Title	A NOTE ON THE HOMOMORPHIC IMAGES OF THE CENTER OF C*-ALGEBRAS
Author(s)	TAKAHASHI, Sin-ei
Citation	Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 6: 29-32
Issue Date	1974
URL	http://hdl.handle.net/10109/2862
Rights	

Bull. Fac. Sci. Ibaraki Univ.
Ser. A, No. 6 (1974), 29-32

A NOTE ON THE HOMOMORPHIC IMAGES OF THE CENTER OF C*-ALGEBRAS

Sin-ei Takahasi

The purpose of this note is to extend the result of J. Vesterstrøm concerning the homomorphic images of the center of C*-algebras [1].

Let A be a C*-algebra and Prim A its structure space with the Jacobson topology. Let $C^b(\mathrm{Prim}\ A)$ be the commutative C*-algebra of all bounded continuous complex functions on Prim A and let $C_0(\mathrm{Prim}\ A)$ be the set of all elements of $C^b(\mathrm{Prim}\ A)$ vanishing at infinity. Let B be the enveloping von Neumann algebra of A and Z_B the center of B. For every $f \in P(A)$ (the set of all pure states of A) and $z \in Z_B$, the operator $\pi_f(z)$ is scalar, so that there is a complex function ϕ_z^0 defined on $P(A)$ with $\pi_f(z) = \phi_z^0(f)I$ where I is the identity operator on H_f. Let λ^1 and λ^2 be the canonical map $P(A) \to \hat{A}$ and $\hat{A} \to \mathrm{Prim}\ A$, respectively, where $\hat{A}$ is the spectrum of A. Then, there exist two complex functions ϕ_z^1 and ϕ_z^2 on $\hat{A}$ and Prim A, respectively, such that $\phi_z^0 = \phi_z^1\lambda^1$ and $\phi_z^1 = \phi_z^2\lambda^2$. Let Z' be the set of all $z \in Z_B$ such that $zA \subset A$. In [2], Dixmier proved that the mapping $z \to \phi_z$ $(= \phi_z^2)$ is a *-isomorphism of the C*-algebra Z' onto the C*-algebra $C^b(\mathrm{Prim}\ A)$.

<u>THEOREM 1</u>. <u>Let</u> A <u>be a C*-algebra which satisfies the following condition</u>:

$$\pi(z) \neq 0 \quad \underline{\text{for every}} \quad \pi \in \hat{A}. \qquad (*)$$

<u>Then we have</u> $\phi(Z) = C_0(\mathrm{Prim}\ A)$.

<u>PROOF</u>. Let $z \in Z$. Then, for any $\varepsilon > 0$, the set

$\{\pi \in \hat{A}: ||\pi(z)|| \geqq \varepsilon\}$ is compact (Prop. 3.3.7 in [3]). Since the topology of $\hat{A}$ is the inverse image of topology of Prim A by λ^2, the set $\{J \in \text{Prim } A: |\phi_z(J)| \geqq \varepsilon\}$ is compact. Thus, we have $\phi(Z) \subset C_0(\text{Prim } A)$.

Conversely, let $z \in Z'$ with $\phi_z \in C_0(\text{Prim } A)$. Suppose that $z \notin A$. We can assume that $z \geqq 0$. Set $A' = A + Z'$. A' is a C*-algebra and A is a closed two-sided ideal of A' (Th. 8 in [2]). Then, there exists $f_0 \in P(A')$ such that $f_0|A = 0$ and $f_0(z) = \varepsilon_0 > 0$. By Theorem 10 in [2], there exists a unique extension $\phi'_z \in C^b(\text{Prim } A)$ with $\phi_z|\text{Prim } A = \phi'_z$. Set $J_0 = \text{Ker } \pi_{f_0}$. Then $J_0 \in \text{Prim } A'$. Since Prim A is dense in Prim A' (Th. 10 in [2]), for any $\varepsilon > 0$, there exists $J_\varepsilon \in \text{Prim } A$ with $|\phi_z(J_\varepsilon) - \phi'_z(J_0)| < \varepsilon$. Note that $\phi'_z(J_0) = \varepsilon_0$. Since ϕ_z vanishes at infinity, the family $\{J_\varepsilon: 0 < \varepsilon < \varepsilon_0/2\}$ has limit points in Prim A if $\{J_\varepsilon: 0 < \varepsilon < \varepsilon_0/2\}$ has infinite elements (c. f. [4]). Set $K = \{J \in \text{Prim } A: \phi_z(J) = \phi'_z(J_0)\}$. Then we have

$$K \cap \overline{U_\lambda(J_0)} \neq \phi \text{ for any neighborhood } U_\lambda(J_0) \text{ of } J_0 \quad (1)$$

where $\{\lambda\}$ is a direct set. Note that if $\{J_\varepsilon: 0 < \varepsilon < \varepsilon_0/2\}$ has only finite elements, (1) also holds. Let $J_\lambda \in \overline{U_\lambda(J_0)} \cap K$ for each λ. Since ϕ_z vanishes at infinity on Prim A, K is the compact set, so that $\{J_\lambda\}$ has a limit point J' in K. Let $f' \in P(A)$ with $\text{Ker } \pi_{f'} = J'$. Let $z' \in Z$ and $\phi'_{z'}$ be the extension of $\phi_{z'}$ on Prim A'. Then $\phi'_{z'}(J_0) = 0$ since $f_0|A = 0$. For any $\varepsilon > 0$, there exists λ_0 such that $|\phi_{z'}(J_\lambda)| < \varepsilon$ for $\lambda_0 \leqq \lambda$. Therefore, we have $\phi_{z'}(J') = 0$, so that $f'(z') = 0$. This is a contradiction to $\pi_{f'}|Z \neq 0$. Then, we have $z \in A \cap Z' = Z$ and the result follows.

Let A be a C*-algebra satisfying the condition (*), and let Z be its center. We define $\eta_A(J) = J \cap Z$ for each $J \in \text{Prim } A$. Then η_A is a continuous map of Prim A onto Prim Z. Let η_A^* be the map of $C_0(\text{Prim } Z)$ into $C_0(\text{Prim } A)$ induced by η_A.

LEMMA 2. η_A^* is the *-isomorphism of $C_0(\text{Prim } Z)$ onto $C_0(\text{Prim } A)$.

PROOF. Let μ be the canonical map of Z onto $C_0(\text{Prim } Z)$. Let ϕ be the *-isomorphism of Z onto $C_0(\text{Prim } A)$ defined in Theorem 1. We show that $\phi = \eta_A^*\mu$. Let $z \in Z$ and $J \in \text{Prim } A$. There is $\pi \in \hat{A}$ with $J = \text{Ker } \pi$. Set $\rho = \pi|Z$. Then, we have

$$\phi_z(J)I = \rho(z) = \mu(z)(\text{Ker } \rho)I = \eta_A^*(\mu(z))(J)I,$$

where I is the identity operator on H_ρ. Thus, we have $\phi = \eta_A^*\mu$ and so η_A^* is the *-isomorphism of $C_0(\text{Prim } Z)$ onto $C_0(\text{Prim } A)$.

THEOREM 3. Let A and A' be C*-algebras with centers Z and Z'. Let ψ be a *-homomorphism of A onto A'. Then, if A satisfies the condition (*), then the following three statements are equivalent:

(i) $\psi(Z) = Z'$.

(ii) $(\psi|Z)^\vee : \text{Prim } Z' \to \text{Prim } Z$ is injective, where $(\psi|Z)^\vee(J') = (\psi|Z)^{-1}(J')$ $(J' \in \text{Prim } Z')$.

(iii) If J_1' and J_2' are primitive ideals of A', which can be separated by functions of $C_0(\text{Prim } A')$, then $\check{\psi}(J_1')$ and $\check{\psi}(J_2')$ can be separated by functions of $C_0(\text{Prim } A)$, where $\check{\psi}$ is the map of $\text{Prim } A'$ into $\text{Prim } A$ such that $\check{\psi}(J') = \psi^{-1}(J')$ $(J' \in \text{Prim } A')$.

PROOF. By Lemma 2, J_1' and J_2' are separated by functions of $C_0(\text{Prim } A')$ if and only if they are separated by $\eta_{A'}$. Similarly, $\check{\psi}(J_1')$ and $\check{\psi}(J_2')$ are separated by functions of $C_0(\text{Prim } A)$ if and only if they are separated by η_A. Note that $(\psi|Z)^\vee \eta_{A'} = \eta_A\check{\psi}$. Therefore, the equivalence (ii)$\Leftrightarrow$(iii) follows. The equivalence (ii)$\Leftrightarrow$(i) is well known.

References

[1] J. Vesterstrøm, On the homomorphic image of the center of a C*-algebra, Math. Scand. 29 (1971), 134-136.

[2] J. Dixmier, Ideal center of a C*-algebra, Duke Math. J. 35 (1968), 375-382.

[3] J. Dixmier, Les C*-algèbres et leur représentations, Gauthier-Villas, Paris, 1964.

[4] G. Bachman and L. Narici, Functional Analysis, New York and London, 1966.

Sin-ei Takahasi
Department of Mathematics,
Faculty of Sience,
Ibaraki University, Mito.

(Received November 15, 1973)