Bull. Fac. Sci., Ibaraki Univ., Math., No. 8, 1976

Some Oscillation and Asymptotic Properties for Linear Differential Equations

V. A. STAIKOS* and Ch. G. PHILOS*

In this paper we consider the n-th order ($n>1$) general ordinary differential equation

(E) $$(r_{n-1}(t)(r_{n-2}(t)(\cdots(r_1(t)(r_0(t)x(t))')'\cdots)')')'+a(t)x(t)=b(t), \qquad t\geqq t_0$$

where the functions r_i ($i=0, 1,\dots, n-1$) are positive at least on the interval $[t_0, \infty)$. The continuity of the functions a, b and r_i ($i=0, 1, 2,\dots, n-1$) as well as sufficient smoothness to guarantee the existence of solutions of (E) on an infinite subinterval of $[t_0, \infty)$ will be assumed without mention. In what follows the term "solution" is always used only for such solutions $x(t)$ of (E) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$ is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

We give here some conditions to ensure that

$$\lim_{t\to\infty} x(t)=0$$

for all oscillatory solutions of the equation (E). However for $b=0$, the same conditions guarantee that all eventually nontrivial solutions of the differential equation

(E)$_0$ $$(r_{n-1}(t)(r_{n-2}(t)(\cdots(r_1(t)(r_0(t)x(t))')'\cdots)')')'+a(t)x(t)=0$$

are nonoscillatory. The technique used is an adaptation of that of Singh [3] which concerns the particular case $r_0=1$, $r_1=r$ and $r_2=\cdots=r_{n-1}=1$.

THEOREM 1. *Consider the differential equation* (E) *subject to the conditions*:

(R_0) $$\liminf_{t\to\infty} r_0(t)>0,$$

(A) $$\int^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |a(s)|\,ds\,ds_{n-1}\cdots ds_2 ds_1<\infty$$

Received January 24, 1976.

* Department of Mathematics, University of Ioannina, Ioannina, Greece.

and

(B)
$$\int^{\infty}\frac{1}{r_1(s_1)}\int_{s_1}^{\infty}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}|b(s)|\,ds\,ds_{n-1}\cdots ds_2ds_1<\infty\,.$$

Then for every oscillatory solution x of the differential equation (E),

$$\lim_{t\to\infty} x(t)=0\,.$$

Proof. Let x be an oscillatory solution of the differential equation (E). Without loss of generality we suppose that x is a solution of (E) on the whole interval $[t_0, \infty)$.

Consider the functions $D_r^{(k)}x$ $(k=0, 1,\dots, n-1)$ which are defined on the interval $[t_0, \infty)$ as follows:

$$D_r^{(0)}x=r_0x$$

and

$$D_r^{(k)}x=r_k(D_r^{(k-1)}x)' \qquad (k=1, 2,\dots, n-1)\,.$$

Now, we assume that

(1)
$$\limsup_{t\to\infty}|(D_r^{(0)}x)(t)|>d$$

for some $d>0$. Moreover, because of condition (R_0), there exists a constant $c>0$ with

(2)
$$r_0(t)\geqq c \qquad \text{for every} \quad t\geqq t_0\,.$$

Thus, by conditions (A) and (B), we have that for some $T\geqq t_0$,

(3)
$$\int_T^{\infty}\frac{1}{r_1(s_1)}\int_{s_1}^{\infty}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}|a(s)|\,ds\,ds_{n-1}\cdots ds_2ds_1\leqq\frac{c}{2}$$

and

(4)
$$\int_T^{\infty}\frac{1}{r_1(s_1)}\int_{s_1}^{\infty}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}|b(s)|\,ds\,ds_{n-1}\cdots ds_2ds_1\leqq\frac{d}{2}\,.$$

Since the solution x is oscillatory, the same holds for the functions $D_r^{(k)}x$ $(k=0, 1,\dots, n-1)$ and consequently we can choose $\tau_1>\tau_2>\cdots>\tau_{n-1}>t_1>T$

with

$$(D_r^{(0)}x)(t_1)=0$$

and

$$(D_r^{(k)}x)(\tau_k)=0 \qquad (k=1, 2,\dots, n-1).$$

Furthermore, we consider, by (1), a $T_0>\tau_1$ with

$$|(D_r^{(0)}x)(T_0)|>d \tag{5}$$

and next a $t_2>T_0$ with

$$(D_r^{(0)}x)(t_2)=0.$$

Now, on repeated integration from equation (E) we have

$$\pm(D_r^{(0)}x)(t)+\int_t^{t_2}\frac{1}{r_1(s_1)}\int_{s_1}^{\tau_1}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{\tau_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\tau_{n-1}}a(s)x(s)ds\,ds_{n-1}\cdots ds_2ds_1=$$

$$=\int_t^{t_2}\frac{1}{r_1(s_1)}\int_{s_1}^{\tau_1}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{\tau_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\tau_{n-1}}b(s)ds\,ds_{n-1}\cdots ds_2ds_1$$

for every $t\in[t_1, t_2]$ and consequently, since $t_2>\tau_1>\tau_2>\cdots>\tau_{n-1}>t_1>T$, we get

$$|(D_r^{(0)}x)(t)|\leq\int_{t_1}^{t_2}\frac{1}{r_1(s_1)}\int_{s_1}^{t_2}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{t_2}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{t_2}|a(s)|\,|x(s)|ds\,ds_{n-1}\cdots ds_2ds_1$$

$$+\int_{t_1}^{t_2}\frac{1}{r_1(s_1)}\int_{s_1}^{t_2}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{t_2}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{t_2}|b(s)|ds\,ds_{n-1}\cdots ds_2ds_1.$$

This immediately gives

$$1\leq\frac{M}{M^*}\int_{t_1}^{t_2}\frac{1}{r_1(s_1)}\int_{s_1}^{t_2}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{t_2}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{t_2}|a(s)|ds\,ds_{n-1}\cdots ds_2ds_1$$

$$+\frac{1}{M^*}\int_{t_1}^{t_2}\frac{1}{r_1(s_1)}\int_{s_1}^{t_2}\frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^{t_2}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{t_2}|b(s)|ds\,ds_{n-1}\cdots ds_2ds_1,$$

where $M = \max_{t\in[t_1,t_2]} |x(t)|$ and $M^* = \max_{t\in[t_1,t_2]} |(D_r^{(0)}x)(t)|$.

Thus, by virtue of (3) and (4), we obtain

$$1 \leqq \frac{M}{M^*} \cdot \frac{c}{2} + \frac{d}{2M^*}.$$

But, by (5) and (2), we have

$$M^* > d \quad \text{and} \quad M^* \geqq c \cdot M$$

and consequently the contradiction

$$1 \leqq \frac{M}{M^*} \cdot \frac{c}{2} + \frac{d}{2M^*} < \frac{1}{2} + \frac{1}{2} = 1.$$

We have just proved that (1) fails and hence

$$\lim_{t\to\infty} |(D_r^{(0)}x)(t)| = 0$$

which, by condition (R_0), gives

$$\lim_{t\to\infty} x(t) = 0.$$

THEOREM 2. *Consider the differential equation* $(E)_0$ *subject to the conditions* (R_0) *and* (A).

Then every eventually nontrivial solution of the differential equation $(E)_0$ *is nonoscillatory.*

PROOF. Let x be an eventually nontrivial oscillatory solution of $(E)_0$ on the whole interval $[t_0, \infty)$. As in the proof of Theorem 1, we can consider c and T satisfying (2) and (3). Similarly, we can choose again $\tau_1 > \tau_2 > \cdots > \tau_{n-1} > t_1 > T$ in exactly the same way. Next, since x is eventually nontrivial and oscillatory, T_0 and t_2 can be chosen so that $t_2 > T_0 > \tau_1$ and

$$|(D_r^{(0)}x)(T_0)| > 0 \quad \text{and} \quad (D_r^{(0)}x)(t_2) = 0.$$

As in the proof of Theorem 1, we obtain

$$|(D_r^{(0)}x)(t)| \leqq \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_2} |a(s)|\,|x(s)|\,ds\,ds_{n-1}\cdots ds_2 ds_1$$

for every $t \in [t_1, t_2]$ and hence

$$M^* \leqq M \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_2} |a(s)|\,ds\,ds_{n-1}\cdots ds_2 ds_1,$$

where $M = \max_{t\in[t_1,t_2]} |x(t)|$ and $M^* = \max_{t\in[t_1,t_2]} |(D_r^{(0)}x)(t)|$.

Therefore, by (2) and (3), we have

$$Mc \leqq M^* \leqq M\frac{c}{2}$$

and consequently $M^*=0$, which is a contradiction since by the definition of T_0,

$$M^* \geqq |(D_r^{(0)}x)(T_0)| > 0.$$

We shall now clarify the importance of Theorems 1 and 2 by applying them in the particular case where for some integer m, $1 \leqq m \leqq n-1$, we have

$$r_j = 1 \quad \text{for} \quad j \neq n-m \quad \text{and} \quad r_{n-m} = r.$$

More precisely, we give two corollaries concerning the differential equation

$$(\mathrm{E}_m) \qquad [r(t)x^{(n-m)}(t)]^{(m)} + a(t)x(t) = b(t).$$

COROLLARY 1. *Consider the differential equation* (E_m) *subject to the conditions*:

$$(\mathrm{A}_m) \qquad \int^{\infty} \frac{t^{n-m-1}}{r(t)} \int_t^{\infty} (s-t)^{m-1} \, |a(s)| \, ds \, dt < \infty$$

and

$$(\mathrm{B}_m) \qquad \int^{\infty} \frac{t^{n-m-1}}{r(t)} \int_t^{\infty} (s-t)^{m-1} \, |b(s)| \, ds \, dt < \infty .$$

Then for every oscillatory solution x of the differential equation (E_m),

$$\lim_{t\to\infty} x(t) = 0 .$$

PROOF. We have the formula

$$\int_u^{\infty}\int_v^{\infty} (s-v)^k p(s) \, ds \, dv = \int_u^{\infty} \frac{(s-u)^{k+1}}{k+1} p(s) \, ds ,$$

where p is a continuous nonnegative function on $[u, \infty)$ and k a nonnegative integer. By this formula, it is a matter of elementary calculus to see that in the considered case the conditions (A) and (B) follow from (A_m) and (B_m) respectively.

COROLLARY 2. *Consider the differential equation* $(\mathrm{E}_m)_0$.

$$(\mathrm{E}_m)_0 \qquad [r(t)x^{(n-m)}(t)]^{(m)} + a(t)x(t) = 0,$$

subject to the condition (A_m).

Then every eventually nontrivial solution of the differential equation $(\mathrm{E}_m)_0$ *is nonoscillatory.*

REMARK. The above corollaries 1 and 2 are generalizations of the results in [3] still in the particular case $m=n-1$. ([1], [2])

References

[1] A. G. Kartsatos, Oscillation properties of solutions of even order differential equations, Bull. Fac. Sci., Ibaraki Univ. **2** (1969), 9–14.

[2] T. Kusano and H. Onose, Nonoscillation theorems for differential equations with deviating arguments, Pacific J. Math., to appear.

[3] Bhagat Singh, Forced oscillations in general ordinary differential equations, Tamkang J. Math. **6** (1975), 5–11.