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An estimate for derivative of the de la Vallée Poussin mean

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Abstract

The de la Vallée Poussin mean for exponential weights on $(-\infty, \infty)$ was investigated in [6]. In the present paper we discuss its derivatives. An estimate for the Christoffel function plays an important role.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$. We consider an exponential weight

$$w(x) = \exp(-Q(x))$$

on $\mathbb{R}$, where Q is an even and nonnegative function on $\mathbb{R}$. Throughout this paper we always assume that w belongs to a relevant class $\mathcal{F}(C^2+)$ (see section 2). A function $T = T_w$ defined by

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{1.1}$$

is very important. We call w a Freud-type weight if T is bounded, and otherwise, w is called an Erdös-type weight. For $x > 0$, the Mhaskar-Rakhmanov-Saff number (MRS number) $a_x = a_x(w)$ of $w = \exp(-Q)$ is defined by a positive root of the equation

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du. \tag{1.2}$$

When $w = \exp(-Q) \in \mathcal{F}(C^2+)$, Q' is positive and increasing on $(0, \infty)$, so that

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$$\lim_{x\to\infty} a_x = \infty \quad \text{and} \quad \lim_{x\to+0} a_x = 0 \tag{1.3}$$

and

$$\lim_{x\to\infty} \frac{a_x}{x} = 0 \quad \text{and} \quad \lim_{x\to+0} \frac{a_x}{x} = \infty \tag{1.4}$$

hold. Note that those convergences are all monotonically.

Let $\{p_n\}$ be orthogonal polynomials for a weight w, that is, p_n is the polynomial of degree n such that

$$\int_{\mathbb{R}} p_n(x)p_m(x)w(x)^2dx = \delta_{mn}.$$

Note that when $w(x) = \exp(-|x|^2)$, then $\{p_n\}$ are Hermite polynomials.

For $1 \le p \le \infty$, we denote by $L^p(I)$ the usual L^p space on an interval I in $\mathbb{R}$. For a function f with $fw \in L^p(\mathbb{R})$, we set

$$s_n(f)(x) := \sum_{k=0}^{n-1} b_k(f)p_k(x) \quad \text{where} \quad b_k(f) = \int_{\mathbb{R}} f(t)p_k(t)w(t)^2dt$$

for $n \in \mathbb{N}$ (the partial sum of Fourier series). The de la Vallée Poussin mean $v_n(f)$ of f is defined by

$$v_n(f)(x) := \frac{1}{n}\sum_{j=n+1}^{2n} s_j(f)(x).$$

In [6], we proved the following; Let $1 \le p \le \infty$ and $w \in \mathcal{F}(C^2+)$. Assume that

$$T(a_n) \le C\left(\frac{n}{a_n}\right)^{2/3} \tag{1.5}$$

for some $C > 1$. Then there exists another constant $C > 1$ such that if $fw \in L^p(\mathbb{R})$, then

$$\|v_n(f)\frac{w}{T^{1/4}}\|_{L^p(\mathbb{R})} \le C\|fw\|_{L^p(\mathbb{R})} \tag{1.6}$$

holds for all $n \in \mathbb{N}$, and if $T^{1/4}fw \in L^p(\mathbb{R})$, then

$$\|v_n(f)w\|_{L^p(\mathbb{R})} \le C\|T^{1/4}fw\|_{L^p(\mathbb{R})} \tag{1.7}$$

holds for all $n \in \mathbb{N}$. It is also known that

$$\left\|P'\frac{w}{T^{1/2}}\right\|_{L^p(\mathbb{R})} \le C\left(\frac{n}{a_n}\right)\|Pw\|_{L^p(\mathbb{R})}, \tag{1.8}$$

for all $P \in \mathcal{P}_n$, where $\mathcal{P}_n$ is the set of all polynomials of degree at most n (see [5, Theorem 6.1]). Since $v_n(f) \in \mathcal{P}_{2n-1}$, combining (1.7) with (1.8), we have

$$\left\|v_n'(f)\frac{w}{T^{1/2}}\right\|_{L^p(\mathbb{R})} \le C\left(\frac{n}{a_n}\right)\|T^{1/4}fw\|_{L^p(\mathbb{R})} \tag{1.9}$$

with some $C > 1$. Here we use the fact that a_n and a_{2n} are comparable (see Lemma 2.1 (1) below). The inequality (1.9) suggests us the following: if $fw \in L^p(\mathbb{R})$, then

$$\left\| v_n'(f) \frac{w}{T^{3/4}} \right\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right) \|fw\|_{L^p(\mathbb{R})} \tag{1.10}$$

and, if $T^{3/4} fw \in L^p(\mathbb{R})$, then

$$\|v_n'(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right) \|T^{3/4} fw\|_{L^p(\mathbb{R})} \tag{1.11}$$

holds?

In the present paper, we will show that (1.10) holds for all $1 \leq p \leq \infty$ and (1.11) is true for $2 \leq p \leq \infty$ at the least. More generally, as for the jth derivative $v_n^{(j)}(f)$ of $v_n(f)$, the following theorems are established.

Theorem 1.1. Let $k \geq 2$ be an integer and let $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < (k+3)/(k+2)$, and let $1 \leq p \leq \infty$. Then there exists a constant $C > 1$ such that if $1 \leq j \leq k$, and if $fw \in L^p(\mathbb{R})$, then

$$\|v_n^{(j)}(f) \frac{w}{T^{(2j+1)/4}} \|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^j \|fw\|_{L^p(\mathbb{R})} \tag{1.12}$$

holds for all $n \in \mathbb{N}$.

The definition of a class $\mathcal{F}_\lambda(C^4+)$ is given in section 2.

Theorem 1.2. Let k and w be as in Theorem 1.1, and let $2 \leq p \leq \infty$. Then there exists a constant $C > 1$ such that if $1 \leq j \leq k$, and if $T^{(2j+1)/4} fw \in L^p(\mathbb{R})$, then

$$\|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^j \|T^{(2j+1)/4} fw\|_{L^p(\mathbb{R})} \tag{1.13}$$

holds for all $n \in \mathbb{N}$.

Theorem 1.3. Let k and w be as in Theorem 1.1, and let $1 \leq p \leq 2$. Then there exists a constant $C > 1$ such that for every $1 \leq j \leq k$ and every $T^{(2j+1)/4} fw \in L^2(\mathbb{R})$, we have

$$\|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^j a_n^{(2-p)/(2p)} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})} \tag{1.14}$$

for all $n \in \mathbb{N}$.

We note that when w is a Freud-type weight, then $1 \leq T(x) \leq C$, so that,

$$\|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^j \|fw\|_{L^p(\mathbb{R})} \tag{1.15}$$

follows from Theorem 1.1. In [3, Chapter 3], Mhaskar discussed the first derivative of the de la Vallée Poussin mean for Freud-type weights. Our contribution is to deal with not only Freud-type but also Erdös-type weights. In the proofs of above theorems, we use Mhaskar's argument. In addition, there are two keys: one is to use mollification of exponential weights (see Lemma 2.4 below) which was obtained in [5], and another is to estimate the Christoffel functions which are done in section 3. Unfortunately, we do not know whether (1.13) holds true or not for $1 \le p < 2$, however, we will give another estimate which holds for all $1 \le p \le \infty$ in section 4. A related inequality to (1.14) is also given in section 6.

Throughout this paper, we write $f(x) \sim g(x)$ for a subset $I \subset \mathbb{R}$ if there exists a constant $C \ge 1$ such that $f(x)/C \le g(x) \le Cf(x)$ holds for all $x \in I$. Similarly, $a_n \sim b_n$ means that $a_n/C \le b_n \le Ca_n$ holds for all $n \in \mathbb{N}$. We will use the same letter C to denote various positive constants; it may vary even within a line. Roughly speaking, $C > 1$ implies that C is sufficiently large, and differently, $C > 0$ means C is a sufficiently small positive number.

2. Definitions and Lemmas

We say that an exponential weight $w = \exp(-Q)$ belongs to class $\mathcal{F}(C^2+)$, when $Q : \mathbb{R} \to [0, \infty)$ is a continuous and even function and satisfies the following conditions:
(a) $Q'(x)$ is continuous in $\mathbb{R}$ and $Q(0) = 0$.
(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
(c) $\lim_{x\to\infty} Q(x) = \infty$.
(d) The function T in (1.1) is quasi-increasing in $(0, \infty)$(i.e. there exists $C > 1$ such that $T(x) \le CT(y)$ whenever $0 < x < y$), and there exists $\Lambda \in \mathbb{R}$ such that

$$T(x) \ge \Lambda > 1, \quad x \in \mathbb{R} \setminus \{0\}.$$

(e) There exists $C > 1$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C\frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}.$$

There also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$, and $C > 1$ such that

$$C\frac{Q''(x)}{|Q'(x)|} \ge \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J.$$

Let $\lambda > 0$. We write $w \in \mathcal{F}_\lambda(C^2+)$ if there exist $K > 1$ and $C > 1$ such that for all $|x| \ge K$,

$$\frac{|Q'(x)|}{Q(x)^\lambda} \le C \tag{2.1}$$

holds. We also write $w \in \mathcal{F}_\lambda(C^3+)$, if $Q \in C^3(\mathbb{R} \setminus \{0\})$ and

$$\left|\frac{Q^{(3)}(x)}{Q''(x)}\right| \le C\left|\frac{Q''(x)}{Q'(x)}\right| \quad \text{and} \quad \frac{|Q'(x)|}{Q(x)^\lambda} \le C$$

hold for every $|x| \geq K$. Moreover, we write $w \in \mathcal{F}_\lambda(C^4+)$, if $Q \in C^4(\mathbb{R} \setminus \{0\})$ and

$$\left|\frac{Q^{(3)}(x)}{Q''(x)}\right| \sim \left|\frac{Q''(x)}{Q'(x)}\right|, \quad \left|\frac{Q^{(4)}(x)}{Q^{(3)}(x)}\right| \leq C\left|\frac{Q''(x)}{Q'(x)}\right| \quad \text{and} \quad \frac{|Q'(x)|}{Q(x)^\lambda} \leq C$$

hold for every $|x| \geq K$. Clearly $\mathcal{F}_\lambda(C^4+) \subset \mathcal{F}_\lambda(C^3+) \subset \mathcal{F}_\lambda(C^2+) \subset \mathcal{F}(C^2+)$.

A typical example of Freud-type weight is $w(x) = \exp(-|x|^\alpha)$ with $\alpha > 1$. It belongs to $\mathcal{F}_\lambda(C^4+)$ for $\lambda = 1$. For $u \geq 0,\ \alpha > 0$ with $\alpha + u > 1$ and $l \in \mathbb{N}$, we set

$$Q(x) := |x|^u(\exp_l(|x|^\alpha) - \exp_l(0)),$$

where $\exp_l(x) := \exp(\exp(\exp(\cdots(\exp x))))$ (l-times). Then $w(x) = \exp(-Q(x))$ is an Erdös-type weight, which belongs to $\mathcal{F}_\lambda(C^4+)$ for any $\lambda > 1$ (see[1]).

In the following lemmas we fix $w \in \mathcal{F}(C^2+)$.

Lemma 2.1. Fix $L > 0$. Then we have

(1) $a_t \sim a_{Lt}$ on $t > 0$ (see [2, Lemma 3.5 (a)]).

(2) $Q(a_t) \sim Q(a_{Lt})$, $Q'(a_t) \sim Q'(a_{Lt})$ and $T(a_{Lt}) \sim T(a_t)$ on $t > 0$ (see [2, Lemma 3.5 (b)]).

(3) $\dfrac{1}{T(a_t)} \sim \left|1 - \dfrac{a_{Lt}}{a_t}\right|$ on $t > 0$ (see [2, Lemma 3.11 (3.52)]).

(4) $\dfrac{t}{\sqrt{T(a_t)}} \sim Q(a_t)$ and $\dfrac{t\sqrt{T(a_t)}}{a_t} \sim |Q'(a_t)|$ on $t > 0$ (see [2, Lemma 3.4 (3.18) and (3.17)]) .

(5) Assume that w is an Erdös-type weight. Then for every $\eta > 0$, there exists a constant $C_\eta > 1$ such that

$$a_x \leq C_\eta x^\eta \quad (x \geq 1) \tag{2.2}$$

(see [4, Proposition 3 (3.8)]).

Lemma 2.2. ([2, Theorem 1.9 (a)]) Let $1 \leq p \leq \infty$. Then

$$\|Pw\|_{L^p(\mathbb{R})} \leq 2\|Pw\|_{L^p([-a_n,a_n])} \tag{2.3}$$

for every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$.

Lemma 2.3. (1) There exist constants $C_1 > 1$ and $c_0 > 0$ such that if $|x - t| < c_0/T(x)$ then $T(t)/C_1 \leq T(x) \leq C_1T(t)$ holds (cf. [2, Theorem 3.2 (e)] see also [6, Lemma 3.4]).

(2) There exist a constant $C_2 > 1$ such that for any $n \in \mathbb{N}$, if $|t|, |x| < a_{2n}$ and $|x - t| \leq a_n/n$ then $T(t)/C_2 \leq T(x) \leq C_2T(t)$ holds (see [6, (4.6)]).

Lemma 2.4. ([5, Theorem 4.1 and (4.11)]) Let $m = 1, 2$ and let $w \in \mathcal{F}_\lambda(C^{2+m}+)$ with $0 < \lambda < (m+2)/(m+1)$. For every $\alpha \in \mathbb{R}$, we can construct a new weight $w^* \in \mathcal{F}_\lambda(C^{1+m}+)$ such that

$$w^*(x) \sim T(x)^\alpha w(x) \text{ and } T^*(x) \sim T(x) \tag{2.4}$$

on $\mathbb{R}$, and

$$a_{x/c} \le a_x^* \le a_{cx} \tag{2.5}$$

holds on $\mathbb{R}$ with some constant $c > 1$, where T^* and a_x^* are corresponding ones defined in (1.1) and (1.2) with respect to a weight w^* respectively.

Using the above lemma, we obtain the following assertions. First one is a generalization of (1.8). Second assertion was shown in [5, Corollary 6.2] under some additional assumption.

Lemma 2.5. Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$ and let $1 \le p \le \infty$. For $j \in \mathbb{N}$, there exists a constant $C_3 > 1$ such that for every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$, we have

$$\left\| P^{(j)} \frac{w}{T^{j/2}} \right\|_{L^p(\mathbb{R})} \le C_3 \left(\frac{n}{a_n} \right)^j \|Pw\|_{L^p(\mathbb{R})} \tag{2.6}$$

and if we further assume that $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < 4/3$, then there exists a constant $C_4 > 1$ such that

$$\left\| P^{(j)} w \right\|_{L^p(\mathbb{R})} \le C_4 \left(\frac{n}{a_n} \right)^j \|T^{j/2} Pw\|_{L^p(\mathbb{R})} \tag{2.7}$$

also holds.

Proof. For $i = 1, \cdots, j$, let $w_i^* \in \mathcal{F}_\lambda(C^2+)$ be a weight obtained in Lemma 2.4 for $\alpha = -(i-1)/2$. Then, since $P^{(j)} \in \mathcal{P}_{n-j}$, by (1.8) for w_j^* and by (2.4) and (2.5), there exists a constant $C > 1$ such that

$$\left\| P^{(j)} \frac{w_j^*}{T^{1/2}} \right\|_{L^p(\mathbb{R})} \le C \left(\frac{n-j+1}{a_{(n-j+1)/c}} \right) \|P^{(j-1)} w_j^*\|_{L^p(\mathbb{R})}.$$

Since $w_j^*(x) \sim T(x)^{-1/2} w_{j-1}^*(x)$, we also see

$$\left\| P^{(j)} \frac{w_j^*}{T^{1/2}} \right\|_{L^p(\mathbb{R})} \le C \left(\frac{n-j+1}{a_{(n-j+1)/c}} \right) \left\| P^{(j-1)} \frac{w_{j-1}^*}{T^{1/2}} \right\|_{L^p(\mathbb{R})}.$$

Repeating this process, we have

$$\begin{aligned} \left\| P^{(j)} \frac{w}{T^{j/2}} \right\|_{L^p(\mathbb{R})} &\le C \left\| P^{(j)} \frac{w_j^*}{T^{1/2}} \right\|_{L^p(\mathbb{R})} \\ &\le C^{j+1} \left(\frac{n-j+1}{a_{(n-j+1)/c}} \right) \cdots \left(\frac{n}{a_{n/c}} \right) \|Pw\|_{L^p(\mathbb{R})} \\ &\le C_3 \left(\frac{n}{a_n} \right)^j \|Pw\|_{L^p(\mathbb{R})}, \end{aligned}$$

where we use Lemma 2.1 (1).

For (2.7), we first remark that if $w \in \mathcal{F}_\lambda(C^3+)$, then

$$\|P'w\|_{L^p(\mathbb{R})} \le C_4 \left(\frac{n}{a_n}\right) \|T^{1/2}Pw\|_{L^p(\mathbb{R})} \tag{2.8}$$

holds true (see [5, (1.4) and its proof]). This is the case $j = 1$. To show general case $j > 1$, we consider a weight $w_i^{**} \in \mathcal{F}_\lambda(C^3+)$ in Lemma 2.4 for $w \in \mathcal{F}_\lambda(C^4+)$ with $\alpha = (i-1)/2$ $(i = 1, \cdots, j)$. Applying $P^{(j-i)}$ and w_i^{**} to (2.8) and repeating this process for $i = 1, \cdots, j$, we obtain (2.7) as in (2.6). This completes the proof.

Lemma 2.6. Let $k \in \mathbb{N} \cup \{0\}$ and $w \in \mathcal{F}_\lambda(C^2+)$ with $0 < \lambda < (k+2)/(k+1)$. Then there exist constants $C_5 > 1$ and $\delta > 0$ such that

$$T(a_n) \le C_5 n^{2/(2k+3)-\delta} \tag{2.9}$$

holds for all $n \in \mathbb{N}$.

Proof. We may assume that $w = \exp(-Q)$ is an Erdős-type weight. By (2.1), $|Q'(x)|/Q(x)^\lambda \le C$ with some constant $C > 1$. Hence Lemma 2.1 (4) gives us

$$\frac{n\sqrt{T(a_n)}}{a_n}\left(\frac{n}{\sqrt{T(a_n)}}\right)^{-\lambda} \le C,$$

that is, $T(a_n) \le C a_n^{2/(\lambda+1)} n^{2(\lambda-1)/(\lambda+1)}$. Since $\lambda < (k+2)/(k+1)$, we can choose $\delta > 0$ and $\eta > 0$ such that $2(\lambda-1)/(\lambda+1) + \delta + 2\eta < 2/(2k+3)$. Hence (2.9) follows from Lemma 2.1 (5). This completes the proof.

We remark that (2.9) implies (1.5). Hence if $w \in \mathcal{F}_\lambda(C^2+)$ with $0 < \lambda < (k+2)/(k+1)$, then (1.6), (1.7), (1.8) and (1.9) hold true.

Lemma 2.7. Let $w \in \mathcal{F}_\lambda(C^2+)$ with $0 < \lambda < 2$. Then there exists a constant $C_6 > 1$ such that for every $n \in \mathbb{N}$, if $|t|, |x| < a_{2n}$ and if $|t - x| < a_n/(n\sqrt{T(x)})$ then

$$w(t)/C_6 \le w(x) \le C_6 w(t) \tag{2.10}$$

Proof. By Lemma 2.3 (2) , we have $T(t)/C_2 \le T(x) \le C_2 T(t)$, and by (1.3) we can write $|t| = a_s$. Then $a_s \le a_{2n}$ implies $s \le 2n$. Hence (1.4) and Lemma 2.1(1) show $sa_n/(na_s) \le C_7$ with some constant $C_7 > 1$. Since $|Q'(t)| \le Cs\sqrt{T(a_s)}/a_s$ by Lemma 2.1 (4), we have

$$\begin{aligned}|Q'(t)||t-x| &\le C\frac{s\sqrt{T(a_s)}}{a_s}\frac{a_n}{n}\frac{1}{\sqrt{T(x)}} \\ &\le C\frac{a_n}{n}\frac{s}{a_s}\frac{\sqrt{T(t)}}{\sqrt{T(x)}} \le CC_7\sqrt{C_2}.\end{aligned}$$

Similarly, we see $|Q'(x)|t - x| \le CC_7$. Hence if we put $C_6 = e^{CC_7\sqrt{C_2}}$, then $|Q'(t)||t - x| \le \log C_6$ and $|Q'(x)||t-x| < \log C_6$ hold true. From the mean value theorem for differential calculus, there exists θ between x and t such that

$$\frac{w(x)}{w(t)} = \exp(Q(t) - Q(x)) = \exp(Q'(\theta)(t - x)).$$

Since Q' is increasing, $|Q'(\theta)(x - t)| \le \max\{|Q'(x)|, |Q'(t)|\}|x - t| \le \log C_6$, which shows (2.10) immediately. This completes the proof.

3. Estimates for Christoffel functions

By definition, the partial sum of Fourier series is given by

$$s_n(f)(x) = \int_{\mathbb{R}} K_n(x,t) f(t) w(t)^2 dt, \tag{3.1}$$

where

$$K_n(x,t) = \sum_{k=0}^{n-1} p_k(x) p_k(t). \tag{3.2}$$

It is known that by the Cristoffel-Darboux formula

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)}{x - t} \tag{3.3}$$

holds, where γ_n and γ_{n-1} are the leading coefficients of p_n and p_{n-1}, respectively. Then

$$a_n \sim \frac{\gamma_{n-1}}{\gamma_n} \tag{3.4}$$

also holds (see [2, Lemma 13.9]).

The Christoffel function $\lambda_n(x) = \lambda_n(w, x)$ is defined by

$$\lambda_n(x) := \frac{1}{K_n(x,x)} = \left(\sum_{k=0}^{n-1} p_k(x)^2 \right)^{-1}.$$

Then

$$\lambda_n(x) = \inf_{P \in \mathcal{P}_{n-1}} \frac{1}{P(x)^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt. \tag{3.5}$$

holds on $\mathbb{R}$. We use derivative versions of (3.5). The following equality is also established.

Proposition 3.1. Let $0 \le j < n$. Then for every $x \in \mathbb{R}$, we have

$$\left(\sum_{k=0}^{n-1} (p_k^{(j)}(x))^2 \right)^{-1} = \inf_{P \in \mathcal{P}_{n-1}} \frac{1}{(P^{(j)}(x))^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt. \tag{3.6}$$

Proof. In [3, Theorem 1.3.2], we see

$$\left(\sum_{k=0}^{n-1}\Phi(p_k)^2\right)^{-1} = \inf_{P\in\mathcal{P}_{n-1}}\frac{1}{(\Phi(P)^2}\int_{\mathbb{R}}|P(t)w(t)|^2dt$$

for any linear functional Φ on polynomials. (3.6) follows if we consider $\Phi(P) = P^{(j)}(x)$.

The following estimate plays an important role in our later argument. We use C_m $(m = 1, \cdots, 6)$, which are constants in lemmas of the previous section.

Proposition 3.2. Let $k \geq 2$ be an integer and let $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < (k+3)/(k+2)$. Then there exists a constant $C_8 > 1$ such that for every $1 \leq j \leq k$ and every $n \in \mathbb{N}$,

$$\frac{w(x)^2}{T(x)^{(2j+1)/2}}\sum_{k=0}^{n-1}(p_k^{(j)}(x))^2 \leq C_8\left(\frac{n}{a_n}\right)^{2j+1}. \tag{3.7}$$

Proof. It is enough to show (3.7) for sufficiently large n. By Proposition 3.1, (3.7) follows from

$$\left(\frac{a_n}{n}\right)^{2j+1}\frac{w(x)^2}{T(x)^{(2j+1)/2}} \leq C_8\frac{1}{(P^{(j)}(x))^2}\int_{\mathbb{R}}|P(t)w(t)|^2dt \tag{3.8}$$

for $P \in \mathcal{P}_{n-1}$. Now to show (3.8), take $P \in \mathcal{P}_{n-1}$ arbitrarily. By Lemma 2.2, we can choose $\zeta \in \mathbb{R}$ such that $|\zeta| \leq a_{n-1}$ and satisfies

$$\|wP\|_{L^\infty(\mathbb{R})} \leq 2|w(\zeta)P(\zeta)|. \tag{3.9}$$

Let $0 < c_1 \leq 1$. Lemma 2.6 gives us $T(a_n) \leq C_5 n^{1-\delta'}$ with some $\delta' > 0$, so that if $t \in \mathbb{R}$ satisfies

$$|t-\zeta| \leq c_1\frac{a_n}{n}\frac{1}{\sqrt{T(\zeta)}}, \tag{3.10}$$

then

$$|t| \leq |\zeta| + |\zeta - t| \leq |\zeta| + c_1\frac{a_n}{n}\frac{1}{\sqrt{T(\zeta)}} \leq a_{n-1} + \frac{a_n}{n} \leq a_n + \frac{C_5}{n^{\delta'}}\frac{a_n}{T(a_n)}.$$

Since there exists a constant $C > 1$ such that $a_n + a_n/(CT(a_n)) \leq a_{2n}$ by Lemma 2.1 (3), if we take $n_0 \in \mathbb{N}$ such that $n_0^{\delta'} > CC_5$, then

$$|t| \leq a_{2n} \tag{3.11}$$

for all $n \geq n_0$. Hence by Lemma 2.7, $w(t)/C_6 \leq w(\zeta) \leq C_6 w(t)$ holds. By monotonicity of w, $w(u)/C_6 \leq w(\zeta) \leq C_6 w(u)$ also holds for every u between t and ζ. Moreover, since T is quasi-increasing, Lemma 2.3 (2) shows $\sqrt{T(u)} \le C\sqrt{T(\zeta)}$ with some $C > 1$. Then using (2.6) for $p = \infty$ and $j = 1$, we have

$$\begin{aligned}
|P(\zeta)| - |P(t)| \le |P(t) - P(\zeta)| &= \left| \int_\zeta^t P'(u)du \right| \\
&\le CC_6 \frac{\sqrt{T(\zeta)}}{w(\zeta)} \left| \int_\zeta^t \frac{1}{\sqrt{T(u)}} w(u)P'(u)du \right| \\
&\le CC_6 |t-\zeta| \frac{\sqrt{T(\zeta)}}{w(\zeta)} \left\| \frac{w}{\sqrt{T}} P' \right\|_{L^\infty(\mathbb{R})} \\
&\le CC_6C_3 |t-\zeta| \frac{\sqrt{T(\zeta)}}{w(\zeta)} \frac{n}{a_n} \|wP\|_{L^\infty(\mathbb{R})} \\
&\le 2c_1CC_6C_3 |P(\zeta)|
\end{aligned}$$

by (3.9) and (3.10). Consequently, if we take $c_1 > 0$ so small that $2c_1CC_6C_3 < 1/2$, we have

$$|P(t)| \ge \frac{1}{2}|P(\zeta)| \text{ if } |t-\zeta| \le c_1 \frac{a_n}{n} \frac{1}{\sqrt{T(\zeta)}}. \tag{3.12}$$

Since $C_2T(t) \ge T(\zeta)$ and $C_6w(t) \ge w(\zeta)$, (3.9) and (3.12) show

$$\begin{aligned}
\int_{\mathbb{R}} \sqrt{T(t)} |P(t)|^2 w(t)^2 dt &\ge \frac{\sqrt{T(\zeta)}}{\sqrt{C_2}} \int_{|t-\zeta| \le c_1 a_n/(n\sqrt{T(\zeta)})} |P(t)|^2 w(t)^2 dt \\
&\ge \frac{\sqrt{T(\zeta)}}{\sqrt{C_2}} \frac{|P(\zeta)|^2}{4} \frac{w(\zeta)^2}{C_6^2} c_1 \frac{a_n}{n} \frac{1}{\sqrt{T(\zeta)}} \\
&\ge \frac{c_1}{4\sqrt{C_2}} \frac{1}{C_6^2} \frac{a_n}{n} \frac{\|wP\|^2_{L^\infty(\mathbb{R})}}{4} \\
&=: \frac{1}{C_0} \frac{a_n}{n} \|wP\|^2_{L^\infty(\mathbb{R})}.
\end{aligned}$$

We note that in the above argument we only use the fact that $w \in \mathcal{F}_\lambda(C^3+)$. If $w \in \mathcal{F}_\lambda(C^4+)$, we can construct $w^* \in \mathcal{F}_\lambda(C^3+)$ such that $w^*(x) \sim T(x)^{-1/4}w(x)$ by Lemma 2.4. Then it follows from (2.6) for $p = \infty$ that for every $x \in \mathbb{R}$,

$$\begin{aligned}
\int_{\mathbb{R}} \sqrt{T^*(t)} |P(t)|^2 w^*(t)^2 dt &\ge \frac{1}{C_0} \frac{a_n^*}{n} \|w^*P\|^2_{L^\infty(\mathbb{R})} \\
&\ge \frac{1}{C_0C_3} \frac{a_n^*}{n} \left(\frac{a_{n-1}^*}{n-1} \right)^{2j} \left\| \frac{w^*}{(T^*)^{j/2}} P^{(j)} \right\|^2_{L^\infty(\mathbb{R})} \\
&\ge \frac{1}{C_0C_3} \left(\frac{a_n^*}{n} \right)^{2j+1} \frac{w^*(x)^2}{T^*(x)^j} |P^{(j)}(x)|^2,
\end{aligned}$$

and hence by (2.4) and (2.5) we see

$$\begin{aligned}\int_{\mathbb{R}} |P(t)|^2 w^2(t)dt &\geq \frac{1}{C}\int_{\mathbb{R}} \sqrt{T^*(t)}|P(t)|^2 w^*(t)^2 dt \\ &\geq \frac{1}{CC_0C_3}\left(\frac{a_n^*}{n}\right)^{2j+1} \frac{w^*(x)^2}{T^*(x)^j}|P^{(j)}(x)|^2 \\ &\geq \frac{1}{C}\left(\frac{a_{n/c}}{n}\right)^{2j+1} \frac{w(x)^2}{T(x)^{(2j+1)/2}}|P^{(j)}(x)|^2.\end{aligned}$$

This together with Lemma 2.1 (1) shows (3.8) and the proof is completed.

4. Proof of Theorem 1.1

In the remaining sections, we again use C_m $(m = 1, \cdots, 6)$ without notice, which are constants in lemmas in section 2.

Let $1 \leq p \leq \infty$, $k \geq 2, w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < (k+3)/(k+2)$ and let $1 \leq j \leq k$. Due to Lemma 2.4, there is $w^* \in \mathcal{F}_\lambda(C^3+)$ such that $w^*(x) \sim T(x)^{-(2j+1)/4}w(x)$. Take $fw \in L^p(\mathbb{R})$ arbitrarily. Since $v_n^{(j)}(f) \in \mathcal{P}_{2n-1-j}$, applying w^* to (2.7), we have

$$\begin{aligned}&\left\| v_n^{(j)}(f)\frac{w}{T^{(2j+1)/4}}\right\|_{L^p(\mathbb{R})} \leq C\|v_n^{(j)}(f)w^*\|_{L^p(\mathbb{R})} \\ &\leq C\left(\frac{2n-j}{a_{2n-j}^*}\right)^j \|(T^*)^{j/2}v_n(f)w^*\|_{L^p(\mathbb{R})} \\ &\leq C\left(\frac{n}{a_{2n/c}}\right)^j \left\| v_n(f)\frac{w}{T^{1/4}}\right\|_{L^p(\mathbb{R})} \\ &\leq C\left(\frac{n}{a_n}\right)^j \|fw\|_{L^p(\mathbb{R})}.\end{aligned}$$

Here we use Lemma 2.1 (1), (2.4) and (2.5). The last inequality follows from (1.6). This completes the proof of Theorem 1.1.

By a similar argument as above, we also have

$$\|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^j T(a_n)^{(2j+1)/4}\|fw\|_{L^p(\mathbb{R})} \tag{4.1}$$

for all $1 \leq p \leq \infty$. In fact, take $w^* \in \mathcal{F}_\lambda(C^3+)$ such that $w^*(x) \sim T^{j/2}(x)w(x)$. Then by (2.7) for w and by Lemma 2.4 and Lemma 2.2 for w^*, we have

$$\begin{aligned}\|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} &\le C\left(\frac{n}{a_n}\right)^j \|T^{j/2}v_n(f)w\|_{L^p(\mathbb{R})}\\ &\le C\left(\frac{n}{a_n}\right)^j \|v_n(f)w^*\|_{L^p([-a_{2n}^*,a_{2n}^*])}\\ &\le C\left(\frac{n}{a_n}\right)^j \left\|v_n(f)T^{(2j+1)/4}\frac{w}{T^{1/4}}\right\|_{L^p([-a_{2cn},a_{2cn}])}\\ &\le C\left(\frac{n}{a_n}\right)^j T(a_n)^{(2j+1)/4}\left\|v_n(f)\frac{w}{T^{1/4}}\right\|_{L^p([-a_{2cn},a_{2cn}])}\\ &\le C\left(\frac{n}{a_n}\right)^j T(a_n)^{(2j+1)/4}\|fw\|_{L^p(\mathbb{R})}.\end{aligned}$$

Note that by Lemma 2.1 (2), $T(x) \le CT(a_{2cn}) \le CT(a_n)$ holds for all $x \in [-a_{2cn}, a_{2cn}]$, because T is quasi-increasing.

5. Proof of Theorem 1.2

Let $k \ge 2, w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < (k+3)/(k+2)$ and let $1 \le j \le k$. We first show (1.13) for the case $p = \infty$. Suppose that $T^{(2j+1)/4}fw \in L^\infty(\mathbb{R})$. Since $v_n^{(j)}(f) \in \mathcal{P}_{2n}$, by Lemma 2.2, it is sufficient to show

$$|v_n^{(j)}(f)(x)w(x)| \le C\left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})} \tag{5.1}$$

for every $|x| \le a_{2n}$. Now we set

$$A_n := \{t \in \mathbb{R}; |t-x| < \frac{a_{2n}}{2n}\}, \quad B_n := \{t \in \mathbb{R}; \frac{a_{2n}}{2n} \le |t-x| < \frac{c_0}{T(x)}\}$$

and $C_n := \mathbb{R} \setminus (A_n \cup B_n)$, where $c_0 > 0$ is a constant in Lemma 2.3 (1). Then as in the proof of (3.11), there exists $n_0 \in \mathbb{N}$ such that if $n \ge n_0$ and $t \in A_n$, then $|t| \le a_{4n}$ holds. Hence Lemma 2.3 (2) implies

$$T(t)/C_2 \le T(x) \le C_2T(t) \quad (t \in A_n). \tag{5.2}$$

Since T is bounded on $[-a_{4n_0}, a_{4n_0}]$, we may assume that (5.2) holds for all $n \in \mathbb{N}$. Also by Lemma 2.3 (1),

$$T(t)/C_1 \le T(x) \le C_1T(t) \quad (t \in B_n) \tag{5.3}$$

holds true. Let $g(t) := f(t)\chi_{A_n}(t)$, where χ_A is the characteristic function of a set A and put $h(t) = f(t) - g(t)$. Since

$$\int_{\mathbb{R}}\left(\sum_{k=0}^{m-1} p_k^{(j)}(x)p_k(t)\right)^2 w(t)^2dt = \sum_{k=0}^{m-1}(p_k^{(j)}(x))^2,$$

(3.2), (5.2) and the Schwarz inequality show that

$$
\begin{aligned}
&|s_m^{(j)}(g)(x)w(x)| \\
&\le w(x)\int_{\mathbb{R}}\left|g(t)\sum_{k=0}^{m-1}p_k^{(j)}(x)p_k(t)w(t)^2\right|dt \\
&\le \left(\sum_{k=0}^{m-1}(p_k^{(j)}(x))^2w(x)^2\right)^{1/2}\left(\int_{A_n}|f(t)w(t)|^2dt\right)^{1/2} \\
&\le C_2^{(2j+1)/4}\left(\sum_{k=0}^{m-1}\frac{w(x)^2}{T(x)^{(2j+1)/2}}(p_k^{(j)}(x))^2\right)^{1/2}\left(\int_{A_n}|T(t)^{(2j+1)/4}f(t)w(t)|^2dt\right)^{1/2} \\
&\le C\left(\sum_{k=0}^{m-1}\frac{w(x)^2}{T(x)^{(2j+1)/2}}(p_k^{(j)}(x))^2\right)^{1/2}\|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})}\left(\frac{a_{2n}}{2n}\right)^{1/2}.
\end{aligned}
$$

Since $v_n^{(j)}(g)(x) = (1/n)\sum_{m=n+1}^{2n}s_m^{(j)}(g)(x)$, Proposition 3.2 gives us

$$
|v_n^{(j)}(g)(x)w(x)| \le C\left(\frac{n}{a_n}\right)^j\|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})} \tag{5.4}
$$

for all $x \in \mathbb{R}$ with $|x| \le a_{2n}$.

To estimate $v_n^{(j)}(h)$, we use (3.3). For $i = 0, 1, \cdots, j$, we put

$$
\begin{aligned}
&v_{n,i}(h)(x) \\
&:= \frac{1}{n}\sum_{m=n+1}^{2n}\frac{\gamma_{m-1}}{\gamma_m}\int_{\mathbb{R}}h(t)\frac{p_m^{(j-i)}(x)p_{m-1}(t)-p_{m-1}^{(j-i)}(x)p_m(t)}{(x-t)^{i+1}}w(t)^2dt \\
&= \frac{1}{n}\sum_{m=n+1}^{2n}\frac{\gamma_{m-1}}{\gamma_m}(b_{m-1}(h_i)p_m^{(j-i)}(x)-b_m(h_i)p_{m-1}^{(j-i)}(x)),
\end{aligned}
$$

where

$$
h_i(t) := \frac{h(t)}{(x-t)^{i+1}} \quad \text{and} \quad b_k(h_i) := \int_{\mathbb{R}}h_i(t)p_k(t)w(t)^2dt \quad (k \in \mathbb{N}\cup\{0\}).
$$

Then

$$
v_n^{(j)}(h)(x) = \sum_{i=0}^{j}(-1)^i\binom{j}{i}v_{n,i}(h)(x). \tag{5.5}
$$

By (3.4), the Schwarz inequality and Proposition 3.2, we have

$$
\begin{aligned}
&|v_{n,i}(h)(x)w(x)| \\
&\le \frac{1}{n}\sum_{m=0}^{2n}\left|\frac{\gamma_{m-1}}{\gamma_m}2p_m^{(j-i)}(x)b_m(h_i)w(x)\right| \\
&\le C\frac{a_n}{n}\left(w(x)^2\sum_{m=0}^{2n}(p_m^{(j-i)}(x))^2\right)^{1/2}\left(\sum_{m=0}^{2n}|b_m(h_i)|^2\right)^{1/2} \\
&\le C\frac{a_n}{n}\left(\frac{w(x)^2}{T(x)^{(2(j-i)+1)/2}}\sum_{m=0}^{2n}(p_m^{(j-i)}(x))^2\right)^{1/2}\left(T(x)^{(2(j-i)+1)/2}\sum_{m=0}^{2n}|b_m(h_i)|^2\right)^{1/2} \\
&\le C\left(\frac{n}{a_n}\right)^{(2(j-i)-1)/2}\left(T(x)^{(2(j-i)+1)/2}\sum_{m=0}^{2n}|b_m(h_i)|^2\right)^{1/2}.
\end{aligned}
$$

The Bessel inequality implies that

$$
\sum_{m=0}^{2n}|b_m(h_i)|^2 \le \int_{\mathbb{R}}\left|\frac{h(t)}{(x-t)^{i+1}}\right|^2 w(t)^2dt = \int_{B_n\cup C_n}\frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}}dt
$$

and hence, by (5.3), we have

$$
\begin{aligned}
&T(x)^{(2(j-i)+1)/2}\int_{B_n}\frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}}dt \\
&\le C_1^{(2(j-i)+1)/2}\int_{B_n}\frac{|T(t)^{(2(j-i)+1)/4}f(t)w(t)|^2}{(x-t)^{2(i+1)}}dt \\
&\le C\|T^{(2(j-i)+1)/4}fw\|^2_{L^\infty(\mathbb{R})}\int_{|x-t|>\frac{a_{2n}}{2n}}\frac{1}{(x-t)^{2(i+1)}}dt \\
&\le C\|T^{(2(j-i)+1)/4}fw\|^2_{L^\infty(\mathbb{R})}\left(\frac{n}{a_n}\right)^{2i+1} \\
&\le C\|T^{(2j+1)/4}fw\|^2_{L^\infty(\mathbb{R})}\left(\frac{n}{a_n}\right)^{2i+1},
\end{aligned}
$$

because $T \ge 1$. On the other hand, if $|x| \le a_{2n}$ then $T(x) \le CT(a_n)$, so that

$$
\begin{aligned}
&T(x)^{(2(j-i)+1)/2}\int_{C_n}\frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}}dt \\
&\le C\|fw\|^2_{L^\infty(\mathbb{R})}T(x)^{(2(j-i)+1)/2}\int_{\frac{c_0}{T(x)}\le|x-t|}\frac{1}{(x-t)^{2(i+1)}}dt \\
&\le C\|fw\|^2_{L^\infty(\mathbb{R})}T(x)^{(2(i+j)+3)/2} \\
&\le C\|T^{(2j+1)/4}fw\|^2_{L^\infty(\mathbb{R})}T(a_n)^{(2(i+k)+3)/2}.
\end{aligned}
$$

Moreover

$$T(a_n)^{(2(i+k)+3)/2} \le C\left(\frac{n}{a_n}\right)^{2i+1} \tag{5.6}$$

holds. In fact, to show this we may assume that w is an Erdös-type weight by (1.4). Then by Lemma 2.1 (5) and Lemma 2.6, we have

$$T(a_n)^{(2k+3)/2} \le Cn^{(2/(2k+3)-\delta)((2k+3)/2)} \le Cn^{1-\delta'} \le C\left(\frac{n}{a_n}\right).$$

Similarly

$$\begin{aligned} T(x)^{(2(i+k)+3)/2} &\le CT(a_n)^{(4k+3)/2} \le Cn^{(2/(2k+3)-\delta)((4k+3)/2)} \\ &\le Cn^{2-\delta''} \le C\left(\frac{n}{a_n}\right)^2 \le C\left(\frac{n}{a_n}\right)^{2i+1} \end{aligned}$$

holds for $i \ge 1$. Combining the above estimates, we thus have

$$\begin{aligned} |v_{n,i}(h)(x)w(x)| &\le C\left(\frac{n}{a_n}\right)^{(2(j-i)-1)/2}\left(T(x)^{(2(j-i)+1)/2}\sum_{m=0}^{2n}|b_m(h_i)|^2\right)^{1/2} \\ &\le C\left(\frac{n}{a_n}\right)^{(2(j-i)-1)/2}\|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})}\left(\frac{n}{a_n}\right)^{(2i+1)/2} \\ &\le C\left(\frac{n}{a_n}\right)^{j}\|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

It follows from (5.5) that

$$|v_n^{(j)}(h)(x)w(x)| \le C\left(\frac{n}{a_n}\right)^{j}\|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})}.$$

This together with (5.4) shows (5.1).

We will prove (1.13) for $p = 2$ in the next section. Then using the Riesz-Thorin interpolation theorem for an operator

$$F \ : \ f \ \mapsto \ wv_n^{(j)}\left(\frac{f}{w}\right),$$

we obtain (1.13) for all $2 \le p \le \infty$. This completes the proof of Theorem 1.2.

6. Proof of Theorem 1.3

Let $1 \le p \le 2$ and $T^{(2j+1)/4}fw \in L^2(\mathbb{R})$. We use the same notations as in the previous section. Then as in the estimate of $s_m^{(j)}(g)$ in the previous section, we have

$$|s_m^{(j)}(g)(x)w(x)| \le C\left(\frac{n}{a_n}\right)^{(2j+1)/2}\left(\int_{A_n}|T(t)^{(2j+1)/4}f(t)w(t)|^2dt\right)^{1/2} \tag{6.1}$$

for $|x| \le a_{2n}$. Hence Lemma 2.2 and the Hölder inequality imply

$$\int_{\mathbb{R}} |s_m^{(j)}(g)(x)w(x)|^p dx \le 2^p \int_{|x|\le a_{2n}} |s_m^{(j)}(g)(x)w(x)|^p dx$$
$$\le C\int_{|x|\le a_{2n}} \left(\frac{n}{a_n}\right)^{p(2j+1)/2} \left(\int_{A_n} |T(t)^{(2j+1)/4} f(t)w(t)|^2 dt\right)^{p/2} dx$$
$$\le C\left(\frac{n}{a_n}\right)^{p(2j+1)/2} \int_{|x|\le a_{2n}} \left(\int_{|u|\le \frac{a_{2n}}{2n}} |T(x-u)^{(2j+1)/4} f(x-u)w(x-u)|^2 du\right)^{p/2} dx$$
$$\le C\left(\frac{n}{a_n}\right)^{p(2j+1)/2} a_n^{(2-p)/2}$$
$$\times \left\{\int_{|x|\le a_{2n}} \left(\int_{|u|\le \frac{a_n}{n}} |T(x-u)^{(2j+1)/4} f(x-u)w(x-u)|^2 du\right) dx\right\}^{p/2}$$
$$\le C\left(\frac{n}{a_n}\right)^{p(2j+1)/2} a_n^{(2-p)/2} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}^p \left(\int_{|u|\le \frac{a_n}{n}} du\right)^{p/2}$$
$$\le C\left(\frac{n}{a_n}\right)^{pj} a_n^{(2-p)/2} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}^p,$$

so that we have

$$\|v_n^{(j)}(g)w\|_{L^p(\mathbb{R})} \le C\left(\frac{n}{a_n}\right)^j a_n^{(2-p)/(2p)} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}. \tag{6.2}$$

Next we estimate $v_{n,i}(h)$. Similarly as above, we have

$$\int_{\mathbb{R}} |v_{n,i}(h)(x)w(x)|^p dx \le 2\int_{|x|\le a_{2n}} |v_{n,i}(h)(x)w(x)|^p dx$$
$$\le C\left(\frac{n}{a_n}\right)^{p(2(j-i)-1)/2} a_n^{(2-p)/2}$$
$$\times \left\{\int_{|x|\le a_{2n}} \left(\int_{B_n\cup C_n} \frac{|T(t)^{(2(j-i)+1)/4} f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt\right) dx\right\}^{p/2}.$$

Also as in the argument of previous section,

$$\int_{|x|\le a_{2n}} \left(\int_{B_n} \frac{|T^{(2(j-i)+1)/4}(t)f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt\right) dx$$
$$\le \int_{\mathbb{R}} \left(\int_{\frac{a_n}{n}\le |u|} \frac{|T^{(2(j-i)+1)/4}(x-u)f(x-u)w(x-u)|^2}{u^{2(i+1)}} du\right) dx$$
$$\le C\left(\frac{n}{a_n}\right)^{2i+1} \|T^{(2(j-i)+1)/4} fw\|_{L^2(\mathbb{R})}^2 \le C\left(\frac{n}{a_n}\right)^{2i+1} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}^2.$$

On the other hand, by (5.6) we have

$$\begin{aligned}
&\int_{|x|\le a_{2n}} \left(T(x)^{(2(j-i)+1)/2} \int_{C_n} \frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \right) dx \\
&\le CT(a_{2n})^{(2(j-i)+1)/2} \int_{\mathbb{R}} \left(\int_{\frac{c_0}{T(a_{2n})}\le |u|} \frac{|f(x-u)w(x-u)|^2}{u^{2(i+1)}} du \right) dx \\
&\le C\|fw\|^2_{L^2(\mathbb{R})} T(a_{2n})^{(2(j-i)+1)/2} \int_{\frac{c_0}{T(a_{2n})}\le |u|} \frac{1}{u^{2(i+1)}} du \\
&\le CT(a_{2n})^{(2j+2i+3)/2} \|fw\|^2_{L^2(\mathbb{R})} \\
&\le C \left(\frac{n}{a_n} \right)^{2i+1} \|T^{(2j+1)/4} fw\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently we have

$$\|v_{n,i}(h)w\|_{L^p(\mathbb{R})} \le C \left(\frac{n}{a_n} \right)^j a_n^{(2-p)/2p} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})} \tag{6.3}$$

for $0 \le i \le j$, so that

$$\|v_n^{(j)}(h)w\|_{L^p(\mathbb{R})} \le C \left(\frac{n}{a_n} \right)^j a_n^{(2-p)/2p} \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})}$$

follows. This together with (6.2) shows (1.14). This completes the proof of Theorem 1.3.

Under the same assumptions in Theorem 1.3, the following estimate is also established. Let $\beta > 1$ and $1 \le p \le 2$. Then

$$\|v_n^{(j)}(f) \frac{w}{(1+|x|)^{(2-p)\beta/(2p)}}\|_{L^p(\mathbb{R})} \le C \left(\frac{n}{a_n} \right)^j \|T^{(2j+1)/4} fw\|_{L^2(\mathbb{R})} \tag{6.4}$$

holds for every $T^{(2j+1)/4} fw \in L^2(\mathbb{R})$ and every $n \in \mathbb{N}$. In fact, in the proof of Theorem 1.3, we used

$$\begin{aligned}
&\int_{|x|\le a_{2n}} \left(\int_{|x-t|\le \frac{a_{2n}}{2n}} |T(t)^{(2j+1)/4} f(t)w(t)|^2 dt \right)^{p/2} dx \\
&\le a_n^{(2-p)/2} \left\{ \int_{|x|\le a_{2n}} \left(\int_{|x-t|\le \frac{a_n}{n}} |T(t)^{(2j+1)/4} f(t)w(t)|^2 du \right) dx \right\}^{p/2},
\end{aligned}$$

which follows from the Hölder inequality. Instead of this, we use

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{(1+|x|)^{(2-p)\beta/2}} \left(\int_{|x-t|\le \frac{a_{2n}}{2n}} |T(t)^{(2j+1)/4} f(t)w(t)|^2 dt \right)^{p/2} dx \\
&\le \left(\int_{\mathbb{R}} \frac{1}{(1+|x|)^{\beta}} dx \right)^{(2-p)/2} \left\{ \int_{\mathbb{R}} \left(\int_{|x-t|\le \frac{a_n}{n}} |T(t)^{(2j+1)/4} f(t)w(t)|^2 dt \right) dx \right\}^{p/2}.
\end{aligned}$$

Then as in (6.2), we obtain

$$\|v_n^{(j)}(g)\frac{w}{(1+|x|)^{(2-p)\beta/(2p)}}\|_{L^p(\mathbb{R})} \le C\left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})}.$$

For the estimate of $v_{n,i}(h)$, we take $w^* \in \mathcal{F}_\lambda(C^3+)$ such that $w^*(x) \sim w(x)/(1+|x|)^{(2-p)\beta/(2p)}$ (see [5, Theorem 4.2]). Then by Lemma 2.2,

$$\int_R \left|v_{n,i}(h)\frac{w(x)}{(1+|x|)^{(2-p)\beta/(2p)}}\right|^p dx \le 2^p \int_{|x|\le a_{2n}^*} \left|v_{n,i}(h)\frac{w(x)}{(1+|x|)^{(2-p)\beta/(2p)}}\right|^p dx.$$

By an estimate similar to (6.3), we obtain

$$\|v_{n,i}(h)\frac{w}{(1+|x|)^{(2-p)\beta/(2p)}}\|_{L^p(\mathbb{R})} \le C\left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})},$$

which shows (6.4).

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