lambda_1 t + i(\omega t + \theta)]. \tag{9}
\]

Substituting Eq. (9) into Eq. (5) gives

\[
\frac{1}{\gamma} (\lambda_1 + i\omega) \approx -1 + \mu f'(x_c) \exp[-(\lambda_1 + i\omega)t_R], \tag{10}
\]

where \( f'(x) \) denotes the derivative of \( f(x) \). With increasing \( \mu \), \( \lambda_1 \) changes from negative to positive owing to the bifurcation and it becomes zero at the bifurcation point, where it gives
the following equation:

\[ 1 + i \frac{\omega}{\gamma} = \mu f'(x_c) \exp(-i\omega t_R). \] (11)

The real and imaginary parts of this equation are respectively written as

\[ 1 = \mu f'(x_c) \cos \omega t_R, \] (12)

\[ \gamma^{-1} \omega = -\mu f'(x_c) \sin \omega t_R. \] (13)

The above equations determine the values of \( \mu \) and \( \omega \) at the bifurcation point. Since the condition \( f'(x_c) < 0 \) is required for the supercritical Hopf bifurcation, we then obtain

\[ \cos \omega t_R < 0, \] (14)

i.e., \( 2n\pi + \pi/2 < \omega t_R < 2n\pi + 3\pi/2 \). Substituting Eq. (12) into Eq. (13) gives

\[ \gamma^{-1} \omega = -\tan \omega t_R. \] (15)

Equation (15) with Eq. (14) determines \( \omega \).

In the limit \( \gamma \to \infty \), Eq. (15) is reduced to

\[ \tan \omega t_R = 0. \] (16)

From Eqs. (12) and (16), \( \omega \) and \( \mu \) are determined as

\[ \omega = \omega_{2l+1} = \frac{(2l + 1)\pi}{t_R}, \] (17)

\[ \eta = \mu_b f'[x_c(\mu_b)] = \frac{1}{\cos \omega_{2l+1} t_R} = -1, \] (18)

where \( \mu_b \) denotes the \( \mu \) at the bifurcation, and the variable \( \eta \), which indicates the derivative of the map at the fixed point, is introduced. Since \( \mu_b \) is independent of the order of modes, all odd modes with \( \omega = \omega_{2l+1} \) are simultaneously unstabilized at \( \mu = \mu_b \). Then, there is no order of priority among the odd modes.

Let us consider the case that \( \gamma \) takes a large real value such as \( 1 \ll \gamma < \infty \), i.e., \( 0 < 1/\gamma \ll 1 \). To consider a first-order approximation, we put \( \omega \) as

\[ \omega = \omega_n + \delta \omega_n, \] (19)

and Eq. (15) is reduced to

\[ \gamma^{-1}(\omega_n + \delta \omega_n) \approx -\delta \omega_n t_R \tan' \omega_n t_R, \] (20)

where \( \tan' \) denotes the derivative of the tangent function, \( \frac{d}{dx} \tan x \). From the relation that
\[ \tan' \omega_{n}t_{R} = 1 + \tan \omega_{n}t_{R} = 1 \] due to \( \tan \omega_{n}t_{R} = 0 \) at \( n = 2l + 1 \), we obtain
\[ \delta \omega_{n} \approx -\frac{\omega_{n}}{\gamma t_{R} + 1} \approx -\frac{\omega_{n}}{\gamma t_{R}}, \]  
(21)
where we make use of the approximation \( \gamma t_{R} \gg 1 \) in the right-hand-most side. Thus, the angular frequency of the \( n \)th mode is approximated by
\[ \hat{\omega}_{n} \equiv \omega_{n} + \delta \omega_{n} \approx \omega_{n}(1 - \frac{1}{\gamma t_{R} + 1}), \]  
(22)
and \( \eta \) and \( \mu \) at mode bifurcations are given by
\[ \eta_{n} = \mu_{n}f'(x_{c}(\mu_{n})) = \frac{1}{\cos \hat{\omega}_{n}t_{R}} \approx -1 + \frac{1}{2}(\delta \omega_{n}t_{R})^{2} \approx -1 - \frac{1}{2}(\frac{\omega_{n}}{\gamma})^{2}. \]  
(23)
Therefore, the fundamental mode with \( \omega = \hat{\omega}_{1} \) takes the largest \( \eta_{1} \) in the set \( \{\eta_{2l+1} \mid l = 0, 1, 2, \ldots\} \), because \( \eta_{n} \) takes a negative value and decreases with \( n \). From the discussion in Appendix, \( \eta \) monotonically decreases with increasing \( \mu \) in the physically important range so that the fundamental mode takes the smallest \( \mu_{1} \) in the set \( \{\mu_{2l+1} \mid l = 0, 1, 2, \ldots\} \). Then, the fundamental mode is first excited, when \( \mu \) is increased adiabatically.

3.2 Analysis of the two-delay system
3.2.1 Fundamental analysis

The fixed point of the two-delay system (1) is defined by
\[ x_{c} = \mu(\alpha_{1} + \alpha_{2})f(x_{c}), \]  
(24)
then \( \mu(\alpha_{1} + \alpha_{2}) \) corresponds to the \( \mu \) of the single-delay system. Hereafter, we consider the case that \( \alpha_{2} \) is fixed at \( \alpha_{2} = 1 \). The solution near the fixed point can be written as
\[ x \approx x_{c} + a \exp[\lambda t + i(\omega t + \theta)]. \]  
(25)
Substituting Eq. (25) into Eq. (1) provides
\[ 1 + \frac{1}{\gamma}(\lambda + i\omega) \approx \mu f'(x_{c})(\alpha_{1} \exp[-(\lambda + i\omega)t_{R1}] + \alpha_{2} \exp[-(\lambda + i\omega)t_{R2}]). \]  
(26)
At the bifurcation point \( \lambda = 0 \), it is reduced to
\[ 1 + i\omega = \mu f'(x_{c})[\alpha_{1} \exp(-i\omega t_{R1}) + \alpha_{2} \exp(-i\omega t_{R2})]. \]  
(27)
The real and imaginary parts of this equation are respectively written as
\[ \eta = \mu f'(x_{c}) = \frac{1}{\alpha_{1} \cos \omega t_{R1} + \alpha_{2} \cos \omega t_{R2}} < 0, \]  
(28)
\[ \gamma^{-1}\omega = -\mu f''(x_c)(\alpha_1 \sin \omega t_{R1} + \alpha_2 \sin \omega t_{R2}). \] (29)

In Eq. (28), we consider the fact that \( f''(x_c) < 0 \). Substituting Eq. (28) into Eq. (29), we get

\[ \gamma^{-1}\omega = -\frac{\alpha_1 \sin \omega t_{R1} + \alpha_2 \sin \omega t_{R2}}{\alpha_1 \cos \omega t_{R1} + \alpha_2 \cos \omega t_{R2}}. \] (30)

This equation determines the angular frequencies of oscillation modes, which are potentially excited. Let us denote the series of the angular frequencies obtained as \( \tilde{\omega}_1, \tilde{\omega}_3, \tilde{\omega}_5, \ldots \), where \( \tilde{\omega}_{2l+1} < \tilde{\omega}_{2l+3} \). In the limit \( \alpha_1 \to 0 \), each \( \tilde{\omega}_{2l+1} \) converges on \( \tilde{\omega}_{2l+1} \) in Eq. (22) with \( t_R = t_{R2} \). From Eq. (28), we obtain \( \tilde{\eta}_{2l+1} \) for each mode with \( \omega = \tilde{\omega}_{2l+1} \) as a function of \( t_{R1}, t_{R2} \), and \( \alpha_1 \), when \( \alpha_2 \) is fixed. We assume that the mode that is first excited with increasing \( \mu \) takes the largest \( \eta \), i.e., the smallest \( \mu \), and it is given by

\[ \tilde{\eta}_{(2l+1)\text{max}}(t_{R1}, t_{R2}, \alpha_1) = \max_{\tilde{\eta}_{2l+1}}[\tilde{\eta}_{2l+1}(t_{R1}, t_{R2}, \alpha_1)]. \] (31)

where \((2l + 1)_{\text{max}}\) is the mode number of the mode with \( \tilde{\eta}_{(2l+1)\text{max}} \), say, the optimal mode. The normalized frequency of the optimal mode, \( F_m \), is defined by \( F_m(t_{R1}, t_{R2}, \alpha_1) = \tilde{\omega}_{(2l+1)\text{max}}/\omega_1 \) and is approximated as \( F_m(t_{R1}, t_{R2}, \alpha_1) \approx (2l + 1)_{\text{max}} \) when \( \alpha_1/\alpha_2 \ll 1 \) and \( \gamma \gg 1 \).

### 3.2.2 Approximation for \( 1 \ll \gamma < \infty \) and \( \alpha_1 \ll \alpha_2 \)

Let us consider the case that \( 1 \ll \gamma < \infty \) and \( \alpha_1 \ll \alpha_2 \). We decompose the angular frequency \( \tilde{\omega}_n \) as

\[ \tilde{\omega}_n = \omega_n + \delta \tilde{\omega}_n, \] (32)

where \( \omega_n \) is given by Eq. (17) with \( t_R = t_{R2} \); \( \delta \tilde{\omega}_n \) is further divided into

\[ \delta \tilde{\omega}_n = \delta \omega_n + \delta' \omega_n \] (33)

and \( \delta \omega_n \) is given by Eq. (21) with \( t_R = t_{R2} \). Then, Eq. (30) is rewritten as

\[ \gamma^{-1}(\omega_n + \delta \tilde{\omega}_n) \approx -\frac{\alpha_1 \sin(\omega_n + \delta \tilde{\omega}_n)t_{R1} + \alpha_2 \sin(\omega_n + \delta \tilde{\omega}_n)t_{R2}}{\alpha_1 \cos(\omega_n + \delta \tilde{\omega}_n)t_{R1} + \alpha_2 \cos(\omega_n + \delta \tilde{\omega}_n)t_{R2}}. \] (34)

Taking into account the fact that \( \sin \omega_n t_{R2} = 0 \) and \( \cos \omega_n t_{R2} = -1 \), we obtain an approximation up to the first order of \( 1/\gamma \) and that of \( \alpha_1 \) as

\[ \gamma^{-1}\omega_n \alpha_2 \approx \alpha_1 \sin \omega_n t_{R1} - \alpha_2 \delta \tilde{\omega}_n t_{R2}. \] (35)

Thus, \( \delta \tilde{\omega}_n \) is given by

\[ \delta \tilde{\omega}_n \approx -\frac{\omega_n}{\gamma t_{R2}} + \frac{\alpha_1}{\alpha_2 t_{R2}} \sin \omega_n t_{R1} \]

\[ \approx \delta \omega_n + \frac{\alpha_1}{\alpha_2 t_{R2}} \sin \omega_n t_{R1}, \] (36)

and we obtain \( \delta' \omega_n = \frac{\alpha_1}{\alpha_2 t_{R2}} \sin \omega_n t_{R1} \).
Substituting $\omega = \tilde{\omega}_n = \omega_n + \delta \tilde{\omega}_n$ into Eq. (28), we get

$$\tilde{\eta}_n = \tilde{\mu}_n f'(x_c)$$

$$\approx \frac{1}{-\alpha_2 + \alpha_1 \cos \omega_n t R_1}.$$  \hspace{1cm} (37)

In the case of $\alpha_1 \ll 1/\gamma$, we have to consider an approximation up to second orders of $\gamma^{-1}$. After some calculations, we obtain

$$\tilde{\eta}_n \approx \left[ -\alpha_2 + \alpha_1 \cos \omega_n t R_1 + \frac{\alpha_2}{2} \left( \frac{\omega_n}{\gamma} \right)^2 \right]^{-1}$$

$$\approx -\frac{1}{\alpha_2} \left[ 1 + \frac{\alpha_1}{\alpha_2} \cos \omega_n t R_1 + \frac{1}{2} \left( \frac{\omega_n}{\gamma} \right)^2 \right]^{-1}.$$  \hspace{1cm} (38)

In the limit $\alpha_1 \to 0$, the most right-hand side in Eq. (38) converges to that of the single-delay equation in Eq. (23). Thus, in the case that $\alpha_1/\alpha_2 \ll 1/\gamma^2$, the fundamental mode ($n = 1$) dominates others and is observed in the entire range of the ratio $t R_1/t R_2$, because $\tilde{\eta}_1$ always takes maximum values. As will be discussed in detail in Sect. 3.2.3, if $\alpha_1/\alpha_2$ is on the same order as or much larger than $1/\gamma^2$, the third harmonic overcomes the fundamental mode in some range of $t R_1/t R_2$, in which the third harmonic should be first excited instead of the fundamental mode with increasing $\mu$. With increasing $\alpha_1/\alpha_2$, the fifth, seventh, and higher harmonics successively follow in the same way. Therefore, an nth harmonic dominates in some ranges of $t R_1/t R_2$, if $\alpha_1/\alpha_2$ goes beyond its critical value, which is roughly estimated as $\alpha_1/\alpha_2 \approx \omega_n^2/\gamma^2$. Finally, at $\alpha_1 = \alpha_2$, any oscillation mode less than the cutoff should be observed in certain ranges of $t R_1/t R_2$.

### 3.2.3 Case I: $t R_2$ is fixed

Let us consider the case that $t R_2$ is fixed. To obtain the optimal mode for given $t R_1$, $t R_2$, and $\alpha_1$, first, we numerically solve Eq. (30) to obtain the angular frequency of the mode $\tilde{\omega}_n$. Next, we calculate $\tilde{\eta}_n$ using Eq. (28), and finally determine the optimal mode given by Eq. (31). Figures 3(a) and 3(b) show the normalized frequency of the optimal mode, $F_m$, as a function of $t R_1/t R_2$ at representative values of $\alpha_1$ for $\gamma = 210$ and $\gamma = 105$, respectively.

According to the theoretical estimation given above, $F_m(t R_1/t R_2)$ has the following tendencies. Since the cutoff frequency becomes higher with increasing $\gamma$, much higher modes are observed for larger values of $\gamma$. At $\alpha_1 = 1$, many higher modes are observed, but the orders of modes become smaller with decreasing $\alpha_1$ and only the fundamental modes are observed in the limit $\alpha_1 \to 0$.

As a result, for almost every $t R_1/t R_2$, the highest mode is observed in the case that $\gamma = 210$ and $\alpha_1 = 1$ compared with the other cases. In a region of $t R_1/t R_2$ close to zero, $F_m$ is inversely
proportional to the ratio \( t_{R1}/t_{R2} \) up to the cutoff frequency; more precisely, it decreases step-wise with \( t_{R1}/t_{R2} \) like \( F_m \approx 2l + 1 \approx t_{R2}/t_{R1} \), where \( 2l + 1 \) is the odd number nearest to \( t_{R2}/t_{R1} \). In the neighborhood of the point, at which \( t_{R1}/t_{R2} \) is given by the ratio of small odd (even) to even (odd) numbers (\( t_{R1}/t_{R2} = \text{odd/even or even/odd} \)), i.e., the mismatch ratio of the two delay times, twin towers consisting of higher odd harmonics, say tower structure, are observed, especially near \( t_{R1}/t_{R2} = 1/2, 1/4, \) and \( 2/3 \). In the neighborhood of the point, at which \( t_{R1}/t_{R2} \) is given by the ratio of small odd numbers, i.e., \( t_{R1}/t_{R2} = (2k+1)/(2l+1) \), the \((2l+1)\)-th mode is observed, e.g., third-harmonic region at approximately \( t_{R1}/t_{R2} = 1/3 \) and fifth-harmonic regions at approximately \( t_{R1}/t_{R2} = 1/5 \) and \( 3/5 \). The results at \( \alpha_1 = 1 \) are essentially the same as those observed for the optical resonator with two delays.\(^5,6\)

In contrast, only the fundamental mode is observed at \( \alpha_1 = 0.001 \) at \( \gamma = 105 \). Although it is not shown in the figure, only the fundamental mode is observed at \( \alpha_1 = 0.0005 \) at \( \gamma = 210 \). In the case of \( \alpha_1 = 0.01 \) and \( \gamma = 210 \) and that of \( \alpha_1 = 0.01 \) and \( \gamma = 105 \), a few lower odd harmonics are observed and the third harmonic appears nearly over the range of \( 1/4 < t_{R1}/t_{R2} < 1/2 \).

Figures 4(a) and 4(b) show the phase diagrams of the optimal modes in the parameter space of the ratio of delay times, \( t_{R1}/t_{R2} \), and the ratio of strength, \( \alpha_1/\alpha_2 \), for \( \gamma = 210 \) and 105, respectively. In those figures, “H” indicates harmonics higher than the fifth harmonic. At \( \alpha_1/\alpha_2 = 1 \), the third harmonic is observed in the small neighborhood of \( t_{R1}/t_{R2} = 1/3 \), while the fifth harmonic is found in small regions at approximately \( t_{R1}/t_{R2} = 1/5 \) and \( 3/5 \). The higher harmonics denoted by “H” occupy large areas in \( t_{R1}/t_{R2} \) at \( \gamma = 210 \), but they are reduced to relatively smaller areas at \( \gamma = 105 \).

When \( \alpha_1/\alpha_2 \) is decreased, the third harmonic lives longer than the others except the fundamental mode. The major domain of the third harmonic approximately covers a range of over \( 1/4 < t_{R1}/t_{R2} < 1/2 \) in a particular range of \( \alpha_1/\alpha_2 \), whose upper and lower limits are, as will be discussed later, estimated by thick broken lines in Figs. 4(a) and 4(b). However the domain of the third harmonic becomes wider under the lower limit because the domains of the fifth harmonic cease to exist. For \( \alpha_1/\alpha_2 = 0.0005 \) and below at \( \gamma = 210 \), and for \( \alpha_1/\alpha_2 = 0.002 \) and below at \( \gamma = 105 \), only the fundamental mode is observed.

To clarify the reason why the third harmonic appears in the range of \( 1/4 < t_{R1}/t_{R2} < 1/2 \) at particular values of \( \alpha_1/\alpha_2 \), let us examine the change in \( \bar{n}_{2l+1} \) with \( t_{R1}/t_{R2} \). Figures 5(a)-5(d) show \( \bar{n}_{2l+1} \) as functions of the ratio \( t_{R1}/t_{R2} \) for \( \gamma = 210 \) at \( \alpha_1 = 1, 0.005, 0.001, \) and \( 0.0005 \), respectively. In those figures, \( \bar{n}_{(2l+1)} \) is also indicated by a blue line.

At \( \alpha_1 = 1 \), we find the reason why the tower structures are constructed. In the neigh-
dominant role in the selection of the optimal mode, although \( \tilde{\eta}_{2l+1} \) takes maximum values. On the other hand, in the neighborhood of \( t_{R1}/t_{R2} = (2k + 1)/(2l + 1) \), a series of higher harmonic modes, which dominate one after another, are observed, especially in the neighborhood of \( t_{R1}/t_{R2} = 1/2, 1/4, \) and \( 3/4 \). Indeed, when \( t_{R1}/t_{R2} \) is approaching the center of the area, the optimal mode number increases to the threshold number, but after that it suddenly decreases. This is the mechanism of forming a tower structure that consists of higher modes.

At \( \alpha_1 = 0.005 \), the third and fifth harmonics together with the fundamental mode play dominant roles in the selection of the optimal mode, although \( \tilde{\eta}_1 \), i.e., the seventh harmonic, takes maximum values in a small region at around \( t_{R1}/t_{R2} = 1/7 \). The fifth harmonic still occupies the regions near \( t_{R1}/t_{R2} = 1/5 \) and \( 3/5 \). The domain of the third harmonic is extended nearly over the range of \( 1/4 < t_{R1}/t_{R2} < 1/2 \) and the fundamental mode occupies the large remaining parts, particularly the top third range of \( t_{R1}/t_{R2} \). At \( \alpha_1 = 0.001 \), the third harmonic and the fundamental mode survive and the others are eliminated. However, the domain of the third harmonic becomes smaller and shifts to a lower side, such as \( 0.2 \leq t_{R1}/t_{R2} \leq 0.4 \), compared with that at \( \alpha_1 = 0.005 \). At \( \alpha_1 = 0.0005 \), the fundamental mode dominates in the entire range, although \( \tilde{\eta}_3 \) takes values slightly smaller than \( \tilde{\eta}_1 \) in the neighborhood of \( t_{R1}/t_{R2} = 1/3 \). As a result, the third harmonic together with the fundamental mode plays a dominant role in the selection of excited modes in low ranges of \( \alpha_1/\alpha_2 \).

Let us consider the domain of the third-harmonic mode. First, we compare the third-harmonic mode with the fundamental mode. From the lowest-order approximation in Eq. (38), \( \tilde{\eta}_1 \) is always larger than \( \tilde{\eta}_3 \), if the following condition is satisfied in the entire range of \( t_{R1} \):

\[
\tilde{\eta}_1 - \tilde{\eta}_3 \approx -\frac{1}{\alpha_2}
\left[
\frac{1}{2} \left( \frac{\omega_1}{\gamma} \right)^2 - \frac{1}{2} \left( \frac{\omega_3}{\gamma} \right)^2 + \frac{\alpha_1}{\alpha_2} (\cos \omega_1 t_{R1} - \cos \omega_3 t_{R1}) \right] > 0.
\] (39)

It is roughly estimated using the values at \( t_{R1} = t_{R2}/3 \):

\[
\frac{1}{2\gamma^2} (\omega_2^2 - \omega_1^2) > \frac{\alpha_1}{\alpha_2} (\cos \omega_1 t_{R2}/3 - \cos \omega_3 t_{R2}/3),
\] (40)

where \( \cos \omega_1 t_{R2}/3 = \cos \pi/3 = 1/2 \) and \( \cos \omega_3 t_{R2}/3 = -1 \) from Eq. (17) with \( t_R = t_{R2} \). Then, it is rewritten as

\[
\frac{1}{\gamma^2} \frac{8\pi^2}{3} \frac{t_{R2}^2}{\omega_2^2} > \frac{\alpha_1}{\alpha_2}.
\] (41)

On the other hand, in the case of \( \alpha_1/\alpha_2 \gg \omega_2^2/\gamma^2 \), the condition \( \tilde{\eta}_3 > \tilde{\eta}_1 \) is, from Eq. (39),
reduced into

\[ \cos \omega_3 t_{R1} < \cos \omega_1 t_{R1}, \]  

(42)

and \( \tilde{\eta}_3 \) is larger than \( \tilde{\eta}_1 \) in the range of \( 0 < t_{R1}/t_{R2} < 1/2 \).

In the same way, from the comparison of the third-harmonic mode with the fifth-harmonic mode, we find that, if \( \alpha_1/\alpha_2 \) takes sufficiently large values, i.e., \( \alpha_1/\alpha_2 \gg \omega_5^2/\gamma^2 \), \( \tilde{\eta}_3 \) is larger than \( \tilde{\eta}_5 \) in the range of \( 1/4 < t_{R1}/t_{R2} < 1/2 \) from the condition

\[ \cos \omega_3 t_{R1} < \cos \omega_5 t_{R1}. \]  

(43)

However, in the case that the condition [obtained by a similar calculation at \( t_{R1} = t_{R2}/5 \) to Eqs.(39)-(41)]

\[ \frac{1}{2\gamma^2}(\omega_5^2 - \omega_7^2) = \frac{1}{2\gamma^2} \frac{16\pi^2}{t_{R2}^2} > \frac{\alpha_1}{\alpha_2}(1 + \cos 3\pi/5) \]  

(44)

is satisfied, \( \tilde{\eta}_3 \) is larger than \( \tilde{\eta}_5 \) in the range of \( 0 < t_{R1}/t_{R2} < 1/2 \).

Furthermore, if \( \alpha_1/\alpha_2 \gg \omega_7^2/\gamma^2 \), we get the approximation

\[ \tilde{\eta}_3 - \tilde{\eta}_7 = \frac{\alpha_1}{\alpha_2}(\cos \omega_7 t_{R1} - \cos \omega_3 t_{R1}), \]  

(45)

so that \( \tilde{\eta}_3 \) is larger than \( \tilde{\eta}_7 \) in the range of \( 1/5 < t_{R1}/t_{R2} < 2/5 \), while \( \tilde{\eta}_3 \) is smaller than \( \tilde{\eta}_7 \) in the range of \( 2/5 < t_{R1}/t_{R2} < 1/2 \). However, \( \tilde{\eta}_3 \) is larger than \( \tilde{\eta}_7 \) in the range of \( 1/5 < t_{R1}/t_{R2} < 1/2 \), if the condition (obtained at \( t_{R1} = 3t_{R2}/7 \))

\[ \frac{1}{2\gamma^2}(\omega_7^2 - \omega_3^2) = \frac{1}{2\gamma^2} \frac{40\pi^2}{t_{R2}^2} > \frac{\alpha_1}{\alpha_2}(1 + \cos 9\pi/7) \]  

(46)

is satisfied.

In the same way, one can compare the third-harmonic mode with any higher harmonic modes. As a result, it is estimated from Eqs.(41), (44), and (46) that the third-harmonic mode becomes dominant in the range of \( 1/4 < t_{R1}/t_{R2} < 1/2 \), if the following condition is satisfied:

\[ \frac{40\pi^2}{2(1 + \cos 9\pi/7)\gamma^2 t_{R2}^2} > \frac{\alpha_1}{\alpha_2} > \frac{16\pi^2}{2(1 + \cos 3\pi/5)\gamma^2 t_{R2}^2}. \]  

(47)

As shown in Fig. 4, Eq. (47) well estimates the characteristic area of the third-harmonic mode in the phase diagram.

3.2.4 Case II: \( t_{R1} \) is fixed

Let us consider the case that \( t_{R1} \) is fixed. Fig. 6 (a) and (b) show the normalized frequency of the optimal mode, \( F_m \), as a function of \( t_{R2}/t_{R1} \) instead of \( t_{R1}/t_{R2} \) at representative values of
Fig. 3. (Color online) Theoretical estimation of the normalized frequency of the optimal mode, $F_m$, as a function of the ratio of delay times, $t_{R1}/t_{R2}$, for representative values of $\alpha_1$ in Case I. (a) $\gamma = 210$. (b) $\gamma = 105$. 

Fig. 4. (Color online) Theoretically obtained phase diagrams, which show the optimal modes in the parameter space of the ratio of delay times, $t_{R1}/t_{R2}$, and the ratio of strength, $\alpha_1/\alpha_2$, in Case I. The labels “1”, “3”, and “5” stand for the fundamental mode, third, and fifth harmonics, respectively, and the label “H” denotes harmonics higher than the fifth harmonic. The simulations are carried out at $\alpha_1/\alpha_2 = k \times 10^{-j}$ in any combination of $k = 1, 1.2, 1.5, 2, 2.5, 3, 4, 5, 6, 7, 8, 9$ and $j = 1, 2, 3, 4$ as well as at $\alpha_1/\alpha_2 = 1$. Thick broken lines indicate the upper and lower values of $\alpha_1/\alpha_2$ given by Eq. (47). (a) $\gamma = 210$. (b) $\gamma = 105$. 

$\alpha_1$ for $\gamma = 210$ and 105, respectively. In Fig. 7 (a) and (b), the phase diagrams of the optimal modes are given in the parameter space of the ratio of delay times, $t_{R2}/t_{R1}$, and the ratio of strength, $\alpha_1/\alpha_2$, for $\gamma = 210$ and 105, respectively. 

The results show similar properties to those in Case I. If $\alpha_1$ is sufficiently large, each $(2l + 1)$-th harmonic is observed in the neighborhood of $t_{R2}/t_{R1} = 2l + 1$. Since the limit $t_{R2}/t_{R1} \to \infty$ corresponds to the limit $t_{R1}/t_{R2} \to 0$ in Case I, then higher odd-harmonic modes seem to be observed one by one when $t_{R2}/t_{R1}$ goes to infinity. However, such domains of
odd harmonics higher than the fifth harmonic do not appear in the phase diagrams in Fig. 7, because we show the results in the range of $1 \leq t_{R2}/t_{R1} \leq 5.5$.

The phase diagram at $\gamma = 210$ shows a complex structure. At large values of $\alpha_1$, odd-harmonic modes higher than the fifth harmonic are observed in regions around the mismatch ratios of the delay times, e.g., $t_{R2}/t_{R1} = 3/2, 2, 5/2, 7/2, 4, \text{and} \ 9/2$. The phase diagram becomes simpler at $\gamma = 105$. Indeed, higher harmonic modes occupy smaller areas at around $t_{R2}/t_{R1} = 3/2, 2, \text{and} \ 4$ even at large values of $\alpha_1$ in the phase diagram of $\gamma = 105$. Note that the fifth harmonic, which appears in the neighborhood of $t_{R2}/t_{R1} = 5/3$, is not caused by mismatched boundary conditions. This fifth harmonic is more stable than the higher harmonic modes generated by mismatched boundary conditions. Actually, it lives longer than the higher harmonic modes when $\alpha_1$ is decreased.

In the middle ranges of $\alpha_1$ in the phase diagrams, where the higher harmonic modes disappear, habitat isolation among odd-harmonic modes is observed such that the $(2l + 1)$-th
harmonic appears roughly in the range of $2l < t_{R2}/t_{R1} < 2l + 2$. Therefore, the third harmonic appears in the range of $2 < t_{R2}/t_{R1} < 4$. With decreasing $\alpha_1$, those domains of third and fifth harmonics are shifted to infinity, and the fundamental mode seems to occupy the entire range of the ratio $t_{R2}/t_{R1}$ in the limit $\alpha_1 \to 0$.

To clarify the mode selection rules in the lower ranges of $\alpha_1/\alpha_2$, let us study the change in $\tilde{n}_{2l+1}$ with $t_{R2}/t_{R1}$. Figures 8(a)-8(c) show $\tilde{n}_{2l+1}$ as functions of $t_{R2}/t_{R1}$ for $\gamma = 210$ at $\alpha_1 = 0.01$, 0.001, and 0.0001, respectively. At $\alpha_1 = 0.01$, $\tilde{n}_3$ takes maximum values in the range of nearly over $2 < t_{R2}/t_{R1} < 4$, while $\tilde{n}_1$ and $\tilde{n}_3$ become dominant in the ranges below and above it, respectively. At $\alpha_1 = 0.001$, the domain in which $\tilde{n}_3$ becomes largest shifts to the upper side, i.e., $2.5 < t_{R2}/t_{R1} < 5$, although $\tilde{n}_1$ takes slightly smaller values than $\tilde{n}_3$ for $t_{R2}/t_{R1} > 2.5$ and $\tilde{n}_5$ takes values close to those of $\tilde{n}_3$ for $t_{R2}/t_{R1} > 4$. At $\alpha_1 = 0.0001$, the fundamental mode is dominant at least in the range of $1 \leq t_{R2}/t_{R1} \leq 5.5$. However, from Eq. (38), all $\tilde{n}_{2l+1}$ should converge on the same value in the limit $t_{R2}/t_{R1} \to \infty$:

$$\tilde{n}_n \to -\frac{1}{\alpha_2} \left(1 + \frac{\alpha_1}{\alpha_2}\right).$$

(48)

This means that the dominance of the fundamental mode is not always ensured in the range of $t_{R2}/t_{R1} \gg 1$, even if $\alpha_1/\alpha_2 \ll 1/\gamma$, because $\tilde{n}_1$ is not much larger than the others. Therefore, the selection of the excited mode may change sensitively depending on the initial condition and changing rate of $\mu$, when $\mu$ is adiabatically increased. This occurs in numerical simulation, as will be shown in Sect. 4.

To estimate the domain of the third-harmonic mode, the same analysis as that in Case I in Sect. 3.2.3 can be applied to Case II, but one should carefully consider the dependence of $\omega_n$ on $t_{R2}$ [see Eq. (17) with $t_R = t_{R2}$]. After some consideration, one finds that the third-harmonic mode dominates others in the range of $2 < t_{R2}/t_{R1} < 4$, if the condition

$$\frac{40\pi^2}{2(1 + \cos 9\pi/7)y^2(7t_{R1}/3)^3} > \frac{\alpha_1}{\alpha_2} > \frac{16\pi^2}{2(1 + \cos 3\pi/5)y^2(5t_{R1})^3}$$

(49)

is satisfied. As shown in Fig. 7, this estimation is in good agreement with the numerical result.

4. Numerical Results of the Model System with Two Delays

4.1 Outline of numerical calculations

In this section, we show the results of numerical simulations of the two-delay model (1). In numerical simulation, we assume that the optimal mode with the smallest $\mu_n$ at the bifurcation, i.e., the largest $\tilde{n}_n$, is first excited, if the control parameter $\mu$ is increased adiabatically. The control parameter $\mu$ monotonically increases at a very small rate $\frac{d\mu}{dt} = 2.0 \times 10^{-7}$ and the period $T$ of the first excited oscillation mode is detected. The frequency $f$ is obtained as
Fig. 6. (Color online) Theoretical estimation of the normalized frequency of the optimal mode, $F_m$, as a function of the ratio of delay times, $t_{R2}/t_{R1}$, for representative values of $\alpha_1$ in Case II. (a) $\gamma = 210$. (b) $\gamma = 105$.

Fig. 7. (Color online) Theoretically obtained phase diagrams of the optimal modes in the parameter space of the ratio of delay times, $t_{R2}/t_{R1}$, and the ratio of strength, $\alpha_1/\alpha_2$, in Case II. Thick broken lines indicate the upper and lower values of $\alpha_1/\alpha_2$ given by Eq. (49). (a) $\gamma = 210$. (b) $\gamma = 105$.

$f = 1/T$. We focus on Case II with a fixed $t_{R1}$ and discuss the relationship between the order of excited oscillation modes and the ratio of delay times, $t_{R2}/t_{R1}$. The normalized frequency is defined by $F_m \equiv f/f_R$, where $f$ is the frequency of the excited mode and $f_R = 1/(2t_{R2})$. Even if the fundamental mode is excited, its frequency is usually slightly different from $f_R$ owing to the effect of decay and nonlinearity of the system. For the same reason, the frequency of the $n$th-harmonic mode observed numerically is not exactly an integer multiple of $f_R$ but slightly different from it. Hence, $F_m$ takes real values, which nearly approximate odd integers.
For the theoretical analyses, we assume that the delay times $t_{R1}$ and $t_{R2}$ are of $O(1)$ and that the decay constant $\gamma$ takes large values. On the other hand, in numerical simulation, we take different choices of parameters: the delay times $t_{R1}$ and $t_{R2}$ take large values, but the decay constant is of $O(1)$. This is really possible owing to the scaling law explained below.

For simplicity, let us consider the system with the single delay given by Eq. (5). With the change in variables from $(t, \gamma)$ to $(\tau, \tilde{\gamma})$ under the condition $\tilde{\gamma}\tau = \gamma t$, Eq. (5) is rewritten as
\[
\frac{1}{\tilde{\gamma}} \frac{dx}{d\tau} = -x + \mu f[x(\tilde{\gamma}(\tau - \tau_{R})/\gamma)],
\]
where $\tau_{R} = \gamma t_{R}/\tilde{\gamma}$. The variable $x$ can be written as a function of $\tau$, i.e., $\tilde{x}(\tau) = x(\tilde{\gamma}\tau/\gamma)$, then the equation is rewritten as
\[
\frac{1}{\tilde{\gamma}} \frac{d\tilde{x}}{d\tau} = -\tilde{x} + \mu f[\tilde{x}(\tau - \tau_{R})],
\]
which is the same as Eq. (5). Therefore, a choice of the parameters such that $t_{R} \approx O(1)$ and $\gamma \gg 1$ is equivalent to another choice that $\tau_{R} \gg 1$ and $\tilde{\gamma} \approx O(1)$, if the relation $\tilde{\gamma}\tau = \gamma t$ is
satisfied. The scaling law for the two-delay system is obtained straightforwardly. This scaling law will again be used in the numerical study of the clarinet model in Sect. 5.

In the numerical simulation in Case II, we take the parameters as \( \tau_{R1} = 70, \tau_{R1} \leq \tau_{R2} \), and \( \tilde{\gamma} = 1.0, 0.5 \) instead of the choice that \( t_{R1} = 1/3, t_{R1} \leq t_{R2} \), and \( \gamma = 210, 105 \). For convenience, we replace \( \tilde{\gamma}, \tau, \) and \( \tilde{x} \) by \( \gamma, t, \) and \( x \), hereafter, respectively.

4.3 Numerical results of Case II

Figures 9(a) and 9(b) show the normalized frequency of the first excited mode (the optimal mode), \( F_{m} \), as a function of the ratio \( t_{R2}/t_{R1} \) at representative values of \( \alpha_1 \) with the fixed \( \alpha_2(=1) \) for \( \gamma = 1 \) and 0.5, respectively. For each \( \gamma \), the changes in \( F_{m} \) with \( t_{R2}/t_{R1} \) are almost the same as those of the theoretical results shown in Fig. 6, although \( F_{m} \) obtained by the numerical simulation changes irregularly in some ranges of \( t_{R2}/t_{R1} \) at \( \alpha_1 = 0.01 \) and 0.001 for both \( \gamma = 1 \) and 0.5.

Figures 10(a) and 10(b) show the phase diagrams of the excited modes in the parameter space of the ratio of delay times, \( t_{R2}/t_{R1} \), and the ratio of strength, \( \alpha_1/\alpha_2 \), at \( \gamma = 1 \) and 0.5, respectively. The phase diagrams are essentially the same as those of the theoretical prediction shown in Fig. 7.

However, in the lower-right area of the diagrams, especially for \( \gamma = 1 \), the selection of the excited mode seems to be irregular: the fundamental mode, third harmonics, fifth harmonics, and even higher harmonics appear somewhat randomly. As discussed in Sect. 3.2.4, \( \tilde{\eta}_1, \tilde{\eta}_3, \) and \( \tilde{\eta}_5 \) are close to each other in this area so that the selection of the excited mode very sensitively depends on the values of the static parameters and changing rate of \( \mu \). As a result, not only the fundamental mode but also other modes, i.e., third, fifth, and further higher harmonics, are observed even in the area in which the fundamental mode is theoretically dominant.

5. Function of the Register Hole of the Clarinet

5.1 Delay difference equation model of single-reed instruments

In this subsection, we introduce a delay difference equation model of the clarinet. Fig. 11 shows the cross section of the operating portion of the clarinet, i.e., the mouthpiece. As well known in the field of musical acoustics, the dynamics of woodwind reed instruments including the clarinet is described in terms of conceptually separate linear and nonlinear mechanisms: an air column and a reed valve (see Fig. 11). A nonlinear element, the reed valve, controls the volume flow supplied from the mouth. The volume flow, which passes
Fig. 9. (Color online) Normalized frequency of the first excited mode, $F_m$, vs ratio of delay times, $t_{R2}/t_{R1}$, for representative values of $\alpha_1$ in Case II. (a) $\gamma = 1$, $t_{R1} = 70$. (b) $\gamma = 0.5$, $t_{R1} = 70$.

Fig. 10. (Color online) Phase diagrams of the first excited modes in the parameter space of $t_{R2}/t_{R1}$ and of $\alpha_1/\alpha_2$ in Case II. (a) $\gamma = 1$, $t_{R1} = 70$. (b) $\gamma = 0.5$, $t_{R1} = 70$.

through the reed slit, generates acoustic pressure in the mouthpiece, which drives a linear element, the air column. The linear element acts as delay feedback and affects the operation of the nonlinear element in turn. As a result, the dynamics of the woodwind reed instrument can be described using a delay equation model.

The property of the air column is characterized by a characteristic function called the reflection function $r(t)$, which is an acoustic analogue of the S-matrix of the scattering theory in the time domain.\textsuperscript{16,18} Actually, the reflection function describes a response of the air column observed at its entrance, when a delta-function pulse is input at time $t = 0$. Note that it is assumed in this case that the air column is terminated at the entrance by a perfect absorber, such that a uniform, semi-infinite tube of the same cross section is joined to it. Using the re-
flection function, we obtain the relationship between the sound pressure \( p(t) \) and the volume flow \( u(t) \) in the mouthpiece:

\[
p = Z_0 u + p_{inc},
\]

where \( p_{inc} \) is defined by

\[
p_{inc} = \int_0^{\infty} r(\tau) [p(t - \tau) + Z_0 u(t - \tau)] d\tau
\]

and \( Z_0 \) denotes the characteristic impedance defined by \( Z_0 = \rho c_0 / S_c \), where \( \rho \), \( c_0 \), and \( S_c \) are the air density, the speed of sound, and the area of the cross section at the entrance, respectively.

The volume flow passing through the reed slit is obtained with a quasi-static approximation based on Bernoulli’s principle. Supposing that the flow velocity in the mouth is zero because it is negligible, then Bernoulli’s principle is written as

\[
P_0 = p + \frac{1}{2} \rho v^2,
\]

where \( v \) denotes the velocity of the flow passing through the reed slit and \( P_0 \) is the blowing pressure in the mouth. The volume flow in the mouthpiece, \( u \), is given as \( u = whv \), where \( w \) and \( h \) are the width and height of the reed slit, respectively.

The reed is driven by the difference in pressure between the mouth and the mouthpiece. Under the quasi-static approximation, the reed displacement is proportional to the difference between \( p \) and \( P_0 \):

\[
k(h - h_0) = p - P_0 \equiv -\Delta p,
\]

where \( h_0 \) is the height of the reed at rest and \( k \) is the effective stiffness of the reed. When the reed is closed, i.e., \( h = 0 \), the pressure is \( p = p_M \), which is obtained from

\[
kh_0 = P_0 - p_M \equiv \Delta p_M.
\]

From Eqs.(54)-(56), the volume flow passing through the reed slit, \( u = whv \), is given by

\[
u = \begin{cases} 
  u_0 \left( 1 - \frac{\Delta p}{\Delta p_M} \right) \sqrt{\frac{\Delta p}{\Delta p_M}} & (\Delta p < \Delta p_M) \\
  0 & (\Delta p \geq \Delta p_M),
\end{cases}
\]

where \( u_0 \) is defined by

\[
u_0 = wh_0 \sqrt{\frac{2kh_0}{\rho}}.
\]

Since the reed is closed for \( \Delta p \geq \Delta p_M \), no flow enters the mouthpiece, i.e., \( u = 0 \), as shown in the second case in Eq. (57).
Since \( P_{\text{inc}} \) is calculated from the previous data of \( p \) and \( u \) [see Eq. (53)], combining Eq. (57) with Eq. (52) provides the pressure \( p(t) \) and the volume flow \( u(t) \) at the present time \( t \). Actually, the intersection of Eq. (57) with Eq. (52) is obtained as one of the roots of a cubic equation of \( p \), when the reed is open, although \( p(t) = P_{\text{inc}} \) and \( u = 0 \), when the reed is closed. To calculate the time evolution of the system, the data of dynamical variables are discretized as \( t_i = i\Delta t \), \( r_i = r(t_i) \), \( p_i = p(t_i) \), and \( u_i = u(t_i) \), then \( P_{\text{inc},i} \) at \( i = n \) is calculated from the previous \( p_i \) and \( u_i \) \((i \leq n - 1)\) as

\[
P_{\text{inc},n} = \sum_{i=1}^{\infty} r_i (p_{n-i} + Z_0u_{n-i}).
\]  

(59)

Then, the intersection of Eq. (57) with Eq. (52) gives the pressure \( p_i \) and the volume flow \( u_i \) at \( i = n \), which provide \( P_{\text{inc},i} \) at \( i = n + 1 \). The repetition of this process provides the time evolution of the system. Therefore, the model of the clarinet is written using a delay difference equation.\(^{19,20}\)

### 5.2 Reflection function for the closed pipe with a register hole

The bore of our model is the closed pipe with a register hole (see Fig. 1). As shown in the introduction, the register hole is placed at a distance of \( l_r \approx 0.14 \text{m} \) from the tip of the mouthpiece. The effective pipe length \( l_p \) adjusted to generate the pitch of a given note in the first register ranges of over 1.9 \( \leq l_p/l_r \leq 4.1 \).

For simplicity, we use a simplified reflection function, which has two inverted peaks, that is, one with a short delay time \( t_{R1} \) corresponding to the reflection at the register hole and the other with a long delay time \( t_{R2} \) corresponding to that at the open end. Since \( t_{R1} \) and \( t_{R2} \) are obtained as \( t_{R1} = 2l_r/c_0 \) and \( t_{R2} = 2l_p/c_0 \), respectively, then the ratio of \( l_p \) to \( l_r \) is equivalent to the ratio of the two delay times \( t_{R2}/t_{R1} \). The reflection function that we use is written as

\[
r(t) = -\tilde{\alpha}_1 f_r(t - t_{R1}, \tau_r) - \tilde{\alpha}_2 f_r(t - t_{R2}, \tau_r),
\]  

(60)
where the function \( f_r(t - t_{Ri}, \tau_Y) \), which represents a reflection peak, is defined by

\[
f_r(t - t_{Ri}, \tau_Y) = \frac{1}{\tau_Y} \exp\left(-\frac{1}{\tau_Y} (t - t_{Ri})\right) H(t - t_{Ri}),
\]

(61)

where \( H(t) \) is the Heaviside function. The parameters \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) play the same roles as \( \alpha_1 \) and \( \alpha_2 \), respectively, and the relaxation time \( \tau_Y \) corresponds to the reciprocal of the decay constant \( \gamma \), i.e., \( \tau_Y = 1/\gamma \) [compare Eq. (61) with Eq. (4)]. From the requirement that the average pressure in the mouthpiece is equal to the pressure of the atmosphere, the reflection function satisfies the following condition:

\[
\int_0^\infty r(t) dt = -1,
\]

(62)

which is reduced into

\[
\tilde{\alpha}_1 + \tilde{\alpha}_2 = 1.
\]

(63)

5.3 Numerical results

Since the register hole is placed at a distance of \( l_r \approx 0.14 \text{m} \) from the tip of the mouthpiece, the delay time caused by the register hole \( t_{R1} \) is taken as \( t_{R1} = 8.25 \times 10^{-4} \text{ s} \), when \( c_0 = 340 \text{m/s} \). The delay time due to the open end, \( t_{R2} \), is put in the range of \( 1 \leq t_{R2}/t_{R1} \leq 5.5 \). The relaxation time \( \tau_Y \) is given as \( \tau_Y = 4 \times 10^{-5} \text{ s} \), which is close to that of the reflection function for a cylindrical pipe with the same radius as the clarinet.\(^ {21-23} \) The relaxation time \( \tau_Y \) is related to the decay constant \( \gamma \) as \( \tau_Y = 1/\gamma \) and we get \( t_{R1}/\tau_Y = 20.625 \), which corresponds through the scaling law to Case II of the two-delay system (1) at \( t_{R1} = 1/3 \) and \( \gamma = 61.875 \). The values of the parameters used for numerical calculations are given in Table I, where \( r_d \) denotes the pipe radius, and the area of the cross section of the pipe is given by \( S_c = \pi r_d^2 \). The blowing pressure \( P_0 \) is increased from 0 to \( 1.0 \times 10^4 \text{ Pa} \) at a rate of 10\text{ Pa/s}, and we observed the period \( T \) of the oscillation excited first.

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_d )</td>
<td>( 7.5 \times 10^{-3} \text{ m} )</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1.2\text{kg/m}^3</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>340\text{m/s}</td>
</tr>
<tr>
<td>( w )</td>
<td>( 1.4 \times 10^{-2} \text{ m} )</td>
</tr>
<tr>
<td>( h_0 )</td>
<td>( 6 \times 10^{-2} \text{ m} )</td>
</tr>
<tr>
<td>( k )</td>
<td>( 12486993.75 \text{ Pa/m} )</td>
</tr>
</tbody>
</table>
In the case of $\tilde{\alpha}_1 = 0$, namely, the single-delay model, the system of Eqs. (57), (52), and (60) under the discrete time approximation provides the time evolution of the model without the register hole. It bifurcates with increasing $P_0$ from a stationary solution to the fundamental oscillation. In particular, its reduction to a 1D map undergoes the period-doubling bifurcation as the reduced map of the two-delay system.\(^{(19)}\) Therefore, it is expected that the clarinet model has essentially the same properties as the two-delay model \(^{(1)}\).

Let us see the results in the case of $\tilde{\alpha}_1 \neq 0$. Fig. 12 shows waveforms of the fundamental mode and third harmonic. Such rectangular waves are typical of single-reed instruments with a cylindrical pipe.\(^{(21,22)}\) Fig. 13 shows the normalized frequency of excited modes, $F_m$, as a function of $t_{R2}/t_{R1}$ at representative values of $\tilde{\alpha}_1/\tilde{\alpha}_2$. Fig. 14 gives the phase diagram of the excited modes in the parameter space of $t_{R2}/t_{R1}$ and $\tilde{\alpha}_1/\tilde{\alpha}_2$.

Comparing Figs. 13 and 14 with Figs. 6 and 7, one can find that essentially the same selection rule of excited modes works for the clarinet model as for the two-delay model \(^{(1)}\). That is, the higher harmonic modes are observed for large values of $\tilde{\alpha}_1/\tilde{\alpha}_2$, especially the tower structures around the mismatch ratios of the delay times. However, the highest mode number, i.e., 13th, at $\tilde{\alpha}_1/\tilde{\alpha}_2 = 1$ is smaller than that in Case II, i.e., 19th, at $\alpha_1/\alpha_2 = 1$ and at $\gamma = 105$. This is because the effective decay constant, $\gamma = 61.875$, obtained with the scaling law is smaller than that in Case II, $\gamma = 105$, so that the cutoff frequency becomes smaller.

The reflection function for a realistic model of the clarinet can be calculated theoretically.\(^{(23,24)}\) Although we do not show the result here, it is known that the ratio of the peak height of the register hole reflection to that of the open end reflection is of around 0.1, which corresponds to the case of $\tilde{\alpha}_1/\tilde{\alpha}_2 = 0.1$. As shown in Figs. 13 and 14, the third harmonic is observed in the range of $2.1 < t_{R2}/t_{R1} < 4.2$ at $\tilde{\alpha}_1/\tilde{\alpha}_2 = 0.1$, which almost corresponds to the working range of the register hole of the clarinet, i.e., $1.9 < l_p/l_r < 4.1$. For $\tilde{\alpha}_1/\tilde{\alpha}_2 \geq 0.2$, the range of the third harmonic becomes smaller owing to the disturbance of higher harmonics, while it is shifted to the upper sides with decreasing $\tilde{\alpha}_1/\tilde{\alpha}_2$. Thus, the simplified clarinet model with two delays well reproduces the function of the register hole. This result is the answer to the questions why the diameter of the register hole is smaller than those of the other tone holes but not extremely small and why such a small register hole works in the wide range of the register.

6. Summary and Discussion

In this work, we have studied the selection rules of oscillation modes for systems with two delays. In particular, we have focused on the case that the strength of the short time delay...
Fig. 12. Pressure wave forms of the fundamental mode at $\bar{\alpha}_1/\bar{\alpha}_2 = 0$ and of the third harmonic at $\bar{\alpha}_1/\bar{\alpha}_2 = 1$ for the clarinet model with $t_{R2}/t_{R1} = 3$ and $P_0 = 7000\text{Pa}$.

Fig. 13. (Color online) Normalized frequency of the first excited mode, $\tilde{F}_m$, vs ratio of delay times, $t_{R2}/t_{R1}$, for the clarinet model.

Fig. 14. (Color online) Phase diagrams of the first excited modes in the parameter space of $t_{R2}/t_{R1}$ and $\bar{\alpha}_1/\bar{\alpha}_2$ for the clarinet model.
is quite small compared with that of the long time delay, in relation with the function of the register hole of the clarinet.

First, we have studied the simple one-dimensional differential equation with two delays (1), which is suitable for theoretical analyses. When the two delays have the same strength, \( \alpha_1 = \alpha_2 \), we get essentially the same results as in previous studies.\(^5,6\) Namely, if the ratio of the two delay times \( t_{R1}/t_{R2} \) is well approximated by the ratio of small odd numbers, i.e., \( t_{R1}/t_{R2} = (2k + 1)/(2l + 1) \), the \((2l + 1)\)-th-harmonic mode is selected. However, in the neighborhood of the mismatched boundary conditions, i.e., \( t_{R1}/t_{R2} = \text{odd/even or even/odd} \), higher odd-harmonic modes forming the so-called tower structure are observed. Thus, the selection rules of the oscillation modes are very complicated, as pointed out by Ikeda and Mizuno.\(^5\)

With decreasing \( \alpha_1 \), the tower structure becomes increasingly smaller and the fundamental mode is only observed in the limit \( \alpha_1 \to 0 \). In this process, the third-harmonic mode lives longer than the others except the fundamental mode. Thus, the domain of the third-harmonic mode becomes wider in \( t_{R1}/t_{R2} \) and occupies the range of \( 1/4 < t_{R1}/t_{R2} < 1/2 \) when \( \alpha_1/\alpha_2 \) takes particular values, which are sufficiently small but not extremely small. It is well explained by the theoretical estimation developed in Sect. 3.

This property of the two-delay systems is related with the function of the register hole of the clarinet. That is, the effective length of the clarinet \( l_p \), when tone holes are opened, is determined by the pitch of the chosen note in the first register. The operation of the register hole covers the range of \( 1.9 \leq l_p/l_\tau \leq 4.1 \), where \( l_\tau \) denotes the distance of the register hole from the tip of the mouthpiece. The pipe of the clarinet roughly generates the two delays: the short time delay \( t_{R1} \) with the weak strength \( \tilde{\alpha}_1 \) is caused by the reflection at the register hole, and the long time delay \( t_{R2} \) with the strength \( \tilde{\alpha}_2 \) is induced by the open end reflection of the pipe with the length \( l_p \). Thus, the register hole works in the range of \( 1.9 \leq t_{R2}/t_{R1} \leq 4.1 \), which is in good agreement with the numerical results obtained by the two-delay model of the clarinet at \( \tilde{\alpha}_1/\tilde{\alpha}_2 \approx 0.1 \), i.e., \( 2.1 \leq t_{R2}/t_{R1} \leq 4.2 \), and also with the theoretical and numerical results of the two-delay system (1) mentioned above. Therefore, the function of the register hole is well captured by the two-delay models. Namely, the small radius of the register hole is necessary to raise the pitch of the first register notes in a wide range more than an octave to the corresponding second register notes by a twelfth (19 semitones), i.e., fundamental modes to third harmonics.

More practically, the reflection caused by the real bore of the clarinet with many opened tone holes is rather complicated and should be characterized using a multiple-delay system with more than two delays. Nevertheless, the two-delay models seem to capture well the
underlying mechanism of the register hole. We postpone the study of the function of the register hole in terms of multiple-delay systems as well as that of the other tone holes using a real bore model for future works.

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Appendix: Properties of the Fixed Point

The fixed point of the map \( x_{n+1} = \mu f(x_n) \) is given by

\[
x_c = \mu f(x_c) = \mu \exp\left(-\frac{(x_c - x_0)^2}{\Delta x^2}\right).
\]  
(A-1)

Then, \( \mu \) can be written as a function of \( x_c \),

\[
\mu(x_c) = x_c \exp\left(\frac{(x_c - x_0)^2}{\Delta x^2}\right),
\]  
(A-2)

and the derivative of \( \mu \) is given by

\[
\mu'(x_c) = \left(1 + \frac{2x_c(x_c - x_0)}{\Delta x^2}\right) \exp\left(\frac{(x_c - x_0)^2}{\Delta x^2}\right),
\]  
(A-3)

If the following condition is satisfied for any values of \( x_c \):

\[
1 + \frac{2x_c(x_c - x_0)}{\Delta x^2} > 0 \Rightarrow 2x_c^2 - 2x_0 x_c + \Delta x^2 > 0;
\]  
(A-4)

\( \mu \) is a monotonically increasing function of \( x_c \). The above equation is replaced with the condition of the discriminant of the quadratic polynomial in the case of a lack of a real solution:

\[
x_0^2 - 2\Delta x^2 < 0.
\]  
(A-5)

When \( x_0 = 0.2 \) and \( \Delta x = 0.5 \), the above equation is always satisfied.

Let us consider the relationship of \( \eta \) with \( \mu \) and \( x_c \). From Eqs.(28) and (A-1), \( \eta \) is given by

\[
\eta = \mu f'(x_c) = -\frac{2(x_c - x_0)}{\Delta x^2} \mu f(x_c)
\]  
(A-6)

and its derivative is

\[
\eta'(x_c) = -\frac{4x_c - 2x_0}{\Delta x^2}.
\]  
(A-7)

If \( x_c > \frac{x_0}{2} > 0 \), \( \eta \) is a decreasing function of \( x_c \). On the other hand, \( x_c \) is, from Eq. (A-6), larger
than $x_0$ at the bifurcation point $\eta = \mu f'(x_c) = -1$. Since $\mu$ is a monotonically increasing function of $x_c$, $\eta$ decreases with $\mu$ near the bifurcation point.
References


