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Kronecker Function Rings of Semigroups

Dedicated to Professor Masayoshi Nagata on his sixtieth birthday

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We review first [12, Theorem 7] for convenience. Let A be a commutative ring $\ni 1$. We denote the total quotient ring of A by $q(A)$. A non-zerodivisor of A is called a regular element of A. Let $\mathfrak{a}$ be an ideal of A. We denote the set of regular elements of A contained in $\mathfrak{a}$ by $\mathrm{Reg}(\mathfrak{a})$. If $\mathrm{Reg}(\mathfrak{a}) \neq \phi$, then $\mathfrak{a}$ is called a regular ideal of A. Let $f \in A[X]$. We denote the ideal of A generated by the coefficients of f by $c(f)$. If $c(f)$ is a regular ideal for each regular element f of $A[X]$, then A is said to have property (C). If each regular ideal $\mathfrak{a}$ of A is generated by $\mathrm{Reg}(\mathfrak{a})$, then A is called a Marot ring. A multiplicative system of A consisting of regular elements is called a regular multiplicative system of A. A quotient ring of A by a regular multiplicative system is called a regular quotient ring of A. A subring of $q(A)$ containing A is called an overring of A. Let P be a prime ideal of A. The set $\{x \in q(A); ax \in A$ for some element $a \in A-P\}$ is denoted by $A_{[P]}$. An overring of A which is a valuation ring of $q(A)$ is called a valuation overring of A. We are able to define $*$-operation on a commutative ring A. Also we are able to define the Kronecker function ring A_* of A with respect to $*$. We proved the fundamental properties of A_* on [10]. Let $*$ be an e.a.b. $*$-operation on A. We set $U^* = \{\text{regular } f \in A[X]; c(f)^* = A\}$. Then U^* is a multiplicative system of $A[X]$.

THEOREM 1 ([12, Theorem 7]). *Let A be a Marot ring with property* (C). *If $*$ is an* e.a.b. *$*$-operation on A, then the following conditions are equivalent*:

(1) *A is a Prüfer $*$-multiplication ring*;

(2) $A[X]_{U^*} = A_*$;

(3) *$A[X]_{U^*}$ is a Prüfer ring*;

(4) *A_* is a regular quotient ring of $A[X]$*;

(5) *Each prime ideal of $A[X]_{U^*}$ is the contraction of a prime ideal of A_**;

(5r) *Each regular prime ideal of $A[X]_{U^*}$ is the contraction of a prime ideal of A_**;

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(6r) *Each regular prime ideal of* $A[X]_{U^*}$ *is the extension of a prime ideal of* A;

(7) *Each valuation overring of* A_* *is of the form* $A[X]_{[PA[X]]}$, *where* P *is a prime ideal of* A *such that* $A_{[P]}$ *is a valuation ring of* $q(A)$;

(8) A_* *is a flat* $A[X]$*-module.*

Moreover there exists a Prüfer Marot ring A *with property* (C) *which satisfies the following condition: Let* $*$ *be any* e.a.b. $*$*-operation on* A. *Then there exists a prime ideal of* $A[X]_{U^*}$ *which is not the extension of a prime ideal of* A.

Let $*$ be a $*$-operation on a ring A. Also we set $U^* = \{\text{regular } f \in A[X];\ c(f)^* = A\}$.

LEMMA 2. *Let* $*$ *be a* $*$*-operation on a Marot ring* A *with property* (C). *Then* U^* *is a multiplicative system of* $A[X]$.

PROOF. Because the Dedekind-Mertens Lemma holds for $A[X]$ (*cf.* [4, Corollary (28.3)]).

LEMMA 3. *Let* $*$ *be a* $*$*-operation on a Marot ring* A *with property* (C). *Assume that* A *is either a Prüfer* $*$*-multiplication ring or* $A[X]_{U^*}$ *is a Prüfer ring. Then we have* $(\mathfrak{a}\mathfrak{a}^{-1})^* = A$ *for each finitely generated regular ideal* $\mathfrak{a}$ *of* A.

PROOF. We have

$$A[X]_{U^*} = (\mathfrak{a}A[X]_{U^*})(\mathfrak{a}A[X]_{U^*})^{-1} = (\mathfrak{a}A[X]_{U^*})(\mathfrak{a}^{-1}A[X]_{U^*}) = (\mathfrak{a}\mathfrak{a}^{-1})A[X]_{U^*}.$$

Therefore there exists an element $u \in U^*$ contained in $(\mathfrak{a}\mathfrak{a}^{-1})A[X]$. It follows $c(u) \subset \mathfrak{a}\mathfrak{a}^{-1}$, and hence $(\mathfrak{a}\mathfrak{a}^{-1})^* = A$.

PROPOSITION 4. *Let* A *be a Marot ring with property* (C). *Assume that either* A *is a Prüfer* v*-multiplication ring or* $A[X]_{U_v}$ *is a Prüfer ring. Then the* $*$*-operation* v *is an* a.b. $*$*-operation, and hence the* 9 *conditions of Theorem* 1 *hold for* A *and the operation* v.

PROOF. Let $\mathfrak{a}$ be a finitely generated regular ideal of A. We have $(\mathfrak{a}\mathfrak{a}^{-1})^v = A$ by Lemma 3. It follows that the operation v is a.b. ([9, Lemma 4]).

PROPOSITION 5. *Let* $*$ *be a* $*$*-operation on an integral domain* D. *Assume that either* D *is a Prüfer* $*$*-multiplication ring or* $D[X]_{U^*}$ *is a Prüfer ring. Then the operation* $*$ *is* a.b., *and hence the* 9 *conditions of Theorem* 1 *hold for* D *and* $*$.

PROOF. By Lemma 3 we have $(\mathfrak{a}\mathfrak{a}^{-1})^* = D$ for each finitely generated nonzero ideal $\mathfrak{a}$ of D. It follows that $*$ is a.b.

REMARK 6. *Let* $*$ *be a* $*$*-operation on a Marot ring* A *with property* (C). *Assume that* A *is a Prüfer* $*$*-multiplication ring and that* $A[X]_{U^*}$ *equals to its total quotient ring. Then* $*$ *is not necessarily* e.a.b.

COUNTER EXAMPLE. Consider the ring A and the operation $*$ on A of [9, Remark 7]. Let f be a regular element of $A[X]$. We have $((b_0,\dots,b_m)c(f))^*=A$ for regular elements $b_0,\dots,b_m$ of A. It follows that $1/f\in A[X]_{U^*}$, and hence $A[X]_{U^*}=q(A[X]_{U^*})$.

The operation $*$ of the above Counter Example differs from v by Proposition 4. In fact we have $(u)^*=(u)\not\ni v\in(u)^v$.

Next let S be a torsion-free cancellative commutative additive semigroup $\supsetneqq\{0\}$. We set $G=\{s-s'; s, s'\in S\}$. Let H be the units of S, and let M be the non-units of S. On [13, Section 10] we defined $*$-operation on S. Let D be a domain. Also we defined the Kronecker function ring S_*^D of S with respect to an e.a.b. $*$-operation $*$ on S. And we proved fundamental properties of S_*^D.

REMARK 7. (1) (cf. [7, p. 75]) *If we replace D by a ring A in* [13, Lemma 10.2)], *the statement is false.*

(2) (A part of [13, Proposition (10.4), (2)]) *Let $*$ be an* e.a.b. *$*$-operation on S. Then we have $S_*^D=S_*^{q(D)}$ for each domain D.*

Let $f\in D[X; S]$. We have $f=\sum_1^n a_iX^{s_i}$ with $a_i\neq 0$ $(1\leqq i\leqq n)$ and $s_i\neq s_j$ $(i\neq j)$. Then the set $\{s_1,\dots,s_n\}$ is denoted by Supp (f).

The following assertion is stated in [13, Proposition (10.4), (3)] without proof. And it seems that the assertion can not be proved simply by analogous ways to rings.

THEOREM 8. *Let $*$ be an* e.a.b. *$*$-operation on S. Then S_*^D is a Bezout ring for each domain D.*

PROOF. Set $k=q(D)$. We have $S_*^D=S_*^k$ by Remark 7, (2). If k is an infinite field, the assertion can be proved by an analogous way to rings (cf. [4, Theorem (32.7), (b)]). Let k be any field. Let $0\neq f\in k[X; S]$. Set $\mathrm{Supp}(f)=\{s_1,\dots,s_n\}$. Then we have $fS_*^k=(X^{s_1},\dots,X^{s_n})S_*^k$. Now let ξ and η be nonzero elements of S_*^k. We set $\xi=f/g$ and $\eta=h/g$ $(f, g, h\in k[X; S])$. We set $\mathrm{Supp}(f)=\{s_1,\dots,s_n\}$, $\mathrm{Supp}(h)=\{t_1,\dots,t_m\}$ and set $\mathrm{Supp}(f)\cup\mathrm{Supp}(h)=\{u_1,\dots,u_l\}$ with $u_i\neq u_j$ $(i\neq j)$. We have

$$\begin{aligned}(\xi,\eta)S_*^k&=\left(\frac{X^{s_1}}{g},\dots,\frac{X^{s_n}}{g},\eta\right)S_*^k\\&=\left(\frac{X^{s_1}}{g},\dots,\frac{X^{s_n}}{g},\frac{X^{t_1}}{g},\dots,\frac{X^{t_m}}{g}\right)S_*^k\\&=\left(\frac{X^{u_1}}{g},\dots,\frac{X^{u_l}}{g}\right)S_*^k=((\sum_1^l X^{u_i})/g)S_*^k.\end{aligned}$$

It follows that $(\xi,\eta)S_*^k$ is a principal ideal of S_*^k.

Henceforth in this paper let k be any field, and we assume that $S\subsetneqq G$.

Let $F(S)$ be the set of fractional ideals of S. We denote the set of finitely generated fractional ideals of S by $F_f(S)$. Let $*$ be a $*$-operation on S. We set $\mathfrak{a}^* = \mathrm{div}^* \mathfrak{a}$ for each $\mathfrak{a} \in F(S)$. We set $\{\mathrm{div}^* \mathfrak{a}; \mathfrak{a} \in F(S)\} = D^*(S)$ and $\{\mathrm{div}^* \mathfrak{a}; \mathfrak{a} \in F_f(S)\} = D_f^*(S)$. These are semigroups under the addition: $\mathrm{div}^* \mathfrak{a} + \mathrm{div}^* \mathfrak{b} = \mathrm{div}^* (\mathfrak{a}+\mathfrak{b})$. We set $D^*(S)/\{\mathrm{div}^* (\alpha); \alpha \in G\} = C^*(S)$ and $D_f^*(S)/\{\mathrm{div}^* (\alpha); \alpha \in G\} = C_f^*(S)$. These are semigroups too. We set $\mathrm{div}^v (\mathfrak{a}) = \mathrm{div} (\mathfrak{a})$ for $\mathfrak{a} \in F(S)$, $D^v(S) = D(S)$, $D_f^v(S) = D_f(S)$, $C^v(S) = C(S)$ and $C_f^v(S) = C_f(S)$. If $D_f^*(S)$ is a group, S is called a Prüfer $*$-multiplication semigroup. If $C_f(S) = 0$, then S is called a pseudo-Bezout semigroup. A pseudo-Bezout semigroup is also called GCD-semigroup ([7]). G/H is denoted by GD(S), and is called the group of divisibility of S. If $C(S) = 0$, then S is called a pseudo-principal semigroup.

Let $\mathfrak{a} \in F(S)$. If $\mathfrak{a} + \mathfrak{b} = S$ for some $\mathfrak{b} \in F(S)$, then $\mathfrak{a}$ is a principal fractional ideal of S. The proof is straightforward.

LEMMA 9. *There exists a valuation oversemigroup of S the center on S of which is M.*

PROOF. $Mk[X; S]$ is a prime ideal of $k[X; S]$. There exists a valuation overring W of $k[X; S]$ the center of which on $k[X; S]$ is $Mk[X; S]$. Then the restriction $W \cap G$ is a desired oversemigroup of S.

LEMMA 10. *S is a valuation semigroup if and only if either $\alpha \in S$ or $-\alpha \in S$ for each $\alpha \in G$.*

PROOF. The sufficiency. By Lemma 9 there exists a valuation v of G with center M on S. Suppose that $v(\alpha) \geqq 0$ and $\alpha \bar{\in} S$. We have $-\alpha \in S$, and hence $v(-\alpha) \geqq 0$. It follows $v(-\alpha) = 0$, hence $-\alpha \in H$; a contradiction.

LEMMA 11. *Assume that S is integrally closed. Let $s, t \in S$. If $(n-1)s + t \in (ns, nt)$ for some $n > 1$, then (s, t) is a principal ideal of S.*

PROOF. An analogy to rings (cf. [4, Proposition (24.2)]).

LEMMA 12. *Assume that S is integrally closed, and let $s, t \in S$. If $(ns, nt)^b = (ns, nt)$ for a natural number n, then $(ns, nt) = n(s, t)$.*

PROOF. Let $\{V_\lambda; \lambda \in \Lambda\}$ be the set of valuation oversemigroups of S. Let $i + j = n$ for natural numbers i and j. Then either $is + jt \in ns + V_\lambda$ or $is + jt \in nt + V_\lambda$ for each $\lambda \in \Lambda$. It follows $is + jt \in (ns, nt)V_\lambda$. We have

$$is + jt \in \bigcap_\lambda (ns, nt)V_\lambda = (ns, nt)^b = (ns, nt).$$

LEMMA 13. *Assume that each element of $F_f(S)$ is principal. Then S is a valuation semigroup.*

PROOF. There exists a valuation oversemigroup V of S with center M on S.

Let $0 \neq \alpha \in V$. We have $\alpha = s_1 - s_2$ with $s_i \in S$. (s_1, s_2) is a principal ideal (s_0) of S for $s_0 \in S$. $s_1 = s_0 + t_1$ and $s_2 = s_0 + t_2$ for $t_i \in S$. Either $s_0 \in s_1 + S$ or $s_0 \in s_2 + S$; say $s_0 \in s_1 + S$. Then $S_2 = s_1 + t_3$ for $t_3 \in S$, and $\alpha = -t_3$. It follows $\alpha \in H \subset S$. The case $s_0 \in s_2 + S$ is similar.

LEMMA 14. *Assume that S is integrally closed and $(s, t)^b = (s, t)$ for each $s, t \in S$. Then S is a valuation semigroup.*

PROOF. Let $s, t \in S$. We have $(2s, 2t) = 2(s, t)$ by Lemma 12. (s, t) is a principal ideal by Lemma 11. S is a valuation semigroup by Lemma 13.

LEMMA 15. *Assume that $*$ is e.a.b. and $S^k_* = S^k_v$ for each $*$-operation $*$ on S. Then we have $\mathfrak{a}^v = \mathfrak{a}$ for each $\mathfrak{a} \in F_f(S)$.*

PROOF. Let ι be the identity mapping of $F(S)$. We have $S^k_\iota = S^k_v$. It follows $\mathfrak{a} = \mathfrak{a}^v$ for each $\mathfrak{a} \in F_f(S)$.

LEMMA 16. *Assume that S is integrally closed. If S is a Prüfer b-multiplication semigroup, then each element $\mathfrak{a}$ of $F_f(S)$ is princiapal.*

PROOF. We have $(\mathfrak{a} + \mathfrak{b})^b = S$ for $\mathfrak{b} \in F_f(S)$. Then $\mathfrak{a} + \mathfrak{b} = S$ by Lemma 9. Therefore $\mathfrak{a}$ is principal.

PROPOSITION 17. *The following conditions are equivalent:*

(1) *Each finitely generated ideal of S is principal;*

(2) *S is integrally closed semigroup and a Prüfer b-multiplication semigroup;*

(3) *S is a valuation semigroup;*

(4) *$*$ is e.a.b. and $S^k_* = S^k_v$ for each $*$-operation $*$ on S;*

(5) *S is integrally closed, and $\mathfrak{a}^v = \mathfrak{a}$ for each $\mathfrak{a} \in F_f(S)$;*

(6) *S is integrally closed, and $(s, t)^b = (s, t)$ for each $s, t \in S$;*

(7) *S is integrally closed, and $\mathfrak{a}^b = \mathfrak{a}$ for each $\mathfrak{a} \in F_f(S)$;*

(8) *S is integrally closed, and $(s, t)^v = (s, t)$ for each $s, t \in S$;*

(9) *$*$ is a.b. and $S^k_* = S^k_v$ for each $*$-operation $*$ on S.*

PROOF. (8)$\Rightarrow$(6): Because $\mathfrak{a}^* \subset \mathfrak{a}^v$ for each $*$-operation $*$. (1)$\Rightarrow$(3): By Lemma 13. (6)$\Rightarrow$(3): By Lemma 14. (4)$\Rightarrow$(5): By Lemma 15. (2)$\Rightarrow$(1): By Lemma 16. (5)$\Rightarrow$(8), (7)$\Rightarrow$(6), (3)$\Rightarrow$(1), (3)$\Rightarrow$(7), (3)$\Rightarrow$(5), (3)$\Rightarrow$(2), (9)$\Rightarrow$(4) and (3)$\Rightarrow$(9) are straightforward.

Let $*$ be a $*$-operation on S. We set $U^* = \{f \in k[X; S]; e(f)^* = S\}$.

LEMMA 18. *U^* is a multiplicative system of $k[X; S]$.*

PROOF. There exists a natural number m such that $(m+1)e(f) + e(g) = me(f) + e(fg)$ ([7, Proposition 6.2] or [13, Lemma (10.2)]). It follows that U^* is a multiplicative system of $k[X; S]$.

REMARK 19. (1) *We have $U^* \subset U^v$ for each *-operation * on S;*

(2) *Assume that S is integrally closed. Then we have $U^b = k[X; S] - Mk[X; S]$ and $U^b \subset U^*$ for each *-operation *.*

PROOF. (2): By Lemma 9.

We define that the ideal of $k[X; S]$ (or $k[X; S]_{U^*}$ or S^k_*) generated by the empty set ϕ of S is zero.

Next we will see the semigroup version of Theorem 1.

LEMMA 20. *Let * be an* e.a.b. **-operation on S. Let $\mathfrak{A}$ be an ideal of S^k_*, and let $\mathfrak{a} = \mathfrak{A} \cap S$. Then we have $\mathfrak{A} \cap k[X; S] = \mathfrak{a}k[X; S]$.*

PROOF. Let $0 \neq f \in \mathfrak{A} \cap k[X; S]$. We have $fS^k_* = (s_1, \dots, s_n)S^k_*$, where $\{s_1, \dots, s_n\} = \mathrm{Supp}(f)$. It follows that $(s_1, \dots, s_n) \subset \mathfrak{a}$, $f \in \mathfrak{a}k[X; S]$ and hence $\mathfrak{A} \cap k[X; S] \subset \mathfrak{a}k[X; S]$.

A valuation semigroup of the form S_p is called essential for S, where P is a prime ideal of S. A valuation ring of the form D_p is called essential for D, where P is a prime ideal of D.

LEMMA 21. *Let * be an* e.a.b. **-operation on S. If $k[X; S]_{U^*}$ is a Prüfer ring, the condition* (7) *of the following Theorem* 25 *holds.*

PROOF. Let W be a valuation overring of S^k_* with center $\mathfrak{P}$ on S^k_*. Set $\mathfrak{P} \cap k[X; S] = \mathfrak{p}$ and $\mathfrak{p} \cap S = P$. Then $\mathfrak{p} = Pk[X; S]$ by Lemma 20. Since $k[X; S]_{U^*}$ is Prüfer, we have $W = k[X; S]_{\mathfrak{p}}$, and hence $W = k[X; S]_{Pk[X;S]}$. Set $W \cap G = V$. Then V is a valuation oversemigroup of S. If $\alpha \in V$, we have $X^\alpha = f/g$ for $f, g \in k[X; S]$ with $g \notin Pk[X; S]$. It follows $\alpha \in S_p$, and hence $V \subset S_p$.

LEMMA 22. *If the condition* (7) *of Theorem* 25 *holds, then S^k_* is a flat $k[X; S]$-module.*

PROOF. Let $\mathfrak{m}$ be a maximal ideal of S^k_*. $(S^k_*)_\mathfrak{m}$ is a valuation overring of S^k_*. The center of $(S^k_*)_\mathfrak{m}$ on $k[X; S]$ is $\mathfrak{m} \cap k[X; S]$. By our hypothesis we have $(S^k_*)_\mathfrak{m} = k[X; S]_{Pk[X;S]}$ for a prime ideal P of S. Therefore the center of $(S^k_*)_\mathfrak{m}$ on $k[X; s]$ is $Pk[X; S]$, and hence $\mathfrak{m} \cap k[X; S] = Pk[X; S]$. It follows $(S^k_*)_\mathfrak{m} = k[X; S]_{k[X;S] \cap \mathfrak{m}}$. Then S^k_* is a flat $k[X; S]$-module by [15, Theorem 2].

LEMMA 23. *Let * be an* e.a.b. **-operation on S. If S^k_* is a flat $k[X; S]$-module, then $k[X; S]_{U^*} = S^k_*$.*

PROOF. Let $\mathfrak{m}$ be a maximal ideal of $k[X; S]_{U^*}$, and let $\mathfrak{p} = \mathfrak{m} \cap k[X; S]$. Suppose that $\mathfrak{m}S^k_* = S^k_*$. We will derive a contradiction. We have $(f_1, \dots, f_n)S^k_* = S^k_*$ for $f_i \in \mathfrak{p}$. If k is an infinite field, there exists nonzero elements $a_1, \dots, a_n$ of k

such that $\mathrm{Supp}(f)=\cup_1^n \mathrm{Supp}(f_i)$, where $f=a_1f_1+\cdots+a_nf_n$. f belongs to $\mathfrak{p}$ and $(f_1,\dots,f_n)S_*^k=fS_*^k$. It follows $f\in U^*$, and hence $\mathfrak{m}=k[X;S]_{U^*}$; a contradiction. If k is a finite field, the characteristic p of k is a prime number. Set $\cup_1^n \mathrm{supp}(f_i)=\{t_1,\dots,t_l\}$ with $t_i\neq t_j$ for $i\neq j$, and set $f=\sum_1^l X^{t_i}$. The proof of Theorem 8 shows that $S_*^k=fS_*^k$, and hence $f\in U^*$. Since each f_i is a nonunit of $k[X;S]_{U^*}$, we have $\{t_1,\dots,t_l\}\subset M$. Set $\mathrm{Supp}(f_i)=\{s(i,1),\dots,s(i,l_i)\}$ for each i. If a number $m(1)$ is large enough, there exist no i, j, k such that $s(1,i)=s(2,j)+p^{m(1)}t_k$. It follows $\mathrm{Supp}(f_1)\cap\mathrm{Supp}(f_2f^{\exp(m(1))})=\phi$, where $\exp(m(1))$ denotes $p^{m(1)}$. Similarly if a number $m(2)$ is large enough, we have $\mathrm{Supp}(f_1+f_2f^{\exp(m(1))})\cap\mathrm{Supp}(f_3f^{\exp(m(2))})=\phi$. $\cdots$. Thus we choose numbers $m(3),\dots,m(n-1)$ similarly. We set $f_1+f_2f^{\exp(m(1))}+\cdots+f_nf^{\exp(m(n-1))}=g$. g belongs to $\mathfrak{p}$. Since $\mathrm{Supp}(f_if^{\exp(m(i-1))})\subset\mathrm{Supp}(g)$, we have $(f_1,f_2f^{\exp(m(1))},\dots,f_nf^{\exp(m(n-1))})S_*^k=gS_*^k$. Since f is a unit of S_*^k, we have $(f_1,f_2,\dots,f_n)S_*^k=gS_*^k$, and hence $S_*^k=gS_*^k$. It follows $g\in U^*$, and hence $\mathfrak{m}=k[X;S]_{U^*}$; a contradiction. We have proved $\mathfrak{m}S_*^k\subsetneq S_*^k$. Let $\mathfrak{m}'$ be a maximal ideal of S_*^k containing $\mathfrak{m}S_*^k$, and let $\mathfrak{p}=\mathfrak{m}'\cap k[X;S]$. Since $(S_*^k)_{\mathfrak{m}'}=k[X;S]_{\mathfrak{p}}$, we have $(S_*^k)_{\mathfrak{m}'}=(k[x;S]_{U^*})_{\mathfrak{m}}$, and hence $S_*^k=k[X;S]_{U^*}$.

LEMMA 24. *Let $*$ be an e.a.b. $\mathring{*}$-operation on S. If each prime ideal of $k[X;S]_{U^*}$ is the extension from S, then S is a Prüfer $\mathring{*}$-multiplication semigroup.*

PROOF. If $(k[X;S]_{U^*})_S$ is not a field, there exists a nonzero prime ideal $\mathfrak{P}$ of $k[X;S]_{U^*}$ such that $\mathfrak{P}\cap S=\phi$. Then $\mathfrak{P}$ is not the extension from S; a contradiction. Therefore $(k[X;S]_{U^*})_S$ is a field. Let $\mathfrak{a}$ be an ideal of S generated by $s_1,\dots,s_n$. Set $f=\sum_1^n X^{s_i}$. We have $1/f=\dfrac{h}{X^tg}$ for $h\in k[X;S]$, $t\in S$ and $g\in U^*$. $fh=x^tg$. Then $(\mathfrak{a}+e(h))^*=(t)$ by [13, Lemma (10.3)]. Therefore $\mathrm{div}^*\,\mathfrak{a}$ is an invertible element of $D_f^*(S)$, and hence S is a Prüfer $*$-multiplication semigroup.

THEOREM 25 (The semigroup version of Theorem 1). *Let k be a field and $*$ an e.a.b. $\mathring{*}$-operation on S. Then the following conditions are equivalent:*

(1) *S is a Prüfer $*$-multiplication semigroup;*

(2) $k[X;S]_{U^*}=S_*^k$;

(3) *$k[X;S]_{U^*}$ is a Prüfer ring;*

(4) *S_*^k is a quotient ring of $k[X;S]$;*

(5) *Each prime ideal of $k[X;S]_{U^*}$ is the contraction of a prime ideal of S_*^k;*

(6) *Each prime ideal of $k[X;S]_{U^*}$ is the extension of a prime ideal of S;*

(7) *Each valuation overring of S_*^k is of the form $k[X;S]_{Pk[X;S]}$, where P is a prime ideal of S such that S_p is a valuation oversemigroup of S;*

(8) *S_*^k is a flat $k[X;S]$-module.*

PROOF. (3)⇒(7): By lemma 21. (7)⇒(8): By Lemma 22. (8)⇒(2): By Lemma 23. (6)⇒(1): By Lemma 24. (4)⇒(2): S_*^k is of the form $k[X;S]_T$ If $f\in T$, then $1/f\in S_*^k$, and hence $f\in U^*$. It follows $k[X;S]_T\subset k[X;S]_{U^*}$, and hence $S_*^k\subset$

$k[X; S]_{U^*}$. (1)⇔(4): By [13, Theorem (10.9), (2)]. (5)⇒(6): Let $\mathfrak{p}$ be a prime ideal of $k[X; S]_{U^*}$. We have $\mathfrak{p}=k[X; S]_{U^*} \cap \mathfrak{P}$ for a prime ideal $\mathfrak{P}$ of S_*^k. Set $\mathfrak{p} \cap S = P$. Then $\mathfrak{p}=Pk[X; S]_{U^*}$ by Lemma 20. (2)⇒(3) and (2)⇒(5): straightforward.

On [14] we stated without proofs that conditions (1), (2), (3), (4), (7) and (8) of Theorem 25 are equivalent. Moreover we had posed a question there that if 8 conditions of Theorem 25 are equivalent or not.

COROLLARY 26. *Assume S is integrally closed. The following conditions are equivalent:*

(1) *S is a valuation semigroup;*

(2) *$k[X; S]_{Mk[X;S]}=S_b^k$;*

(3) *$k[X; S]_{Mk[X;S]}$ is a valuation ring;*

(4) *S_b^k is a quotient ring of $k[X; S]$;*

(5) *Each prime ideal of $k[X; S]_{Mk[X;S]}$ is the contraction of a prime ideal of S_b^k;*

(6) *Each prime ideal of $k[X; S]_{Mk[X;S]}$ is the extension of a prime ideal of S;*

(7) *Each valuation overring of S_b^k is of the form $k[X; S]_{Pk[X;S]}$, where P is a prime ideal of S such that S_P is a valuation semigroup;*

(8) *S_b^k is a valuation ring.*

PROOF. S is a Prüfer b-multiplication semigroup if and only if S is a valuation semigroup by the equivalence of (2) and (3) of Proposition 17. We have $k[X; S]_{U^b}=k[X; S]_{Mk[X;S]}$ by Remark 19, (2). The equivalence of (1), (2),..., (7) follows by Theorem 25. (8)⇒(1): Because $S_b^k \cap G=S$.

PROPOSITION 27 (The semigroup version of Proposition 5). *Let $*$ be a $*$-operation on S. Assume that either S is a Prüfer $*$-multiplication semigroup or $k[X; S]_{U^*}$ is a Prüfer ring. Then $*$ is* a.b., *and hence* 8 *conditions of Theorem* 25 *hold.*

PROOF. If $\mathfrak{a} \in F_f(S)$, we have $(\mathfrak{a}k[X; S]_{U^*})^{-1}=\mathfrak{a}^{-1}k[X; S]_{U^*}$. Since $(\mathfrak{a}k[X; S]_{U^*})(\mathfrak{a}k[X; S]_{U^*})^{-1}=k[X; S]_{U^*}$, we have $(\mathfrak{a}+\mathfrak{a}^{-1})k[X; S]_{U^*}=k[X; S]_{U^*}$. Therefore there exists $u \in U^*$ contained in $(\mathfrak{a}+\mathfrak{a}^{-1})k[X; S]$. Then $(\mathfrak{a}+\mathfrak{a}^{-1})^*=S$.

If the operator v is a.b., then S is called regularly integrally closed. If v is e.a.b., then v is a.b.

COROLLARY 28. *If S is a pseudo-Bezout semigroup, the operation v satisfies* 8 *conditions of Theorem* 25.

LEMMA 29. *Assume that S is regularly integrally closed. If S admits a family $\{V_\lambda; \lambda \in \Lambda\}$ of essential valuation semigroups such that $\bigcap_\lambda V_\lambda=S$, then we* *have* $\cap_\lambda V_\lambda^* = S_v^k$, *where* V_λ^* *denotes the natural extension of* V_λ *to* $q(k[X; S])$.

PROOF. Let $\mathfrak{a}$ be an ideal of S generated by $a_1, \dots, a_n$. Each V_i is of the form $S_{P(i)}$ for a prime ideal $P(i)$ of S. We have $\mathfrak{a} + V_i = s_i + V_i$ with $s_i \in S$. Then $a_j = s_i + e_{ij} - t_i$ for $e_{ij} \in S$ and $t_i \in S - P(i)$. Since $\mathfrak{a} \subset (s_i - t_i)$, we have $\mathfrak{a}^v \subset \cap_i (s_i - t_i) \subset \cap_i (s_i + V_i) = \cap_i (\mathfrak{a} + V_i) = \mathfrak{a}^*$, where $*$ is the w-operation on S induced by the representation $S = \cap_\lambda V_\lambda$. It follows $\mathfrak{a}^v = \mathfrak{a}^*$, and hence $S_v^k = S_*^k$. By [13, Proposition (10.6)] we have $S_v^k = \cap_\lambda V_\lambda^*$.

THEOREM 30. *Let* $*$ *be an* e.a.b. $*$*-operation on a regularly integrally closed semigroup* S. *If one of the* 8 *conditions of Theorem* 25 *holds, then* $S_v^k = S_*^k$.

PROOF. Let $\{W_\lambda; \lambda \in \Lambda\}$ be the set of valuation overrings of S_*^k. $\cap_\lambda W_\lambda = S_*^k$. Set $W_\lambda \cap G = V_\lambda$.. Then W_λ is the natural extension V_λ^* of V_λ. Each V_λ is essettial for S by our hypothesis. It follows $\cap_\lambda V_\lambda^* = S_v^k$ by Lemma 29, and hence $S_*^k = S_v^k$.

Next we will see the semigroup version of [1, Theorem 5]. We call a discrete valuation (resp. semigroup and ring) of rank one ([4, §17]) a discrete valuation (resp. semigroup and ring). If $D_\mathfrak{p}$ is a discrete valuation ring for each prime ideal $\mathfrak{p}$ of D, then the domain D is called an almost Dedekind ring.

LEMMA 31. *Assume that* S *is integrally closed. If* S_b^k *is almost Dedekind, then* S *is a discrete valuation semigroup.*

PROOF. Set $\mathrm{GD}(S) = \bar{G}$ and $\{\bar{m}; m \in M\} \cup \{\bar{0}\} = \bar{P}$, where $\bar{m}$ denotes $m + H$. Then $\bar{P}$ is a positive set of $\bar{G}$ ([4, §15]). $\bar{G}$ is a torsion-free abelian group. By [4, Theorem (15.6)] we see that $\bar{G}$ is a totally ordered group, and each element of $\bar{P}$ is non-negative. Let v be the natural mapping of G to $\bar{G}$. Then v is a valuation of G which is non-negative on S. The natural extension v^* of v is non-negative on S_b^k. It follows v^* is discrete, and hence $\bar{G} = Z\bar{\alpha}$ for $\bar{0} < \bar{\alpha} \in \bar{G}$. Then $G = H \oplus Z\alpha$. It follows $\alpha \in S$. If $v(\beta) \geqq \bar{0}$, we have $\bar{\beta} = n\bar{\alpha}$ for $n \geqq 0$, and hence $\beta \in S$. Thus S is the valuation semigroup of the valuation v.

THEOREM 32 (The semigroup version of [1, Theorem 5, 6]). *Assume that* S *is integrally closed. The following conditions are equivalent*:

(1) *S is a discrete valuation semigroup*;

(2) *Each ideal of S is principal*;

(3) $k[X; S]_{Mk[X;S]}$ *is a discrete valuation ring*;

(4) S_b^k *is an almost Dedekind ring*;

(5) S_b^k *is a Dedekind ring*;

(6) S_b^k *is a Noetherian ring*;

(7) S_b^k *is a Krull ring*;

(8) S_b^k *is a discrete valuation ring.*

PROOF. (1)$\Rightarrow$(3): Let v be the valuation associated with S. Then

$k[X; S]_{Mk[X;S]}$ is the valuation ring associated with v^*. (3)⇒(8): Because S_b^k is an overring of $k[X; S]_{Mk[X;S]}$. (6)⇒(7): Because S_b^k is integrally closed. (7)⇒(5): Because S_b^k is Prüfer (cf. [4, Theorem (43.16)]). (4)⇒(1): By Lemma 31. (2)⇒(1): S is a valuation semigroup by Proposition 17. Since M is principal, S is a discrete valuation semigroup. (8)⇒(6) and (5)⇒(4) are straightforward.

The semigroup version of [1, Theorm 4] is contained in Corollary 26.

If there exists a set $\{V_\lambda; \lambda \in \Lambda\}$ of discrete valuation semigroups of G such that $\cap_\lambda V_\lambda = S$ and s is a unit of V_λ for almost all $\lambda \in \Lambda$ for each $s \in S$, then S is called a Krull semigroup.

LEMMA 33 ([2]). (1) *S is a Krull semigroup if and only if S is completely integrally closed and satisfies the ascending chain condition for divisorial ideals cf S;*

(2) *If S is a Krull semigroup under a family $\{V_\lambda; \lambda \in \Lambda\}$ of valuation oversemigroups, then S is of the form $H \oplus S_1$ with $S_1 = q(S_1) \cap (\sum_\lambda \oplus Z_0)$, where $\sum_\lambda \oplus Z_0$ denotes the direct sum of copies of non-negative integers of the cardinality $|\Lambda|$. Conversely a semigroup S of the form is a Krull semigroup;*

(3) *Let $\{V_\lambda; \lambda \in \Lambda\}$ be the family of discrete valuation oversemigroups which are essential for a Krull semigroup S. Then S is a Krull semigroup under $\{V_\lambda; \lambda \in \Lambda\}$.*

THEOREM 34. *Assume S is regularly integrally closed. Then the following conditions are equivalent:*

(1) *S is a Krull semigroup;*

(2) *S_v^k is a principal idela domain;*

(3) *S_v^k is a Noetherian ring;*

(4) *S_v^k is a Krull ring.*

PROOF. (1)⇒(4); There exists a family $\{V_\lambda; \lambda \in \Lambda\}$ *of essential valuation* oversemigroups of S under which S is Krull. We have $S_v^k = \cap_\lambda V_\lambda^*$ by Lemma 29. Therefore S_v^k is a Krull ring. (4)⇒(3); Since S_v^k is Prüfer, it is a Dedekind ring. (3)⇒(2); Because S_v^k is a Bezout ring. (4)⇒(1); Assume that S_v^k is a Krull ring under a family $\{W_\lambda; \lambda \in \Lambda\}$ of valuation overrings of S_v^k. Set $W_\lambda \cap G = V_\lambda$. Then S is a Krull semigroup under $\{V_\lambda; \lambda \in \Lambda\}$. (2)⇒(4); Straightforward.

If S is a Krull semigroup with $C(S)=0$, then S is called a factorial semigroup. S is a factorial semigroup if and only if S is a UFS of [7]. (The proof is similar to rings.) S is a factorial semigroup if and only if each element of S is uniquely expressed as a finite sum of irreducible elements up to associates and order.

PROPOSITION 35. (1) ([2, p. 1460]) *$D(S)$ is a group if and only if S is completely integrally closed;*

(2) *S is regularly integrally closed if and only if* div $\mathfrak{a}$ *is an invertible*

element of $D(S)$ for each $\mathfrak{a}\in F_f(S)$.

PROOF. An analogy to rings (cf. [4, Theorems (34.3) and (34.6)]).

If follows that if S is completely integrally closed, then S is regularly integrally closed. Especially if S is a Krull semigroup, then S is regularly integrally closed.

PROPOSITION 36 (The semigroup version of [3, Theorem (2.3)]). *Assume that S is pseudo-Bezout. Then the following conditions are equivalent:*

(1) *S is a factorial semigroup;*

(2) *S_v^k is a principal ideal domain.*

PROOF. (2)⇒(1); S is a Krull semigroup by Theorem 34. S satisfies the ascending chain condition for principal ideals of S by Lemma 33, (1). Since S is pseudo-Bezout, we see that S is a factorial semigroup. (1)⇒(2): By Theorem 34.

PROPOSITION 37 (The semigroup version of [3, Theorem (2.4)]). *Assume that S is a Krull semigroup. Then each valuation overring of S_v^k is the natural extension of a discrete valuation semigroup of G which is essential for S.*

PROOF. We confer Theorem 34 and its Proof. There exists a family $\{V_\lambda; \lambda\in\Lambda\}$ of essential valuation oversemigroups of S under which S is Krull. Let W be a valuation overring of S_v^k. Then $W=(S_v^k)_{\mathfrak{p}}$, where $\mathfrak{p}$ is the center of W on S_v^k. Since S_v^k is a principal ideal domain, $\mathfrak{p}$ is a minimal prime ideal $\neq(0)$ of S_v^k. Since S_v^k is a Krull ring under $\{V_\lambda^*; \lambda\in\Lambda\}$, we have $(S_v^k)_{\mathfrak{p}}=V_\lambda^*$ for some λ. Thus W is the natural extension of V_λ.

Assume that there exists a family $\{V_\lambda; \lambda\in\Lambda\}$ of valuation semigroups of G such that $\bigcap_\lambda V_\lambda=S$. If $\bigcap_{\lambda\neq\lambda'} V_\lambda\supsetneqq S$ for each λ', the representation $S=\bigcap_\lambda V_\lambda$ is called irredundant. We define irredundant representation for domains similarly.

PROPOSITION 38 (The semigroup version of [6, Proposition 2.1]). *Assume that S admits an irredundant representation $S=\bigcap_\lambda V_\lambda$. Let $*$ be the w-operation induced by the representation. Then $\bigcap_\lambda V_\lambda^*$ is an irredundant representation for S_*^k.*

PROOF. If $V_\mu^*\supset\bigcap_{\lambda\neq\mu} V_\lambda^*$ for some μ, we have $V_\mu\supset\bigcap_{\lambda\neq\mu} V_\lambda$.

PROPOSITION 39 (The semigroup version of [6, Proposition 2.3]). *Assume that S is regularly integrally closed. If S_v^k admits an irredundant representation, then S admits an irredundant representation.*

PROOF. Let $S_v^k=\bigcap_\lambda W_\lambda$ be an irredundant representation for S_v^k. Set $V_\lambda=W_\lambda\cap G$. Then W_λ is the natural extension of V_λ. We have $S=\bigcap_\lambda V_\lambda$. Suppose $S=\bigcap_{\lambda\neq\mu} S_\lambda$ for some μ. The representation $S=\bigcap_{\lambda\neq\mu} S_\lambda$ induces a w-operation $*$ on S. $S_*^k=\bigcap_{\lambda\neq\mu} V_\lambda^*$. Since $S_*^k\subset S_v^k$, we see that $\bigcap_\lambda V_\lambda^*$ is not an irredundant representation for S_v^k; a contradiction.

PROPOSITION 40. *Assume that S is integrally closed. Then the following conditions are equivalent:*

(1) *S is a Noetherian semigroup;*

(2) *S is regularly integrally closed, and S_v^k is a principal integral domain with only a finite number of prime ideals.*

PROOF. (1)⇒(2); S is a Krull semigroup by Lemma 33, (1). S_v^k is a principal ideal domain by Theorem 34. We have $M=(s_1,\dots, s_m)$ for $s_i \in S$, where we may assume that each s_i is irreducible. S does not have other irreducible elements than $s_1,\dots, s_m$ up to associates. Therefore S is a Krull semigroup under a finite family of valuation semigroups of G. Then S_v^k is a Krull ring under a finite family of valuation rings by the proof of Theorem 34. It follows that S_v^k has only a finite number of prime ideals. (2)⇒(1); S is a Krull semigroup under a finite family of valuation semigroups. We may assume that $H=\{0\}$ by Lemma 33, (2). There exists a number n such that $S=G\cap(\sum_1^n \oplus Z_0)$ where $\sum_1^n \oplus Z_0$ is the direct sum of n copies of non-negative integers. The following Lemma 41 shows that S is Noetherian.

Lemma 41 is stated at [2, Remark l] and is proved at [5, Theorem 15.11]. We will give an another proof.

LEMMA 41. *Assume that $S=G\cap(\sum_1^n \oplus Z_0)$ for a natural number n. Then S is Noetherian.*

PROOF. For example, let $n=5$. Let p_i be the i-projection of elements of $\sum_1^n \oplus Z_0$. Let $\mathfrak{a}$ be an ideal of S. There exists an element $s_i \in \mathfrak{a}$ such that $p_i(s_i)= \min\{p_i(s);\ s\in\mathfrak{a}\}$ for each i. Set $\max\{p_j(s_i);\ i, j\}=H_1$. Let a number $h \leqq H_1$. If $\{s\in\mathfrak{a};\ p_i(s)=h\}$ is not empty, there exists an elements $s_{i'}(h;\ i)\in\mathfrak{a}$ such that $p_{i'}(s_{i'}(h;\ i))=\min\{p_{i'}(s);\ s\in\mathfrak{a},\ p_i(s)=h\}$ for each i, i'. Set $\max\{p_j(s_{i'}(h;\ i));\ i, i', h, j\}=H_2$. Let $h_1, h_2 \leqq H_2$. If $\{s\in\mathfrak{a};\ p_{i_1}(s)=h_1,\ p_{i_2}(s)=h_2\}$ is not empty, there exists an element $s_l(h_1, h_2;\ i_1, i_2)\in\mathfrak{a}$ such that $p_l(s_l(h_1, h_2;\ i_1, i_2))=\min\{p_l(s);\ s\in\mathfrak{a},\ p_{i_1}(s)=h_1,\ p_{i_2}(s)=h_2\}$ for each i_1, i_2, l. Set $\max\{p_j(s_l(h_1, h_2;\ i_1, i_2));\ i_1, i_2, h_1, h_2, l, j\}=H_3$. Let numbers $h_1, h_2, h_3 \leqq H_3$. If $\{s\in\mathfrak{a};\ p_{i_1}(s)=h_1,\ p_{i_2}(s)=h_2,\ p_{i_3}(s)=h_3\}$ is not empty, there exists an element $s_l\{h_1, h_2, h_3;\ i_1, i_2, i_3)\in\mathfrak{a}$ such that $p_l(s_l(h_1, h_2, h_3;\ i_1, i_2, i_3))=\min\{p_l(s);\ s\in\mathfrak{a},\ p_{i_1}(s)=h_1,\ p_{i_2}(s)=h_2,\ p_{i_3}(s)=h_3\}$ for each i_1, i_2, i_3, l. Set $\max\{p_j(s_l(h_1, h_2, h_3;\ i_1, i_2, i_3));\ i_1, i_2, i_3, h_1, h_2, h_3, l, j\}=H_4$. Similarly we may consider elements $s_l(h_1, h_2, h_3, h_4;\ i_1, i_2, i_3, i_4)\in\mathfrak{a}$ for each $i_1, i_2, i_3, i_4, h_1, h_2, h_3, h_4, l$. Similarly we may determine a number H_5. Set $B=\{s_{i_1},\ s_l(h_1, i_1),\ s_l(h_1, h_2;\ i_1, i_2),\ s_l(h_1, h_2, h_3;\ i_1, i_2, i_3),\ s_l(h_1, h_2, h_3, h_4;\ i_1, i_2, i_3, i_4);\ i_1, i_2, i_3, i_4, h_1, h_2, h_3, h_4, l\}\cup\{s\in\mathfrak{a};\ p_1(s)<H_5,\ p_2(s)<H_5,\ p_3(s)<H_5,\ p_4(s)<H_5,\ p_5(s)<H_5\}$. We will show that $\mathfrak{a}$ is generated by the finite set B. Let $s\in\mathfrak{a}$, and set $p_i(s)=e_i$. We may assume that

$e_1 \leq e_2 \leq e_3 \leq e_4 \leq e_5$. If $e_1 \geq H_1$, then $s - s_1 \in G \cap (\sum \oplus Z_0)$, and hence $s \in (s_1)$. If $e_1 < H_1$ and $e_2 \geq H_2$, then $s - s_2(e_1; 1) \in (\sum \oplus Z_0) \cap G$, and hence $s \in (s_2(e_1; 1))$. If $e_1 < H_1$, $e_2 < H_2$ and $e_3 \geq H_3$, then $s - s_3(e_1, e_2; 1, 2) \in G \cap (\sum \oplus Z_0)$, and hence $s \in (s_3(e_1, e_2; 1, 2))$. If $e_1 < H_1$, $e_2 < H_2$, $e_3 < H_3$ and $e_4 \geq H_4$, then $s - s_4(e_1, e_2, e_3; 1, 2, 3) \in G \cap (\sum \oplus Z_0)$, and hence $s \in (s_4(e_1, e_2, e_3; 1, 2, 3))$. If $e_1 < H_1$, $e_2 < H_2$, $e_3 < H_3$, $e_4 < H_4$ and $e_5 \geq H_5$, then $s - s_5(e_1, e_2, e_3, e_4; 1, 2, 3, 4) \in G \cap (\sum \oplus Z_0)$, and hence $s \in (s_5(e_1, e_2, e_3, e_4; 1, 2, 3, 4))$.

PROPOSITION 42. *Assume that S is regularly integrally closed.*

(1) *S_v^k is a valuation ring if and only if S is a valuation semigroup;*

(2) *S_v^k is a discrete valuation ring if and only if S is a discrete valuation semigroup.*

PROOF. (1) If S_v^k is a valuation ring, then S is a valuation semigroup since $S_v^k \cap G = S$. If S is a valuation semigroup, S_b^k is a valuation ring by Corollary 26. It follows that S_v^k is a valuation ring. (2) If S is a discrete valuation semigroup, then S_b^k is a discrete valuation ring by Theorem 32.

THEOREM 43 (The semigroup version of [11, Theorem 2]).

(1) *If S is a Krull semigroup, then $k[X; S]_{U^v}$ is a principal ideal domain;*

(2) *If $k[X; S]_{U^v}$ is a Krull ring, then S is a Krull semigroup.*

PROOF. (1): Each valuation overring of S_v^k is the natural extension of an essential valuation oversemigroup of S by Proposition 37. It follows $k[X; S]_{U^v} = S_v^k$ by Theorem 25. Then $k[X; S]_{U^v}$ is a principal ideal domain by Theorem 34. (2): Because $k[X; S]_{U^v} \cap G = S$.

The semigroup version of [11, Theorem 1] is contained in Proposition 27.

PROPOSITION 44. *Assume that S is regularly integrally closed. Then the following conditions are equivalent:*

(1) *S_v^k is a pseudo-principal ring;*

(2) *Each element of $D(S)$ is the difference of two elements of $D_f(S)$.*

PROOF. (1)⇒(2); Let $\mathfrak{a}$ be an ideal of S. Let ξ be the greatest common divisor of $\mathfrak{a}S_v^k$ in S_v^k. We have $\xi = f/g$ for $f, g \in k[X; S]$ with $e(f)^v \subset e(g)^v$. We have $\operatorname{div} e(g) + \operatorname{div} \mathfrak{b} = 0$ for $\mathfrak{b} \in F(S)$ by Proposition 35, (2). If $s \in \mathfrak{a}$, then $X^s \in \xi S_v^k$. It follows $s \in (e(f) + \mathfrak{b})^v$, and hence $(e(f) + \mathfrak{b})^v \supset \mathfrak{a}$ and $(e(f) + \mathfrak{b})^v \supset \mathfrak{a}^v$. Next if $\mathfrak{a} \subset (s)$ for $s \in S$, we have $\xi S_v^k \subset X^s S_v^k$. It follows $e(f)^v \subset s + e(g)^v$, and hence $(e(f) + \mathfrak{b})^v = \mathfrak{a}^v$. (2)⇒(1); Let $\mathfrak{U}$ be a non-zero ideal of S_v^k. Set $\mathfrak{U} - \{0\} = \{\xi_\lambda; \lambda \in \Lambda\}$, and let $\xi_\lambda = f_\lambda / g_\lambda$ for $f_\lambda, g_\lambda \in k[X; S]$ with $e(f_\lambda)^v \subset e(g_\lambda)^v$. We have $\operatorname{div} e(g_\lambda) + \operatorname{div} \mathfrak{b}_\lambda = 0$ for $\mathfrak{b}_\lambda \in F(S)$ for each λ. Set $\cup_\lambda (e(f_\lambda) + \mathfrak{b}_\lambda) = \mathfrak{a}$. Then $\operatorname{div} \mathfrak{a} = \operatorname{div} e(f) - \operatorname{div} e(g)$. Since $\mathfrak{a} \subset S$, we have $f/g \in S_v^k \cdot f/g$ is the greatest common divisor of $\mathfrak{U}$ in S_v^k.

PROPOSITION 45. *The following conditions are equivalent*:

(1) *S is a pseudo-Bezout semigroup*;

(2) *S is regularly integrally closed, and* $\mathrm{GD}(S) \cong \mathrm{GD}(S_v^k)$ *canonically.*

PROOF. (1)⇒(2); Let $\bar{\alpha} \in G/H$ with $\alpha \in G$. We denote the element $\overline{X^\alpha}$ of $\mathrm{GD}(S_v^k)$ by $\phi(\bar{\alpha})$. Let $0 \neq f \in k[X; S]$. We have $e(f)^v = (s)$ for $s \in S$. Then $\phi(\bar{s}) = \bar{f}$. It follows ϕ is an isomorphism of $\mathrm{GD}(S)$ to $\mathrm{GD}(S_v^k)$. (2)⇒(1); Let $\mathfrak{a}$ be an ideal of S generated by $s_1, \dots, s_n$. Set $\sum_1^n X^{s_i} = f$. We have $\bar{f} = \phi(\bar{s})$ for $s \in G$, and hence $e(f)^v = (s)$. It follows $\mathfrak{a}^v = (s)$.

REMARK 46 ([13, Theorem (10.9), (1)]). *If S is a Prüfer *-multiplication semigroup, then* $\mathrm{GD}(S_*^k) \cong \mathrm{D}_f^*(S)$ *canonically.*

PROPOSITION 47. *The following conditions are equivalent*:

(1) *S is a pseudo-principal semigroup*;

(2) *S is regularly integrally closed and* S_v^k *is pseudo-principal, and* $\mathrm{GD}(S) \cong \mathrm{GD}(S_v^k)$ *canonically.*

PROOF. (1)⇒(2); We have $\mathrm{GD}(S) \cong \mathrm{GD}(S_v^k)$ canonically by Proposition 45. S_v^k is pseudo-principal by Proposition 44. (2)⇒(1); Let $\mathfrak{a}$ be an ideal of S. We have $\mathrm{div}\,\mathfrak{a} = \mathrm{div}\,\mathfrak{b} - \mathrm{div}\,\mathfrak{c}$ for $\mathfrak{b}, \mathfrak{c} \in F_f(S)$ by Proposition 44. Then $\mathrm{div}\,\mathfrak{a}$ is principal by proposition 45.

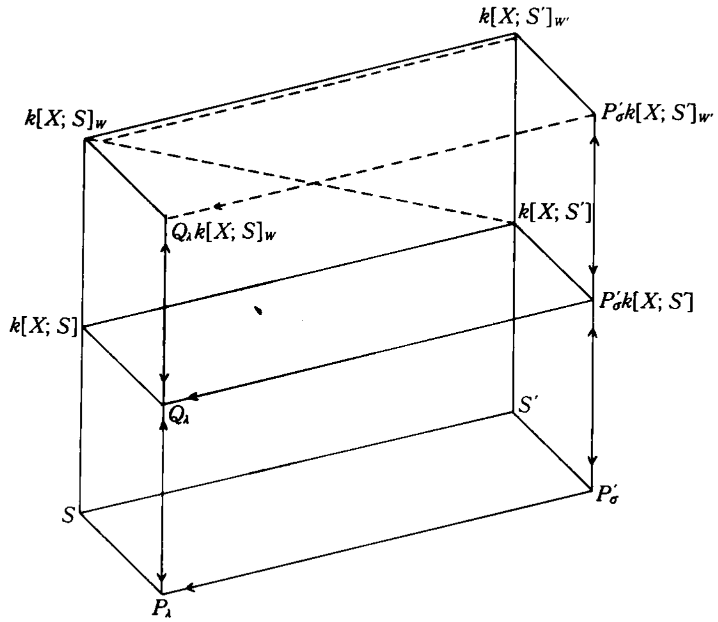
PROPOSITION 48. *Assume that S is integrally closed, and let W be a multiplicative system of* $k[X; S]$. *If each prime ideal of* $k[X; S]_W$ *is the extension of a prime ideal of S, then* $k[X; S]_W$ *is a Bezout ring.*

PROOF. Let $\{Q_\lambda; \lambda \in \Lambda\}$ be the set of prime ideals of $k[X; S]$ which does not intersect with W. Set $Q_\lambda \cap S = P_\lambda$ for each λ. Then $Q_\lambda = P_\lambda k[X; S]$. Let $\{V_\sigma; \sigma \in \Sigma\}$ be the set of valuation oversemigroups of S centers on S of which are among $\{P_\lambda; \lambda \in \Lambda\}$. Let v_σ be the valuation associated with V_σ. Set $\bigcap_\sigma V_\sigma = S'$. Then $S \subset S'$. Let P'_σ be the center of v_σ on S' for each σ. Let W' be the complement of $\bigcup_\sigma P'_\sigma k[X; S']$ in $k[X; S']$. Let $\sigma \in \Sigma$. Then $P'_\sigma \cap S = P_\lambda$ and $Q_\lambda = P_\lambda k[X; S] = (P'_\sigma k[X; S']) \cap k[X; S]$ for some λ. If an element $w \in W$ is contained in $P'_\sigma k[X; S']$, we have $w \in Q_\lambda$; a contradiction. If follows $W \subset W'$, and hence $k[X; S]_W \subset k[X; S']_{W'}$. Let U be a valuation ring the center on $k[X; S]$ of which is Q_λ for some λ. $U \cap G$ is a valuation semigroup the center on S of which is P_λ. It follows $U \cap G = V_\sigma$ for some σ. We have $V_\sigma \supset S'$ and $U \supset S'$. It follows $k[X; S]_W \supset S'$ and $k[X; S]_W \supset k[X; S']$. We have

(#); $k[X; S']_{W'}$ is a quotient ring of $k[X; S]_W$.

Each prime ideal of $k[X; S]_W$ is of the form $Q_\lambda k[X; S]_W$ for some λ. We have $P_\lambda = P'_\sigma \cap S$ for some σ. We have both $(P'_\sigma k[X; S']) \cap k[X; S] = P_\lambda k[X; S]$ and $k[X; S] \cap ((P'_\sigma k[X; S']_{W'}) \cap k[X; S]_W) = P_\lambda k[X; S]$. It follows $(P'_\sigma k[X; S']_{W'})$

$\cap k[X; S]_W = P_\lambda k[X; S]_W$. That is, each prime ideal of $k[X; S]_W$ is the contraction of a prime ideal of $k[x; S']_{W'}$. We see that $k[X; S]_W = k[X; S']_{W'}$ by (#). Let $*$ be the w-operation on S' induced by the representation $S' = \cap_\sigma V_\sigma$. Set $\{f \in k[X; S']; e(f)^* = S'\} = U^*$. If $0 \neq f \in k[X; S]$, then $f \in U^*$ if and only if for each σ we have $v_\sigma(t) = 0$ for some $t \in \mathrm{Supp}(f)$. It follows that $W' = U^*$, and hence $k[X; S']_{W'} = k[X; S]_{U^*}$. Therefore $k[X; S]_W = k[X; S']_{U^*}$. It follows that $k[X; S]_W = S^k_*$ by Theorem 25, and hence $k[X; S]_W$ is a Bezout ring.



The above Proposition 48 is a semigroup version of [8, Lemma (3.0)].

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