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SOME NOTES ON FIXED POINT THEOREMS IN ν-GENERALIZED METRIC SPACES

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Abstract

We study ν-generalized metric spaces. We first study the concept of Cauchy sequence. We next give a proof of the Banach contraction principle in ν-generalized metric spaces. The proof is similar to the proof of the original Banach contraction principle in metric spaces. Also, we give proofs of Kannan's and Ćirić's fixed point theorems in ν-generalized metric spaces.

1. Introduction

Throughout this paper we denote by $\mathbf{N}$ the set of all positive integers.

In 2000, Branciari in [3] introduced the following, very interesting concept. See also [6, 8] and others.

Definition 1 (Branciari [3]). Let X be a set, let d be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbf{N}$. Then (X, d) is said to be a *ν-generalized metric space* if the following hold:

(N1) $d(x, y) = 0$ iff $x = y$ for any $x, y \in X$.
(N2) $d(x, y) = d(y, x)$ for any $x, y \in X$.
(N3) $d(x, y) \le d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$ for any $x, u_1, u_2, \ldots, u_\nu, y \in X$ such that $x, u_1, u_2, \ldots, u_\nu, y$ are all different.

It is obvious that (X, d) is a metric space if and only if (X, d) is a 1-generalized metric space. Very recently, in [11], we found that not every generalized metric space has the compatible topology. See also [12]. In [1], we discussed the completeness of ν-generalized metric spaces.

In this paper, we study ν-generalized metric spaces. We first study the concept of Cauchy sequence. We next give a proof of the Banach contraction principle in ν-generalized metric spaces. The proof is similar to the proof of the original Banach

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contraction principle in metric spaces. Also, we give proofs of Kannan's and Ćirić's fixed point theorems in ν-generalized metric spaces.

2. Preliminaries

In this section, we study the concept of Cauchy sequence. As mentioned above, in general, ν-generalized metric spaces do not necessarily have the compatible topology. So we have to define the concept concerning the convergence. See [1, 3, 13] and others.

DEFINITION 2. Let (X, d) be a ν-generalized metric space.

- A sequence $\{x_n\}$ in X is said to be *Cauchy* iff
$$\lim_{n\to\infty} \sup_{m>n} d(x_m, x_n) = 0$$
holds.
- A sequence $\{x_n\}$ in X is said to be 2-*Cauchy* iff
$$\lim_{n\to\infty} \sup\{d(x_n, x_{n+1+2j}) : j = 0, 1, 2, \dots\} = 0$$
holds.
- A sequence $\{x_n\}$ in X is said to *converge* to x iff $\lim_n d(x, x_n) = 0$ holds.
- A sequence $\{x_n\}$ in X is said to *converge to x in the strong sense* iff $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to x.

DEFINITION 3. Let (X, d) be a ν-generalized metric space. Then X is *complete* iff every Cauchy sequence converges.

It is obvious that every Cauchy sequence is 2-Cauchy. If we assume something additional, then the converse holds.

LEMMA 4. *Let (X, d) be a ν-generalized metric space. Let $\{x_n\}$ be a 2-Cauchy sequence such that x_n are all different and*
$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$
Then $\{x_n\}$ is Cauchy.

PROOF. Fix $\varepsilon > 0$. Then there exists $\ell \in \mathbf{N}$ such that
$$d(x_n, x_{n+1+2j}) < \varepsilon \qquad \text{and} \qquad d(x_n, x_{n+2}) < \varepsilon$$
for any $j \in \mathbf{N} \cup \{0\}$ and $n \in \mathbf{N}$ with $n \geq \ell$. Fix $j \in \mathbf{N} \cup \{0\}$ and $n \in \mathbf{N}$ with $n \geq \ell$. Then in the case where $\nu = 1$, we have
$$d(x_n, x_{n+2+2j}) \leq d(d_n, x_{n+1+2j}) + d(x_{n+1+2j}, x_{n+2+2j}) < 2\varepsilon.$$

In the other case, where $\nu \geq 2$, we have by (N3)

$$\begin{aligned} d(x_n, x_{n+2+2j}) &\leq d(x_n, x_{n+1+2j}) + d(x_{n+1+2j}, x_{n+2j+2\nu}) \\ &\quad + \sum_{i=1}^{\nu-1} d(x_{n+2j+2i}, x_{n+2j+2i+2}) \\ &< (\nu+1)\varepsilon. \end{aligned}$$

Therefore $\{x_n\}$ is Cauchy. □

The following is essentially proved in [1].

Lemma 5 ([1]). *Let (X, d) be a ν-generalized metric space. Let $\{x_n\}$ be a sequence in X such that x_n are all different and*

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

Then the following hold:

- *$\{x_n\}$ is Cauchy provided ν is odd.*
- *$\{x_n\}$ is 2-Cauchy provided ν is even.*

By using these lemmas, we can prove the following, which is useful in this paper.

Proposition 6. *Let (X, d) be a ν-generalized metric space. Let $\{x_n\}$ be a sequence in X such that x_n are all different,*

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \qquad \text{and} \qquad \lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Then $\{x_n\}$ is Cauchy.

Proof. In the case where ν is odd, the conclusion obviously holds by Lemma 5. In the other case, where ν is even, $\{x_n\}$ is 2-Cauchy by Lemma 5 again. So by Lemma 4, $\{x_n\}$ is Cauchy. □

The following is connected with the continuity of d. See also [8].

Proposition 7 ([13]). *Let (X, d) be a ν-generalized metric space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to u and v in the strong sense, respectively. Then*

$$d(u, v) = \lim_{n\to\infty} d(x_n, y_n)$$

holds.

3. Fixed point theorems

We give proofs of some fixed point theorems in ν-generalized metric spaces. We note that the proofs in this section are similar to the proofs of the original theorems in metric spaces.

LEMMA 8. *Let (X,d) be a ν-generalized metric space and let T be a mapping on X. Assume that*

$$\sum_{n=1}^{\infty} d(T^n u, T^{n+1} u) < \infty \tag{1}$$

for some $u \in X$. Assume also either of the following:

- *ν is odd.*
- *ν is even and*

$$\lim_{n \to \infty} d(T^n u, T^{n+2} u) = 0 \tag{2}$$

holds.

Then $\{T^n u\}$ is Cauchy.

PROOF. We consider the following two cases:

- There exists $k, \ell \in \mathbf{N}$ such that $k < \ell$ and $T^k u = T^\ell u$.
- $T^n u$ are all different.

In the first case, we note that $T^k u$ is a fixed point of $T^{\ell - k}$. So we have

$$\sum_{n=1}^{\infty} d(T^n u, T^{n+1} u) \geq \sum_{j=1}^{\infty} d(T^{k+j(\ell-k)} u, T^{k+1+j(\ell-k)} u) = \sum_{j=1}^{\infty} d(T^k u, T^{k+1} u).$$

Hence $d(T^k u, T^{k+1} u) > 0$ contradicts (1). Thus $T^k u$ is a fixed point of T. Therefore $\{T^n u\}$ is Cauchy. In the second case, by Lemma 5 and Proposition 6, $\{T^n u\}$ is Cauchy. □

We give a proof of the following fixed point theorem which is a generalization of the Banach contraction principle [2, 4].

THEOREM 9 (Branciari [3]). *Let (X,d) be a complete ν-generalized metric space and let T be a contraction on X, that is, there exists $r \in [0,1)$ such that*

$$d(Tx, Ty) \leq r d(x, y)$$

for any $x, y \in X$. Then T has a unique fixed point z of T. Moreover, for any $x \in X$, $\{T^n x\}$ converges to z in the strong sense.

REMARK. See also [9, 10, 11].

Proof. Fix $u \in X$. Then we have

$$\sum_{n=1}^{\infty} d(T^n u, T^{n+1} u) \leq \sum_{n=1}^{\infty} r^n d(u, Tu) < \infty.$$

We also have

$$\lim_{n\to\infty} d(T^n u, T^{n+2} u) \leq \lim_{n\to\infty} r^{n-2} d(u, T^2 u) = 0.$$

Therefore by Lemma 8, $\{T^n u\}$ is Cauchy. Since X is complete, $\{T^n u\}$ converges to some $z \in X$. By Proposition 7, we have

$$d(z, Tz) = \lim_{n\to\infty} d(T^n u, Tz) \leq \lim_{n\to\infty} r d(T^{n-1} u, z) = r d(z, z) = 0,$$

which implies z is a fixed point of T. Let y be a fixed point of T. Then we have

$$d(z, y) = d(Tz, Ty) \leq r d(z, y)$$

and hence $z = y$ holds. So the fixed point z is unique. □

We also give a proof of the following fixed point theorem which is a generalization of Kannan's fixed point theorem [7].

Theorem 10. *Let (X, d) be a complete ν-generalized metric space and let T be a Kannan mapping on X, that is, there exists $\alpha \in [0, 1/2)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$$

for any $x, y \in X$. Then T has a unique fixed point z of T. Moreover, for any $x \in X$, $\{T^n x\}$ converges to z in the strong sense.

Proof. Since

$$d(Tx, T^2 x) \leq \alpha d(x, Tx) + \alpha d(Tx, T^2 x),$$

we have

$$d(Tx, T^2 x) \leq r d(x, Tx)$$

for any $x \in X$, where $r := \alpha/(1 - \alpha) \in [0, 1)$. Fix $u \in X$. Then we have

$$\sum_{n=1}^{\infty} d(T^n u, T^{n+1} u) \leq \sum_{n=1}^{\infty} r^n d(u, Tu) < \infty,$$

which implies $\lim_n d(T^n u, T^{n+1} u) = 0$. We also have

$$\lim_{n\to\infty} d(T^n u, T^{n+2} u) \leq \lim_{n\to\infty} (\alpha d(T^{n-1} u, T^n u) + \alpha d(T^{n+1} u, T^{n+2} u)) = 0.$$

Therefore by Lemma 8, $\{T^n u\}$ is Cauchy. Since X is complete, $\{T^n u\}$ converges to some $z \in X$. By Proposition 7, we have

$$d(z, Tz) = \lim_{n\to\infty} d(T^n u, Tz) \le \lim_{n\to\infty} (\alpha d(T^{n-1}u, T^n u) + \alpha d(z, Tz)) = \alpha d(z, Tz),$$

which implies $d(z, Tz) = 0$. Hence z is a fixed point of T. Let y be a fixed point of T. Then we have

$$d(z, y) = d(Tz, Ty) \le \alpha d(z, Tz) + \alpha d(y, Ty) = 0$$

and hence $z = y$ holds. So the fixed point z is unique. □

We finally give a proof of the following fixed point theorem which is a generalization of Ćirić's fixed point theorem [5] and Theorems 9 and 10.

THEOREM 11. *Let (X, d) be a complete v-generalized metric space and let T be a mapping on X such that there exists $r \in [0, 1)$ satisfying*

$$d(Tx, Ty) \le r \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{3}$$

for any $x, y \in X$. Then T has a unique fixed point z of T. Moreover, for any $x \in X$, $\{T^n x\}$ converges to z in the strong sense.

PROOF. Fix $u \in X$ and put

$$A(m, n) = \{T^j u : j \in \mathbf{N} \cup \{0\}, m \le j \le n\},$$

$$A(m, \infty) = \{T^j u : j \in \mathbf{N} \cup \{0\}, m \le j\},$$

$$D(m, n) = \sup\{d(x, y) : x, y \in A(m, n)\}$$

and

$$D(m, \infty) = \sup\{d(x, y) : x, y \in A(m, \infty)\}$$

for $m, n \in \mathbf{N} \cup \{0\}$ with $m \le n$, where T^0 is the identity mapping on X. Thus $D(m, n)$ and $D(m, \infty)$ are the diameter of $A(m, n)$ and $A(m, \infty)$, respectively. By (3), we note

$$D(m, n) \le rD(m-1, n) \tag{4}$$

for $m, n \in \mathbf{N}$ with $m \le n$. We also note by (3)

$$\max\{d(u, T^j u) : 1 \le j \le n\} = D(0, n) \tag{5}$$

for $n \in \mathbf{N}$. We consider the following two cases:

- There exists $k, \ell \in \mathbf{N}$ such that $k < \ell$ and $T^k u = T^\ell u$.
- $T^n u$ are all different.

In the first case, we note

$$D(k, \ell - 1) = D(k + 1, \ell) \le rD(k, \ell) = rD(k, \ell - 1)$$

by (4). Hence

$$D(k, \infty) = D(k, \ell - 1) = 0,$$

which implies T^k is a fixed point of T. We next consider the second case. Fix $n \in \mathbf{N}$ with $n > \nu$. By (5), there exists $\ell \in \mathbf{N}$ with $\ell \le n$ such that $d(u, T^\ell u) = D(0, n)$. If $\ell > \nu$, then we have

$$\begin{aligned} D(0, n) &= d(u, T^\ell u) \\ &\le \sum_{j=0}^{\nu-1} d(T^j u, T^{j+1} u) + d(T^\nu u, T^\ell u) \\ &\le \nu D(0, \nu) + D(\nu, \ell) \\ &\le \nu D(0, \nu) + r^\nu D(0, \ell) \\ &\le \nu D(0, \nu) + r^\nu D(0, n) \end{aligned}$$

and hence

$$D(0, n) \le \frac{\nu}{1 - r^\nu} D(0, \nu). \tag{6}$$

If $\ell \le \nu$, then (6) obviously holds. Since $n \in \mathbf{N}$ is arbitrary, $\{D(0, n)\}$ is bounded, which is equivalent to $D(0, \infty) < \infty$. By (3), we note

$$D(m, \infty) \le rD(m - 1, \infty) \le \cdots \le r^m D(0, \infty)$$

for $m \in \mathbf{N}$, which implies that $\{T^n u\}$ is Cauchy. Since X is complete, $\{T^n u\}$ converges to some $z \in X$. By Proposition 7, we have

$$\begin{aligned} d(z, Tz) &= \lim_{n \to \infty} d(T^n u, Tz) \\ &\le \lim_{n \to \infty} r \max\{d(T^{n-1} u, z), d(T^{n-1} u, T^n u), d(z, Tz), d(T^{n-1} u, Tz), d(T^n u, z)\} \\ &= rd(z, Tz) \end{aligned}$$

and hence $d(z, Tz) = 0$, thus, z is a fixed point of T. We have shown that there exists a fixed point in both cases. As in the proof of Theorem 9, we can prove that the fixed point is unique. □

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