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One-Parameter Semigroups of the Gaussian and Poisson Integral Transforms on the W^2-space

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Introduction. Let W^2 be the totality of Lebesgue measurable functions such that

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{x^2+1} dx < \infty.$$

When f and g are in W^2, we define the inner product of f and g by

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \frac{1}{x^2+1} dx.$$

Introducing the operations of addition and scalar multiplication in W^2 as usual, we can show that the space W^2 is the Hilbert space with norm $\|f\| = [(f, f)]^{1/2}$. Let $t > 0$ and let

$$(T(t)f)(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right) f(y)dy,$$

$$T(0)f = f$$

and

$$(P(t)f)(x) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{t}{(x-y)^2+t^2} f(y)dy,$$

$$P(0)f = f$$

for f in W^2. If we choose the $L^2(R)$-space instead of W^2, we know that the family $\{T(t): 0 \leqq t < \infty\}$ is a semigroup of class (C_0) on $L^2(R)$ and also the family $\{P(t): 0 \leqq t < \infty\}$ is a semigroup of class (C_0) on $L^2(R)$, and that the infinitesimal generator of $\{T(t): 0 \leqq t < \infty\}$ is the differential operator $\frac{d^2}{dx^2}$ and the infinitesimal generator of $\{P(t): 0 \leqq t < \infty\}$ is the composition of the differntial operator $\frac{d}{dx}$ and the Hilbert transform C, that is, $\frac{d}{dx} \cdot C$, and that

$$\frac{d^2}{dx^2} = -\left[\frac{d}{dx} \cdot C\right]^2$$

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holds for f in the domain $\left(\left[\frac{d}{dx}\cdot C\right]^2\right)$. (See [3]) Hence it is the purpose of this short note to show that similar relation also holds between the families of linear operators $\{T(t): 0\leqq t<\infty\}$ and $\{P(t): 0\leqq t<\infty\}$ on W^2.

§1. The semigroup $\{T(t): 0\leqq t<\infty\}$ associated with the Gaussian kernel

In this section let us show that the family $\{T(t): 0\leqq t<\infty\}$ of linear operators on W^2 is a semigroup of class (C_0).

LEMMA 1. *$T(t)$ is a linear bounded operator on W^2 to itself and*

$$\omega_0=\lim_{t\to\infty}\frac{1}{t}\log\|T(t)\|=0.$$

PROOF. Let f be in W^2. By the Schwartz inequality we have

$$\begin{aligned}|(T(t)f)(x)|^2 &\leqq (4\pi t)^{-1/2}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4t}\right)|f(y)|^2dy\\ &\quad\cdot(4\pi t)^{-1/2}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4t}\right)dy\\ &=(4\pi t)^{-1/2}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4t}\right)|f(y)|^2dy.\end{aligned}$$

Hence by the Fubini theorem we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{|(T(t)f)(x)|^2}{x^2+1}dx &\leqq \int_{-\infty}^{\infty}|f(y)|^2\left[\int_{-\infty}^{\infty}(4\pi t)^{-1/2}\exp\left(-\frac{(x-y)^2}{4t}\right)\frac{dx}{x^2+1}\right]dy\\ &\leqq\|f\|^2\sup_{y\in R}\left\{(y^2+1)\int_{-\infty}^{\infty}(4\pi t)^{-1/2}\exp\left(-\frac{(x-y)^2}{4t}\right)\frac{dx}{x^2+1}\right\}.\end{aligned}\tag{1.1}$$

Let us prove that

$$\sup_{y\in R}\left\{(1+y^2)\int_{-\infty}^{\infty}(4\pi t)^{-1/2}\exp\left(-\frac{(x-y)^2}{4t}\right)\frac{dx}{x^2+1}\right\}\tag{1.2}$$

is a finite value. By the Parseval theorem we see that

$$\begin{aligned}&\int_{-\infty}^{\infty}(4\pi t)^{-1/2}\exp\left(-\frac{(x-y)^2}{4t}\right)\frac{dx}{x^2+1}\\ &=\frac{1}{\pi}\int_{-\infty}^{\infty}\exp(2\pi ixy-4\pi^2tx^2-2\pi|x|)dx\\ &=\frac{2}{\pi}\int_0^{\infty}\cos(2\pi yx)\exp(-4\pi^2tx^2-2\pi x)dx\end{aligned}\tag{1.3}$$

To prove that (1.2) is a finite value, by (1.3) it suffices to prove that

$$\sup_{y\in R}\left\{2\pi y^2\int_0^\infty \cos(2\pi yx)\exp(-4\pi^2tx^2-2\pi x)dx\right\} \tag{1.4}$$

is a finite value. We see that

$$\left|2\pi y^2\int_0^\infty \cos(2\pi yx)\exp(-4\pi^2tx^2-2\pi x)dx\right|$$

$$\leqq 1+2\pi\int_0^\infty(1+8\pi tx+16\pi^2t^2x^2+2t)e^{-2\pi x}dx \tag{1.5}$$

$$=2+6t+8t^2.$$

From the above arguments and (1.1) it follows that

$$\int_{-\infty}^\infty \frac{|(T(t)f)(x)|^2}{x^2+1}dx \leqq \|f\|^2(3+6t+8t^2).$$

Hence we see that $\|T(t)\|$ is uniformly bounded in t at the neighborhood of $t=0$ and

$$\omega_0=\lim_{t\to\infty}\frac{1}{t}\log\|T(t)\|=0. \qquad \text{Q.E.D.}$$

Let us de.iote the differentiation $\frac{d}{dx}$ by D.

THEOREM 1. *The family $\{T(t): 0\leqq t<\infty\}$ of linear operators on W^2 to itself is a semigroup of class (C_0) and its infinitesimal generator A with domain $D(A)$ is given by the following form;*

$D(A)=\{f\in W^2: f(x), f'(x)$ are absolutely continuous and $f'(x), f''(x)\in W^2\}$
and

$$Af=D^2f$$

for f in $D(A)$.

PROOF. 1. It is easy to prove that the semigroup property and the strong continuity of $\{T(t): 0\leqq t<\infty\}$.

2. Infinitesimal generator A and its domain $D(A)$: From Lemma 1 and [2], the resolvent $R(\lambda; A)$ of the infinitesimal generator A is given by

$$R(\lambda;A)f=\int_0^\infty e^{-\lambda t}T(t)f\,dt,\ (f\in W^2) \tag{1.6}$$

for $\lambda>\omega_0=0$. For simplicity let $\lambda=1$. Then

$$D(A)=\{g=R(1;A)f: f\in W^2\}$$

Take a function f in W^2 and let $g=R(1;A)f$. Let us show that $g(x)$, $g'(x)$ are absolutely continuous and $g', g''\in W^2$. From (1.6) and by the Fubini theorem we obtain that

$$g(x)=\frac{1}{2}\int_{-\infty}^{\infty}f(y)e^{-|x-y|}dy$$

$$=\frac{1}{2}\int_{x}^{\infty}f(y)e^{-y}dye^{x}+\frac{1}{2}\int_{-\infty}^{x}f(y)e^{y}dye^{-x}. \tag{1.7}$$

Hence

$$g'(x)=\frac{1}{2}\int_{x}^{\infty}f(y)e^{-y}dye^{x}-\frac{1}{2}\int_{-\infty}^{x}f(y)e^{y}dye^{-x} \tag{1.8}$$

and

$$g''(x)=-f(x)+g(x) \tag{1.9}$$

for almost all $x\in R$.

From (1.7), (1.8) it is seen that $g(x)$, $g'(x)$ are absolutely continuous.

Next let us prove that $g'(x)$ is in W^2. It holds that

$$|g'(x)|\leqq\frac{1}{2}\int_{-\infty}^{\infty}|f(y)|e^{-|x-y|}dy.$$

By the Schwartz inequality and the Fubini theorem we see that

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{|g'(x)|^2}{x^2+1}dx\\ &\leqq\frac{1}{4}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\left[\int_{-\infty}^{\infty}\frac{|f(y)|}{(1+(x-y)^2)^{1/2}}(1+(x-y)^2)^{1/2}e^{-|x-y|}dy\right]^2dx\\ &\leqq\frac{1}{4}\int_{0}^{\infty}(1+x^2)e^{-2x}dx\int_{-\infty}^{\infty}|f(y)|^2\left\{\int_{-\infty}^{\infty}\frac{1}{1+x^2}\frac{1}{1+(x-y)^2}dx\right\}dy\\ &=\frac{3}{8}\int_{-\infty}^{\infty}\frac{|f(y)|^2}{y^2+4}dy<\infty.\end{aligned}$$

From this fact we also see that $g(x)$ is in W^2 and $g''(x)$ is in W^2. Hence we find from (1.9) and from the same manner as in [7. Pages 243–244] that $D(A)=\{f\in W^2: f$ and f' are absolutely continuous and $f',f''\in W^2\}$ and $Af=f''$ for f in $D(A)$. Q.E.D.

§2. The semigroup $\{P(t): 0\leqq t<\infty\}$ associated with the Poisson kernel

For simplicity let

$$n(x)=\frac{f(x)}{x-i},\quad g(x)=\frac{f(x)}{(u-i)^2}$$

for f in W^2. Let Cn denote the Hilbert transform of n, that is,

$$(Cn)(x)=\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{n(x+u)-n(x-u)}{u}du.$$

Let Fn denote the Fourier transform of n, that is,

$$(Fn)(x) = \underset{a\to\infty}{\text{l.i.m}}\,\frac{1}{2}\int_{-a}^{a} n(y)e^{-ixy}dy,$$

where l.i.m denotes limit in the mean. Then we have

$$(F(Cn))(x) = i(\operatorname{sgn} x)(Fn)(x).$$

Let us introduce the transformation

$$(C_G f)(x) = (x-i)(Cn)(x)$$

for f in W^2.

THEOREM 2. *The family $\{P(t): 0 \leqq t < \infty\}$ of linear operators on W^2 is a semigroup of class (C_0). Its infinitesimal generator A with domain $D(A)$ is given by the following form;*
$D(A) = \{f \in W^2 : (Cn)(x)$ is absolutely continuous and $(DCn)(x) \in L^2(R)\}$
and

$$Af = DC_G f$$

for f in $D(A)$.
If f belongs to the domain $D([DC_G]^2)$, then

$$DC_G DC_G f = -D^2 f.$$

PROOF. By [5. Theorem 1.1] it suffices to obtain the infinitesimal generator A and its domain $D(A)$. From [5. (1.4)] we see that if t is sufficiently small

$$t^{-1}(P(t)f-f)(v) = (v-i)t^{-1}\Big(-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(u)}{(u-i+it)(u-v+it)}du$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(u)}{(u-i-it)(u-v-it)}du$$
$$-\frac{f(v)}{v-i}\Big)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{(u-i-it)(u-i+it)}du$$
$$=(v-i)t^{-1}\Big(-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{n(u)}{u-v+it}du+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{n(u)}{u-v-it}du-n(v)\Big)$$
$$+(v-i)t^{-1}\Big(\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(u)\Big(\frac{1}{u-i}-\frac{1}{u-i+it}\Big)\frac{du}{u-v+it}$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(u)\Big(\frac{1}{u-i-it}-\frac{1}{u-i}\Big)\frac{du}{u-v-it}\Big)$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{(u-i-it)(u-i+it)}du$$

$$=(v-i)t^{-1}\left(\int_{-\infty}^{\infty}\frac{1}{\pi}\frac{t}{(v-u)^2+t^2}n(u)du-n(v)\right)$$

$$+(v-i)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{n(u)}{u-i+it}\frac{du}{u-v+it}+(v-i)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{n(u)}{u-i-it}\frac{du}{u-v-it}$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{(u-i-it)(u-i+it)}du. \tag{2.1}$$

It is easily seen that

$$\int_{-\infty}^{\infty}\frac{f(u)}{(u-i-it)(u-i+it)}du \longmapsto \int_{-\infty}^{\infty}g(u)du \quad \text{as} \quad t\longmapsto 0. \tag{2.2}$$

Let

$$g(t, u)=\frac{n(u)}{u-i+it}.$$

By [6. Proof of Theorem 101] we see that

$$\left(\int_{-\infty}^{\infty}\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}g(t, u)\frac{du}{u-v+it}+\frac{1}{2\pi}\int_{-\infty}^{\infty}g(-t, u)\frac{du}{u-v-it}\right.\right.$$

$$\left.\left.-\frac{1}{2\pi}\int_{-\infty}^{\infty}g(u)\frac{du}{u-v+it}-\frac{1}{2\pi}\int_{-\infty}^{\infty}g(u)\frac{du}{u-v-it}\right|^2dv\right)^{1/2}$$

$$\leqq\frac{1+M}{2}(\|g(t, \cdot)-g\|_2+\|g(-t, \cdot)-g\|_2)\longrightarrow 0 \tag{2.3}$$

as $t\to 0$, where $\|\cdot\|_2$ denotes the L^2-norm and M is a constant number. On the other hand we see from [6. Theorem 92] that

$$\left(\int_{-\infty}^{\infty}\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}g(u)\frac{du}{u-v+it}+\frac{1}{2\pi}\int_{-\infty}^{\infty}g(u)\frac{du}{u-v-it}-(Cg)(v)\right|^2dv\right)^{1/2}$$

$$=\left(\int_{-\infty}^{\infty}\left|\frac{1}{\pi}\int_{-\infty}^{\infty}g(u)\frac{(u-v)du}{(u-v)^2+t^2}-(Cg)(v)\right|^2dv\right)^{1/2}\longmapsto 0 \tag{2.4}$$

as $t\to 0$. By (2.1)–(2.4) and by [3] we see that
$D(A)=\{f\in W^2$: $(Cn)(x)$ is absolutely continuous and $(DCn)(x)\in L^2(R)\}$ and

$$(Af)(v)=(v-i)(DCn)(v)+(v-i)(Cg)(v)+\frac{1}{\pi}\int_{-\infty}^{\infty}g(u)du$$

for f in $D(A)$. From [1. Pages 128–129] we obtain

$$(Cn)(v)=(v-i)(Cg)(v)+\frac{1}{\pi}\int_{-\infty}^{\infty}g(u)du \quad \text{and} \quad Af=DC_Gf.$$

Let us prove the latter part. If f belongs to the domain $D(A^2)$, it follows from $f(x)=(x-i)n(x)$ that $n(x)$, $n'(x)$ are absolutely continuous and $n'(x)$, $n''(x)\in L^2(R)$. We obtain $DCn=CDn$ from [3] and so

$$(DC_G f)(v) = (v-i)(CDn)(v) + (Cn)(v).$$

By the inversion formula of the Hilbert transform and by [1. Pages 128–129] we have

$$(C_G(DC_G f))(v) = (v-i)[C(CDn)](v) + (v-i)C\left[\frac{1}{\cdot - i}(Cn)(\cdot)\right](v)$$

$$= -(v-i)(Dn)(v) - n(v) + K(Cn),$$

where $K(Cn)$ is a constant number. We lastly obtain

$$(DC_G DC_G f)(v) = -2n'(v) - (v-i)n''(v) = -f''(v). \qquad \text{Q.E.D.}$$

It is easy to show the following facts;

(1) If f is a sufficiently regular function in $L^2(R)$, it is shown in [3] that $DCf = CDf$, but here it holds that

$$DC_G f - C_G Df = \int_{-\infty}^{\infty} \frac{f(u)}{(u-i)^2} du.$$

(2) The operator DC_G is connected with the infinitesimal generator D^2 of $T(t)$ by

$$DC_G f = -(-D^2)^{1/2} f$$

for f in $D(D^2)$. (See [7. Page 268])

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