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A stochastic method for all seasons

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Abstract

It is well known that the numerical solution of stochastic ordinary differential equations leads to a step size reduction when explicit methods are used. This has led to a plethora of implicit methods with a wide variety of stability properties. However, for stochastic problems whose eigenvalues lie near the negative real axis, explicit methods with extended stability regions can be very effective. In this paper we extend these ideas to the stochastic realm and present a family of weak order two explicit stochastic Runge-Kutta methods with extended stability intervals that can be used to solve a variety of non-stiff and stiff problems.

1 Introduction

While it has been customary to treat the numerical solution of stiff ordinary differential equations (ODEs) by implicit methods, there is a class of explicit methods with extended stability regions that are well suited to solving stiff problems whose eigenvalues lie near the negative real axis. Such problems include parabolic partial differential equations when solved by the method of lines.

An original contribution was by van der Houwen and Sommeijer [28] who constructed s-stage explicit Runge-Kutta (RK) methods whose stability functions are shifted Chebyshev polynomials, $T_s(1+z/s^2)$. These have stability intervals along the negative real axis $[-2s^2, 0]$. The corresponding RK methods satisfy a three term recurrence relation which make them efficient to implement, but their drawback is that they are of order 1. Lebedev [14, 15] and Medovikov [18] constructed high order methods by computing the zeros of the optimal stability polynomials for maximal stability. But, the method is sensitive to the ordering of these zeros and there are no recurrence relationships.

Abdulle and Medovikov [3] developed a new strategy to construct Chebyshev-like methods with nearly optimal stability domain of order two. These methods are based on a weighted orthogonal polynomial and so the numerical methods satisfy a three-term recurrence relationship. In this case the stability interval is $[-l_s, 0]$ where $l_s \approx 0.81s^2$. These ideas were extended by Abdulle [1] who constructed a family of s-stage damped Chebyshev-like methods of order 4 that possess nearly optimal stability along the negative real axis and a three-term recurrence relationship. For these methods, $l_s \approx 0.35s^2$.

One of the drawbacks with Chebyshev methods is that the stability region can collapse to $s-1$ single points on the negative real axis due to the mini-max property of Chebyshev polynomials. Accordingly we require the modulus of the stability polynomial to be bounded by $\eta < 1$. The stability interval shrinks slightly but a strip around the negative real axis is included in stability region. With $\eta = 0.95$, $l_s \approx 0.81s^2$ for the second order Chebyshev methods.

In the case of stochastic differential equations (SDEs) the issues are much more complex. Nevertheless, Abdulle and Cirilli [2] have developed a family of explicit stochastic orthogonal Runge-Kutta Chebyshev (SROCK) methods with extended mean square (MS) stability regions. These methods have strong order 1 for Stratonovich problems driven by a single Wiener process, but only strong order 0.5 for non-commutative problems driven by multi-dimensional Wiener noise. They reduce to the first order Chebyshev methods when there is no noise. Such an approach is important because there are very few good numerical methods for solving stiff SDEs. If the stiffness is only in the deterministic component, then we can use semi-implicit methods that are implicit in the deterministic component of the method and explicit in the stochastic components. However, it is difficult to construct effective methods that are implicit in both the deterministic and stochastic components – see the balanced methods in [4, 19]. This is because the Wiener samples can be positive or negative with equal probability and can cause the iteration matrix for the nonlinear system solver to become singular. On the other hand, it is known in [6, 25] that the semi-implicit midpoint rule has special properties that makes it suitable for certain classes of stiff problems.

Despite the claimed performance of the SROCK methods, one of their drawbacks is the low strong order of 0.5 in the multi-dimensional Wiener process case. In fact it is a complicated process to construct high order methods for multi-dimensional noise

SDEs. Nevertheless, Burrage and Burrage [5] have given a general framework for deriving order conditions for stochastic Runge-Kutta (SRK) methods for Stratonovich SDEs. It generalizes the rooted tree theory of Butcher [7] by having $d+1$ coloured nodes of a tree when the stochastic differential equation (SDE) is driven by d Wiener processes.

Rößler [24] has used this rooted tree theory to construct strong order 1 RK methods with reduced complexity to multi-dimensional Wiener noise and constructed strong order 1.5 RK methods for scalar, diagonal and additive noise problems. In the weak order setting, Platen [8, 20], Tocino and Vigo-Aguiar [27] have constructed derivative free methods of weak order 2. Rößler [22, 23] and Komori [9, 11] have applied a rooted tree analysis to derive weak order conditions for a family of SRK methods and constructed a method of weak order 2 for non-commutative SDEs which is derivative free.

In this paper we shall put all these ideas together. We will construct a family of s-stage SRK methods with weak order 2 for multi-dimensional Wiener noise and with extended MS stability regions. The method will reduce to the second order Chebyshev methods of Abdulle and Medovikov [3] when the noise terms are set to zero. In Section 2 we will give some background material on Chebyshev-like methods for ODEs. In Section 3 we will give background material on SDEs. In Section 4 we will give a framework of SRK methods, while in Section 5 we will derive our new class of methods based on the stability analysis. Section 6 will present numerical results and Section 7 our conclusions.

2 Chebyshev-like methods for ODEs

Consider the system of initial value ODEs given by

$$\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0. \tag{2. 1}$$

The class of s-stage RK methods for solving (2. 1) is

$$\begin{aligned} \boldsymbol{y}_i &= \boldsymbol{y}_n + h\sum_{j=1}^{s} a_{ij}\boldsymbol{f}(t_n + c_j h, \boldsymbol{y}_j), \qquad i = 1, 2, \ldots, s, \\ \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h\sum_{j=1}^{s} b_j \boldsymbol{f}(t_n + c_j h, \boldsymbol{y}_j), \end{aligned} \tag{2. 2}$$

and is characterised by the Butcher tableau

$$\begin{array}{c|c} \boldsymbol{c} & A \\ \hline & \boldsymbol{b}^\top \end{array}$$

where $\boldsymbol{b} \stackrel{\text{def}}{=} (b_1\ b_2\ \cdots\ b_s)^\top$, A is a $s \times s$ matrix (a_{ij}), $\boldsymbol{c} \stackrel{\text{def}}{=} A\boldsymbol{e}$ and $\boldsymbol{e} \stackrel{\text{def}}{=} (1\ 1\ \cdots\ 1)^\top$. A RK method is explicit if A is strictly lower triangular.

If (2. 2) is applied to the linear, scalar test problem

$$y'(t) = \lambda y(t), \quad \Re(\lambda) \le 0, \tag{2. 3}$$

then

$$y_{n+1} = R(h\lambda) y_n$$

where

$$R(z) \stackrel{\text{def}}{=} 1 + z\boldsymbol{b}^\top (I - Az)^{-1}\boldsymbol{e}. \tag{2. 4}$$

Here R is called the stability function and for explicit methods $R(z)$ is a polynomial of at most degree s, namely

$$R(z) = 1 + \sum_{j=1}^{s} z^j \boldsymbol{b}^\top A^{j-1} \boldsymbol{e}. \tag{2. 5}$$

The stability region of (2. 2) is

$$S \stackrel{\text{def}}{=} \{z \mid |R(z)| \leq 1\}.$$

A method whose stability domain contains the whole left half of the complex plane is said to be A-stable, but such methods are by necessity implicit.

Van der Houwen and Sommeijer [28] constructed RK methods of order 1 that have maximal stability along the negative real axis, namely $[-l_s, 0]$, $l_s = 2s^2$. These methods have stability polynomial given by

$$R(z) = T_s(1 + z/s^2), \tag{2. 6}$$

where $T_n(x)$ is the Chebyshev polynomial of degree n defined by

$$T_n(\cos\theta) \stackrel{\text{def}}{=} \cos(n\theta)$$

or by the three term recurrence relation

$$T_0(x) \stackrel{\text{def}}{=} 1, \quad T_1(x) \stackrel{\text{def}}{=} x, \quad T_j(x) \stackrel{\text{def}}{=} 2xT_{j-1}(x) - T_{j-2}(x), \quad j \geq 2.$$

The corresponding RK method whose stability function is given by (2. 6) is

$$\begin{aligned}
&K_0 \stackrel{\text{def}}{=} \boldsymbol{y}_n, \qquad K_1 \stackrel{\text{def}}{=} \boldsymbol{y}_n + \frac{h}{s^2}\boldsymbol{f}(K_0), \\
&K_j \stackrel{\text{def}}{=} 2\frac{h}{s^2}\boldsymbol{f}(K_{j-1}) + 2K_{j-1} - K_{j-2} \ (j = 2, 3, \ldots, s), \\
&\boldsymbol{y}_{n+1} = K_s.
\end{aligned} \tag{2. 7}$$

The Chebyshev method given by (2. 7) can be written as an $s+1$-stage RK method with

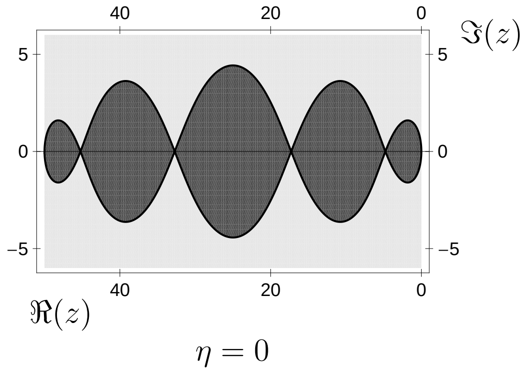
$$a_{ij} = \tilde{a}_{ij}, \qquad b_j = \tilde{a}_{s+1,j}, \qquad c_j = \left(\frac{j-1}{s}\right)^2 \qquad (1 \leq j \leq s),$$

where

$$\tilde{a}_{ij} \stackrel{\text{def}}{=} \begin{cases} \dfrac{i-1}{s^2} & (j = 1), \\ \dfrac{2(i-j)}{s^2} & (1 < j < i). \end{cases}$$

One of the drawbacks associated with this family of methods is that the stability region reduces to a single point at $s-1$ intermediate points in $[-2s^2, 0]$. This can be overcome by introducing a damping parameter η that allows a strip around the negative real axis to be included in the stability domain at a cost of a slightly shortening of the stability interval. This can be achieved by setting

$$R_s(z) = \frac{T_s(\omega_0 + \omega_1 z)}{T_s(\omega_0)}, \qquad \omega_0 \stackrel{\text{def}}{=} 1 + \eta/s^2, \qquad \omega_1 \stackrel{\text{def}}{=} \frac{T_s(\omega_0)}{T'(s)(\omega_0)}. \tag{2. 8}$$



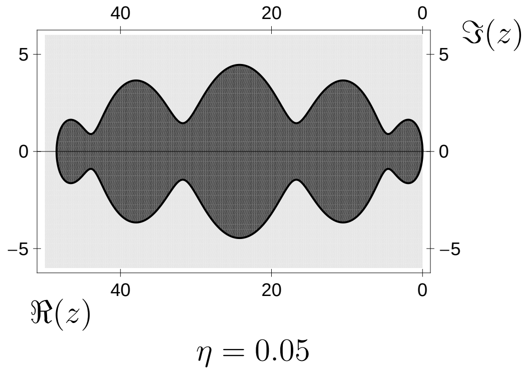


Figure 1: Stability region with $s = 5$ and $\eta = 0$ or 0.05

See Figure 1.

The corresponding RK method can be written as a three term recurrence relation

$$\begin{aligned} &K_0 \stackrel{\text{def}}{=} \boldsymbol{y}_n, \qquad K_1 \stackrel{\text{def}}{=} \boldsymbol{y}_n + h\frac{\omega_1}{\omega_0}\boldsymbol{f}(K_0), \\ &K_j \stackrel{\text{def}}{=} 2\frac{T_{j-1}(\omega_0)}{T_j(\omega_0)}(h\omega_1\boldsymbol{f}(K_{j-1}) + \omega_0 K_{j-1}) - \frac{T_{j-2}(\omega_0)}{T_j(\omega_0)}K_{j-2} \quad (2 \le j \le s), \\ &\boldsymbol{y}_{n+1} = K_s. \end{aligned} \tag{2. 9}$$

Despite giving more robust stability regions, these methods are still only order 1. Suppose now we require

$$R_s(z) = 1 + z + \frac{1}{2}z^2 + \sum_{j=3}^{s}\alpha_{sj}z^j$$

such that

$$|R_s(z)| \le 1 \text{ for } z \in [-l_s, 0], \qquad l_s \text{ as large as possible.}$$

Riha [21] showed that for a given s such polynomials exist, are unique, satisfy an equal ripple property on $s-1$ points and have equal two complex zeros. Lebedev [16] gave analytic expressions in terms of elliptic integrals. Abdulle and Medovikov [3] relaxed optimal stability and constructed approximations to these optimal stability polynomials using orthogonal polynomials such that

$$R_s(x) = w(x)P_{s-2}(x),$$

where if we write

$$w(x) \stackrel{\text{def}}{=} \bar{w}(a_s + x/d_s), \qquad P_j(x) \stackrel{\text{def}}{=} \bar{P}_j(a_s + x/d_s),$$

$\bar{w}(x)$ is of degree two with complex zeros and satisfied $\bar{w}(a_s) = 1$, then the orthogonal polynomials $\bar{P}_0(x), \bar{P}_1(x), \ldots, \bar{P}_{s-2}(x)$ are orthogonal with respect to the weight function $\bar{w}^2(x)/\sqrt{1-x^2}$ on $[-1, 1]$, $\bar{P}_0(a_s) = \bar{P}_1(a_s) = \cdots \bar{P}_{s-2}(a_s) = 1$, and satisfy a three-term recurrence relation. This leads to the method

$$\begin{aligned} &K_0 \stackrel{\text{def}}{=} \boldsymbol{y}_n, \qquad K_1 \stackrel{\text{def}}{=} \boldsymbol{y}_n + h\mu_1\boldsymbol{f}(K_0), \\ &K_j \stackrel{\text{def}}{=} h\mu_j\boldsymbol{f}(K_{j-1}) + (\theta_j + 1)K_{j-1} - \theta_j K_{j-2} \quad (j = 2, 3, \ldots, s-2), \\ &K_{s-1} \stackrel{\text{def}}{=} K_{s-2} + h\sigma_s\boldsymbol{f}(K_{s-2}), \qquad K_s^* \stackrel{\text{def}}{=} K_{s-1} + h\sigma_s\boldsymbol{f}(K_{s-1}), \\ &K_s \stackrel{\text{def}}{=} K_s^* - h\sigma_s(1 - \tau_s/\sigma_s^2)(\boldsymbol{f}(K_{s-1}) - \boldsymbol{f}(K_{s-2})), \qquad \boldsymbol{y}_{n+1} = K_s. \end{aligned} \tag{2. 10}$$

The computation of K_{s-1}, K_s^* can be viewed as a finishing procedure. When (2. 10) is applied to (2. 3), then

$$K_j = P_j(z)y_n \quad (j = 0, \ldots, s-2), \qquad K_s = w(z)K_{s-2}, \qquad y_{n+1} = R_s(z)y_n,$$

where

$$w(z) = 1 + 2\sigma_s z + \tau_s z^2 \tag{2. 11}$$

and

$$\begin{aligned} &P_0(z) = 1, \qquad P_1(z) = 1 + \mu_1 z, \\ &P_j(z) = (\mu_j z + \theta_j + 1)P_{j-1}(z) - \theta_j P_{j-2}(z), \quad j = 2, 3, \ldots, s-2. \end{aligned} \tag{2. 12}$$

If the zeros of w are $\alpha_s + i\beta_s$ and $\alpha_s - i\beta_s$, then

$$\sigma_s = \frac{a_s - \alpha_s}{d_s\left((a_s - \alpha_s)^2 + \beta_s^2\right)}, \qquad \tau_s = \frac{1}{d_s^2\left((a_s - \alpha_s)^2 + \beta_s^2\right)}, \qquad d_s = \frac{l_s}{1 + a_s}.$$

The value for l_s depends on what damping (2. 10) has. Away from $z = 0$ it is appropriate to require

$$|R_s(z)| \leq \eta < 1, \qquad z \leq -\varepsilon \quad (\varepsilon: \text{ small positive parameter})$$

and a number of authors set $\eta = 0.95$. In this case $l_s \approx 0.81s^2$ (rather than $0.82s^2$ with $\eta = 1$), and Abdulle and Medovikov [3] give the following values:

Table 1: Zeros of $w(x)$ and parameters

s	α_s	β_s	a_s	d_s	σ_s	τ_s
5	0.876008	0.138447	1.009632	9.48582	0.380486	0.300179
10	0.968456	3.399721D-2	1.001578	39.7252	0.370095	0.281274
20	0.992172	8.455313D-3	1.000433	160.722	0.367831	0.277039
50	0.998801	1.342920D-3	1.000114	1011.69	0.367929	0.276983
100	0.999704	3.355449D-4	1.000032	4049.18	0.367908	0.277012

Finally, the parameters μ_j, θ_j are determined from (2. 12) and by inserting 2 different values for $z \neq 0$, say r_1 and r_2, into (2. 12) and solving

$$(\mu_j r_i + \theta_j + 1)P_{j-1}(r_i) - \theta_j P_{j-2}(r_i) = P_j(r_i), \quad i = 1, 2$$

assuming the system is nonsingular.

Abdulle [1] extended this idea in the obvious way to construct Chebyshev-like methods of order 4, but since our SRK methods reduce to the order two method in the no noise case we do not extend on this analysis.

3 Methods for SDEs

Consider now the SDE

$$\mathrm{d}\boldsymbol{y}(t) = \sum_{j=0}^{d} \boldsymbol{g}_j(\boldsymbol{y}(t)) \circ \mathrm{d}w_j(t) \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \tag{3. 1}$$

which we will assume is in Stratonovich form. Here $w_0(t) = t$ and the $w_j(t),\ j = 1, 2, \ldots, d$ are independent Wiener processes which have the properties

$$E[w_j(t)] = 0, \qquad Var[w_j(t) - w_j(s)] = t - s, \quad t > s$$

and nonoverlapping Wiener increments are independent of one another.

If (3. 1) is interpreted in the Itô sense, then the simplest numerical method for simulating (3. 1) is the Euler-Maruyama (EM) method given by

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{g}_0(\boldsymbol{y}_n) + \sum_{j=1}^{d} \triangle w_j^{(n)} \boldsymbol{g}_j(\boldsymbol{y}_n), \tag{3. 2}$$

where $\triangle w_j^{(n)} \stackrel{\text{def}}{=} w_j(t_n + h) - w_j(t_n) \sim N(0, h) = \sqrt{h}N(0, 1)$. The EM method is known to have strong (pathwise) order 1/2 and weak (moment) order 1.

If a counterpart of (3. 2) is implicit in the deterministic component, then we can define the semi-implicit EM method with the same order properties as the EM method namely

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{g}_0(\boldsymbol{y}_{n+1}) + \sum_{j=1}^{d} \triangle w_j^{(n)} \boldsymbol{g}_j(\boldsymbol{y}_n). \tag{3. 3}$$

However if another counterpart of (3. 2) is implicit in the deterministic and stochastic components, there is no obvious generalisation of (3. 3) as introducing $\boldsymbol{y}_{n+1}$ into the stochastic components can cause unboundedness in the solution of the ensuring nonlinear systems. This has led to more esoteric implicit methods such as the balanced methods [19, 26]. However, rather than considering semi-implicit methods such as (3. 3), we are going to construct explicit methods with extended stability regions, using the ideas from Section 2.

As with the deterministic case, the quality of a stochastic method can be partly characterised by its stability region, associated with the scalar linear test equation

$$\mathrm{d}y(t) = ay(t)\mathrm{d}t + by(t)\mathrm{d}w, \qquad y(0) = y_0. \tag{3. 4}$$

In the Itô case the solution is

$$y(t) = \mathrm{e}^{\left(a - b^2/2\right)t + bw(t)} y_0,$$

while in the Stratonovich case, the solution is

$$y(t) = \mathrm{e}^{at + bw(t)} y_0.$$

In these cases, the solution is MS stable ($\lim_{t\to\infty} E[|y(t)|^2] = 0$) if

$$\text{Itô} : 2\Re(a) + |b|^2 < 0; \ \text{Stratonovich} : \Re(a) + (\Re(b))^2 < 0.$$

If an SRK method is applied to (3. 4),

$$E[|y_{n+1}|^2] = R(p, q)E[|y_n|^2],$$

where R is a multinomial in $p \stackrel{\text{def}}{=} ha$, $q \stackrel{\text{def}}{=} \sqrt{h}b$ if the method is explicit. Analogous to the deterministic case, the MS stability region of a method is defined as

$$S = \{p, q \in \boldsymbol{C} : \ |R(p, q)| \leq 1\}.$$

In the case of the EM method

$$R(p,q) = |1+p|^2 + |q|^2$$

and in the (p,q) plane with $p, q \in \boldsymbol{R}$, the stability region is a circle of radius 1 centred on (-1,0). On the other hand if the semi-implicit method (3. 3) is applied to (3. 4) then

$$R(p,q) = \frac{1+q^2}{(1-p)^2}, \quad p, q \in \boldsymbol{R}, \quad p \leq 0, \qquad S = \{(p,q): \ 2p + q^2 \leq p^2, \ p \leq 0\}.$$

Unlike the ODE setting the accuracy of an SDE numerical method is characterised by two order properties: strong (pathwise) order and weak (moment) order. More formally a method has strong order p_s or weak order p_w if there exists a constant C_s or C_w such that

$$E(|y_N - y(T)|) \leq C_s h^{p_s}, \qquad |E[F(y_N)] - E[F(y(T))]| \leq C_w h^{p_w}$$

with $T = Nh$ and h sufficiently small and for all functions $F : \boldsymbol{R}^n \mapsto \boldsymbol{R}$ that are $2(p_w+1)$ times continuously differentiable and for which all partial derivatives have polynomial growth.

In the completely general case when the $g_j(y)$ do not commute, then it is very difficult to construct strong order methods of order 1.5 or greater (see [5], for example). This is because of the plethora of order conditions and the need to simulate exactly the iterated stochastic integrals. However, if the noise terms are, for example, diagonal, then strong order 1.5 methods can be constructed [24]. In the weak order setting, these issues are not severe and so general weak order two methods can be constructed (but rarely weak order three); see [11].

We have already remarked on the difficulties in constructing methods that can cope with stiff SDEs. One very recent, and effective, approach was by Abdulle and Cirilli [2] who derived a family of explicit s-stage SROCK methods with extended MS stability regions. By controlling the number of stages large, stiff problems can be effectively solved without resource to the linear algebra overheads associated with implicit methods. When there is no noise, these methods reduce to the Chebyshev RK methods of order 1 (either undamped or damped). However, the drawbacks of these SROCK methods is that they only have strong order 1 in the $d = 1$ case, and for the multi-dimensional noise case, the strong order is only 0.5. Furthermore, namely all the analyses (order, stability) in this paper are performed in the $d = 1$ case. We extend these ideas to construct a family of s-stage SROCK2 methods that have weak order two for completely general multi-dimensional noise case and that reduce to the family of second order Chebyshev methods (ROCK2) presented in [3].

4 A general SRK framework

Komori [9] and Rößler [22] consider a general SRK weak order framework for solving (3. 1). Let $\boldsymbol{y}^{(j_a,j_b)}$, $j_a, j_b = 0, 1, \ldots, d$ be a vector of s internal stage components. Then the general form is

$$\begin{aligned} Y_i^{(j_a,j_b)} &= \zeta_i^{(j_a,j_b)} g_{j_b}\left(y_n + \sum_{j_c,j_d=0}^{d} \left(\boldsymbol{\alpha}_i^{(j_a,j_b,j_c,j_d)}\right)^\top \boldsymbol{Y}^{(j_c,j_d)} \right), \\ y_{n+1} &= y_n + \sum_{j_a,j_b=0}^{d} \boldsymbol{b}_{j_a j_b}^\top \boldsymbol{Y}^{(j_a,j_b)} \end{aligned} \tag{4. 1}$$

for $i = 1, 2, \ldots, s$ and $j_a, j_b = 0, 1, \ldots, d$. Here, the $\boldsymbol{b}_{j_a j_b}^\top$ and $\left(\boldsymbol{\alpha}_i^{(j_a j_b j_c j_d)}\right)^\top$ are row vectors of length s and $\zeta_i^{(j_a, j_b)}$ is a random variable independent of $\boldsymbol{y}_n$ and satisfies

$$E\left[\left(\zeta_i^{(j_a,j_b)}\right)^{2k}\right] = \begin{cases} K_1 h^{2k} & (j_b = 0), \\ K_2 h^k & (j_b \neq 0) \end{cases}$$

for constants K_1, K_2 and $k = 1, 2, \ldots$. Such a method can be generally viewed as having $s(d+1)^2$ stages. Note that we have made the interpretation simpler by assuming a scalar problem thus avoiding tensor notation.

Komori [10, 11] has shown that when (4. 1) is of weak order 2, there are 17 order conditions if $d = 1$, 29 order conditions if the noise is commutative and 38 order conditions if it is non-commutative, by setting many of $\boldsymbol{b}_{j_a j_b}$ and $\boldsymbol{\alpha}_i^{(j_a j_b j_c j_d)}$ at zero vectors.

In the present paper, we consider a simpler setting in order to decrease the number of evaluations of the diffusion coefficients [12]. In fact to get weak order 2 the structure in (4. 1) is in some sense too rich. For example, in the $d = 1$ case we can assume (4. 1) takes the form

$$\begin{aligned} Y_i^{(0)} &= h g_0 \left(y_n + \left(\boldsymbol{\alpha}_i^{(0)}\right)^\top \boldsymbol{Y}^{(0)} + \left(\boldsymbol{\alpha}_i^{(2)}\right)^\top \boldsymbol{Y}^{(1)} \right), \\ Y_i^{(1)} &= \zeta_i^{(1,1)} g_1 \left(y_n + \left(\boldsymbol{\alpha}_i^{(1)}\right)^\top \boldsymbol{Y}^{(0)} + \left(\boldsymbol{\alpha}_i^{(3)}\right)^\top \boldsymbol{Y}^{(1)} \right), \\ y_{n+1} &= y_n + \boldsymbol{b}_0^\top \boldsymbol{Y}^{(0)} + \boldsymbol{b}_1^\top \boldsymbol{Y}^{(1)}, \end{aligned} \qquad (4.\ 2)$$

while in the non-commutative, multi-dimensional Wiener noise case we can have

$$\begin{aligned} Y_i^{(0,0)} &= h g_0 \left(y_n + \left(\boldsymbol{\alpha}_i^{(0)}\right)^\top \boldsymbol{Y}^{(0,0)} + \left(\boldsymbol{\alpha}_i^{(2)}\right)^\top \sum_{j=1}^{d} \boldsymbol{Y}^{(j,j)} \right), \\ Y_i^{(j,j)} &= \zeta_i^{(j,j)} g_j \left(y_n + \left(\boldsymbol{\alpha}_i^{(1)}\right)^\top \boldsymbol{Y}^{(0,0)} + \left(\boldsymbol{\alpha}_i^{(3)}\right)^\top \boldsymbol{Y}^{(j,j)} + \left(\boldsymbol{\alpha}_i^{(4)}\right)^\top \sum_{\substack{l=1 \\ l \neq j}}^{d} \boldsymbol{Y}^{(l,l)} \right) \\ &\qquad (j = 1, 2, \ldots, d), \\ Y_i^{(j,l)} &= \zeta_i^{(j,l)} g_l \left(y_n + \left(\boldsymbol{\alpha}_i^{(5)}\right)^\top \boldsymbol{Y}^{(0,0)} + \left(\boldsymbol{\alpha}_i^{(6)}\right)^\top \sum_{\substack{m=1 \\ m \neq l}}^{d} \boldsymbol{Y}^{(l,m)} \right) \\ &\qquad (l \neq j, \quad j, l = 1, 2, \ldots, d), \\ y_{n+1} &= y_n + \boldsymbol{b}_0^\top \boldsymbol{Y}^{(0,0)} + \boldsymbol{b}_1^\top \sum_{j=1}^{d} \boldsymbol{Y}^{(j,j)} + \boldsymbol{b}_2^\top \sum_{l=1}^{d} \boldsymbol{Y}^{(k(l),l)}, \end{aligned} \qquad (4.\ 3)$$

where $k(l)$ is a value in $\{1, 2, \ldots, l-1, l+1, \ldots, m\}$. Clearly, putting $d = 1$ into (4. 3) gives (4. 2).

In order to construct weak order 2 methods the $\zeta^{(j,l)}$ are chosen as follows:

$$\zeta_i^{(j,l)} = \begin{cases} \triangle w_l & (j = l), \\ \triangle w_j \triangle \tilde{w}_l / \sqrt{h} & (l > j > 0 \text{ and } i = s-2), \\ -\triangle \tilde{w}_j \triangle w_l / \sqrt{h} & (j > l > 0 \text{ and } i = s-2), \\ \sqrt{h} & (j \neq l \text{ and } i \neq s-2), \end{cases} \qquad (4.\ 4)$$

where the $\triangle\tilde{w}_l$ are independent 2 point discrete random variables

$$P(\triangle\tilde{w}_j = \pm\sqrt{h}) = 1/2$$

and the $\triangle w_j$ are independent 3 point discrete random variables

$$P(\triangle w_j = \pm\sqrt{3h}) = 1/6, \qquad P(\triangle w_j = 0) = 2/3.$$

In the sequel, we will make the number of nonzero roles concerning stochastic parts as small as possible. For this, in addition to the assumption for $\zeta_i^{(j,l)}$ we suppose

$$b_{2,i} = 0 \quad (i < s-2), \qquad \alpha^{(6)}_{i_a i_b} = 0 \quad (i_a, i_b < s-2 \text{ or } i_a \le i_b). \tag{4. 5}$$

Moreover, we define

$$A^{(j)} \stackrel{\text{def}}{=} \left(\boldsymbol{\alpha}_1^{(j)} \ \boldsymbol{\alpha}_2^{(j)} \ \cdots \ \boldsymbol{\alpha}_s^{(j)}\right)^\top, \qquad \boldsymbol{c}^{(j)} \stackrel{\text{def}}{=} A^{(j)}\boldsymbol{e}, \qquad C^{(j)} \stackrel{\text{def}}{=} \mathrm{diag}\left(c_1^{(j)}, c_2^{(j)}, \ldots, c_s^{(j)}\right)$$

for $j = 0, 1, \ldots, 6$. With these conditions we now give, for completeness, the weak order 2 conditions for the 1 Wiener process case and for the completely general multi-dimensional Wiener case: for $d = 1$ case

1. $\boldsymbol{b}_0^\top \boldsymbol{e} = 1$,	2. $\boldsymbol{b}_0^\top \boldsymbol{c}^{(0)} = 1/2$,	3. $\boldsymbol{b}_0^\top \boldsymbol{c}^{(2)} = 1/2$,
4. $\boldsymbol{b}_0^\top C^{(2)}\boldsymbol{c}^{(2)} = 1/2$,	5. $\boldsymbol{b}_0^\top A^{(2)}\boldsymbol{c}^{(3)} = 1/4$,	6. $\boldsymbol{b}_1^\top \boldsymbol{e} = 1$,
7. $\boldsymbol{b}_1^\top \boldsymbol{c}^{(1)} = 1/2$,	8. $\boldsymbol{b}_1^\top \boldsymbol{c}^{(3)} = 1/2$,	9. $\boldsymbol{b}_1^\top A^{(3)}\boldsymbol{c}^{(1)} = 1/4$,
10. $\boldsymbol{b}_1^\top A^{(1)}\boldsymbol{c}^{(2)} = 0$,	11. $\boldsymbol{b}_1^\top C^{(1)}\boldsymbol{c}^{(3)} = 1/4$,	12. $\boldsymbol{b}_1^\top A^{(3)}A^{(3)}\boldsymbol{c}^{(3)} = 1/24$,
13. $\boldsymbol{b}_1^\top A^{(3)}C^{(3)}\boldsymbol{c}^{(3)} = 1/12$,	14. $\boldsymbol{b}_1^\top C^{(3)}A^{(3)}\boldsymbol{c}^{(3)} = 1/8$,	15. $\boldsymbol{b}_1^\top C^{(3)}C^{(3)}\boldsymbol{c}^{(3)} = 1/4$,
16. $\boldsymbol{b}_1^\top A^{(3)}\boldsymbol{c}^{(3)} = 1/6$,	17. $\boldsymbol{b}_1^\top C^{(3)}\boldsymbol{c}^{(3)} = 1/3$,	

additionally for multi-dimensional Wiener case

18. $\boldsymbol{b}_1^\top \boldsymbol{c}^{(4)} = 1/2$,	19. $\boldsymbol{b}_1^\top C^{(4)}A^{(4)}\boldsymbol{c}^{(4)} = 0$,	20. $\boldsymbol{b}_1^\top C^{(4)}\boldsymbol{c}^{(4)} = 1/2$,
21. $\boldsymbol{b}_1^\top A^{(3)}A^{(4)}\boldsymbol{c}^{(3)} = 1/8$,	22. $\boldsymbol{b}_1^\top A^{(4)}A^{(4)}\boldsymbol{c}^{(4)} = 0$,	23. $\boldsymbol{b}_1^\top A^{(4)}A^{(3)}\boldsymbol{c}^{(4)} = 0$,
24. $\boldsymbol{b}_1^\top A^{(3)}C^{(4)}\boldsymbol{c}^{(4)} = 1/4$,	25. $\boldsymbol{b}_1^\top A^{(4)}C^{(3)}\boldsymbol{c}^{(4)} = 0$,	26. $\boldsymbol{b}_1^\top C^{(3)}A^{(4)}\boldsymbol{c}^{(3)} = 1/8$,
27. $\boldsymbol{b}_1^\top C^{(4)}A^{(3)}\boldsymbol{c}^{(4)} = 1/4$,	28. $\boldsymbol{b}_1^\top C^{(3)}C^{(4)}\boldsymbol{c}^{(4)} = 1/4$,	29. $\boldsymbol{b}_1^\top A^{(3)}\boldsymbol{c}^{(4)} = 1/4$,
30. $\boldsymbol{b}_1^\top A^{(4)}\boldsymbol{c}^{(3)} = 1/4$,	31. $\boldsymbol{b}_1^\top A^{(4)}\boldsymbol{c}^{(4)} = 0$,	32. $\boldsymbol{b}_1^\top C^{(3)}\boldsymbol{c}^{(4)} = 1/4$,
33. $b_{2,s-2} = 0$,	34. $\boldsymbol{b}_2^\top \boldsymbol{e} = 0$,	35. $\boldsymbol{b}_2^\top \boldsymbol{c}^{(5)} = 0$,
36. $\alpha^{(6)}_{s,s-1} = 0$,	37. $\boldsymbol{b}_2^\top \boldsymbol{c}^{(6)} = 1/2$,	38. $\boldsymbol{b}_2^\top C^{(6)}\boldsymbol{c}^{(6)} = 0$.

Since ROCK2 is embedded in (4. 3) when there is no noise, $A^{(0)}$ and $\boldsymbol{b}_0$ are given by the Chebyshev formulation in (2. 10). We now assume that the $A^{(j)}$, $j = 1, 2, \ldots, 6$ take the partitioned form

$$\left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline A_1^{(j)} & A_2^{(j)} \end{array}\right)$$

while

$$\boldsymbol{b}_j^\top = \left[\boldsymbol{b}_j^{(1)} \ \boldsymbol{b}_j^{(2)}\right]^\top, \quad j = 1, 2.$$

In fact, for $A^{(6)}$ and $\boldsymbol{b}_2^\top$ we have already taken

$$A_1^{(6)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad A_2^{(6)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & 0 \end{pmatrix},$$

$$\boldsymbol{b}_2^\top = \begin{pmatrix} \mathbf{0}_{s-4}^\top & 0 & * & * & * \end{pmatrix}.$$

Here, $*$ denotes, possibly, a nonzero element.

Now if we want to make the number of nonzero roles in $A^{(3)}$ and $A^{(4)}$ as small as possible, then there is a unique solution [10, 11] so that

$$A_1^{(3)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad A_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 \\ 1/12 & 1/4 & 0 & 0 \\ -5/4 & 1/4 & 2 & 0 \end{pmatrix},$$

$$A_1^{(4)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad A_2^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{b}_1^\top = \begin{pmatrix} \mathbf{0}_{s-4}^\top & 1/8 & 3/8 & 3/8 & 1/8 \end{pmatrix}.$$

Similarly, for $A^{(2)}$ we will assume

$$A_1^{(2)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad A_2^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}.$$

Finally, in order to achieve good stability properties, we will assume

$$A_1^{(1)} = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ * & * & \cdots & * \\ * & * & \cdots & * \end{pmatrix}, \qquad A_2^{(1)} = \begin{pmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$$

as well as

$$A_1^{(5)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_{s-2,1}^{(0)} & \alpha_{s-2,2}^{(0)} & \cdots & \alpha_{s-2,s-4}^{(0)} \\ \alpha_{s-2,1}^{(0)} & \alpha_{s-2,2}^{(0)} & \cdots & \alpha_{s-2,s-4}^{(0)} \\ \alpha_{s-2,1}^{(0)} & \alpha_{s-2,2}^{(0)} & \cdots & \alpha_{s-2,s-4}^{(0)} \end{pmatrix}, \qquad A_2^{(5)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_{s-2,s-3}^{(0)} & 0 & 0 & 0 \\ \alpha_{s-2,s-3}^{(0)} & 0 & 0 & 0 \\ \alpha_{s-2,s-3}^{(0)} & 0 & 0 & 0 \end{pmatrix}.$$

It is remarkable that Condition 35 is automatically satisfied from Conditions 33, 34 and the assumptions on $\boldsymbol{b}_2$ and $A^{(5)}$.

5 Mean square stability

We will now apply our SROCK2 method to the linear, scalar multiplicative noise problem

$$\mathrm{d}y(t) = \lambda_0 y(t)\mathrm{d}t + \sum_{j=1}^{d} \lambda_j y(t) \circ \mathrm{d}w_j(t), \tag{5. 1}$$

where $\lambda_j,\ j=0,1,\ldots,d$ are real values. Because of the structure we can easily see that

$$Y_j^{(0,0)} = P_{j-1}(h\lambda_0)y_0, \quad j=1,2,\ldots,s-3.$$

We now compute successively $Y_i^{(0,0)}, Y_i^{(j,j)}, Y_i^{(j,l)}$ for $i=s-2,s-1,s$ and y_{n+1}, using the order conditions to try and get a simple form for these expressions. Once we have found the form

$$y_{n+1} = Ry_n,$$

the MS stability function is given by

$$\hat{R} = E[R^2].$$

$\hat{R}$ in fact will be a function of $p \stackrel{\text{def}}{=} h\lambda_0$, $q_j \stackrel{\text{def}}{=} h\lambda_j^2,\ j=1,2,\ldots,d$.

5.1 How to determine $\boldsymbol{\alpha}_i^{(1)}$ and $\boldsymbol{\alpha}_i^{(2)}$

In order to determine the values of $\boldsymbol{\alpha}_i^{(1)}$ and $\boldsymbol{\alpha}_i^{(2)}$ for $i=s-3,s-2,s-1,s$, let us begin with the scalar noise problem. By applying (4. 2) to (5. 1) when $d=1$, we obtain

$$\begin{aligned}
R &= R(p,\triangle w_1,\lambda_1)\\
&= \left(1+2\sigma_s p+\tau_s p^2\right)P_{s-2}(p)+\triangle w_1\lambda_1\left(\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3\right)\\
&\quad +(\triangle w_1\lambda_1)^2\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)+(\triangle w_1\lambda_1)^3\left(\beta_{30}+\beta_{31}p\right)+(\triangle w_1\lambda_1)^4\beta_{40},
\end{aligned}$$

and thus

$$\begin{aligned}
\hat{R} &= \hat{R}(p,q_1)\\
&= \left(1+2\sigma_s p+\tau_s p^2\right)^2\left(P_{s-2}(p)\right)^2\\
&\quad +q_1\Big\{2\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)\left(1+2\sigma_s p+\tau_s p^2\right)P_{s-2}(p)\\
&\qquad +\left(\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3\right)^2\Big\}\\
&\quad +3q_1^2\Big\{2\beta_{40}\left(1+2\sigma_s p+\tau_s p^2\right)P_{s-2}(p)+\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)^2\\
&\qquad +2\left(\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3\right)\left(\beta_{30}+\beta_{31}p\right)\Big\}\\
&\quad +9q_1^3\Big\{2\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)\beta_{40}+\left(\beta_{30}+\beta_{31}p\right)^2\Big\}+27q_1^4\beta_{40}^2,
\end{aligned} \tag{5. 2}$$

where

$$
\begin{aligned}
&Q_{s-3}(p) \overset{\text{def}}{=} 1 + \sum_{j=1}^{s-3} \alpha_{s-3,j}^{(1)} p P_{j-1}(p), \\
&Q_i(p) \overset{\text{def}}{=} 1 + \sum_{j=1}^{i-1} \alpha_{ij}^{(1)} p P_{j-1}(p), \quad i = s-2, s-1, s, \\
&\beta_{10} \overset{\text{def}}{=} \sum_{i=s-3}^{s} b_{1,i} Q_i(p), \qquad \beta_{11} \overset{\text{def}}{=} \sum_{i=s-2}^{s} \sum_{j=s-3}^{i-1} b_{0,i} \alpha_{ij}^{(2)} Q_j(p), \\
&\beta_{12} \overset{\text{def}}{=} \sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} \sum_{k=s-3}^{j-1} b_{0,i} \alpha_{ij}^{(0)} \alpha_{jk}^{(2)} Q_k(p), \\
&\beta_{13} \overset{\text{def}}{=} b_{0,s} \alpha_{s,s-1}^{(0)} \alpha_{s-1,s-2}^{(0)} \alpha_{s-2,s-3}^{(2)} Q_{s-3}(p), \\
&\beta_{20} \overset{\text{def}}{=} \sum_{i=s-2}^{s} \sum_{j=s-3}^{i-1} b_{1,i} \alpha_{ij}^{(3)} Q_j(p), \\
&\beta_{21} \overset{\text{def}}{=} \sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} \sum_{k=s-3}^{j-1} \left(b_{0,i} \alpha_{ij}^{(2)} \alpha_{jk}^{(3)} + b_{1,i} \alpha_{ij}^{(1)} \alpha_{jk}^{(2)} \right) Q_k(p), \\
&\beta_{22} \overset{\text{def}}{=} \Big(b_{1,s} \alpha_{s,s-1}^{(1)} \alpha_{s-1,s-2}^{(0)} \alpha_{s-2,s-3}^{(2)} + b_{0,s} \alpha_{s,s-1}^{(0)} \alpha_{s-1,s-2}^{(2)} \alpha_{s-2,s-3}^{(3)} \\
&\qquad\quad + b_{0,s} \alpha_{s,s-1}^{(2)} \alpha_{s-1,s-2}^{(1)} \alpha_{s-2,s-3}^{(2)} \Big) Q_{s-3}(p), \\
&\beta_{30} \overset{\text{def}}{=} \sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} \sum_{k=s-3}^{j-1} b_{1,i} \alpha_{ij}^{(3)} \alpha_{jk}^{(3)} Q_k(p), \\
&\beta_{31} \overset{\text{def}}{=} \Big(b_{0,s} \alpha_{s,s-1}^{(2)} \alpha_{s-1,s-2}^{(3)} \alpha_{s-2,s-3}^{(3)} + b_{1,s} \alpha_{s,s-1}^{(1)} \alpha_{s-1,s-2}^{(2)} \alpha_{s-2,s-3}^{(3)} \\
&\qquad\quad + b_{1,s} \alpha_{s,s-1}^{(3)} \alpha_{s-1,s-2}^{(1)} \alpha_{s-2,s-3}^{(2)} \Big) Q_{s-3}(p), \\
&\beta_{40} \overset{\text{def}}{=} b_{1,s} \alpha_{s,s-1}^{(3)} \alpha_{s-1,s-2}^{(3)} \alpha_{s-2,s-3}^{(3)} Q_{s-3}(p).
\end{aligned} \tag{5. 3}
$$

Incidentally, from the assumption on $A^{(0)}$ and $A^{(2)}$ we have the following relationship between elements of $A^{(0)}$ and $P_j(p)$:

$$
P_{i-1}(p) = 1 + \sum_{j=1}^{i-1} \alpha_{ij}^{(0)} p P_{j-1}(p), \qquad i = 1, 2, \ldots, s-1. \tag{5. 4}
$$

This will be useful to determine the parameter values.

In (5. 2), there is the term $\beta_{31} p^3$. When it is isolated, it causes problems in stability since it includes $p^3 Q_{s-3}(p)$. We consider avoiding it. First, let us assume

$$
\alpha_{s-3,j}^{(1)} = \alpha_{s-2,j}^{(0)}, \qquad j = 1, 2, \ldots, s-3. \tag{5. 5}
$$

Then, we obtain

$$
Q_{s-3}(p) = P_{s-3}(p) \tag{5. 6}
$$

and

$$
\begin{aligned}
\beta_{12} + \beta_{13} p = &\sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} b_{0,i} \alpha_{ij}^{(0)} \alpha_{j,s-3}^{(2)} P_{s-3}(p) \\
&+ b_{0,s} \alpha_{s,s-1}^{(0)} \alpha_{s-1,s-2}^{(2)} Q_{s-2}(p) + b_{0,s} \alpha_{s,s-1}^{(0)} \alpha_{s-1,s-2}^{(0)} \alpha_{s-2,s-3}^{(2)} p P_{s-3}(p).
\end{aligned} \tag{5. 7}
$$

Since

$$
\begin{aligned}
&\alpha_{s-1,s-2}^{(2)} Q_{s-2}(p) + \alpha_{s-1,s-2}^{(0)} \alpha_{s-2,s-3}^{(2)} p P_{s-3}(p) \\
= \;&\alpha_{s-2,s-3}^{(2)} \left(\frac{\alpha_{s-1,s-2}^{(2)}}{\alpha_{s-2,s-3}^{(2)}} + \frac{\alpha_{s-1,s-2}^{(2)}}{\alpha_{s-2,s-3}^{(2)}} \sum_{j=1}^{s-3} \alpha_{s-2,j}^{(1)} p P_{j-1}(p) + \alpha_{s-1,s-2}^{(0)} p P_{s-3}(p) \right)
\end{aligned}
$$

in the right-hand side of (5. 7), let us assume

$$\begin{aligned} \alpha^{(2)}_{s-1,s-2} &= \alpha^{(2)}_{s-2,s-3}, && (5.\ 8)\\ \alpha^{(1)}_{s-2,j} &= \alpha^{(0)}_{s-1,j}, \qquad j=1,2,\ldots,s-3. && (5.\ 9) \end{aligned}$$

Then, we have

$$\begin{aligned} &Q_{s-2}(p) = P_{s-2}(p) - \alpha^{(0)}_{s-1,s-2} p P_{s-3}(p),\\ &\beta_{12}+\beta_{13}p\\ &= \sum_{i=s-1}^{s}\sum_{j=s-2}^{i-1} b_{0,i}\alpha^{(0)}_{ij}\alpha^{(2)}_{j,s-3}P_{s-3}(p) + b_{0,s}\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}P_{s-2}(p). \end{aligned} \qquad (5.\ 10)$$

We can see that $pP_{s-3}(p)$ disappears from $\beta_{12}+\beta_{13}p$ in (5. 10), whereas it does in (5. 7). This is a merit in (5. 8) and (5. 9).

Next, similarly to (5. 5), let us assume

$$\alpha^{(1)}_{s-1,j} = \alpha^{(1)}_{s,j} = \alpha^{(0)}_{s-2,j}, \qquad j=1,2,\ldots,s-3. \qquad (5.\ 11)$$

Then,

$$\begin{aligned} Q_{s-1}(p) &= P_{s-3}(p) + \alpha^{(1)}_{s-1,s-2}pP_{s-3}(p),\\ Q_{s}(p) &= P_{s-3}(p) + \alpha^{(1)}_{s,s-2}pP_{s-3}(p) + \alpha^{(1)}_{s,s-1}pP_{s-2}(p). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3\\ &= \Bigg[\sum_{\substack{i=s-3\\ i\neq s-2}}^{s} b_{1,i} + \left(-b_{1,s-2}\alpha^{(0)}_{s-1,s-2} + \sum_{i=s-1}^{s} b_{1,i}\alpha^{(1)}_{i,s-2} + \sum_{i=s-2}^{s} b_{0,i}\alpha^{(2)}_{i,s-3} + b_{0,s}\alpha^{(2)}_{s,s-1}\right)p\\ &\quad +b_{0,s}\left(\sum_{i=s-2}^{s-1}\alpha^{(0)}_{s,i}\alpha^{(2)}_{i,s-3} + \alpha^{(2)}_{s,s-1}\alpha^{(1)}_{s-1,s-2} - \alpha^{(2)}_{s,s-2}\alpha^{(0)}_{s-1,s-2}\right)p^2\Bigg] P_{s-3}(p)\\ &\quad + \left[b_{1,s-2} + \left(b_{1,s}\alpha^{(1)}_{s,s-1} + \sum_{i=s-1}^{s} b_{0,i}\alpha^{(2)}_{i,s-2}\right)p + b_{0,s}\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}p^2\right]P_{s-2}(p), \end{aligned} \qquad (5.\ 12)$$

$$\begin{aligned} &\beta_{20}+\beta_{21}p+\beta_{22}p^2\\ &= \Bigg\{\sum_{i=s-2}^{s} b_{1,i}\alpha^{(3)}_{i,s-3} + b_{1,s}\alpha^{(3)}_{s,s-1} + \Bigg[-\sum_{i=s-1}^{s} b_{1,i}\alpha^{(3)}_{i,s-2}\alpha^{(0)}_{s-1,s-2} + b_{1,s}\alpha^{(3)}_{s,s-1}\alpha^{(1)}_{s-1,s-2}\\ &\qquad + \sum_{i=s-1}^{s}\sum_{j=s-2}^{i-1}\left(b_{1,i}\alpha^{(1)}_{i,j}\alpha^{(2)}_{j,s-3} + b_{0,i}\alpha^{(2)}_{i,j}\alpha^{(3)}_{j,s-3}\right)\Bigg]p\\ &\quad +b_{0,s}\left[\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-1,s-2}\alpha^{(3)}_{s-2,s-3} + \alpha^{(2)}_{s,s-1}\alpha^{(1)}_{s-1,s-2}\alpha^{(2)}_{s-2,s-3}\right.\\ &\quad \left. -\alpha^{(2)}_{s,s-1}\alpha^{(3)}_{s-1,s-2}\alpha^{(0)}_{s-1,s-2}\right]p^2\Bigg\} P_{s-3}(p), \end{aligned} \qquad (5.\ 13)$$

$$\begin{aligned} &\beta_{30}+\beta_{31}p\\ = &\Bigg[\sum_{i=s-1}^{s}\sum_{j=s-2}^{i-1} b_{1,i}\alpha^{(3)}_{i,j}\alpha^{(3)}_{j,s-3} + \Big(b_{1,s}\alpha^{(1)}_{s,s-1}\alpha^{(2)}_{s-1,s-2}\alpha^{(3)}_{s-2,s-3} + b_{1,s}\alpha^{(3)}_{s,s-1}\alpha^{(1)}_{s-1,s-2}\alpha^{(2)}_{s-2,s-3} \end{aligned}$$

$$+b_{0,s}\alpha^{(2)}_{s,s-1}\alpha^{(3)}_{s-1,s-2}\alpha^{(3)}_{s-2,s-3} - b_{1,s}\alpha^{(3)}_{s,s-1}\alpha^{(3)}_{s-1,s-2}\alpha^{(0)}_{s-1,s-2}\Big) p\Big] P_{s-3}(p) \qquad (5.\ 14)$$

$$+b_{1,s}\alpha^{(3)}_{s,s-1}\alpha^{(3)}_{s-1,s-2}P_{s-2}(p).$$

Now, let us solve the system of the order conditions for the scalar noise case. Since $A^{(0)}$, $A^{(3)}$, $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$ are given, we can solve it as follows [10]:

1) From Conditions 7 and 9, seek $c^{(1)}_{s-1}$ and $c^{(1)}_s$. Then, Condition 11 is automatically satisfied.
2) From Conditions 3 and 4, seek seek $c^{(2)}_{s-1}$ and $c^{(2)}_s$.
3) Substitute the results in 2) into Condition 10, and seek $\alpha^{(1)}_{s,s-2}$.
4) Substitute the results in 2) into Condition 5, and seek $\alpha^{(2)}_{s,s-2}$ or $\alpha^{(2)}_{s,s-1}$.

Thus, we have

$$\begin{aligned}
c^{(1)}_{s-1} &= 1 - \frac{1}{2}\left(c^{(1)}_{s-3} + c^{(1)}_{s-2}\right), \qquad c^{(1)}_{s} = 1 + \frac{1}{2}\left(c^{(1)}_{s-3} - 3c^{(1)}_{s-2}\right), \\
c^{(2)}_{s-1} &= \frac{1 - 2b_{0,s-2}\alpha^{(2)}_{s-2,s-3}}{2\left(b_{0,s-1} + b_{0,s}\right)} \mp \frac{\sqrt{\gamma_0}}{2b_{0,s-1}\left(b_{0,s-1} + b_{0,s}\right)}, \\
c^{(2)}_{s} &= \frac{1 - 2b_{0,s-2}\alpha^{(2)}_{s-2,s-3}}{2\left(b_{0,s-1} + b_{0,s}\right)} \pm \frac{\sqrt{\gamma_0}}{2b_{0,s}\left(b_{0,s-1} + b_{0,s}\right)}, \\
\alpha^{(1)}_{s,s-2} &= \frac{-\alpha^{(1)}_{s,s-1} + 2\left\{b_{0,s-2}\alpha^{(1)}_{s,s-1} - 3\left(b_{0,s-1} + b_{0,s}\right)\alpha^{(1)}_{s-1,s-2}\right\}\alpha^{(2)}_{s-2,s-3}}{2\left(b_{0,s-1} + b_{0,s}\right)\alpha^{(2)}_{s-2,s-3}} \\
&\quad \pm \frac{\alpha^{(1)}_{s,s-1}\sqrt{\gamma_0}}{2\left(b_{0,s-1} + b_{0,s}\right)b_{0,s-1}\alpha^{(2)}_{s-2,s-3}},
\end{aligned} \qquad (5.\ 15)$$

and

$$\alpha^{(2)}_{s,s-1} = \frac{3 - 8b_{0,s-1}\alpha^{(2)}_{s-1,s-2} - 8b_{0,s}\alpha^{(2)}_{s,s-2}}{4b_{0,s}}$$

or

$$\alpha^{(2)}_{s,s-2} = \frac{3 - 8b_{0,s-1}\alpha^{(2)}_{s-1,s-2} - 4b_{0,s}\alpha^{(2)}_{s,s-1}}{8b_{0,s}}, \qquad (5.\ 16)$$

where

$$\gamma_0 \stackrel{\text{def}}{=} b_{0,s-1}b_{0,s}\left\{-4b_{0,s-2}\left(\sum_{i=s-2}^{s} b_{0,i}\right)\left(\alpha^{(2)}_{s-2,s-3}\right)^2 + 4b_{0,s-2}\alpha^{(2)}_{s-2,s-3} + 2\sum_{i=s-1}^{s} b_{0,i} - 1\right\}.$$

Noting

$$b_{0,s-1} = 2\sigma_s - \frac{\tau_s}{\sigma_s} < 0, \qquad b_{0,s} = \frac{\tau_s}{\sigma_s} > 0$$

from Table 1 and Appendix A, we obtain a solution for $\gamma_0 \geq 0$:

$$\begin{aligned}
\alpha^{(2)}_{s-2,s-3} &\leq \frac{b_{0,s-2} - \sqrt{b_{0,s-2}\left(b_{0,s-1} + b_{0,s}\right)\gamma_1}}{b_{0,s-2}(\gamma_1 + 1)}, \\
\alpha^{(2)}_{s-2,s-3} &\geq \frac{b_{0,s-2} + \sqrt{b_{0,s-2}\left(b_{0,s-1} + b_{0,s}\right)\gamma_1}}{b_{0,s-2}(\gamma_1 + 1)}
\end{aligned}$$

when

$$b_{0,s-2}\left(b_{0,s-1}+b_{0,s}\right)\gamma_1 \geq 0,$$

where

$$\gamma_1 \stackrel{\text{def}}{=} 2\left(b_{0,s-}+b_{0,s-1}+b_{0,s}\right)-1.$$

Incidentally, $\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3$ has terms $p^2P_{s-3}(p)$ and $p^2P_{s-2}(p)$ in (5. 12), whereas $\beta_{20}+\beta_{21}p+\beta_{22}p^2$ has a term $p^2P_{s-3}(p)$ in (5. 13). Since these can let $\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3$ or $\beta_{20}+\beta_{21}p+\beta_{22}p^2$ largely oscillate in a long interval, we need to make the absolute values of their coefficients small in order to get large stability regions. As such a parameter setting for all $s \geq 4$, we choose

$$\alpha^{(2)}_{s-2,s-3} = \frac{b_{0,s-2}-\sqrt{b_{0,s-2}\left(b_{0,s-1}+b_{0,s}\right)\gamma_1}}{b_{0,s-2}(\gamma_1+1)}, \tag{5. 17}$$

$$\alpha^{(2)}_{s,s-1} = -\frac{\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-1,s-2}\alpha^{(3)}_{s-2,s-3}}{\alpha^{(1)}_{s-1,s-2}\alpha^{(2)}_{s-2,s-3}-\alpha^{(3)}_{s-1,s-2}\alpha^{(0)}_{s-1,s-2}}. \tag{5. 18}$$

Then, it is remarkable that $\gamma_0 = 0$ for (5. 17) and the coefficient of $p^2P_{s-3}(p)$ vanishes in $\beta_{20}+\beta_{21}p+\beta_{22}p^2$ for (5. 18).

Consequently, from (5. 15), (5. 16) and (5. 18) we obtain a final solution for $\boldsymbol{\alpha}^{(1)}_i$ and $\boldsymbol{\alpha}^{(2)}_i$ $(i = s-3, s-2, s-1, s)$

$$\alpha^{(1)}_{s-1,s-2} = 1-\frac{1}{2}\left(3c^{(0)}_{s-2}+c^{(0)}_{s-1}-\alpha^{(0)}_{s-1,s-2}\right),$$

$$\alpha^{(1)}_{s,s-2} = -\frac{2(b_{0,s-1}+b_{0,s})\alpha^{(2)}_{s-2,s-3}\gamma_3}{(\gamma_1+1)\alpha^{(2)}_{s-2,s-3}-1}+1-\frac{1}{2}\left(c^{(0)}_{s-2}+3c^{(0)}_{s-1}-3\alpha^{(0)}_{s-1,s-2}\right)$$

$$\alpha^{(1)}_{s,s-1} = -\frac{2(b_{0,s-1}+b_{0,s})\alpha^{(2)}_{s-2,s-3}\gamma_3}{(\gamma_1+1)\alpha^{(2)}_{s-2,s-3}-1}, \qquad \alpha^{(2)}_{s-1,s-3} = \frac{1-(\gamma_1+1)\alpha^{(2)}_{s-2,s-3}}{2\left(b_{0,s-1}+b_{0,s}\right)},$$

$$\alpha^{(2)}_{s,s-3} = \frac{1-2b_{0,s-2}\alpha^{(2)}_{s-2,s-3}}{2\left(b_{0,s-1}+b_{0,s}\right)}-\frac{3-8b_{0,s-1}\alpha^{(2)}_{s-2,s-3}}{8b_{0,s}}+\frac{4\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}}{3\gamma_2},$$

$$\alpha^{(2)}_{s,s-2} = \frac{3-8b_{0,s-1}\alpha^{(2)}_{s-2,s-3}}{8b_{0,s}}+\frac{4\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}}{3\gamma_2}, \qquad \alpha^{(2)}_{s,s-1} = -\frac{8\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}}{3\gamma_2},$$

as well as (5. 5), (5. 8), (5. 9), (5. 11) and (5. 17), where

$$\gamma_2 \stackrel{\text{def}}{=} 2\left(2-3c^{(0)}_{s-2}-c^{(0)}_{s-1}+\alpha^{(0)}_{s-1,s-2}\right)\alpha^{(2)}_{s-2,s-3}-\alpha^{(0)}_{s-1,s-2},$$

$$\gamma_3 \stackrel{\text{def}}{=} 4-5c^{(0)}_{s-2}-3c^{(0)}_{s-1}+3\alpha^{(0)}_{s-1,s-2}.$$

By applying Abdulle's parameter values* to this solution, we obtain Figure 2. The solid, dash or dotted line means the behaviour of $\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3$, $\beta_{20}+\beta_{21}p+\beta_{22}p^2$ or $\beta_{30}+\beta_{31}p$, respectively. On the other hand, since β_{40} is very small, it is omitted. Here, note that $\eta = 0.95$.

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*Readers can get them from a fortran code "rock2.f" in http://www.unige.ch/~hairer/software.html.

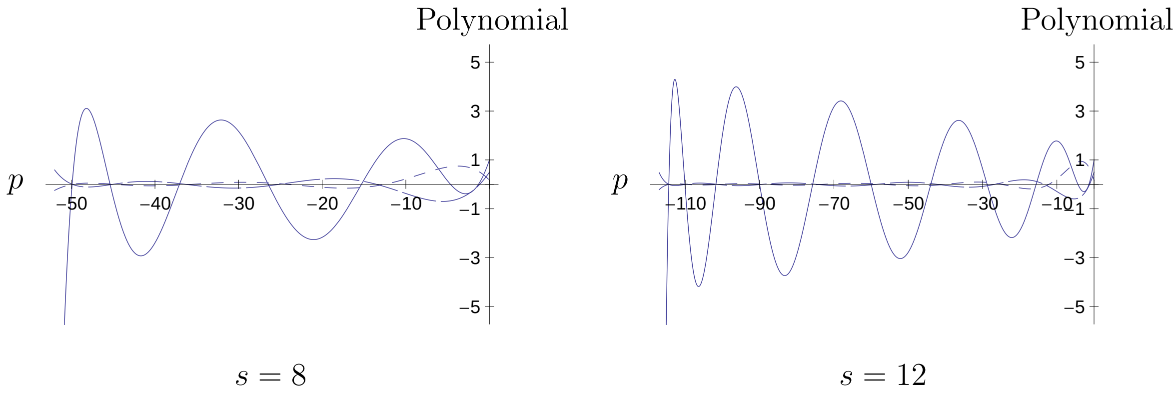


Figure 2: Behaviour of $\beta_{10} + \beta_{11}p + \beta_{12}p^2 + \beta_{13}p^3$, $\beta_{20} + \beta_{21}p + \beta_{22}p^2$ or $\beta_{30} + \beta_{31}p$

5.2 Multi-dimensional noise problem

In this subsection let us deal with the multi-dimensional noise problem. By applying (4. 3) to (5. 1) and by using Condition 33 and the assumption on $A^{(5)}$, we obtain

$$R = R\left(p, \{\triangle w_j\}_{j=1}^d, \{\triangle \tilde{w}_l\}_{l=2}^d, \{\lambda_j\}_{j=1}^d\right) = \left(1 + 2\sigma_s p + \tau_s p^2\right) P_{s-2}(p) + \sum_{j=1}^d G_j$$

and thus

$$\begin{aligned} \hat{R} &= \hat{R}(p, \{q_j\}_{j=1}^d) \\ &= \left(1 + 2\sigma_s p + \tau_s p^2\right)^2 (P_{s-2}(p))^2 \\ &\quad +2\left(1 + 2\sigma_s p + \tau_s p^2\right) P_{s-2}(p) \\ &\qquad \times \left\{ \sum_{j=1}^m q_j \left(\beta_{20} + \beta_{21}p + \beta_{22}p^2\right) + 3\left(\sum_{j=1}^d q_j\right)^2 \beta_{40} \right\} \\ &\quad + \sum_{j=1}^d E\left[G_j^2\right] + 2\sum_{j=1}^{d-1} \sum_{l=j+1}^d E\left[G_j G_l\right], \end{aligned} \tag{5. 19}$$

where $3\beta_{40} = \delta_{220}$ was used in the second expression of the right-hand side to simplify it and where

$$\begin{aligned} G_j &\stackrel{\text{def}}{=} \triangle w_j \lambda_j \left(\beta_{10} + \beta_{11}p + \beta_{12}p^2 + \beta_{13}p^3\right) + (\triangle w_j \lambda_j)^2 \left(\beta_{20} + \beta_{21}p + \beta_{22}p^2\right) \\ &\quad +(\triangle w_j \lambda_j)^3 (\beta_{30} + \beta_{31}p) + (\triangle w_j \lambda_j)^4 \beta_{40} \\ &\quad +\triangle w_j \lambda_j \sum_{\substack{l=1 \\ l \neq j}}^d \left[\triangle w_l \lambda_l \left(\delta_{110} + \delta_{111}p + \delta_{112}p^2\right) + (\triangle w_l \lambda_l)^2 (\delta_{120} + \delta_{121}p)\right] \\ &\quad +(\triangle w_j \lambda_j)^2 \sum_{\substack{l=1 \\ l \neq j}}^d \left[\triangle w_l \lambda_l (\delta_{210} + \delta_{211}p) + (\triangle w_l \lambda_l)^2 \delta_{220}\right], \end{aligned}$$

$$\delta_{110} \stackrel{\text{def}}{=} \sum_{i=s-1}^{s} \sum_{j=s-3}^{s-2} b_{1,i} \alpha_{ij}^{(4)} Q_j(p),$$

$$\delta_{111} \stackrel{\text{def}}{=} \sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} \sum_{k=s-3}^{j-1} b_{1,i} \alpha_{ij}^{(1)} \alpha_{jk}^{(2)} Q_k(p) + b_{0,s} \alpha_{s,s-1}^{(2)} \sum_{i=s-3}^{s-2} \alpha_{s-1,i}^{(4)} Q_i(p),$$

$$\delta_{112} \stackrel{\text{def}}{=} \left(b_{1,s}\alpha^{(1)}_{s,s-1}\alpha^{(0)}_{s-1,s-2} + b_{0,s}\alpha^{(2)}_{s,s-1}\alpha^{(1)}_{s-1,s-2}\right)\alpha^{(2)}_{s-2,s-3}Q_{s-3}(p),$$

$$\delta_{120} \stackrel{\text{def}}{=} \left(\sum_{i=s-1}^{s} b_{1,i}\alpha^{(4)}_{i,s-2}\right)\alpha^{(3)}_{s-2,s-3}Q_{s-3}(p),$$

$$\delta_{121} \stackrel{\text{def}}{=} \left(b_{1,s}\alpha^{(1)}_{s,s-1}\alpha^{(2)}_{s-1,s-2} + b_{0,s}\alpha^{(2)}_{s,s-1}\alpha^{(4)}_{s-1,s-2}\right)\alpha^{(3)}_{s-2,s-3}Q_{s-3}(p),$$

$$\delta_{210} \stackrel{\text{def}}{=} b_{1,s}\alpha^{(3)}_{s,s-1}\sum_{i=s-3}^{s-2}\alpha^{(4)}_{s-1,i}Q_i(p), \qquad \delta_{211} \stackrel{\text{def}}{=} b_{1,s}\alpha^{(3)}_{s,s-1}\alpha^{(1)}_{s-1,s-2}\alpha^{(2)}_{s-2,s-3}Q_{s-3}(p),$$

$$\delta_{220} \stackrel{\text{def}}{=} b_{1,s}\alpha^{(3)}_{s,s-1}\alpha^{(4)}_{s-1,s-2}\alpha^{(3)}_{s-2,s-3}Q_{s-3}(p).$$

Our $\boldsymbol{b}_1$, $A^{(3)}$ and $A^{(4)}$ satisfy Conditions 18–32 [10, 11]. In addition, as we have said, our $A^{(5)}$ satisfies Condition 35. Thus, all we need to do is to seek a solution for Conditions 34, 37 and 38 under the Conditions 33 and 36. From these, we have

$$\alpha^{(6)}_{s,s-2} = \frac{1}{4b_{2,s}}, \qquad \alpha^{(6)}_{s-1,s-2} = -\frac{1}{4b_{2,s}}, \qquad b_{2,s-1} = -b_{2,s}.$$

Here, note that $\hat{R}$ in (5. 19) does not depend on the free parameters $b_{2,s}$. Expressions for $E[G_j^2]$ and $E[G_jG_l]$ will be shown in Appendix B.

Finally, we show MS-stability regions, in which $\hat{R} < 1$. In general, however, such a region lies in the $d+1$-dimensional space with respect to p and q_i, $i = 1, 2, \ldots, d$. For this, let us assume $q_1 = q_2 = \cdots = q_d$ and denote $d \times q_i$ by $\tilde{q}$. Then, in Figure 3 a dark-colored part indicates an MS-stability region, whereas the part enclosed by the two straight lines $\tilde{q} = -p$ and $\tilde{q} = 0$ indicates the region in which the test SDE is stable in mean square. It is remarkable that $s = 4$ is the minimum stage number because our SROCK2 methods are of weak order 2 [10, 11].

6 Numerical results

In the previous section we have derived our SROCK2 methods, which have the free parameters $b_{2,s}$. Now let us set it at 1 and confirm its performance in two numerical examples.

The first example comes from the following heat equation with noise:

$$\mathrm{d}u(t,x) = (D\Delta u(t,x))\mathrm{d}t + ku(t,x) \circ \mathrm{d}w_1(t), \qquad (t,x) \in [0,T] \times [0,1], \tag{6. 1}$$

which was dealt with in [2]. Here, Δ is the Laplacian operator, D is the diffusion coefficient, and k is a noise parameter.

Let us suppose that $u(0,x) = 1$ as an initial condition and $u(t,0) = \frac{\partial u(t,x)}{\partial x}|_{x=1} = 0$ as mixed boundary conditions, and set $D = k = 1$ for simplicity. If we discretize the space interval by $M+1$ equidistant points x_i, $i = 0, 1, \ldots, M$ and define a vector-valued function $\boldsymbol{y}(t)$ by

$$\boldsymbol{y}(t) \stackrel{\text{def}}{=} \left(u(t,x_1)\ u(t,x_2)\ \cdots\ u(t,x_M)\right),$$

then we obtain

$$\mathrm{d}\boldsymbol{y}(t) = A\boldsymbol{y}(t)\mathrm{d}t + \boldsymbol{y}(t) \circ \mathrm{d}w_1(t), \qquad \boldsymbol{y}(0) = (1\ 1\ \cdots 1)^\top \tag{6. 2}$$

by applying the central difference scheme to (6. 1) and by using the relationship

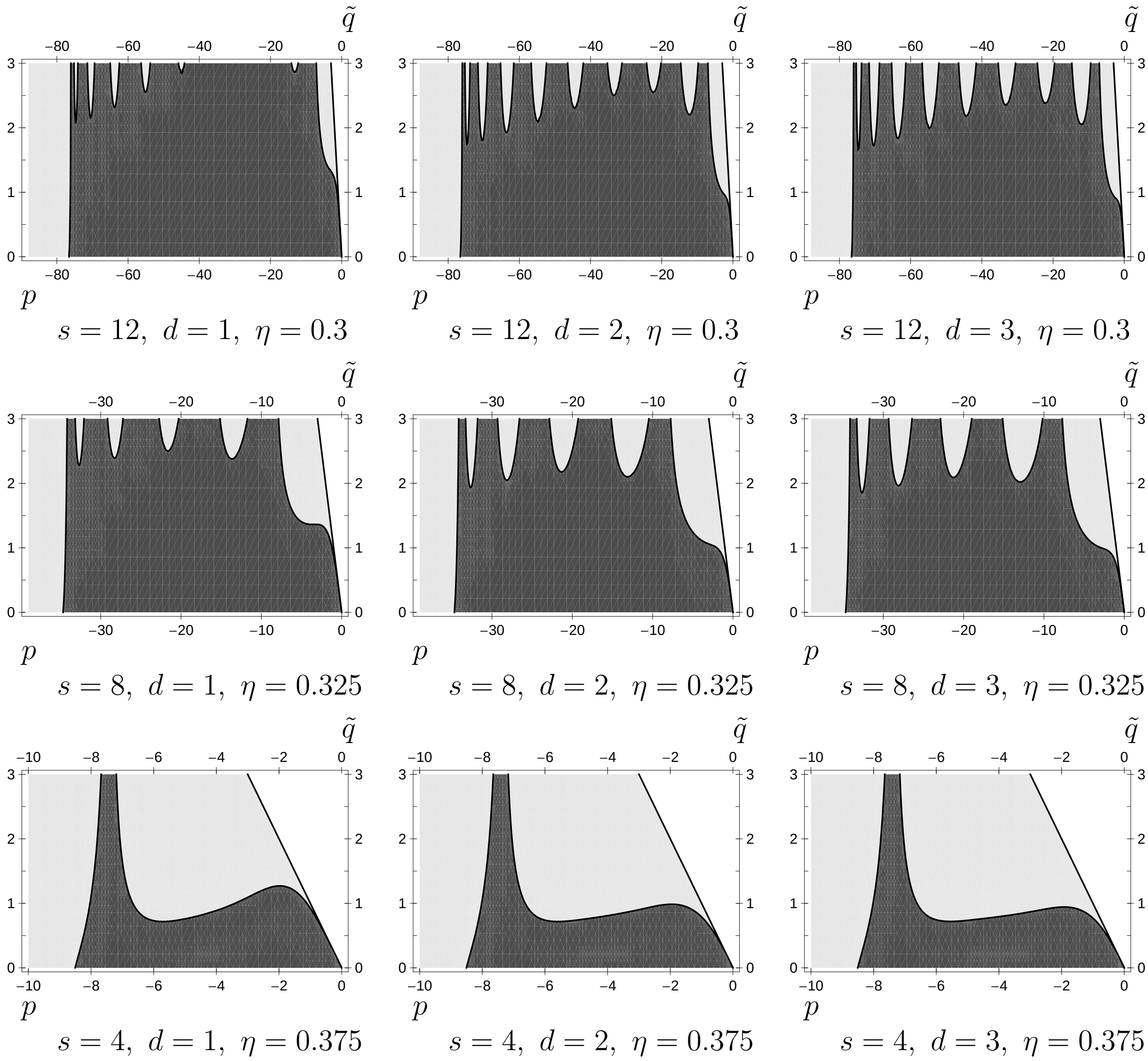


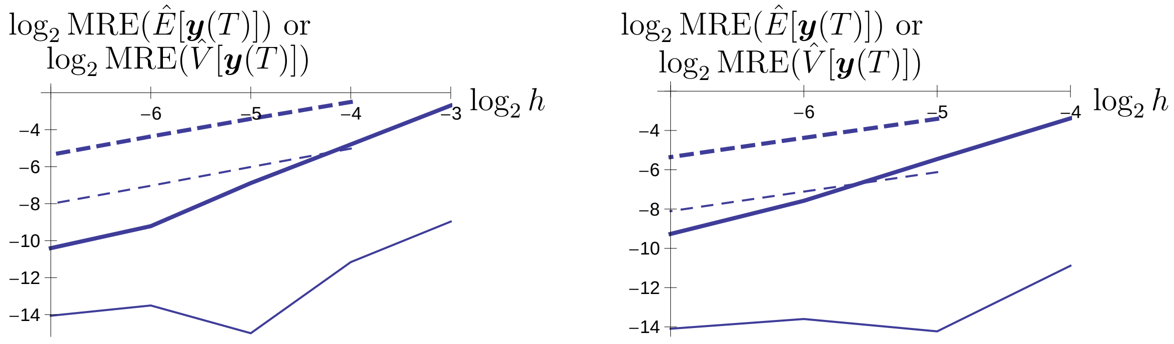
Figure 3: MS stability region of the SROCK2 schemes for some s, d and η

$u(t, x_{M-1}) = u(t, x_{M+1})$ from the boundary conditions, where

$$A = M^2 \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}.$$

It is known that the eigenvalues of A are distributed around the negative real axis in the interval $(-4M^2, 0)$ [2]. Thus, remark that it becomes more difficult to numerically solve (6. 2) as M becomes larger.

Setting $T = 1$, we seek $\boldsymbol{y}_N$ by schemes, and calculate the arithmetic means $\langle y_{N,i} \rangle$ and variances $\langle (y_{N,i} - \langle y_{N,i} \rangle)^2 \rangle$ of the ith element of $\boldsymbol{y}_N$ $(i = 1, 2, \ldots, M)$ as estimates of the means $E[y_i(T)]$ and variances $V[y_i(T)]$, respectively. On the other hand, because (6. 2)



Case of $M = 40$ Case of $M = 100$

Figure 4: Relative errors in the first example

is linear, we can get a system of ODEs with respect to the mean and variance of $\boldsymbol{y}(t)$. In fact, it is given by $\mathrm{d}E[\boldsymbol{y}(t)]/\mathrm{d}t = \tilde{A}E[\boldsymbol{y}(t)]$ and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}(t) = \tilde{A}\Phi(t) + \Phi(t)\tilde{A}^\top + \Phi(t) + E[\boldsymbol{y}(t)](E[\boldsymbol{y}(t)])^\top,$$

where

$$\tilde{A} \stackrel{\text{def}}{=} A + \frac{1}{2}\mathrm{diag}(1, 1, \ldots, 1), \qquad \Phi(t) = E\left[(\boldsymbol{y}(t) - E[\boldsymbol{y}(t)])(\boldsymbol{y}(t) - E[\boldsymbol{y}(t)])^\top\right].$$

We simulate 256×10^6 independent trajectories for a given h. In Monte Carlo simulation for SDEs, statistical independence properties in pseudorandom numbers are very important [13]. In addition, their period needs to be very long. For this, we use the Mersenne twister [17]. By it, for example, we generate a pseudorandom number for $\triangle\tilde{w}_l/\sqrt{h}$ which takes ± 1.

The results are indicated in Fig. 4. In the figure, $\mathrm{MRE}(\hat{\boldsymbol{\varphi}})$ denotes $||\hat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}_0||/||\boldsymbol{\varphi}_0||$, where $\hat{\boldsymbol{\varphi}}$ is an estimator of an unknown vector $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}_0$ is a numerical solution of the ODEs with respect to it. Solid or dotted lines means the SROCK2 or SROCK schemes, respectively, whereas normal or thick lines stand for a relative error of mean or variance. In the case of $M = 40$ or 100, calculations were performed by the SROCK2 schemes with $s = 41$ or 104 and the SROCK schemes with $s = 50$ or 100 [2], respectively. Remark that numerical solutions were not stably obtained by the SROCK when $\log_2 h = -3$ or -4.

The second example comes from the following chemistry problem, which has three species and three reaction channels:

$$S_1 + S_2 \xrightarrow{k_1} S_3, \qquad S_3 \xrightarrow{k_2} S_1 + S_2, \qquad S_3 \xrightarrow{k_3} S_1 + P.$$

Here, P is a product. This leads to the the following Itô SDE

$$\mathrm{d}\boldsymbol{y}(t) = \sum_{j=1}^{3}\boldsymbol{\nu}_j a_j(\boldsymbol{y}(t))\mathrm{d}t + \sum_{j=1}^{3}\boldsymbol{\nu}_j\sqrt{a_j(\boldsymbol{y}(t))}\mathrm{d}w_j(t), \tag{6. 3}$$

where $a_1(\boldsymbol{y}) \stackrel{\text{def}}{=} k_1 y_1 y_2$, $a_2(\boldsymbol{y}) \stackrel{\text{def}}{=} k_2 y_3$, $a_3(\boldsymbol{y}) \stackrel{\text{def}}{=} k_3 y_3$,

$$\boldsymbol{\nu}_1 \stackrel{\text{def}}{=} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\nu}_2 \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{\nu}_3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

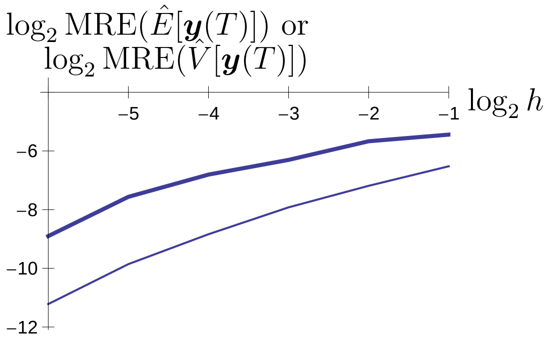


Figure 5: Relative errors in the second example

and y_i means the density of S_i. Note that (6. 3) corresponds to the Stratonovich SDE

$$\begin{aligned} \mathrm{d}\boldsymbol{y}(t) = \boldsymbol{\nu}_1 \left(a_1(\boldsymbol{y}(t)) + \frac{k_1}{4}(y_1(t) + y_2(t)) \right) \mathrm{d}t + \sum_{j=2}^{3} \boldsymbol{\nu}_j \left(a_j(\boldsymbol{y}(t)) + \frac{k_j}{4} \right) \mathrm{d}t \\ + \sum_{j=1}^{3} \boldsymbol{\nu}_j \sqrt{a_j(\boldsymbol{y}(t))} \circ \mathrm{d}w_j(t). \end{aligned} \tag{6. 4}$$

Let us suppose that $\boldsymbol{y}_0 = (0\ 0\ 20)^\top$ as an initial condition and $0 \leq t \leq 10$, and set $k_1 = 0.002$, $k_2 = 0.5$ and $k_3 = 0.04$. Similarly to the first example, we simulate 256×10^6 independent trajectories for a given h, but differently from the example we cannot get a system of ODEs with respect to the mean and variance of $\boldsymbol{y}(t)$ in the closed form. When we seek $\mathrm{MRE}(\hat{\boldsymbol{\varphi}})$, thus, we use as $\boldsymbol{\varphi}_0$ a numerical solution of the SDE for $h = 2^{-7}$. The result for the SROCK2 scheme with 4 stages are indicated in Fig. 5. The positiveness of any element of the numerical solutions is demanded in this example. Thus, if a trajectory with a negative element had appeared in calculations, it was replaced with another new trajectory. Associated with this, it is remarkable that we can see only lower convergence order than expected order in Fig. 5.

7 Conclusions

We have derived the explicit s-stage SROCK2 schemes with weak order 2 for non-commutative SDEs. Because the schemes are equivalent to the ROCK2 schemes when they are applied to ODEs and their parameter values are carefully chosen for stability, they have large MS stability regions along the negative real axis.

The SROCK2 schemes have the following other features.

- The SROCK2 schemes have only $d-1$ random variables ($\triangle\tilde{w}_j$'s) except $\triangle w_j$'s for one step. On the other hand, for example, the EM scheme has only $\triangle w_j$'s for one step. When the step size is set at h/k ($k \geq 2$) to obtain better approximations, however, it in total needs $k \times d$ random variables to proceed with calculation from time $t = nh$ to $t = (n+1)h$. Since the SROCK2 schemes can give good approximations even if the step size is set at h, they need less random variables than the EM scheme in such cases.

- The computational costs for each diffusion coefficient do not depend on the dimension of the Wiener process. For details, see Appendix C. On the other hand, for example, the scheme proposed in [11] depends it. Thus, the SROCK2 schemes have big advantages not only in the number of random variables, but also in computational costs.

In the first mildly stiff problem, the SROCK2 schemes have shown good performance in accuracy and stability. In the second problem concerning positivity of the solutions, the convergence order has been reduced practically, but nevertheless shows very good performance.

Expressions for the implementation of the SROCK2 schemes are given in Appendix C. In addition, source codes for the schemes and examples are obtainable from Komori's homepage:

http://galois.ces.kyutech.ac.jp/˜komori/

Appendix

A Elements of $A^{(0)}$ and $\boldsymbol{b}_0$

As we have seen, $A^{(0)}$ and $\boldsymbol{b}_0$ are given by the Chebyshev formulation in (2. 10). Thus, by setting $g_0(y) = f(y)$ and $g_1(y) = 0$ in (4. 2) and by comparing it with (2. 10), we obtain the following relationship:

$$\begin{aligned}
&\alpha_{ij}^{(0)} \stackrel{\text{def}}{=} (1+\theta_{i-1})\,\alpha_{i-1,j}^{(0)} - \theta_{i-1}\alpha_{i-2,j}^{(0)}, \quad j = 1, 2, \ldots, i-3,\\
&\alpha_{i,i-2}^{(0)} \stackrel{\text{def}}{=} (1+\theta_{i-1})\,\alpha_{i-1,i-2}^{(0)}, \qquad \alpha_{i,i-1}^{(0)} \stackrel{\text{def}}{=} \mu_{i-1}, \qquad \alpha_{ij}^{(0)} \stackrel{\text{def}}{=} 0, \quad j = i, i+1, \ldots, s
\end{aligned}$$

for $i = 1, 2, \ldots, s-1$, and

$$\begin{aligned}
&\alpha_{sj}^{(0)} \stackrel{\text{def}}{=} \alpha_{s-1,j}^{(0)}, \quad j = 1, 2, \ldots, s-2, \qquad \alpha_{s,s-1}^{(0)} \stackrel{\text{def}}{=} \sigma_s, \qquad \alpha_{ss}^{(0)} \stackrel{\text{def}}{=} 0,\\
&b_{0,j} \stackrel{\text{def}}{=} \alpha_{sj}^{(0)}, \quad j = 1, 2, \ldots, s-2, \qquad b_{0,s-1} \stackrel{\text{def}}{=} \alpha_{s,s-1}^{(0)} + \sigma_s - \tau_s/\sigma_s, \qquad b_{0,s} \stackrel{\text{def}}{=} \tau_s/\sigma_s.
\end{aligned}$$

B Expressions for $E[G_j^2]$ and $E[G_jG_l]$

$E[G_j^2]$ and $E[G_jG_l]$ are given as follows:

$$\begin{aligned}
E[G_j^2] = {}& q_j\left(\beta_{10} + \beta_{11}p + \beta_{12}p^2 + \beta_{13}p^3\right)^2\\
&+ 3q_j^2\left[\left(\beta_{20} + \beta_{21}p + \beta_{22}p^2\right)^2 + 2\left(\beta_{10} + \beta_{11}p + \beta_{12}p^2 + \beta_{13}p^3\right)(\beta_{30} + \beta_{31}p)\right]\\
&+ 9q_j^3\left[(\beta_{30} + \beta_{31}p)^2 + 2\left(\beta_{20} + \beta_{21}p + \beta_{22}p^2\right)\beta_{40}\right] + 27q_j^4\beta_{40}^2\\
&+ q_j\sum_{\substack{m=1\\ m\neq j}}^{d} q_m\left[\left(\delta_{110} + \delta_{111}p + \delta_{112}p^2\right)^2\right.\\
&\qquad\qquad \left. +2\left(\beta_{10} + \beta_{11}p + \beta_{12}p^2 + \beta_{13}p^3\right)(\delta_{120} + \delta_{121}p)\right]
\end{aligned}$$

$$
\begin{aligned}
&+2q_j \sum_{\substack{m=1\\m\neq j}}^{d} q_m^2 \left(\delta_{120}+\delta_{121}p\right)^2 + q_j \left(\sum_{\substack{m=1\\m\neq j}}^{d} q_m\right)^2 \left(\delta_{120}+\delta_{121}p\right)^2 \\
&+3q_j^2 \sum_{\substack{m=1\\m\neq j}}^{d} q_m \left[\left(\delta_{210}+\delta_{211}p\right)^2 + 2\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)\delta_{220}\right. \\
&\qquad\qquad \left. +2\left(\beta_{30}+\beta_{31}p\right)\left(\delta_{120}+\delta_{121}p\right)\right] \\
&+6q_j^2 \sum_{\substack{m=1\\m\neq j}}^{d} q_m^2\delta_{220}^2 + 3q_j^2 \left(\sum_{\substack{m=1\\m\neq j}}^{d} q_m\right)^2 \delta_{220}^2 + 18q_j^3 \sum_{\substack{m=1\\m\neq j}}^{d} q_m\beta_{40}\delta_{220},
\end{aligned}
$$

$$
\begin{aligned}
E[G_jG_l] = {}& q_jq_l \left[\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)^2 + \left(\delta_{110}+\delta_{111}p+\delta_{112}p^2\right)^2\right. \\
&\qquad \left. +2\left(\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3\right)\left(\delta_{210}+\delta_{211}p\right)\right] \\
&+3q_jq_l\left(q_j+q_l\right)\left[\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)\beta_{40} + \left(\beta_{30}+\beta_{31}p\right)\left(\delta_{210}+\delta_{211}p\right)\right] \\
&+q_jq_l\left(q_j+q_l+2\sum_{m=1}^{d} q_m\right) \\
&\quad \times \left[\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)\delta_{220} + \left(\delta_{120}+\delta_{121}p\right)\left(\delta_{210}+\delta_{211}p\right)\right] \\
&+q_jq_l \sum_{\substack{m=1\\m\neq j}}^{d} q_m \left(\delta_{210}+\delta_{211}p\right)^2 + 4q_j^2q_l^2\delta_{220}^2 \\
&+2q_jq_l\left(q_j \sum_{\substack{m=1\\m\neq j}}^{d} q_m + q_l \sum_{\substack{m=1\\m\neq l}}^{d} q_m\right)\delta_{220}^2 \\
&+q_jq_l\left(\sum_{m=1}^{d} q_m\right)^2 \delta_{220}^2 + 2q_jq_l \sum_{m=1}^{d} q_m^2\delta_{220}^2.
\end{aligned}
$$

Here, note that $3\beta_{40} = \delta_{220}$ was used in the last four expressions of the right-hand side concerning $E[G_jG_l]$ to simplify them.

C Useful form for the implimetation of SROCK2

From (2. 10) and Appendix A we have

$$
K_{i-1} = y_n + \left(\boldsymbol{\alpha}_i^{(0)}\right)^\top \boldsymbol{Y}^{(0,0)}, \qquad i = 1, 2, \ldots, s-2. \tag{C. 1}
$$

From (5. 5) and this,

$$
Y_{s-3}^{(j,j)} = \triangle w_j g_j(K_{s-3}), \qquad Y_{s-2}^{(0,0)} = hg_0\left(K_{s-3} + \alpha_{s-2,s-3}^{(2)} \sum_{j=1}^{d} Y_{s-3}^{(j,j)}\right).
$$

From (2. 10), (C. 1) and Appendix A,

$$
K_{s-2}^{-} \overset{\text{def}}{=} K_{s-2} - h\mu_{s-2}g_0(K_{s-3}) = y_n + \sum_{i=1}^{s-3} \alpha_{s-1,i}^{(0)} Y_i^{(0,0)}.
$$

From (2. 10) and this,

$$Y_{s-2}^{(j,j)} = \triangle w_j g_j \left(K_{s-2}^{-} + \alpha_{s-2,s-3}^{(3)} Y_{s-3}^{(j,j)} \right).$$

Similarly, we have

$$\begin{aligned}
Y_{s-2}^{(j,l)} &= \zeta_{s-2}^{(j,l)} g_l(K_{s-3}), \\
Y_{s-1}^{(0,0)} &= h g_0 \left(K_{s-2}^{-} + \alpha_{s-1,s-2}^{(0)} Y_{s-2}^{(0,0)} + \sum_{i=s-3}^{s-2} \alpha_{s-1,i}^{(2)} \sum_{j=1}^{d} Y_i^{(j,j)} \right), \\
Y_{s-1}^{(j,j)} &= \triangle w_j g_j \left(K_{s-3} + \alpha_{s-1,s-2}^{(1)} Y_{s-2}^{(0,0)} + \sum_{i=s-3}^{s-2} \alpha_{s-1,i}^{(3)} Y_i^{(j,j)} + \sum_{i=s-3}^{s-2} \alpha_{s-1,i}^{(4)} \sum_{\substack{l=1 \\ l \neq j}}^{d} Y_i^{(l,l)} \right), \\
Y_i^{(k(l),l)} &= \sqrt{h} g_l \left(K_{s-3} + \alpha_{i,s-2}^{(6)} \sum_{\substack{m=1 \\ m \neq l}}^{d} Y_{s-2}^{(l,m)} \right) \qquad (i = s-1, s), \\
Y_s^{(0,0)} &= h g_0 \left(K_{s-2}^{-} + \alpha_{s-1,s-2}^{(0)} Y_{s-2}^{(0,0)} + \sigma_s Y_{s-1}^{(0,0)} + \sum_{i=s-3}^{s-1} \alpha_{s,i}^{(2)} \sum_{j=1}^{d} Y_i^{(j,j)} \right), \\
Y_s^{(j,j)} &= \triangle w_j g_j \left(K_{s-3} + \sum_{i=s-2}^{s-1} \alpha_{s,i}^{(1)} Y_i^{(0,0)} + \sum_{i=s-3}^{s-1} \alpha_{s,i}^{(3)} Y_i^{(j,j)} + \sum_{i=s-3}^{s-2} \alpha_{s,i}^{(4)} \sum_{\substack{l=1 \\ l \neq j}}^{d} Y_i^{(l,l)} \right), \\
y_{n+1} &= K_{s-2}^{-} + \alpha_{s-1,s-2}^{(0)} Y_{s-2}^{(0,0)} + \sigma_s \left(Y_{s-1}^{(0,0)} + Y_s^{(0,0)} \right) - \sigma_s \left(1 - \frac{\tau_s}{\sigma_s^2} \right) \left(Y_s^{(0,0)} - Y_{s-1}^{(0,0)} \right) \\
&\quad + \sum_{i=s-3}^{s} b_{1,i} \sum_{j=1}^{d} Y_i^{(j,j)} + \sum_{i=s-1}^{s} b_{2,i} \sum_{l=1}^{d} Y_i^{(k(l),l)}.
\end{aligned}$$

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