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On $\Lambda(\varphi, M)$-spaces

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1. Introduction and Prelimlnary. In 1951, G. G. Lorentz [7] has introduced a class of function spaces called Λ-spaces. Let $\varphi(t)$ be a positive integrable and almost everywhere equivalent to a non-increasing function defined on $(0, l)$, $l<\infty$. For a measurable function f, we denote by f^*, the *decreasing* (truely, *non-increasing*) *rearrangement of f* [2; p. 260-299], [6; p. 60]. It is defined as follows. Let $\mu_f(\alpha)$ be the Lebesgue measure of the set $\{t: |f(t)|>\alpha\}$ for any real number α. Then $\mu_f(\alpha)$ is right-continuous, *i. e.*, $\lim_{\alpha_n\downarrow\alpha}\mu_f(\alpha_n)=\mu_f(\alpha)$. Now we define the function $f^*(x)$ as the right-inverse of $\mu_f(y)$, *i. e.*,

$$(1.1)\qquad f^*(x)=\inf\{y: \mu_f(y)\leq x\}.$$

The space $\Lambda(\varphi, p)$, $p>1$ is the set of all measurable functions f. We shall define the norm $\|f\|$, such that

$$(1.2)\qquad \|f\|=\left\{\int_0^l\varphi(t)f^*(t)^p dt\right\}^{\frac{1}{p}}<\infty.$$

Then, $\Lambda(\varphi, p)$ equipped with the norm $\|\cdot\|$ defined by (1.2), is a reflexive Banach space where $1<p<\infty$ [7].

Hence we can regard as the p-th power of function is a convex on the positive real line. Now let $M(u)$, $0\leq u<\infty$ be a N-function, φ be as above, and for a measurable function f we put

$$(1.3)\qquad \rho(f)=\int_0^l\varphi(t)M[f^*(t)]dt.$$

In this paper, we shall discuss with a class $\Lambda(\varphi, M)$, which extends that of the spaces $\Lambda(\varphi, p)$, where the function $M(u)$ is a N-function in the sense [5; p. 6]. The set $\Lambda(\varphi, M)$ of all f with $\rho(\alpha f)<\infty$ for some $\alpha>0$ is a *modular space* and ρ is a *modular* on $\Lambda(\varphi, M)$

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in the sense of Nakano [12], *i. e.*,

(1. 4) $\Lambda(\varphi, M)=\{f: \rho(\alpha f)<\infty, \text{ for some } \alpha>0\}$.

In §2, we shall show that $\Lambda(\varphi, M)$ is a modular space and a Banach space with the norm which is induced by the modular (1. 3). In §3, we shall treat with the dual space $\Lambda^*(\varphi, M)$ of $\Lambda(\varphi, M)$ and show that the spaces $\Lambda(\varphi, M)$ are reflexive if M and N, the dual of M, satisfy (Δ_2) and $(\overline{\Delta_2})$-condition, generalizing the Theorem 4 in [7; p. 417].

For two measurable functions f and g, if they are equimeasurable to each other, *i. e.*, $\mu_f(\alpha)=\mu_g(\alpha)$, then we write $f\sim g$.

For two measurable functions f and g defined on $(0, l)$, $f\prec g$ means that $\int_0^x f^*dt\leq\int_0^x g^*dt$ for all x with $0<x<l$.

Here we present several basic properties about f^* and the *pre-order* "$\prec$".

(1. 5) $f\prec g\Longleftrightarrow f^*\prec g^*\Longrightarrow M[f^*]\prec M[g^*]$.

In fact, the equivalence of left hand side is obvious from the definition. For the proof of right hand side, see [7; p. 414].

Furthermore if $\varphi(t)$ is positive decreasing, then

(1. 6) $f^*\prec g^*\Longrightarrow\varphi M[f^*]\prec\varphi M[g^*]$.

For the proof, see [7; p. 414]. Also we have

(1. 7) $(f+g)^*\prec f^*+g^*$.

Because,

$$\int_0^x(f+g)^*dt=\sup_{\substack{e\\ \mu(e)=x}}\int_e |f+g|(t)dt$$

(μ : Lebesgue measure)

$$\leq\sup_e\int_e(|f(t)|+|g(t)|)dt$$

$$\leq\sup_e\int_e|f(t)|dt+\sup_e\int_e|g(t)|dt$$

$$\leq\int_0^x f^*dt+\int_0^x g^*dt.$$

Also we have

(1.8) $f\ g \prec f^* g^*$.

For the proof, see [2; p. 278] or [9; p. 102]. Since M is a convex function, we obtain

(1.9) $M[(\alpha f+\beta g)^*] \prec \alpha M[f^*]+\beta M[g^*]$, where $\alpha>0$, $\beta>0$, $\alpha+\beta=1$, by (1.5) and (1.7).

2. The spaces $\Lambda(\varphi, M)$. Let $M(u)$ be a N-*function* [5; p. 6], that is, there exist a function $p(t)$ which is right-continuous and positive non-decreasing such that $p(0)=0$ and

$$M(u)=\int_0^u p(t)dt, \qquad 0\le u<\infty.$$

THEOREM 1. *$\Lambda(\varphi, M)$ is a modular space with the modular ρ.*

PROOF. First $\Lambda(\varphi, M)$ is a linear space. For any f, $g\in\Lambda(\varphi, M)$,

$$\rho(\alpha f+\beta g)\le\rho\Big(\frac{1}{2}\alpha_0(f+g)\Big)$$

$$\le\frac{1}{2}\Big(\rho(\alpha_0 f)+\rho(\alpha_0 g)\Big),$$

for $\alpha_0=2\max(\alpha, \beta)$. We can see that $\rho(\alpha_0 f)$, $\rho(\alpha_0 g)<\infty$ for f and g. Therefore $\Lambda(\varphi, M)$ is linear.

The following properties are easily shown from the definition of ρ.

(ρ. 1) $0\le\rho(f)\le\infty$(for any measurable function f), and $\rho(|f|)=\rho(f)$ and $\rho(0)=0$;

(ρ. 2) For any $f\in\Lambda(\varphi, M)$, and a positive real number α, $\rho(\alpha f)<\infty$;

(ρ. 3) $\rho(\alpha f)=0$, $\alpha>0$, then $f=0$ a. e.;

(ρ. 4) If $|f|\le|g|$, then $\rho(f)\le\rho(g)$ (*monotone*);

(ρ. 5) $\alpha\ge0$, $\beta\ge0$, $\alpha+\beta=1$, then $\rho(\alpha f+\beta g)\le\alpha\rho(f)+\beta\rho(g)$, (*convex*);

(ρ. 6) $0\le f_n\uparrow f$, then $\rho(f)=\sup_n \rho(f_n)$ (*upper semi-continuous*).

To see (ρ. 6) we used the fact that $0\le f_n\uparrow f$ implies $f_n^*\uparrow f^*$. Thus we conclude our assertion.

Here, M is called to satisfy the (Δ_2)-*condition*, if there exist a constant γ and some $u_0\ge0$, such that

(2.1) $M(2u)\le\gamma M(u)$ for all $u\ge u_0$.

Then we define a class $\Lambda_0(\varphi, M)$ of all measurable functions f that $\rho(f)<\infty$, *i. e.*,

(2.2) $$\Lambda_0(\varphi, M)=\{f:\ \rho(f)<\infty\}.$$

THEOREM 2. *If M satisfies (Δ_2)-condition, then* $\Lambda_0(\varphi, M)=\Lambda(\varphi, M)$.

PROOF. We have always $\Lambda_0(\varphi, M)\subset\Lambda(\varphi, M)$. Since, for any $f\in\Lambda(\varphi, M)$, then $\rho\left(\frac{1}{2}f\right)<\infty$. Thus we have

$$\begin{aligned}\rho(f)&\leq\int_0^l\varphi M[f^*]dt\\ &\leq\gamma\int_0^l\varphi M\left[\frac{1}{2}f^*\right]dt\\ &=\gamma\rho\left(\frac{1}{2}f\right)<\infty,\end{aligned}$$

i. e.,

$$f\in\Lambda_0(\varphi, M).$$

Therefore we conclude our assertion.

Further $\rho(f)$ satisfies some properties as follows. A modular ρ on $\Lambda(\varphi, M)$ is said to be *lower semi-additive*,

(2.3) $$\rho(f\cup g)\leq\rho(f)+\rho(g)\quad\text{for } 0\leq f,\ g\in\Lambda(\varphi, M).$$

In fact, since $\sup\{f^*, g^*\}\prec(f+g)^*$ with the preorder $\prec$, then we have our assertion.

Furthermore, by $(\rho.5)$, $\rho(\alpha f)$ is a *convex function of* α for each f, *i. e.*,

(2.4) $$\rho\left(\frac{\alpha+\beta}{2}f\right)\leq\frac{1}{2}\left(\rho(\alpha f)+\rho(\beta g)\right).$$

Now we can define the norm of $f(\in\Lambda(\varphi, M))$ which is called ***Luxemburg norm***, as follows:

(2.5) $$|||f|||=\inf\{\xi:\ \rho\left(\frac{f}{\xi}\right)\leq 1,\ \xi>0\}.$$

Then, it is known that (2.5) satisfies the norm condition. This norm (2.5) obviously satisfies the following:

(2.6) $$|f|\leq|g|\Longrightarrow|||f|||\leq|||g|||;$$

(2.7) $$0\leq f_n\uparrow f\Longrightarrow|||f_n|||\uparrow|||f|||.$$

We can see from the definition that $|||f|||\leq 1$ is equivalent to $\rho(f)\leq 1$.

THEOREM 3. *$\Lambda(\varphi, M)$ is a Banach space with the induced norm* (2.5) *by the modular* $\rho(f)$.

PROOF. We shall show the completeness. By (2.7), we obtain the property that is called to be ***monotone complete*** (*the weak Fatou property*) in the sense of Amemiya.

$$0\leq f_n \uparrow f,\ \sup|||f_n|||<\infty \Longrightarrow f\in\Lambda(\varphi, M),$$

and hence $\Lambda(\varphi, M)$ is complete by the theorem in [4].

3. The reflexivity of the space $\Lambda(\varphi, M)$. First we shall construct the dual space of $\Lambda(\varphi, M)$. Let $N(v)$ be the N-function complementary to $M(u)$ in the sense of Young [5; p. 11]. That is to say, the function $q(s)$ is the right inverse of $p(t)$ which is defined by the equality:

$$q(s)=\sup_{p(t)\leq s} t, \qquad 0<s<\infty.$$

Then we have

$$N(v)=\int_0^v q(s)ds, \qquad 0<v<\infty.$$

Next we shall give some definitions and propositions which will be needed in the sequel. Now, let $G(x)=\int_0^x g(t)dt$, where $g(t)$ is integrable and positive on $(0, l)$, and $\Phi(x)=\int_0^x \varphi(t)dt$, $0<x<\infty$. The function $G(x)$ is said to be *φ-concave*, if

$$\frac{G(x)-G(a)}{\Phi(x)-\Phi(a)}\geq\frac{G(b)-G(a)}{\Phi(b)-\Phi(a)},\ a<x<b.$$

Lorentz [6; §3.6, Theorem 3.6.3] has shown the following proposition.

PROPOSITION 1. *The function $G(x)=\int_0^x g(t)\,dt$ is φ-concave if and only if $g(t)=\varphi(t)D(t)$ a. e., where $D(t)$ is a positive decreasing function.*

We define for each measurable function $g(t)$,

$$\tau(g)=\inf_{g^*\prec\varphi D}\{\int_0^l \varphi N[D]dt\}, \tag{3.1}$$

where the infimum is taken for all decreasing positive functions $D(t)$ for which $g^* \prec \varphi D$.

The *conjugate modular* $\bar{\rho}$ [12; p. 92] of ρ, is defined by

(3.2) $$\bar{\rho}(g) = \sup_{f \in \Lambda(\varphi, M)} \left\{ \left| \int_0^l f g dt \right| - \rho(f) \right\},$$

for any measurable function g. We consider the *dual space of* $\Lambda(\varphi, M)$, denoted by $\overline{\Lambda(\varphi, M)}$, as follows:

(3.3) $\overline{\Lambda(\varphi, M)} = \{g : \bar{\rho}(\alpha g) < \infty$ for some $\alpha > 0$, and g is measurable$\}$.

Then $\overline{\Lambda(\varphi, M)}$ is also a modular space.

THEOREM 4. *For each* $g \in \overline{\Lambda(\varphi, M)}$, *we have* $\bar{\rho}(g) \leq \tau(g)$.

PROOF. By Young's inequality and (1.8), we have, for any decreasing positive function D with $g^* \prec \varphi D$,

$$\begin{aligned} \left| \int_0^l f g dt \right| &\leq \int_0^l f^* g^* dt \\ &\leq \int_0^l f^* \varphi D dt \\ &\leq \int_0^l \varphi M[f^*] dt + \int_0^l \varphi N[D] dt, \end{aligned}$$

and hence

$$\left| \int_0^l f g dt \right| - \rho(f) \leq \int_0^l \varphi N[D] dt.$$

Thus we have

$$\sup_{f \in \Lambda(\varphi, M)} \left\{ \left| \int_0^l f g dt \right| - \rho(f) \right\} \leq \inf_{g^* \prec \varphi D} \int_0^l \varphi N[D] dt.$$

Hence

$$\bar{\rho}(g) \leq \tau(g).$$

Now we shall show that $\bar{\rho}(g) = \tau(g)$ on $\overline{\Lambda(\varphi, M)}$. To show the result, we need some definitions. For a given g, g^0 is the smallest (in the sense of the relation $\prec$, see §1) function among the functions satisfying $g \prec h = \varphi D$ with a positive decreasing function D, and is called the *level function of* g with respect to φ. Lorentz has shown the following proposition [4; §3.6, Theorem 3.6.4].

PROPOSITION 2. *Let $g(t)$ be integrable and positive, and let D^0 be defined by $g^0=\varphi D^0$. For any $G(x)=\int_0^x g(t)dt$, the function $G^0(x)$ is also of the form $G^0(x)=\int_0^x g^0(t)dt$, $g^0\geq 0$. Then $G^0(x)=G(x)$ holds a. e.* (consequently $g(t)=g^0(t)$ *a. e.*) *except perhaps for the maximal intervals (a, b) of constancy of D; on each such interval (a, b), $\int_a^b g^0 dt=\int_a^b g dt$.*

Thus we obtain that for a given integrable function g the infimum of (3.1) is attained for $D=D^0$:

$$\inf_{g^*\prec\varphi D}\int_0^l \varphi N[D]dt=\int_0^l \varphi N[D^0]dt,$$

and hence

$$\tau(g)=\int_0^l \varphi N[D^0]dt. \tag{3.4}$$

Now we shall define the condition of N-function $N(v)$ which is called ($\bar\Delta_2$)*-condition.* There exist real number α, β; $1<\alpha<\beta$ and some v_0, such that

$$N(\alpha v)\leq\beta N(v) \qquad \text{for all } v\geq v_0. \tag{3.5}$$

If the N-function $M(u)$ satisfies the (Δ_2)-condition, then it's complementary N-function $N(v)$ satisfies $(\bar\Delta_2)$-condition. Moreover, we obtain that, if N-function satisfies $(\bar\Delta_2)$-condition, then there exists a constant $\delta(>1)$,

$$vq(v)\leq\delta N(v) \qquad \text{for all } v\geq v_0. \tag{3.6}$$

THEOREM 5. *Suppose that the* N*-function $M(u)$ satisfies the (Δ_2)-condition, then for $g(\in\Lambda(\varphi, \bar M))$,*

$$\bar\rho(g)=\tau(g).$$

PROOF. Since $g^0=\varphi D^0$ for g^*, in virtue of (3.4), we have

$$\int_0^l f^*g^*dt=\int_0^l f^*\varphi D\ dt.$$

(Because, for some function $f^*=q[D^0(t)]$ as $f(t)$, on each the maximal interval (a, b) of constancy of D^0, D^0 is constant, then $q[D^0(t)]$ is constant: otherwise, $g^*=g^0=\varphi D^0$ a. e., then $f^*g^*=f^*\varphi D^0$.) In the Young's inequality for such f, the equality sign holds.

Therefore we have

$$\int_0^l f^*g^*dt=\rho(f)+\int_0^l \varphi N[D^0]dt;$$

$$\left|\int_0^l fgdt\right|-\rho(f)=\tau(g),$$

i. e.,

$$\bar{\rho}(f)\geq\tau(g).$$

Now, we have from (3. 6) and Young's inequality, for any $f(=q[D])$,

$$\int_0^l \varphi M[f]dt=\int_0^l \varphi D^0 q[D^0]dt-\int_0^l \varphi N[D^0]dt$$

$$\leq(\delta-1)\int_0^l \varphi N[D^0]dt.$$

Therefore there exists a function $f(=q[D^0])$ in $\Lambda(\varphi, M)$, and we obtain from Theorem 4, $\bar{\rho}(g)=\tau(g)$.

($\varphi N[D^0]$ is integrable whenever $\int_0^l fgdt-\rho(f)$ is bounded. Then $g(=\varphi D^0)\in\Lambda(\varphi, M)$. For the proof, see [6; p. 73-74].)

We denote the Banach dual of $\Lambda(\varphi, M)$, by $\Lambda^*(\varphi, M)$, consisting of all linear functionals $F_g(f)=\int_0^l fgdt$ defined on $\Lambda(\varphi, M)$. We introduce the norm $\|g\|_{(\bar{\Lambda})}$ of g, which is called *Orlicz norm* as follows;

$$\|g\|_{(\bar{\Lambda})}=\sup_{\rho(f)\leq 1}\left|\int_0^l fgdt\right|.$$

Then we have [3; p. 80]

$$|||g|||_{\bar{\Lambda}}\leq\|g\|_{(\bar{\Lambda})}\leq 2|||g|||_{\bar{\Lambda}}.$$

Since the norm $\|F_g\|$ of the linear functional F_g is defined as

$$\|F_g\|=\sup_{|||f|||_{\Lambda}\leq 1}\left|\int_0^l fgdt\right|,$$

we obtain that the norm $|||g|||_{\bar{\Lambda}}$ on $\overline{\Lambda(\varphi, M)}$ is equivalent to the norm $\|F_g\|$ of the linear functional F_g on the Banach dual of $\Lambda^*(\varphi, M)$ (note that $|||f|||_{\Lambda}\leq 1\rightleftarrows\rho(f)\leq 1$). Therefore we have $\overline{\Lambda(\varphi, M)}$ is isomorphic to $\Lambda^*(\varphi, M)$. Then we obtain the following theorem.

THEOREM 6. *If both the N-functions M and N satisfy* (Δ_2)-*condition, then* $\Lambda(\varphi, M)$ *is relexive as a Banach space.*

PROOF. From the definition of $\overline{\rho}(g)$

$$\sup_{g\in\overline{\Lambda(\varphi, M)}}\left\{ \left| \int_0^t fgdt \right| -\overline{\rho}(g)\right\} \leq \rho(f).$$

Here, for some $f(=p[D])$ [12; Theorem 2.5],

$$\sup_{g\in\overline{\Lambda(\varphi, M)}}\left\{ \left| \int_0^t fgdt \right| -\overline{\rho}(g)\right\} = \rho(f).$$

Hence we have $\overline{\overline{\Lambda(\varphi, M)}} = \Lambda(\varphi, M)$. Since

$$\Lambda(\varphi, \overline{M}) = \Lambda^*(\varphi, M),$$

we have

$$\Lambda^{**}(\varphi, M) = \overline{\Lambda(\varphi, \overline{M})}^* = \Lambda(\varphi, \overline{\overline{M}}) = \Lambda(\varphi, M).$$

Thus we obtain our assertion.

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