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Author(s)	BABA, Kiyoshi; ODA, Susumu; YOSHIDA, Ken-ichi
Citation	Mathematical Journal of Ibaraki University, 32: 63-67
Issue Date	2000
URL	http://hdl.handle.net/10109/3081
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Math. J., Ibaraki Univ., Vol. 32, 2000

Some Relationship between Anti-Integral Extensions of Noetherian Domains

KIYOSHI BABA*, SUSUMU ODA** AND KEN-ICHI YOSHIDA***

ABSTRACT. Let R be a Noetherian domain with quotient field K and let α be an anti-integral element of degree d over R. Let β be an element of $R[\alpha]$ (resp. $R[\alpha,\alpha^{-1}]$) such that β is an anti-integral element over R and that $R[\alpha]$ (resp. $R[\alpha,\alpha^{-1}]$) is integral over $R[\beta]$). We shall investigate some properties descending from $R[\alpha]$ (resp. $R[\alpha,\alpha^{-1}]$) to $R[\beta]$, *i.e.*, flatness and faithful flatness, and study the ideals $J_{[\alpha]}$, $J_{[\beta]}$, $\tilde{J}_{[\alpha]}$ and $\tilde{J}_{[\beta]}$.

Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic extension field L of the quotient field K of R and let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Then η_i $(1 \le i \le d)$ are uniquely determined by α. Let $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^{d} I_{\eta_i}$. If $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)$, we say that α is *anti-integral* over R. When α is an anti-integral element over R, the extension $R[\alpha]$ is called an *anti-integral* extension of R. Put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Then $J_{[\alpha]} = \mathcal{C}(I_{[\alpha]}\varphi_\alpha(X))$, where $\mathcal{C}(\ \ )$ denotes the ideal generated by the coefficients of the polynomials in ( ), that is, the content ideal of ( ). Let $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$. For a prime ideal $p \in \mathrm{Spec}(R)$, $k(p)$ denotes the residue field R_p/pR_p at p.

All rings considered in this paper are commutative and have identity. We use the notation above unless otherwise specified. Our general reference for unexplained technical terms is [M2].

Received January 18, 2000.

1991 *Mathematics Subject Classification.* Primary 13C20, Secondary 13F99

Key words and phrases. anti-integral, flat, faithfully flat, quasi-finite, blowing-up

* Department of Math., Faculty of Education and Welfare Science, Oita University, Oita 870-1192 JAPAN

e-mail: [redacted]

** Matsusaka C. H. School, Toyohara 1600, Matsusaka, Mie 515-0205, JAPAN

e-mail: [redacted]

*** Department of Applied Math., Faculty of Science, Okayama University of Science, Ridai-cho, Okayama 700-0005, JAPAN

e-mail: [redacted]

1. Simple Sub-Extension of $R[\alpha]$.

We say that $R[\alpha]$ has a *blowing-up* at p if the following two conditions are satisfied :

(1) $pR_p[\alpha] \cap R_p = pR_p$ (equivalently, $pR[\alpha] \cap R = p$),

(2) $R_p[\alpha]/pR_p[\alpha]$ is isomorphic to a polynomial ring $k(p)[T]$.

Let A be an R-algebra of finite type and let $P \in \mathrm{Spec}(A)$. We say the A is *quasi-finite* over R at P if A_P/pA_P is finite over $k(p)$, where $p := P \cap R$. We say that A is *quasi-finite* over R if A is quasi-finite over R at every point $P \in \mathrm{Spec}(A)$. We say also that A is *quasi-finite* over R at $p \in \mathrm{Spec}(R)$ if A_p is quasi-finite over R_p.

It is easy to see the following remark by definition.

Remark 1. Let A be an R-algebra of finite type.

(1)For a prime ideal P of A, A is quasi-finite over R at P if and only if P is isolated in its fiber, *i.e.*, in $\mathrm{Spec}(A \otimes_R k(p))$, where $p := P \cap R$.

(2) A is quasi-finite over R if and only if the fiber $A \otimes_R k(p)$ is finite over $k(p)$ for all $p \in \mathrm{Spec}(R)$.

(3) Let $p \in \mathrm{Spec}(R)$. When $A = R[\alpha]$, a simple extension of R, the following statements are equivalent :

(i) A is quasi-finite over R at $p \in \mathrm{Spec}(R)$,

(ii) A does not have a blowing up at p,

(iii) either $pA \cap R \neq p$ or for every $P \in \mathrm{Spec}(A)$ with $P \cap R = p$, $k(P)$ is algebraic over $k(p)$.

We recall the following fundamental result for later use :

Lemma 2(*cf.*[OSY,(2.6)]). *Assume that α is anti-integral over R. Let $p \in \mathrm{Spec}(R)$. Then the following are equivalent :*

(i) $\mathrm{rank}_{k(p)}(R[\alpha] \otimes_R k(p))$ *is finite ;*

(ii) $\overline{\alpha}$ *is algebraic over* $k(p)$ *;*

(iii) $J_{[\alpha]} \not\subseteq p$ *;*

(iv) $R_p[\alpha]$ *is flat over* R_p *;*

(v) $R_p[\alpha]$ *is quasi-finite over* R_p *;*

(vi) $R[\alpha]$ *does not have a blowing-up at* p.

Proof. The equivalences (i) $\sim$ (v) follow from [OSY,(2.6)]. The equivalence (v) $\Leftrightarrow$ (vi) follows from Remark 1(3). □

Proposition 3. *Assume that α is an anti-integral element of degree d over R. Let β be an element in $R[\alpha]$ such that β is anti-integral over R and that $R[\alpha]$ is integral over $R[\beta]$. Then the following two assertions hold :*

(1) $R[\alpha]$ *is flat over R if and only if* $R[\beta]$ *is flat over R,*

(2) $R[\alpha]$ *is faithfully flat over R if and only if* $R[\beta]$ *is faithfully flat over R.*

Proof. (1) By Lemma 2, we know that $R[\alpha]$ (resp. $R[\beta]$) is flat over R if and only if $R[\alpha]$ (resp. $R[\beta]$) does not have a blowing-up at any point in $\mathrm{Spec}(R)$. So we have only to show that $R[\alpha]$ does not have a blowing-up at any point in

$\mathrm{Spec}(R)$ if and only if $R[\beta]$ does not have a blowing-up at any point in $\mathrm{Spec}(R)$. Let p be a prime ideal of R.

($\Rightarrow$) If $pR_p[\beta] \cap R_p \neq pR_p$, then $R[\beta]$ does not have a blowing-up at p by definition. So we may assume that $pR_p[\beta] \cap R_p = pR_p$. Take a prime ideal Q of $R[\beta]$ such that $Q \cap R = p$. Since $R[\alpha]$ is integral over $R[\beta]$, there exists $P \in \mathrm{Spec}(R[\alpha])$ such that $P \cap R[\beta] = Q$. Since $R[\alpha]$ does not have a blowing-up at p, $k(P)$ is algebraic over $k(p)$. Since $k(P) \supseteq k(Q) \supseteq k(p)$, we have that $k(Q)$ is algebraic over $k(p)$. Thus $R[\beta]$ does not have a blowing-up at p.

($\Leftarrow$) Since $R[\alpha]$ is integral over $R[\beta]$, $R_p[\alpha]$ is integral over $R_p[\beta]$ and hence is finite over $R_p[\beta]$. Thus $R_p[\alpha]$ is quasi-finite over $R_p[\beta]$. Since $R_p[\beta]$ is quasi-finite over R_p by Lemma 2, the composite $R_p \to R_p[\beta] \to R_p[\alpha]$ is quasi-finite. So by Lemma 2, we conclude that $R[\alpha]$ does not have a blowing-up at p.

(2) By virtue of (1), it suffices to show that the canonical map $\psi_\alpha : \mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)$ is surjective if and only if the canonical map $\psi_\beta : \mathrm{Spec}(R[\beta]) \to \mathrm{Spec}(R)$ is surjective. Consider the following commutative diagram :

$$\begin{array}{ccc} \mathrm{Spec}(R[\alpha]) & \xrightarrow{\psi_{\alpha,\beta}} & \mathrm{Spec}(R[\beta]) \\ {\scriptstyle\psi_\alpha}\downarrow & & {\scriptstyle\psi_\beta}\downarrow \\ \mathrm{Spec}(R) & = & \mathrm{Spec}(R) \end{array}$$

Since $R[\alpha]$ is integral over $R[\beta]$, the canonical map $\psi_{\alpha,\beta} : \mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R[\beta])$ is surjective. So $\psi_\alpha = \psi_\beta \cdot \psi_{\alpha,\beta}$ is surjective if and only if ψ_β is surjective. We are done. □

Remark 4. (1) Let β be an element of $R[\alpha]$ and write $\beta = f(\alpha)$ with $f(X) \in R[X]$. If $f(X)$ is monic, then $R[\alpha]$ is integral over $R[\beta]$ because α is a solusion of $f(X) - \beta = 0$.

(2) Assume that R is a Noetherian Krull domain and that α is integral over R. Then both α and $\beta \in R[\alpha]$ are anti-integral elements over R by [OSY,(1.12) and (1.13)]. Thus $R[\alpha]$ is flat (resp. faithfully flat) over R if and only if so is $R[\beta]$ over R by Proposition 3.

Recall the following result.

Lemma 5(*cf.*[OSY,(3.7)]). *Assume that α is an anti-integral element of degree d over R. Then* $\mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)$ *is surjective if and only if* $\sqrt{J_{[\alpha]}} = \sqrt{\tilde{J}_{[\alpha]}}$.

Proposition 6. *Assume that α is an anti-integral element of degree d over R. Let β be an element of $R[\alpha]$ such that β is anti-integral over R and that $R[\alpha]$ is integral over $R[\beta]$. Then the following two equalities hold :*

(1) $\sqrt{J_{[\alpha]}} = \sqrt{J_{[\beta]}}$,

(2) $\sqrt{\tilde{J}_{[\alpha]}} = \sqrt{\tilde{J}_{[\beta]}}$.

Proof. (1) Let p be a prime ideal in $\mathrm{Spec}(R)$. Then $J_{[\alpha]} \not\subseteq p$ if and only if $R_p[\alpha]$ is a flat over R_p by Lemma 2. Proposition 3 yields that $J_{[\alpha]} \not\subseteq p$ if and only if $J_{[\beta]} \not\subseteq p$. Hence $\sqrt{J_{[\alpha]}} = \sqrt{J_{[\beta]}}$.

(2) We know that the contraction map $\mathrm{Spec}(R[\alpha]) \to \mathrm{Spec}(R)$ is surjective if and only if $\sqrt{J_{[\alpha]}} = \sqrt{\tilde{J}_{[\alpha]}}$ by Lemma 5. Let p be a prime ideal in $\mathrm{Spec}(R)$. Then we see that the following :

$$\begin{aligned} \tilde{J}_{[\alpha]} \not\subseteq p \Leftrightarrow R_p[\alpha] &\text{ is a faithfully flat over } R_p \\ \Leftrightarrow R_p[\beta] &\text{ is a faithfully flat over } R_p \\ \tilde{J}_{[\beta]} &\not\subseteq p. \end{aligned}$$

Hence $\sqrt{\tilde{J}_{[\alpha]}} = \sqrt{\tilde{J}_{[\beta]}}$. □

Definition 7. Set

$$\Gamma := \{ \alpha \in L \mid \alpha \text{ is an anti-integral element over } R \}.$$

Two elements α and β in Γ are called to be *connected by a segment* if either of the following conditions hold :

(1) β is in $R[\alpha]$ and $R[\alpha]$ is integral over $R[\beta]$,

(2) α is in $R[\beta]$ and $R[\beta]$ is integral over $R[\alpha]$.

If α and β are connected by a segment, we write $\alpha - \beta$.

Let Γ_α denote the connected component containing α, that is,

$$\Gamma_\alpha := \{ \gamma \in \Gamma \mid \text{ there exist some elements } \beta_1, \dots, \beta_n \in \Gamma \text{ such that } \alpha - \beta_1,\ \beta_1 - \beta_2,\ \dots,\ \beta_n - \gamma \}.$$

Remark 8. (1) Note that the relation " — " is not necessarily an equivalence relation.

(2) For $\alpha, \beta \in \Gamma$, if $\alpha - \beta$ then $\beta - \alpha$.

(3) If $\beta, \gamma \in \Gamma_\alpha$, then $\Gamma_\beta = \Gamma_\gamma = \Gamma_\alpha$.

Theorem 9. *Assume that $\beta, \gamma \in \Gamma_\alpha$. Then the following four statements hold :*

(1) *$R[\gamma]$ is flat over R if and only if $R[\beta]$ is flat over R,*

(2) *$R[\gamma]$ is faithfully flat over R if and only if $R[\beta]$ is faithfully flat over R,*

(3) $\sqrt{J_{[\gamma]}} = \sqrt{J_{[\beta]}}$,

(4) $\sqrt{\tilde{J}_{[\gamma]}} = \sqrt{\tilde{J}_{[\beta]}}$.

Proof. Noting that $\Gamma_\alpha = \Gamma_\gamma = \Gamma_\beta$ by Remark 8(3), our proof follows from Propositions 3 and 6 and the definition of Γ_α

2. Simple SubExtension of $R[\alpha, \alpha^{-1}]$.

Lemma 10. *Assume that α is an anti-integral element of degree d over R. Let β be an element in $R[\alpha, \alpha^{-1}]$ such that $R[\alpha, \alpha^{-1}]$ is integral over $R[\beta]$. Then $\sqrt{J_{[\alpha]}} \supseteq \sqrt{J_{[\beta]}}$.*

Proof. Put $A := R[\alpha, 1/\alpha] \supseteq R[\beta] := B$. Take $p \in \mathrm{Spec}(R)$ with $p \supseteq J_{[\alpha]}$. Then $P := pA$ is a prime ideal of A and $A/P \cong R_p[T, T^{-1}]$, where T denotes an indeterminate over R/p. Put $Q := P \cap B$. Then A is integral over B by the assumption. Hence A/P is algebraic over B/Q. Thus $\mathrm{Tr.deg}_{k(p)} k(Q) > 0$. So it does not hold that $J_{[\beta]} \not\subseteq p$. Therefore $J_{[\alpha]} \subseteq p$. □

Lemma 11. *Let β be an element in $R[\alpha, \alpha^{-1}]$. Assume that β is anti-integral over R and that $R[\alpha, \alpha^{-1}]$ is integral over $R[\beta]$. Then $\sqrt{J_{[\alpha]}} \subseteq \sqrt{J_{[\beta]}}$.*

Proof. Put $A := R[\alpha, 1/\alpha] \supseteq R[\beta] := B$. Take $p \in \mathrm{Spec}(R)$ with $p \supseteq J_{[\beta]}$. Then $Q := pB$ is a prime ideal of B and $B/Q \cong R/p[T]$, where T denotes an indeterminate over R/p. Since A is integral over B, there exists $P \in \mathrm{Spec}(A)$ with $P \cap B = Q$. Since $k(p) \subseteq k(Q) \subseteq k(P)$ and $\mathrm{Tr.deg}_{k(p)}k(Q) > 0$, we have $\mathrm{Tr.deg}_{k(p)}k(P) > 0$. Put $C := R[\alpha]$ and $P' := P \cap C$. Then $\mathrm{Tr.deg}_{k(p)}k(P') > 0$. Note that $P' \cap R = p$. So C has a blowing-up at p. Thus $J_{[\alpha]} \subseteq p$ by Lemma 2. □

Theorem 12. *Assume that α is an element of degree d over R. Let β be an element in $R[\alpha, \alpha^{-1}]$ such that β is anti-integral over R and that $R[\alpha, \alpha^{-1}]$ is integral over $R[\beta]$. Then $\sqrt{J_{[\alpha]}} = \sqrt{J_{[\beta]}}$.*

Proof. Our conclusion follows from Lemmas 10 and 11. □

Recall the following result.

Lemma 13(*cf.*[SOY,Cor.5]). *Assume that α is an anti-integral element over R. Then the following statements are equivalent :*
(1) *$R[\alpha]$ is flat over R,*
(2) *$R[\alpha, \alpha^{-1}]$ is flat over R.*

Theorem 14. *Assume that α is an element of degree d over R. Let β be an element in $R[\alpha, \alpha^{-1}]$ such that β is anti-integral over R and that $R[\alpha, \alpha^{-1}]$ is integral over $R[\beta]$. Then the following statements are equivalent :*
(1) *$R[\alpha]$ is flat over R,*
(2) *$R[\alpha, \alpha^{-1}]$ is flat over R,*
(3) *$R[\beta]$ is flat over R,*
(4) *$R[\beta, \beta^{-1}]$ is flat over R.*

Proof. The inplications (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) follow Lemma 13. (1) $\Leftrightarrow$ (3) follows from Lemma 2 and Theorem 12. □

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