Title	On Uniformly Convex Spaces
Author(s)	HIRAOKA, tadashi
Citation	茨城大学教育学部紀要(1): 77-83
Issue Date	1953-03-12
URL	http://hdl.handle.net/10109/10061
Rights	

On Uniformly Convex Spaces.

By Tadashi HIRAOKA.

Let B denote a Banach space (linear, metric, complete, normed space), with elements $x, y, \cdots\cdots$. We denote the norm of an element x by $\|x\|$.

A Banach space B is said to be *uniformly convex*, if to any positive number ε there exist a positive number $\delta(\varepsilon)$ so that,

$$\|x\|=\|y\|=1, \qquad \|x-y\| \geqq \varepsilon, \qquad x, y \varepsilon B$$

implies $\left\| \dfrac{x+y}{2} \right\| \leqq 1-\delta(\varepsilon)$.

Uniformly convex spaces were investigated by several authorities, and many interesting results were obtained. James A. Clarkson[1] has shown us that spaces l_p and L_p are uniformly convex for p exceeding unity. In this paper, we shall show the same result, with different inequalities concerning the norm from Clarkson's. In the first part of this paper, we shall attempt to prove on inequalities about l_p space, by using the Lemma. Further, we shall prove the corresponding statement to L_p space by exhibiting a set of inequalities for these spaces. At last, from the Theorem 1 and the Minkowski's inequalities we conclude that spaces l_p and L_p are uniformly convex, with $p>1$.

Now, let us prove the following Lemma.

Lemma. For any two complex numbers x, y

(1) *if* $2 \geqq p > 1$, *then* $2^{p-1}(|x|^p+|y|^p) \geqq \{|x+y|^2+(p-1)|x-y|^2\}^{\frac{p}{2}}$,

(2) *if* $p \geqq 2$, *then* $2^{p-1}(|x|^p+|y|^p) \geqq |x+y|^p+(2^{p-1}-1)|x-y|^p$.

Proof. We assume that $|x| \geqq |y|$, and put $1 \geqq |c| = |y|/|x|$. Then we see that (1) is reduced to the next form,

(3) $2^{p-1}(1+|c|^p) \geqq \{|1+c|^2+(p-1)|1-c|^2\}^{\frac{p}{2}}$.

By elementary calculus methods and with a little attentions, we can easily see that we need only consider $0<c<1$. Making the further transformation

1) James A. Clarkson, *Uniformly convex spaces*. Trans. Amer. Math. Soc. 40 (1936). 396-414.

$c=(1-z)/(1+z)$ $(0<z<1)$, we reduce (3) to the form

$$A=\frac{1}{2}\{(1+z)^p+(1-z)^p\}-\{1+(p-1)z^2\}^{\frac{p}{2}}.$$

Here, we must prove that $A\geqq 0$. So expanding each term of A in its Taylor's series, we have

$$A_1=\frac{1}{2}\{(1+z)^p+(1-z)^p\}=1+\frac{p(p-1)}{2!}z^2+\frac{p(p-1)(2-p)(3-p)}{4!}z^4$$
$$+\cdots\cdots+\frac{p(p-1)(2-p)\cdots\cdots(2k-1-p)}{(2k)!}z^{2k}+\cdots\cdots,$$

$$A_2=\{1+(p-1)z^2\}^{\frac{p}{2}}=1+\frac{p(p-1)}{2!}z^2-\frac{p(p-1)^2(2-p)}{2^2\ 2!}z^4$$
$$+\cdots\cdots+\frac{p(p-1)^{2k-1}(2-p)(4-p)\cdots\cdots(4(k-1)-p)}{2^{2k-1}\ (2k-1)!}z^{2(2k-1)}$$
$$-\frac{p(p-1)^{2k}(2-p)(4-p)\cdots\cdots(2(2k-1)-p)}{2^{2k}\ (2k)!}z^{4k}+\cdots\cdots.$$

Hence

$$A=A_1-A_2=\sum_{k=1}^{\infty}\left\{\frac{p(p-1)(2-p)(3-p)\cdots\cdots(2k-1-p)}{(2k)!}z^{2k}\right.$$
$$-\frac{p(p-1)^{2k-1}(2-p)(4-p)\cdots\cdots(4(k-1)-p)}{2^{2k-1}\ (2k-1)!}z^{2(2k-1)}$$
$$\left.+\frac{p(p-1)^{2k}(2-p)(4-p)\cdots\cdots(2(2k-1)-p)}{2^{2k}\ (2k)!}z^{4k}\right\}.$$

Here, if $2\geqq p>1$, by using $2k-1-p\geqq(4(k-1)-p)/2$ and $2k-p\geqq(2(2k-1)-p)/2$ for $k=1,2,3,\cdots\cdots$, then

$$A=\sum_{k=1}^{\infty}\left[\frac{z^{2k}}{(2k-1)!}\left\{\frac{p(p-1)}{2!}(2-p)(3-p)\cdots\cdots(2k-1-p)\frac{1}{k}\right.\right.$$
$$-\frac{p(p-1)}{2}\cdot\frac{2-p}{2}\cdot\frac{4-p}{2}\cdots\cdots\frac{4(k-1)-p}{2}((p-1)z)^{2(k-1)}$$
$$\left.\left.+\frac{p(p-1)}{2}\cdot\frac{2-p}{2}\cdot\frac{4-p}{2}\cdots\cdots\frac{4(k-1)-p}{2}\cdot\frac{2(2k-1)-p}{2}\cdot\frac{1}{2k}((p-1)z)^{2k}\right\}\right]$$
$$\geqq\sum_{k=1}^{\infty}\left[\frac{p(p-1)}{2}\cdot\frac{2-p}{2}\cdot\frac{4-p}{2}\cdots\cdots\frac{4(k-1)-p}{2}\cdot\frac{z^{2k}}{(2k-1)!}\left\{\frac{1}{k}-((p-1)z)^{2(k-1)}\right.\right.$$
$$\left.\left.+\frac{2(2k-1)-p}{2}\cdot\frac{1}{2k}((p-1)z)^{2k}\right\}\right].$$

Further, we use that $(2(2k-1)-p)/4=k-1/2-p/4\geqq k-1$ for $2\geqq p>1$ and $k=1,2,3,\cdots\cdots$. Hence, we have

$$A \geqq \sum_{k=1}^{\infty} \Big[p(p-1)(2-p)(4-p)\cdots\cdots(4(k-1)-p) \frac{z^{2k}}{2^{2k-1}(2k-1)!}$$

$$\times \Big\{ \frac{1}{k} - ((p-1)z)^{2(k-1)} + \frac{k-1}{k}((p-1)z)^{2k} \Big\} \Big]$$

$$= \sum_{k=1}^{\infty} \Big[p(p-1)(2-p)(4-p)\cdots\cdots(4(k-1)-p) \frac{k-1}{2^{2k-1}(2k-1)!} z^{2k}$$

$$\times \Big\{ \frac{1-((p-1)z)^{2(k-1)}}{k-1} - \frac{1-((p-1)z)^{2k}}{k} \Big\} \Big].$$

But { } is non-negative, because for $0<w\leqq 1$ and $t>0$, $f(t)=(1-w^{2t})/t$ is monotone decreasing function of t. Then $A\geqq 0$ is proved and therefore we see that (1) is true.

Next we attempt to prove (2) under the same assumptions with the proof of (1), and then we come to prove the following relation

$$A_1' = \frac{1}{2}\{(1+z)^p+(1-z)^p\} \geqq 1+(2^{p-1}-1)z^p = A_3,$$

where

$$A_1' = 1 + \frac{p(p-1)}{2!} z_2 + \frac{p(p-1)(p-2)(p-3)}{4!} z^4 + \cdots\cdots$$

$$+ \frac{p(p-1)(p-2)\cdots\cdots(p-2k+1)}{(2k)!} z^{2k} + \cdots\cdots,$$

$$A_3 = 1+\{2^{p-1}-1\}z^p = 1 + \Big\{ 1 + \frac{p(p-1)}{2!} + \frac{p(p-1)(p-2)(p-3)}{4!} + \cdots\cdots$$

$$+ \frac{p(p-1)\cdots\cdots(p-2k+1)}{(2k)!} + \cdots\cdots -1 \Big\} z^p.$$

Now, $p\geqq 2$, so for some natural number k_0 we must consider two cases; namely $2k_0\leqq p<2k_0+1$ and $2k_0+1\leqq p<2(k_0+1)$.

1. For some natural number k_0, if $2k_0\leqq p<2k_0+1$, then

$$A_1' = 1 + \Big[1 + \frac{p(p-1)}{2!} z^2 + \frac{p(p-1)(p-2)(p-3)}{4!} z^4 + \cdots\cdots$$

$$+ \frac{p(p-1)(p-2)\cdots\cdots(p-2k_0+1)}{(2k^0)!} z^{2k_0}$$

$$+ \frac{p(p-1)(p-2)\cdots\cdots(p-2k_0+1)(p-2k_0)(p-2k_0-1)}{(2k_0+2)!} z^{2(k_0+1)} + \cdots\cdots -1 \Big]$$

$$\geqq 1 + \Big[1 + \frac{p(p-1)}{2!} + \frac{p(p-1)(p-2)(p-3)}{4!} + \cdots\cdots$$

$$+ \frac{p(p-1)(p-2)\cdots\cdots(p-2k_0+1)}{(2k_0)!} \Big] z^p$$

$$+\left\{\frac{p(p-1)\cdots\cdots(p-2k_0)(p-2k_0-1)}{(2k_0+2)!}+\cdots\cdots-1\right\}z^p=1+(2^{p-1}-1)z^p=A_3,$$

where $\{\quad\}$ is non-positive.

2. If $2k_0+1\leqq p<2(k_0+1)$ for some natural number k_0, then

$$A_1'=1+\frac{p(p-1)}{2!}z^2+\frac{p(p-1)(p-2)(p-3)}{4!}z^4+\cdots\cdots$$

$$+\frac{p(p-1)(p-2)\cdots\cdots(p-2k_0+1)}{(2k_0)!}z^{2k_0}$$

$$+\frac{p(p-1)(p-2)\cdots\cdots(p-2k_0-1)}{(2k_0+2)!}z^{2(k_0+1)}$$

$$+\left\{\frac{(p-1)(p-2)\cdots\cdots(p-2k_0-1)(2k_0+2-p)(2k_0+3-p)}{(2k_0+3)!}\right.$$

$$\left.-\frac{(p-1)(p-2)\cdots\cdots(2k_0+3-p)(2k_0+4-p)}{(2k_0+4)!}\right\}z^{2(k_0+2)}+\cdots\cdots$$

$$\geqq 1+\left\{1-1+\frac{p(p-1)}{2!}+\cdots\cdots+\frac{p(p-1)(p-2)\cdots\cdots(p-2k_0+1)}{(2k_0)!}\right\}z^p$$

$$+\frac{p(p-1)(p-2)\cdots\cdots(p-2k_0-1)}{(2k_0+2)!}z^{2(k_0+1)}$$

$$+\left\{\frac{(p-1)(p-2)\cdots\cdots(p-2k_0-1)(2k_0+2-p)(2k_0+3-p)}{(2k_0+3)!}\right.$$

$$\left.-\frac{(p-1)(p-2)\cdots\cdots(2k_0+3-p)(2k_0+4-p)}{(2k_0+4)!}\right\}z^{2(k_0+2)}+\cdots\cdots.$$

Here, we must prove that

$$\frac{p(p-1)(p-2)\cdots\cdots(p-2k_0+1)}{(2k_0)!}z^{2k_0}+\frac{p(p-1)(p-2)\cdots\cdots(p-2k_0-1)}{(2k_0+2)!}z^{2(k_0+1)}\tag{4}$$

$$+\frac{(p-1)(p-2)\cdots\cdots(2k_0+2-p)(2k_0+3-p)}{(2k_0+3)!}z^{2(k_0+2)}$$

$$\geqq\left\{\frac{p(p-1)(p-2)\cdots\cdots(p-2k_0+1)}{(2k_0)!}+\frac{p(p-1)(p-2)\cdots\cdots(p-2k_0-1)}{(2k_0+2)!}\right.$$

$$\left.+\frac{(p-1)(p-2)\cdots\cdots(2k_0+2-p)(2k_0+3-p)}{(2k_0+3)!}\right\}z^p.$$

So we devide the both sides of (4) by $(p-1)(p-2)\cdots\cdots(p-2k_0+1)z^p/(2k_0)!$ and we put

$$A_0=pz^{2k_0-p}+\frac{p(p-2k_0)(p-2k_0-1)}{(2k_0+1)(2k_0+2)}z^{2(k_0+1)-p}$$

$$+\frac{(p-2k_0)(p-2k_0-1)(2k_0+2-p)(2k_0+3-p)}{(2k_0+1)(2k_0+2)(2k_0+3)}z^{2(k_0+2)-p}$$

$$-\left\{p+\frac{p(p-2k_0)(p-2k_0-1)}{(2k_0+1)(2k_0+2)}\right.$$

$$+\frac{(p-2k_0)(p-2k_0-1)(2k_0+2-p)(2k_0+3-p)}{(2k_0+1)(2k_0+2)(2k_0+3)}\Big\}\geqq 0.$$

And then we come to prove that $A_0\geqq 0$. So we differentiate A_0 with z. Then

$$\frac{dA_0}{dz}=\frac{1}{(2k_0+1)(2k_0+2)(2k_0+3)}\Big\{p(2k_0+1)(2k_0+2)(2k_0+3)(2k_0-p)\,z^{2k_0-p-1}$$

$$+p(p-2k_0)(p-2k_0-1)(2k_0+3)(2k_0+2-p)\,z^{2k_0+1-p}$$

$$+(p-2k_0)(p-2k_0-1)(2k_0+2-p)(2k_0+3-p)(2k_0+4-p)\,z^{2k_0+3-p}\Big\}$$

$$=\frac{z^{2k_0-p-1}}{(2k_0+1)(2k_0+2)(2k_0+3)}\Big\{p(2k_0+1)(2k_0+2)(2k_0+3)(2k_0-p)$$

$$+p(2k_0+3)(2k_0+2-p)(p-2k_0)(p-2k_0-1)\,z^2$$

$$+(p-2k_0)(p-2k_0-1)(2k_0+2-p)(2k_0+3-p)(2k_0+4-p)\,z^4\Big\}$$

$$\leqq\frac{z^{2k_0-p-1}}{(2k_0+1)(2k_0+2)(2k_0+3)}\Big[(p-2k_0)\Big\{(p-2k_0-1)(2k_0+2-p)(p(2k_0+3)$$

$$+(2k_0+3-p)(2k_0+4-p))-p(2k_0+1)(2k_0+2)(2k_0+3)\Big\}\Big]\leqq 0,$$

where $1>p-2k_0-1\geqq 0$, $1\geqq 2k_0+2-p>0$ and $\{\quad\}$ is negative. Further, if $z=1$, then $A_0=0$. Therefore $A_0\geqq 0$ is proved.

Furtheremore by similar methods, we see the next relation; i.e.

$$\sum_{l=2}^{\infty}\Big\{\frac{(p-1)(p-2)\cdots\cdots(2k_0+2l-p)}{(2k_0+2l)!}z^{2(k_0+l)}$$

$$-\frac{(p-1)(p-2)\cdots\cdots(2k_0+2l+1-p)}{(2k_0+2l+1)!}z^{2(k_0+l+1)}\Big\}$$

$$\leqq\sum_{l=2}^{\infty}\Big\{\frac{(p-1)(p-2)\cdots\cdots(2k_0+2l-p)}{(2k_0+2l)!}-\frac{(p-1)(p-2)\cdots\cdots(2k_0+2l+1-p)}{(2k_0+2l+1)!}\Big\}z^{p}.$$

So we understand $A_1'\geqq A_3$ for case 2. Therefore the proof of the Lemma is completed.

Now, this time we consider the following Theorem on l_p and L_p spaces.

Theorem 1. For spaces l_p and L_p, the following inequalities between the norms of two arbitrary elements x and y of the space are valid:

(5) *if* $2\geqq p>1$, *then* $2^{p-1}(\|x\|^p+\|y\|^p\geqq\{\|x+y\|^2+(p-1)\|x-y\|^2\}^{\frac{p}{2}}$,

(6) *if* $p\geqq 2$, *then* $2^{p-1}(\|x\|^p+\|y\|^p\geqq\|x+y\|^p+(2^{p-1}-1)\|x-y\|^p$.

Proof. First, we shall prove respecting l_p space. Let $x=(x_1,x_2,\cdots\cdots)$, $y=(y_1,y_2,\cdots\cdots)$ be the two elements of l_p space. By using the Lemma and the

Minkowski's[2] inequality, d. h. if a_i, b_i are arbitrary two sets of non-negative numbers, finite or infinite in number, and $0 \leqq r \leqq 1$, then

$$(\sum_i a_i^r)^{\frac{1}{r}} + (\sum_i b_i^r)^{\frac{1}{r}} \leqq \{\sum_i (a_i+b_i)^r\}^{\frac{1}{r}},$$

further we put $0 < p/2 = r \leqq 1$, $a_i = |x_i + y_i|^2$ and $b_i = \{(p-1)^{\frac{1}{2}} |x_i - y_i|\}^2$, then we see that the right side of (5) is

$$\{\|x+y\|^2 + (p-1)\|x-y\|^2\}^{\frac{p}{2}}$$

$$= [\{\sum_i |x_i+y_i|^p\}^{\frac{2}{p}} + \{\sum_i ((p-1)^{\frac{1}{2}} |x_i-y_i|)^p\}^{\frac{2}{p}}]^{\frac{p}{2}}$$

$$\leqq \sum_i \{ |x_i+y_i|^2 + ((p-1)^{\frac{1}{2}} |x_i-y_i|)^2\}^{\frac{p}{2}},$$

so from (1)

$$\leqq \sum_i \{ 2^{p-1}(|x_i|^p + |y_i|^p)\}^{\frac{2}{p} \cdot \frac{p}{2}} = \sum_i 2^{p-1}(|x_i|^p + |y_i|^p)$$

$$= 2^{p-1}\{\sum_i |x_i|^p + \sum_i |y_i|^p\} = 2^{p-1}\{(\sum_i |x_i|^p)^{\frac{1}{p} \cdot p} + (\sum_i |y_i|^p)^{\frac{1}{p} \cdot p}\}$$

$$= 2^{p-1}(\|x\|^p + \|y\|^p).$$

Thus (5) is proved.

Next we must prove (6). So from (2), we see easily that

$$\|x+y\|^p + (2^{p-1}-1)\|x-y\|^p = \sum_i \{ |x_i+y_i|^p + (2^{p-1}-1)|x_i-y_i|^p\}$$

$$\leqq \sum \{ 2^{p-1}(|x_i|^p + |y_i|^p)\} = 2^{p-1}(\|x\|^p + \|y\|^p).$$

Thus (6) is proved, and therefore the proof for l_p space is completed.

Now, to extend these results to space L_p with respect to p as seen in l_p, let $[0, 1]$ be the interval over which the functions of our space are to be defined. We consider first two functions $x(t)$, $y(t)$ which are step functions on a division of $[0, 1]$ into equal parts. It is easily verified that for such functions the relations (5) and (6) are reduced to the space l_p case already treated, and as these functions form a dense set in L_p the result follows by continuity of the norm.

Thus we see that the Theorem 1 is true for L_p space, too. Therefore our proof is completed.

At last by using Theorem 1, the following fact is concluded.

2) G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*. Cambridge (1934). 30–32.

Theorem 2. The spaces l_p and L_p are uniformly convex with $p>1$.

Proof. Under the definition of uniform convexity; if $\|x\|=\|y\|=1$ and $\|x-y\|\geqq\varepsilon$ for any positive number ε, then, we obtain the following fact from (5)

$$2^2=\{2^{p-1}(\|x\|^p+\|y\|^p)\}^{\frac{2}{p}}\geqq\|x+y\|^2+(p-1)\|x-y\|^2.$$

Hence

$$\left\|\frac{x+y}{2}\right\|^2\leqq1-(p-1)\left\|\frac{x-y}{2}\right\|^2\leqq1-(p-1)\left(\frac{\varepsilon}{2}\right)^2,$$

then

$$\left\|\frac{x+y}{2}\right\|\leqq1-\left[1-\left\{1-(p-1)\left(\frac{\varepsilon}{2}\right)^2\right\}^{\frac{1}{2}}\right],$$

and so

$$\delta(\varepsilon)=1-\left\{1-(p-1)\left(\frac{\varepsilon}{2}\right)^2\right\}^{\frac{1}{2}}.$$

Therefore we see that l_p and L_p spaces are uniformly convex for $2\geqq p>1$.

Next, on the case $p\geqq2$, we have from (6) and the assumption of definition of unjform convexity

$$2^p=2^{p-1}(\|x\|^p+\|y\|^p)\geqq\|x+y\|^p+(2^{p-1}-1)\|x-y\|^p,$$

then

$$\left\|\frac{x+y}{2}\right\|^p\leqq1-(2^{p-1}-1)\left\|\frac{x-y}{2}\right\|^p,$$

hence

$$\left\|\frac{x+y}{2}\right\|\leqq1-\left[1-\left\{1-(2^{p-1}-1)\left(\frac{\varepsilon}{2}\right)^p\right\}^{\frac{1}{p}}\right]$$

and so

$$\delta(\varepsilon)=1-\left\{1-(2^{p-1}-1)\left(\frac{\varepsilon}{2}\right)^p\right\}^{\frac{1}{p}}.$$

Thus we see that Theorem 2 is true for $p\geqq2$, too, and our proof is completed.

November 10, 1951

Ibaraki University.