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ON THE RELATION BETWEEN THE WEAK PALAIS-SMALE CONDITION AND COERCIVITY BY ZHONG

TOMONARI SUZUKI

Abstract. In this paper, we discuss Zhong’s result of that the weak Palais-Smale condition implies coercivity under some assumption in [Nonlinear Anal., 29 (1997), 1421–1431]. We also give a simple proof of Zhong’s result. Further we generalize the result in Caklovic, Li and Willem [Differential Integral Equations, 3 (1990), 799–800].

1. Introduction

Throughout this paper we denote by \( \mathbb{N} \) the set of all positive integers and by \( \mathbb{R} \) the set of all real numbers.

Let \( f \) be a function from a Banach space \( X \) into \((−∞, +∞]\). We recall that \( f \) is called \( Gâteaux differentiable \) at \( x \in X \) with \( f(x) \in \mathbb{R} \) if there exists a continuous linear functional \( f'(x) \) such that
\[
\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = \langle f'(x), y \rangle
\]
holds for every \( y \in X \). \( f \) is said to be coercive if
\[
\lim_{r \to \infty} \inf_{\|x\| \geq r} f(x) = \infty
\]
holds. Also, \( f \) is said to satisfy the \( \text{weak Palais-Smale condition} \) [17] if there exists a nondecreasing function \( h \) from \([0, ∞)\) into itself satisfying \( \int_0^\infty (1/(1 + h(t)))dt = ∞ \), and the following condition: Every sequence \( \{x_n\} \) in \( X \) such that \( \{f(x_n)\} \) is bounded and
\[
\lim_{n \to \infty} \|f'(x_n)\| (1 + h(\|x_n\|)) = 0
\]
contains a convergent subsequence. This definition seems to be weaker than the definition in [17]. However they are equivalent; see Section 5. In the case of \( h(t) = 0 \) for all \( t \in [0, ∞) \), we call that \( f \) satisfies the \( \text{Palais-Smale condition} \). In the case of \( h(t) = t \) for all \( t \in [0, ∞) \), we call that \( f \) satisfies the \( \text{Cerami-Palais-Smale condition} \) [4].

It is well known that the Palais-Smale condition implies coercivity under some assumption; see Brézis and Nirenberg [2], Caklovic, Li and Willem [3] and others. In 1997, Zhong [17] generalized these results and proved that the weak Palais-Smale condition implies coercivity. However the proof is slightly complicated.

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Key words and phrases. Palais-Smale condition, coercivity, Ekeland’s variational principle, Zhong’s variational principle, \( \tau \)-distance.

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In this paper, we discuss Zhong’s result and we also give a simple proof of it. Further we generalize the result in Caklovic, Li and Willem [3]. We also discuss the conditions of the continuity of \( h, \int_0^\infty (1/(1 + h(t))) dt = \infty \), and the completeness of \( X \).

2. \( \tau \)-Distance

In our discussion, the notion of \( \tau \)-distance plays an important role.

Let \( (X, d) \) be a metric space. Then a function \( p \) from \( X \times X \) into \([0, \infty)\) is called a \( \tau \)-distance on \( X \) [10] if there exists a function \( \eta \) from \( X \times [0, \infty) \) into \([0, \infty)\) and the following are satisfied:

\[
\begin{align*}
(\tau 1) & \quad p(x, z) \leq p(x, y) + p(y, z) \text{ for all } x, y, z \in X; \\
(\tau 2) & \quad \eta(x, 0) = 0 \text{ and } \eta(x, t) \geq t \text{ for all } x \in X \text{ and } t \in [0, \infty), \text{ and } \eta \text{ is concave and continuous in its second variable}; \\
(\tau 3) & \quad \lim_n x_n = x \text{ and } \lim_n \sup \{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0 \text{ imply } p(w, x) \leq \lim\inf_n p(w, x_n) \text{ for all } w \in X; \\
(\tau 4) & \quad \lim_n \sup \{p(x_n, y_m) : m \geq n\} = 0 \text{ and } \lim_n \eta(x_n, t_n) = 0 \text{ imply } \lim_n \eta(y_n, t_n) = 0; \\
(\tau 5) & \quad \lim_n \eta(z_n, p(z_n, x_n)) = 0 \text{ and } \lim_n \eta(z_n, p(z_n, y_n)) = 0 \text{ imply } \lim_n d(x_n, y_n) = 0.
\end{align*}
\]

We note that \( \eta \) is strictly increasing in its second variable. We also note that the metric \( d \) is a \( \tau \)-distance on \( X \). Many useful propositions and examples are stated in [7–16].

Though the following is a corollary of Proposition 2 in [12], we give a proof.

**Proposition 1.** Let \( (X, d) \) be a metric space with a \( \tau \)-distance \( p \). Let \( q \) be a function from \( X \times X \) into \([0, \infty)\). Suppose that

(i) \( q \) satisfies (\( \tau 1 \))\( _q \), i.e., \( q(x, z) \leq q(x, y) + q(y, z) \) for all \( x, y, z \in X \);
(ii) \( q \) is lower semicontinuous in its second variable;
(iii) \( q(x, y) \geq p(x, y) \) for all \( x, y \in X \).

Then \( q \) is also a \( \tau \)-distance on \( X \).

**Proof.** Let \( \eta \) be a function satisfying (\( \tau 2 \))–(\( \tau 5 \)). From the assumption (ii), (\( \tau 3 \))\( _q \) clearly holds. We assume that \( \lim_n \sup \{q(x_n, y_m) : m \geq n\} = 0 \) and \( \lim_n \eta(x_n, t_n) = 0 \). Then from the assumption (iii), we have \( \lim_n \sup \{p(x_n, y_m) : m \geq n\} = 0 \). So by (\( \tau 4 \)), we obtain \( \lim_n \eta(y_n, t_n) = 0 \). This is (\( \tau 4 \))\( _q \). Let us prove (\( \tau 5 \))\( _q \). We assume that \( \lim_n \eta(z_n, q(z_n, x_n)) = 0 \) and \( \lim_n \eta(z_n, q(z_n, y_n)) = 0 \). Then from the assumption (iii) again, we have \( \lim_n \eta(z_n, p(z_n, x_n)) = 0 \) and \( \lim_n \eta(z_n, p(z_n, y_n)) = 0 \). So by (\( \tau 5 \)), we obtain \( \lim_n d(x_n, y_n) = 0 \). This completes the proof. \( \square \)

Now, we give the following example.

**Example 1.** Let \( (X, d) \) be a metric space, and \( h \) a nondecreasing function from \([0, \infty)\) into itself such that \( \int_0^\infty (1/(1 + h(t))) dt = \infty \). Fix \( z_0 \in X \). Then functions \( p \) and \( q \) from \( X \times X \) into \([0, \infty)\) defined by

\[
p(x, y) = \int_{d(z_0, x)}^{d(z_0, x) + d(x, y)} \frac{dt}{1 + h(t)} \quad \text{and} \quad q(x, y) = p(x, y) + p(y, x)
\]

for all \( x, y \in X \) are \( \tau \)-distances on \( X \).
Proof. We know that \( p \) is a \( \tau \)-distance on \( X \); see Proposition 4 in [10]. So, since \( p \) satisfies (\( \tau 1 \)), we have

\[
q(x, z) = p(x, z) + p(z, x) \\
\leq p(x, y) + p(y, z) + p(z, y) + p(y, x) \\
= q(x, y) + q(y, z)
\]

for \( x, y, z \in X \). This is (\( \tau 1 \)). It is obvious that \( q \) is continuous and \( q(x, y) \geq p(x, y) \) for all \( x, y \in X \). So by Proposition 1, we have \( q \) is a \( \tau \)-distance on \( X \). \( \square \)

In [10], using the above \( p \), the author gave the slight generalization and another proof of Zhong’s variational principle [17, 18]. In this paper, we use the above \( q \).

The following is Theorem 4 in [10], which is the \( \tau \)-distance version of Ekeland’s variational principle [5, 6]. Of course, this is one of the generalizations of the Banach contraction principle [1].

**Theorem 1.** Let \( X \) be a complete metric space with a \( \tau \)-distance \( p \). Let \( f \) be a function from \( X \) into \((-\infty, +\infty]\) which is proper lower semicontinuous and bounded from below. Then for \( \varepsilon > 0 \) and \( u \in X \) with \( p(u, u) = 0 \), there exists \( v \in X \) such that \( f(v) \leq f(u) - \varepsilon p(u, v) \) and \( f(w) > f(v) - \varepsilon p(v, w) \) for all \( w \in X \) with \( w \neq v \).

From Example 1 and Theorem 1, we obtain the following.

**Theorem 2.** Let \( X \), \( d \), \( h \), \( z_0 \) be as in Example 1. Suppose that \( X \) is complete. Let \( f \) be a function from \( X \) into \((-\infty, +\infty]\) which is proper lower semicontinuous and bounded from below. Then for \( \varepsilon > 0 \) and \( u \in X \), there exists \( v \in X \) such that

\[
f(v) \leq f(u) - \varepsilon \int_{d(z_0, u)}^{d(z_0, v) + d(u, v)} \frac{dt}{1 + h(t)} - \varepsilon \int_{d(z_0, v)}^{d(z_0, w) + d(v, w)} \frac{dt}{1 + h(t)}
\]

and

\[
f(w) > f(v) - \varepsilon \int_{d(z_0, v)}^{d(z_0, w) + d(v, w)} \frac{dt}{1 + h(t)} - \varepsilon \int_{d(z_0, u)}^{d(z_0, v) + d(u, v)} \frac{dt}{1 + h(t)}
\]

for all \( w \in X \) with \( w \neq v \).

3. Zhong’s Result

In this section, using Theorem 2, we can easily prove the following Zhong’s result in [17]. Compare the proof with Zhong’s. We use Theorem 2 only one time.

**Theorem 3** (Zhong [17]). Let \( X \) be a Banach space, and \( h \) a nondecreasing function from \([0, \infty)\) into itself such that \( \int_0^{\infty} (1/(1 + h(t))) dt = \infty \). Let \( f \) be a function from \( X \) into \((-\infty, +\infty]\) which is proper lower semicontinuous. Assume that \( f \) is Gâteaux differentiable at every point \( x \in X \) with \( f(x) \in \mathbb{R} \). If

\[
\alpha := \lim_{r \to \infty} \inf_{\|x\| \geq r} f(x) \in \mathbb{R},
\]

then there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_n \|x_n\| = \infty \), \( \lim_n f(x_n) = \alpha \), and

\[
\lim_{n \to \infty} \|f'(x_n)\| (1 + h(\|x_n\|)) = 0.
\]

**Remark.** In [17], the continuity of \( h \) is needed. We discuss this condition in Section 5.

In the proof of Theorem 3, we use the following lemma, which is well known.
Lemma 1. Suppose that $c \geq 0$, $\delta > 0$, $v \in X$, $f(v) \in \mathbb{R}$ and either of the following holds:

- $f(w) \geq f(v) - c \|v - w\|$ for all $w \in X$ with $0 < \|v - w\| < \delta$; or
- $f(w) \leq f(v) + c \|v - w\|$ for all $w \in X$ with $0 < \|v - w\| < \delta$.

Then $\|f'(v)\| \leq c$.

Proof of Theorem 3. We shall only show the following: For every $\varepsilon > 0$, there exists $v \in X$ satisfying $\|v\| \geq 1/\varepsilon$, $|f(v) - \alpha| \leq \varepsilon$, and $\|f'(v)\| \left(1 + h(\|v\|)\right) \leq \varepsilon$. Fix $\varepsilon > 0$. Define a function $\theta$ from $[0, \infty)$ into itself by

$$(1) \quad \theta(t) = 1 + 2h(t + 1)$$

for $t \in [0, \infty)$. Then it is obvious that $\theta$ is nondecreasing, and we have

$$\int_0^\infty \frac{dt}{1 + \theta(t)} = \frac{1}{2} \int_0^\infty \frac{dt}{1 + h(t + 1)} = \frac{1}{2} \int_1^\infty \frac{dt}{1 + h(t)} = \infty.$$ 

We also define a function $g$ from $X$ into $(-\infty, +\infty]$ by

$$g(x) = \max \{f(x), \alpha - 2\varepsilon\}$$

for $x \in X$. Then it is obvious that $g$ is proper lower semicontinuous and bounded from below. We next choose $r, r' \in \mathbb{R}$ with $1/\varepsilon < r < r'$, $1 < r$,

$$\inf_{\|x\| \geq r} f(x) > \alpha - \varepsilon, \quad \text{and} \quad \int_r^{r'} \frac{dt}{1 + \theta(t)} = 3.$$ 

We also choose $u \in X$ with $\|u\| > r'$ and $f(u) < \alpha + \varepsilon$. We note that $g(u) = f(u)$ because of $\|u\| > r$. Then by Theorem 2, there exists $v \in X$ such that

$$(2) \quad g(v) \leq g(u) - \varepsilon \int_{\|v\|}^{\|u\| + \|u - v\|} \frac{dt}{1 + \theta(t)} \varepsilon \int_{\|v\|}^{\|u\| + \|u - v\|} \frac{dt}{1 + \theta(t)}$$

and

$$(3) \quad g(w) > g(v) - \varepsilon \int_{\|v\|}^{\|u\| + \|u - w\|} \frac{dt}{1 + \theta(t)} \varepsilon \int_{\|v\|}^{\|u\| + \|u - w\|} \frac{dt}{1 + \theta(t)}$$

for all $w \in X$ with $w \neq v$. Arguing by contradiction, we assume that $\|v\| < r$. From (2), we have

$$\alpha - 2\varepsilon \leq g(v) \leq g(u) - \varepsilon \int_{\|v\|}^{\|u\| + \|u - v\|} \frac{dt}{1 + \theta(t)}$$

$$\leq g(u) - \varepsilon \int_{\|v\|}^{\|u\|} \frac{dt}{1 + \theta(t)} \leq g(u) - \varepsilon \int_r^{r'} \frac{dt}{1 + \theta(t)}$$

$$= f(u) - 3\varepsilon < \alpha - 2\varepsilon.$$ 

This is a contradiction. Therefore we obtain $\|v\| \geq r > 1/\varepsilon$. Thus we have $g(v) = f(v)$ and

$$\alpha - \varepsilon < \inf_{\|x\| \geq r} f(x) \leq f(v) \leq f(u) < \alpha + \varepsilon.$$ 

This implies $|f(v) - \alpha| \leq \varepsilon$. From (3) and nondecreasingness of $\theta$, we have

$$g(w) > g(v) - \left(\frac{\varepsilon}{1 + \theta(\|v\|)} + \frac{\varepsilon}{1 + \theta(\|w\|)}\right) \|v - w\|.$$
for \( w \in X \) with \( w \neq v \). Since \( f \) is lower semicontinuous and \( f(v) > \alpha - 2\varepsilon \), there exists \( \delta \in (0,1) \) such that \( f(w) > \alpha - 2\varepsilon \) for \( w \in X \) with \( \|v - w\| < \delta \). Hence, for \( w \in X \) with \( 0 < \|v - w\| < \delta \), since \( g(w) = f(w) \) and
\[
\|w\| \geq \|v\| - \|v - w\| > \|v\| - \delta > \|v\| - 1 > 0,
\]
we have
\[
f(w) > f(v) - \left( \frac{\varepsilon}{1 + \theta(\|v\|)} + \frac{\varepsilon}{1 + \theta(\|v\| - 1)} \right) \|v - w\|
\geq f(v) - \frac{2\varepsilon}{1 + \theta(\|v\| - 1)} \|v - w\|
= f(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|.
\]
So by Lemma 1, we have \( \|f'(v)\| (1 + h(\|v\|)) \leq \varepsilon \). This completes the proof. \( \square \)

As a direct consequence of Theorem 3, we obtain the following.

**Theorem 4** (Zhong [17]). Let \( X \) be a Banach space. Let \( f \) be a function from \( X \) into \( (-\infty, +\infty) \) which is proper lower semicontinuous and bounded from below. Assume that \( f \) is Gâteaux differentiable at every point \( x \in X \) with \( f(x) \in \mathbb{R} \), and \( f \) satisfies the weak Palais-Smale condition. Then \( f \) is coercive.

**Remark.** We can weaken the condition that \( f \) satisfies the weak Palais-Smale condition as follows: Every sequence \( \{x_n\} \) in \( X \) such that \( \{f(x_n)\} \) is bounded and \( \lim_n \|f'(x_n)\| (1 + h(\|x_n\|)) = 0 \) contains a bounded subsequence.

### 4. Coercivity of \( |f| \)

In this section, we discuss the coercivity of \( |f| \).

The following is a generalization of the result in Caklovic, Li and Willem [3].

**Theorem 5.** Let \( X \) be a Banach space, and \( h \) a nondecreasing function from \( [0, \infty) \) into itself such that \( \int_0^\infty \frac{1}{1 + h(t)} \, dt = \infty \). Let \( f \) be a continuous function from \( X \) into \( \mathbb{R} \). Assume that \( f \) is Gâteaux differentiable at every point \( x \in X \). If there exists \( \gamma \in \mathbb{R} \) such that \( \{x \in X : f(x) = \gamma\} \) is bounded, and
\[
\alpha := \lim_{r \to \infty} \inf_{\|x\| \geq r} |f(x) - \gamma| \in \mathbb{R},
\]
then there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_n \|x_n\| = \infty \), \( \lim_n |f(x_n) - \gamma| = \alpha \), and
\[
\lim_{n \to \infty} \|f'(x_n)\| (1 + h(\|x_n\|)) = 0.
\]

**Proof.** We put \( g(x) = |f(x) - \gamma| \) for all \( x \in X \). We shall only show the following: For every \( \varepsilon > 0 \), there exists \( v \in X \) satisfying \( \|v\| \geq 1/\varepsilon \), \( |g(v) - \alpha| \leq \varepsilon \), and \( \|f'(v)\| (1 + h(\|v\|)) \leq \varepsilon \). Fix \( \varepsilon > 0 \). Define a function \( \theta \) from \( [0, \infty) \) into itself by (1). We next choose \( r, r' \in \mathbb{R} \) with \( 1/\varepsilon < r < r' \), \( 1 < r \), \( g(x) > 0 \) for \( x \in X \) with \( \|x\| \geq r \),
\[
\inf_{\|x\| \geq r} g(x) > \alpha - \varepsilon, \text{ and } \int_r^{r'} \frac{dt}{1 + \theta(t)} = \frac{\alpha + \varepsilon}{\varepsilon}.
\]
We also choose \( u \in X \) with \( \|u\| > r' \) and \( g(u) < \alpha + \varepsilon \). Then by Theorem 2, there exists \( v \in X \) with (2) and (3) for all \( w \in X \) with \( w \neq v \). Arguing by contradiction, we assume that \( \|v\| < r \). From (2), we have

\[
0 \leq g(v) \leq g(u) - \varepsilon \int_r^{r'} \frac{dt}{1 + \theta(t)} = g(u) - (\alpha + \varepsilon) < 0.
\]

This is a contradiction. Therefore we obtain \( \|v\| \geq r > 1/\varepsilon \) and hence \( g(v) > 0 \). We also have

\[
\alpha - \varepsilon < \inf_{\|x\| \geq r} g(x) \leq g(v) < \alpha + \varepsilon.
\]

and hence \( |g(v) - \alpha| \leq \varepsilon \). Since \( f \) is continuous and \( g(v) > 0 \), there exists \( \delta \in (0, 1) \) such that either of the following holds:

- \( g(w) = +f(w) - \gamma \) for \( w \in X \) with \( \|v - w\| < \delta \); or
- \( g(w) = -f(w) + \gamma \) for \( w \in X \) with \( \|v - w\| < \delta \).

As in the proof of Theorem 3, we have

\[
g(w) > g(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|
\]

for \( w \in X \) with \( 0 < \|v - w\| < \delta \). In the former case, we obtain

\[
f(w) > f(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|.
\]

In the latter case, we obtain

\[
f(w) < f(v) + \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|.
\]

So, by Lemma 1, we have \( \|f'(v)\| \left(1 + h(\|v\|)\right) \leq \varepsilon \) in both cases. This completes the proof. \( \Box \)

As a direct consequence of Theorem 5, we obtain the following.

**Theorem 6.** Let \( X \) be a Banach space. Let \( f \) be a continuous function from \( X \) into \( \mathbb{R} \). Assume that \( f \) is Gâteaux differentiable at every point \( x \in X \), and \( f \) satisfies the weak Palais-Smale condition. If there exists \( \gamma \in \mathbb{R} \) such that \( \{x \in X : f(x) = \gamma\} \) is bounded, then \( |f| \) is coercive.

**Remark.** We have the same remark of Theorem 4.

5. **CONTINUITY OF \( h \)**

In this section, we discuss the continuity of \( h \).

Without the assumption of continuity of \( h \), we can prove Theorem 3. However, Theorem 3 is not a generalization of Zhong’s result because the following proposition holds. That is, Theorem 3 in this paper and Theorem 3.7 in [17] are equivalent. Also the two definitions of weak Palais-Smale condition in [17] and in this paper are equivalent.

**Proposition 2.** Let \( h \) be a nondecreasing function from \([0, \infty)\) into itself such that \( \int_0^\infty (1/(1+h(t)))dt = \infty \). Then there exists a continuous nondecreasing function \( \theta \) from \([0, \infty)\) into itself such that \( \int_0^\infty (1/(1+\theta(t)))dt = \infty \) and \( h(t) \leq \theta(t) \) for all \( t \in [0, \infty) \).
For \( t \in \mathbb{R} \), we denote by \([t]\) the maximum integer not exceeding \( t \). Define a function \( \theta \) from \([0, \infty)\) into itself by

\[
\theta(t) = \left(1 - t + [t]\right)h([t] + 1) + (t - [t])h([t] + 2)
\]

for \( t \in [0, \infty) \). Putting \( k = [t] \) and \( s = t - [t] \in [0, 1) \), we have

\[
\theta(k + s) = (1 - s)h(k + 1) + sh(k + 2).
\]

It is obvious that \( \theta \) is continuous and nondecreasing. For \( t \in [0, \infty) \), we have

\[
\theta(t) \geq h([t] + 1) \geq h(t)
\]

because \( t < [t] + 1 \). We also have

\[
\int_0^\infty \frac{dt}{1 + \theta(t)} \geq \int_0^\infty \frac{dt}{1 + h([t] + 2)} \geq \int_0^\infty \frac{dt}{1 + h(t + 2)} = \int_2^\infty \frac{dt}{1 + h(t)} = \infty.
\]

This completes the proof. \( \square \)

Similarly, we can prove the following.

**Proposition 3.** Let \( h \) be a nondecreasing function from \([0, \infty)\) into itself such that \( \int_0^\infty (1/(1 + h(t)))dt < \infty \). Then there exists a continuous nondecreasing function \( \theta \) from \([0, \infty)\) into itself such that \( \int_0^\infty (1/(1 + \theta(t)))dt < \infty \) and \( \theta(t) \leq h(t) \) for all \( t \in [0, \infty) \).

**Proof.** Define a function \( \theta \) from \([0, \infty)\) into itself by

\[
\theta(t) = \begin{cases} 
 h(0), & \text{if } t \leq 1, \\
 (1 - t + [t])h([t] - 1) + (t - [t])h([t]), & \text{if } t \geq 1.
\end{cases}
\]

for \( t \in [0, \infty) \). Then \( \theta \) is continuous, nondecreasing, \( \theta(t) \leq h([t]) \leq h(t) \) for \( t \in [0, \infty) \), and \( h(t - 2) \leq h([t] - 1) \leq \theta(t) \) for \( t \in [2, \infty) \). Hence

\[
\int_2^\infty \frac{dt}{1 + \theta(t)} \leq \int_2^\infty \frac{dt}{1 + h(t - 2)} = \int_0^\infty \frac{dt}{1 + h(t)} < \infty.
\]

This completes the proof. \( \square \)

6. **Counterexamples**

In this section, we give examples, which say that we use conditions \( \int_0^\infty (1/(1 + h(t)))dt = \infty \) and the completeness of \( X \) in Theorem 3 and others.

**Example 2.** Put \( X := \mathbb{R} \) and let \( h \) be a nondecreasing function from \([0, \infty)\) into itself such that \( \int_0^\infty (1/(1 + h(t)))dt < \infty \). Then there exists a differentiable function \( f \) from \( X \) into \( \mathbb{R} \) such that

\[
\lim_{r \to \infty} \inf_{|x| \geq r} f(x) \in \mathbb{R} \quad \text{and} \quad |f'(x)| (1 + h(|x|)) \geq 1
\]

for all \( x \in X \).

**Proof.** By Proposition 3, there exists a continuous nondecreasing function \( \theta \) from \([0, \infty)\) into itself such that \( \int_0^\infty (1/(1 + \theta(t)))dt < \infty \) and \( \theta(t) \leq h(t) \) for all \( t \in [0, \infty) \). Define a function \( f \) from \( X \) into \( \mathbb{R} \) by

\[
f(x) = \int_0^x \frac{-1}{1 + \theta(\max\{t, 0\})} \, dt\]
It is obvious that \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to +\infty} f(x) \in \mathbb{R} \). We also have
\[
|f'(x)| (1 + h(|x|)) = \frac{1}{1 + \theta(\max\{x, 0\})} (1 + h(|x|)) \geq \frac{1 + h(|x|)}{1 + \theta(|x|)} \geq 1
\]
for all \( x \in X \). This completes the proof. \( \square \)

**Example 3.** Let \( X \) be the normed linear space consisting of all functions \( x \) from \( \mathbb{N} \) into \( \mathbb{R} \) (i.e., \( x \) is a real sequence) such that \( \{ n \in \mathbb{N} : x(n) \neq 0 \} \) is a finite subset of \( \mathbb{N} \). Define a norm \( \| \cdot \| \) on \( X \) by \( \|x\| = \sum_{n=1}^{\infty} |x(n)| \) for all \( x \in X \). Define a lower semicontinuous (not continuous), convex, and Gâteaux differentiable function \( f \) from \( X \) into \( \mathbb{R} \) by
\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \exp \left( 2^n x(n) \right)
\]
for \( x \in X \). Then
\[
\lim_{r \to \infty} \inf_{\|x\| \geq r} f(x) = 0 \in \mathbb{R} \quad \text{and} \quad \|f'(x)\| \geq 1
\]
for all \( x \in X \).

**Proof.** It is obvious that \( f \) is convex and \( \lim_{r \to \infty} \inf \{ f(x) : \|x\| \geq r \} = 0 \). By the definition of \( X \), \( f \) is Gâteaux differentiable and its derivative is given by
\[
f'(x) = \sum_{n=1}^{\infty} \exp \left( 2^n x(n) \right) e_n
\]
for all \( x \in X \), where \( \{ e_n \} \) is the canonical basis of \( X \). Thus, we have
\[
\|f'(x)\| = \sup \{ \exp \left( 2^n x(n) \right) : n \in \mathbb{N} \} \geq \exp(0) = 1
\]
for all \( x \in X \). Fix \( x \in X \) and define a sequence \( \{ x_n \} \) in \( X \) by
\[
x_n(k) = \begin{cases} x(k), & \text{if } k \neq n, \\ 1/n, & \text{if } k = n \end{cases}
\]
for \( n \in \mathbb{N} \). Since \( \|x - x_n\| = 1/n \) for large \( n \in \mathbb{N} \), \( \{ x_n \} \) converges to \( x \). Since
\[
\frac{2^{n-1}}{n^2} \leq \frac{1}{2^n} \left( 1 + \frac{2^n}{n} + \left( \frac{2^n}{n} \right)^2 / 2 \right) \leq \frac{1}{2^n} \exp \left( \frac{2^n}{n} \right) \leq f(x_n)
\]
for \( n \in \mathbb{N} \), we have \( \lim_n f(x_n) = \infty \). This implies \( f \) is not continuous everywhere. We finally show that \( f \) is lower semicontinuous. Let \( \{ x_n \} \) be a sequence in \( X \) converging to some \( x \in X \). We fix \( \varepsilon > 0 \) and choose \( \nu \in \mathbb{N} \) such that \( 2^{-\nu} < \varepsilon \) and \( x(n) = 0 \) for every \( n \in \mathbb{N} \) with \( n \geq \nu \). Define functions \( g \) and \( h \) from \( X \) into \( (0, \infty) \) by
\[
g(y) = \sum_{n=1}^{\nu} \frac{1}{2^n} \exp \left( 2^n y(n) \right) \quad \text{and} \quad h(y) = \sum_{n=\nu+1}^{\infty} \frac{1}{2^n} \exp \left( 2^n y(n) \right)
\]
for \( y \in X \). Then it is obvious that \( f = g + h \), \( g \) is continuous and \( h(x) = 2^{-\nu} < \varepsilon \). We have
\[
f(x) = g(x) + h(x) \leq g(x) + \varepsilon = \lim_{n \to \infty} g(x_n) + \varepsilon \leq \liminf_{n \to \infty} f(x_n) + \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, we have \( f(x) \leq \liminf_{n \to \infty} f(x_n) \). Therefore \( f \) is lower semicontinuous. This completes the proof. \( \square \)
References


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