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A NOTE ON ENDMORPHISM RINGS OF H-SEPARABLE EXTENSIONS
YASUKAZU YAMASHIRO

1. Introduction

Let $R \supset S$ be a ring extension. $R$ is said to be an $H$-separable extension over $S$ if as $R - R$-modules $R \otimes_S R^\oplus (R^\oplus \cdots \oplus R)$. There is an equivalent definition of $H$-separable extension in terms of the natural $R - R$-homomorphism $\varphi : R \otimes_S R \rightarrow \text{Hom}(\Delta_C, R_C)$ where $\varphi(a \otimes b)(z) = azb, C$ is the center of $R$ and $\Delta$ is the commutator ring of $S$ in $R$, i.e. $\Delta = R^S = \{z \in R : za = az \text{ for all } a \in S\}$. $R$ is $H$-separable extension over $S$ if and only if $\Delta$ is $C$-finitely generated projective and $\varphi$ is an isomorphism. Let $Q$ be a ring and $P$ a $Q - Q$-bimodule. $P$ is said to be $Q$-centrally projective if as $Q - Q$-bimodules $P^\oplus (Q^\oplus \cdots \oplus Q)$. Furthermore, if $P$ is ring extension over $Q$ then $P \cong Q \otimes_C \Delta^o$ where $\Delta^o$ is the commutator ring of $Q$ in $P$ and $C$ is the center of $P$. For details see [1].

In this paper, we give some one to one correspondence between some class of intermediate rings of $H$-separable extension and some class of intermediate rings of centrally projective extension, by taking the endomorphism ring.

2. Main Results

Let $P \supset Q$ be a ring extension, $\Delta^o$ the commutator ring of $Q$ in $P$ and $C$ the center of $Q$. Furthermore, let $\mathcal{B}_o$ be the set of subrings

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B of $P$ such that $Q \subset B$, $QB_B (\oplus QP_B$ and the multiplication map $P \otimes_Q B \rightarrow P$ splits as a $P - B$-homomorphism, and $\mathcal{D}_r^o$ the set of $C$-subalgebras $D$ of $\Delta^o$ such that $D_D (\oplus \Delta_D^o$ and the multiplication map $\Delta^o \otimes_C D \rightarrow \Delta^o$ splits as a $\Delta^o - D$-homomorphism.

**Lemma.** Let $P$ be $Q$-centrally projective, that is

$$P (\oplus (Q \oplus \cdots \oplus Q)$$

as $Q - Q$-bimodules. Then we have one to one correspondence between $\mathcal{B}_r^o$ and $\mathcal{D}_r^o$, by taking $B \cap \Delta^o$ for $B \in \mathcal{B}_r^o$ and $QD$ for $D \in \mathcal{D}_r^o$.

**Proof:** Let $B \in \mathcal{B}_r^o$ and $p$ the $Q - B$-projection of $P$ to $B$. Since, for any $q \in Q$ and $\delta \in \Delta^o$,

$$qp(\delta) = p(q\delta) = p(\delta q) = p(\delta)q,$$

we have $D \subset \Delta^o$ and $D_D (\oplus \Delta_D^o$, where $D = p(\Delta^o)$. By [2, Proposition 5.5], $P \cong Q \otimes_C \Delta^o$. Then we have

$$B = p(P) = p(Q\Delta^o) = Qp(\Delta^o) = QD.$$

Since $D (\oplus \Delta^o$, we have the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & Q \otimes_C D \\
\downarrow & & \downarrow \alpha \\
0 & \longrightarrow & QD \\
\uparrow & & \\
& & P \\
& & B
\end{array}
$$
where two rows are exact and $\alpha$ is an isomorphism. Therefore $B \cong Q \otimes_C D$. On the other hand, it is easy to check that $D = B \cap \Delta^\circ$. Now, for any $Q - Q$-bimodule $M$, we denote by $M^Q$ the subset \( \{ m \in M;qm = mq \text{ for all } q \in Q \} \) of $M$. Then we have the following isomorphisms

\[
\text{Hom}(Q_{-\Delta^\circ}Q \otimes_C \Delta^\circ_{Q-D},Q_{-\Delta^\circ}Q \otimes_C (\Delta^\circ \otimes_C D)_{Q-D}) \\
\cong \text{Hom}(\Delta^\circ \Delta^\circ_D,\Delta^\circ \text{Hom}(Q_{Q-Q}Q \otimes_C (\Delta^\circ \otimes_C D)_{Q-D}) \\
\cong \text{Hom}(\Delta^\circ \Delta^\circ_D,\Delta^\circ (Q \otimes_C (\Delta^\circ \otimes_C D))_{D}) \\
\cong \text{Hom}(\Delta^\circ \Delta^\circ_D,\Delta^\circ Q^Q \otimes_C (\Delta^\circ \otimes_C D)_{D}) \\
= \text{Hom}(\Delta^\circ \Delta^\circ_D,\Delta^\circ C \otimes_C (\Delta^\circ \otimes_C D)_{D}) \\
\cong \text{Hom}(\Delta^\circ \Delta^\circ_D,\Delta^\circ (\Delta^\circ \otimes_C D)_D).
\]

The inverse map of the composition of these isomorphisms is given by $f \mapsto [q \otimes \delta \mapsto q \otimes f(\delta)]$ where $f \in \text{Hom}(\Delta^\circ \Delta^\circ_D,\Delta^\circ (\Delta^\circ \otimes_C D)_D), q \in Q$ and $\delta \in \Delta^\circ$. Furthermore, since

\[
P \otimes_Q B \cong (Q \otimes_C \Delta^\circ) \otimes_Q (Q \otimes_C D) \cong Q \otimes_C (\Delta^\circ \otimes_C D)
\]

and the multiplication map $P \otimes_Q B \to P$ splits as a $P - B$-homomorphism, there exist a splitting map of the multiplication map $\Delta^\circ \otimes_C D \to \Delta^\circ$ as a $\Delta^\circ - D$-homomorphism. Therefore $D \in \mathcal{D}_\gamma$.

Next let $D \in \mathcal{D}_\gamma$ and $B = QD$. Since $D \langle \oplus \Delta^\circ \rangle$, we have the commutative diagram
where two rows are exact and α is an isomorphism. Therefore $B \cong Q \otimes_C D$. Then we have

$$QB_B \cong Q (Q \otimes_C D)_{Q-D} (\oplus Q (Q \otimes_C D)_{Q-D} \cong_Q P_B$$

and

$$P P \otimes_Q B_B \cong_{Q-\Delta^o} (Q \otimes_C \Delta^o) \otimes_Q (Q \otimes_C D)_{Q-D}$$

$$\cong_{Q-\Delta^o} Q \otimes_C (\Delta^o \otimes_C D)_{Q-D} \oplus Q-\Delta^o Q \otimes_C \Delta^o_{Q-D}$$

$$\cong_P P_B.$$
But $Q \otimes_C D \cong B \cong Q \otimes_C D'$.

Then $Q \otimes_C D'/D = 0$. On the other hand, $D'/D$ is $C$-finitely generated projective, for so are $D'$ and $D$ and $D \langle \oplus D' \rangle$. Hence $D'/D$ is $C$-flat and

\[
\begin{array}{ccc}
0 & \longrightarrow & C \otimes_C D'/D \longrightarrow Q \otimes_C D'/D \\
\| & & \| \\
D'/D & \longrightarrow & 0
\end{array}
\]

Therefore $D'/D = 0$ and $D' = D$. This completes the proof. 

Next theorem is the purpose of this paper. Let $R \supset S$ be a ring extension, $\Delta$ the commutator ring of $S$ in $R$ and $C$ the center of $R$. Furthermore, let $B_1$ be the set of subrings $B$ of $R$ such that $S \subseteq B$, $B B_S \langle \oplus B R_S \rangle$ and the multiplication map $B \otimes_S R \rightarrow R$ splits as a $B - R$-homomorphism, and $D_1$ the set of $C$-subalgebras $D$ of $\Delta$ such that $D D \langle \oplus D \Delta \rangle$ and the multiplication map $D \otimes_C \Delta \rightarrow \Delta$ splits as a $D - \Delta$-homomorphism.

**Theorem.** Let $R \supset S$ be an $H$-separable extension and $M$ a right $R$-module such that faithfully balanced. And let $P = \text{Hom}(M_S, M_S)$, $Q = \text{Hom}(M_R, M_R)$. Then we have one to one correspondence between $B_1$ and $B_r^0$, by taking the endomorphism ring.

**Proof:** First we have

$$P^Q = \text{Hom}(M_S, M_S)^Q = \text{Hom}(Q M, Q M)^S \cong R^S = \Delta.$$  

Considering the ring structure, we have that $P^Q \cong \Delta^o$ (it means that the opposite ring of $\Delta$). By the same way, we have that $Q^Q = C$, that
is the center of $Q$. By [3, proposition 2.1], $P$ is $Q$–centrally projective. By [2, Proposition 5.6], $P \cong Q \otimes_C \Delta^o$.

Let $B \in \mathcal{B}_l$ and $D = R^B$. By [4, (1.3)], $D \in \mathcal{D}_I$ and $D^o \in \mathcal{D}_I^o$. Then we have

$$\tilde{B} = \text{Hom}(M_B, M_B) = \text{Hom}(M_S, M_S)^B = P^B$$

$$\cong (Q \otimes_C \Delta^o)^B = Q \otimes_C \Delta^o = Q \otimes_C D^o.$$ 

By Lemma, $\tilde{B} \in \mathcal{B}_I$. By [4, (1,3)],

$$\text{Hom}(\tilde{B}, M, M) = \text{Hom}(Q - D^o M, Q - D^o M)$$

$$\text{Hom}(Q M, Q M)^{D^o} \cong R^{D^o} = R^D$$

$$= B.$$ 

Next let $B \in \mathcal{B}_I^*$ and $D = \Delta^o \cap B$. By Lemma, $D \in \mathcal{D}_I$, and $D^o \in \mathcal{D}_I$. Then, by [4, (1.3)], we have

$$\tilde{B} = \text{Hom}(B M, B M) = \text{Hom}(Q M, Q M)^B$$

$$= \text{Hom}(Q M, Q M)^{Q - D} = \text{Hom}(Q M, Q M)^D$$

$$\cong R^D = R^{D^o} \in \mathcal{B}_I.$$ 

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By [4, (1.3)], $R^{\hat{B}} = D^c$, and by the above result,

$$\text{Hom}(M_{\hat{B}}, M_{\hat{B}}) \cong Q \otimes_C D^c$$

$$= Q \otimes_C D \cong B.$$

This completes the proof. □

References


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