On Complete Continuity of P. S. Uryson's Operator in Function Spaces

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Abstract

The purpose of this paper is to give conditions of both the continuity and compactness of Uryson's operator \( \int K[s, t, \phi(t)] \, dt \) which acts in modulated function spaces.

1. Introduction. In non-linear integral equations, the complete continuity of an operator from which the equation is produced plays a very important role, for example, the existence of solutions or eigen-functions in the equations. (cf. M. A. Krasnosel'skii and S. Yamamuro)

A sufficient condition of the complete continuity of Uryson's operator acting in the space \( C \subset \mathbb{R}^d \) as the totality of all continuous functions on a compact subset in Euclidean space, have been given by L. A. Ladyzhenskii.

In case the operator acts from the space \( L_{p_1} (p_1 > 1) \) to the space \( L_{p_2} (p_2 > 1) \), M. A. Krasnosel'skii and L. A. Ladyzhenskii have given some sufficient conditions of the complete continuity, but it seems that one result has a defect, so far as we see the fact described in [Amer. Math. Soc. Transl. Ser. 2 vol. 10, p 352].

In this paper, we will consider the operator acting in modulated function spaces with some restrictions, which was defined by H. Nakano, and we give some sufficient conditions for the complete continuity of the operator. (see Theorem 4 and 5)

2. Preliminaries. In this section, we will state an outline of modulated function spaces and fundamental definitions.

Let \( D \) be a bounded subset in Euclidean space and \( \text{mes} (D) = 1 \).

Let \( \Phi (\xi, x) (\xi \geq 0, x \in D) \) be measurable on \( D \) for each \( \xi \geq 0 \) and non-decreasing convex function of \( \xi \) for which satisfies:

1) \( \Phi (0, x) = 0 \) for all \( x \in D \);
2) \( \lim_{\xi \to 0} \Phi (\xi, x) = \Phi (0, x) \) for each \( x \in D \);
3) \( \lim_{\xi \to +\infty} \Phi (\xi, x) = +\infty \) for each \( x \in D \);
4) for any \( x \in D \), there exists \( \alpha = \alpha (x) > 0 \) such that \( \Phi (\alpha, x) > +\infty \).

The modulated function space \( L_\Phi (D) \) is a totality of all measurable functions \( \phi (x) \) on \( D \) such that

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When we define a semi-order (or partial order) in $L_\sigma$ by the relation that $\phi \geq \phi'$ if and only if $\phi (x) \geq \phi'(x)$ except for a set of measure zero, the space $L_\sigma$ is a superuniversally continuous semiordered linear space*.

The above functional $m (\phi)$ on $L_\sigma$ is called a modular on $L_\sigma$ and satisfies the modular conditions:

1) $0 \leq m (\phi) \leq +\infty$ for all $\phi \in L_\sigma$;
2) if $m (\xi \phi) = 0$ for all $\xi \geq 0$, then $\phi = 0$
3) for any $\phi \in L_\sigma$ there exists $\alpha > 0$ such that $m (\alpha \phi) < +\infty$;
4) for every $\phi \in L_\sigma$, $m (\xi \phi)$ is a convex function of $\xi \geq 0$;
5) $|\phi| \leq |\xi|$ implies $m (\phi) \leq m (\xi)$;
6) $|\phi| \cap |\phi| = 0$ implies $m (\phi + \phi) = m (\phi) + m (\phi)$;
7) $0 \leq \phi, \phi \phi = \phi$ implies $m (\phi) = \sup_{x \in \mathcal{X}} m (\phi)$.

Writing the left-derivative of $\Phi (\xi, x)$ at $\xi$ by $\varphi (\xi, x)$ with $\varphi (0, x) = 0$, we have a measurable in $x$ and non-decreasing function $\varphi (\xi, x)$ in $\xi \geq 0$. If we define an inverse function $\psi (\eta, x)$ of $\varphi (\xi, x)$ as $\eta = \varphi (\xi, x)$, such that it is non-decreasing function of $\eta \geq 0$, $\varphi (0, x) = 0$ and

$$
\begin{align*}
\varphi (\eta - 0, x) &\leq \eta \leq \psi (\eta + 0, x) & \text{for } \eta = \varphi (\xi, x), \\
\varphi (\xi - 0, x) &\leq \xi \leq \psi (\xi + 0, x) & \text{for } \xi = \varphi (\eta, x),
\end{align*}
$$

then the function:

$$
\psi (\eta, x) = \int_0^\eta \varphi (\eta, x) \, d\eta
$$

is measurable in $x \in \mathcal{A}$ and satisfies the same conditions as $\Phi (\xi, x)$. Furthermore, we have Young's inequality

$$
\xi \eta \leq \Phi (\xi, x) + \psi (\eta, x)
$$

for $\xi, \eta \geq 0$ and $x \in \mathcal{A}$, with equality if one at least of the relations

$$
\varphi (\xi - 0, x) \leq \eta \leq \varphi (\xi + 0, x), \quad \varphi (\eta - 0, x) \leq \xi \leq \psi (\eta + 0, x)
$$

is satisfied. By the function $\psi (\eta, x)$, the space $L_\psi$ which is called a conjugate space of $L_\sigma$ is defined, and further a modular $\overline{m} (\phi)$ on $L_\psi$, i.e.

$$
\overline{m} (\phi) = \int \psi (|\phi (x)|, x) \, d\sigma
$$

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* A semi-ordered linear space $R$ is said to be superuniversally continuous, if for any system $a_i \geq 0$ ($i \in \mathcal{A}$) there exist countable $a_i$, $i \in \mathcal{A}$ and $a \in R$ for which $a = \bigcap_{i=1}^\infty a_i = \bigcap_{i=1}^\infty a_i$, where $\bigcap_{i=1}^\infty a_i$ means a infimum of $a_i$.

** $\phi \phi = \phi$ means that for any $\lambda, \mu \in \mathcal{A}$ there exists $p \in \mathcal{A}$ such that $\phi \mu \phi \eta \leq \phi \eta$, and $\bigcap_{p \in \mathcal{A}} \phi \phi \eta = \bigcap_{p \in \mathcal{A}} \phi \phi \eta$, where $\bigcap_{p \in \mathcal{A}} \phi \phi \eta$ is a supremum of $\phi \phi \eta$.

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is defined as follows:

$$\hat{m}(\phi) = \sup \left\{ \int f(x) \phi(x) \, dx - m(\phi) \right\}$$

where $\hat{m}$ is called a **conjugate modular** of $m$.

In the space $L_\phi$, defining two kinds of norms:

$$\|\phi\|_\phi = \inf_{m(\xi) \leq 1} \frac{1}{|\xi|} \text{; } \|\phi\|_\phi = \inf_{i \geq 0} \frac{1 + m(\xi \phi)}{\xi},$$

we have $\|\phi\|_\phi \leq \|\phi\|_\phi \leq 2 \|\phi\|_\phi$, and their norms are both monotone complete norms*, so $L_\phi$ is Banach space, because the above modular on $L_\phi$ is monotone complete*.

As examples of such spaces, we can denote the well-known following spaces.

Orlicz space*\textsuperscript{10}, i.e., for a non-decreasing left-continuous function $\varphi(\xi)$ on $[0, \infty)$ with $\varphi(0)=0$, putting

$$\Phi(u) = \int_0^u \varphi(\xi) \, d\xi \quad (u \geq 0)$$

the totality of all measurable functions $\phi(x)$ on $\mathcal{A}$ such that

$$\int f(x) |\phi(x)| \, dx < + \infty \quad \text{for some } \alpha > 0.$$

Space $L_{p(x)}$, i.e., for a measurable function $p(x) \geq 1 (x \in \mathcal{A})$, the totality of all measurable functions $\phi(x)$ on $\mathcal{A}$ such that

$$\int \frac{1}{p(x)} |\alpha \phi(x)|^{p(\alpha)} \, dx < + \infty \quad \text{for some } \alpha > 0.$$

A modular $m(\phi)$ on $L_\phi$ is said to be upper bounded modular, if there exist $\alpha, \gamma > 1$ such that

$$\Phi(\alpha \xi, x) \leq \gamma \Phi(\xi, x) \quad \text{for all } \xi \geq 0, \ x \in \mathcal{A},$$

And, $m$ is said to be lower bounded modular, if there exist $\alpha > \gamma > 1$ such that

$$\Phi(\alpha \xi, x) \geq \gamma \Phi(\xi, x) \quad \text{for all } \xi \geq 0, \ x \in \mathcal{A}.$$

If $m$ is lower (upper) bounded then its conjugate modular $\hat{m}$ is upper (lower) bounded.

$m$ is said to be bounded modular if it is upper and lower bounded modular. If $m$ is a bounded modular, then $L_\phi$ is reflexive as Banach space with the above norms*, for instance, $L_{p(x)} (p > 1)$ and Orlicz spaces defining by complementary Young's functions $\Phi(u)$ and $\psi(v)$ for which satisfy both $(\mathcal{F})$-condition.

* A norm $\| \phi \|$ is called to be monotone complete if $0 \leq \phi \in L_1$ and $\sup_{i \geq 1} \| \phi_i \| < + \infty$ implies the existence of an element $\hat{\phi}$ such that $\phi_i \xrightarrow{i \to \infty} \hat{\phi}$. A monotone completeness of a modular implies a monotone completeness of a norm, and a monotone completeness of a norm implies a completeness in usual sense.$^{1,10}$

** Orlicz spaces are modulated function spaces with constant modulars$^{9}$. 

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Throughout this paper we assume that the modulared function spaces $L_{\sigma_i}$ and their conjugate spaces $L_{\sigma_i}^*$ ($i=1, 2$) have the bounded modulars, and the functions $\varphi_i(1, x), \psi_i(1, x)$ are integrable on $\mathcal{A}$, where $\varphi_i$ and $\psi_i$ are the left-derivatives of $\Phi_i$ and $\Psi_i$ respectively.

The integral operator:

$$\Lambda \phi (s) = \int_\mathcal{A} K [s, t, \phi (t)] \, dt$$

is called the operator of P. S. Uryson, where the function $K [s, t, u]$ is defined for $(s, t) \in \mathcal{A} \times \mathcal{A}$ and for real number $u$.

In this paper, we will deal with the case which $K [s, t, u]$ is continuous in $u$ for fixed $(s, t)$ and measurable in the remainder of the variables for fixed $u$.

A subset $F$ of Banach space $E$ is called to be compact (weakly compact), if every infinite subset contains a subsequence converging (weakly converging) in $E$.

An operator is called to be bounded if it transforms every bounded (in the norm) subset of Banach space $E$ into a set which is bounded (in the norm) in Banach space $E$.

An operator $A$, acting from $E_i$ into $E_j$, is called to be continuous at the point $\phi_i \in E_i$ if, for every sequence $\{\phi_n\}$ converging to $\phi_i$, $\{A \phi_n\}$ converges to $A \phi_i$ in $E_j$. An operator is called to be continuous on $E$ if it is continuous at each point of $E$.

An operator $A$ is called to be compact if it transforms every bounded set into a compact set.

An operator $A$ is called to be completely continuous if it is continuous and compact.

3. In this section, we will consider a sufficient condition of both the boundedness and continuity of Uryson's operator which acts from the space $L_{\sigma_i}$ with a modular $m_i$ into the space $L_{\sigma_j}$ with a modular $m_j$.

**Lemma 1.** If $K [s, t, u]$, $(s, t \in \mathcal{A}, -\infty < u < +\infty)$ is measurable on $\mathcal{A} \times \mathcal{A}$ for fixed $u$ and continuous in $u$ for fixed $(s, t)$, then for any $a \leq b$ there exists a bounded measurable function $h(s, t)$ on $\mathcal{A} \times \mathcal{A}$ such that

$$\sup_{a \leq u \leq b} |K [s, t, u]| = |K [s, t, h(s, t)]|$$

for each $s$ and $t$.

**Proof.** First, we shall show the measurability of the function

$$k(s, t) = \sup_{a \leq u \leq b} |K [s, t, u]|.$$

When we put, for any positive number $\alpha$,

$$E_a = \{s, t \}; \quad k(s, t) \leq \alpha, \quad F_{a, n} = \{(s, t) \}; \quad |K [s, t, u]| < \alpha + 1/n$$

and

$$E_a = \bigcap_{n=1}^{\infty} \bigcap_{u_r} F_{a, n}^*,$$

$a \cap$ means the intersection of sets.

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where \( \{u_r\} \) is a totality of all rational numbers in the closed interval \([a, b]\) and \( n \) is a natural number, we get a measurability of subset \( F_a \) of \( A \times A \). Furthermore, we can see easily an equality \( E_a = F_a \) so that \( E_a \) is a measurable subset of \( A \times A \). Thus \( k(s, t) \) is measurable on \( A \times A \).

Next, we define the function \( h(s, t) \) as, for each \( (s, t) \), a maximum value of \( u \)'s for which hold the relations \( k(s, t) = |K[s, t, u]| \).

For any \( \beta (a \leq \beta \leq b) \), putting

\[
E_{\beta} = \{ (s, t) ; h(s, t) \leq \beta \}
\]

\[
F^a_{\beta} = \{ (s, t) ; \sup_{a \leq u \leq b} |K[s, t, u]| < \sup_{a \leq u \leq b} |K[s, t, u]| \}
\]

and

\[
F_{\beta} = \bigcap_{a \leq \beta + 1/n \leq b} F^a_{\beta}
\]

where \( n \) is a natural number, we have also a measurable subset \( F_{\beta} \) of \( A \times A \) and an equality \( E_{\beta} = F_{\beta} \), and hence \( h(s, t) \) is measurable on \( A \times A \). It is obvious that \( h(s, t) \) is bounded on \( A \times A \).

We state the following:

**Theorem 1.** Let \( K[s, t, u] \) \( (s, t \in A, -\infty < u < +\infty) \) be continuous in \( u \) for fixed \( s, t \), and measurable on \( A \times A \) for fixed \( u \).

If it satisfies the following conditions:

a) for every bounded measurable function \( h(s, t) \) on \( A \times A \)

\[
m_2 \left( \int_A K[s, t, h(s, t)] \, dt \right) < +\infty ;
\]

b) for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( \|\phi - \phi\|_{\varepsilon} < \delta \) implies

\[
m_2 \left( \int_A \left( K[s, t, \phi] - K[s, t, \phi] \right) \, dt \right) < \varepsilon
\]

for \( \text{mes } (F) < \delta (F \subset A) \), then Uryson’s operator \( A \phi \) acts from \( L_{\phi} \) into \( L_{\phi} \), and is bounded and continuous.

**Proof.** We prove at first that \( A \phi \) acts from \( L_{\phi} \) into \( L_{\phi} \) and is bounded. For any \( \phi \in L_{\phi} \), taking \( \varepsilon = 1 \) in b) there exists \( \delta = \delta(1) > 0 \) such that

\[
m_2 \left( \int_A \left( K[s, t, \phi] - K[s, t, \phi] \right) \, dt \right) < 1
\]

for \( \|\phi - \phi\|_{\varepsilon} < \delta \) and \( \text{mes } (F) < \delta \). Since we can select \( \phi_i \in L_{\phi} \) \( (i = 0, 1, \ldots, k) \) such that \( \phi_0 = \phi \), \( \|\phi_k - \phi_{k-1}\|_{\varepsilon} < \delta \) \( (i = 1, 2, \ldots, k) \) and \( \phi_k = 0 \), where \( k = \lceil\|\phi\|_{\varepsilon}/\delta\rceil + 1 \), we have, by the convexity of \( \phi_0 \),

\[
m_2 \left( \frac{1}{k+1} \int_A K[s, t, \phi] \, dt \right)
\]

\[
\leq \frac{1}{k+1} \sum_{i=0}^{k-1} m_2 \left( \int_A \left( K[s, t, \phi_i] - K[s, t, \phi_{i+1}] \right) \, dt \right)
\]

\[
+ \frac{1}{k+1} m_2 \left( \int_A K[s, t, 0] \, dt \right) \leq \frac{k+A}{k+1} < 1 + A
\]

* \([x]\) is the symbol of Gauss.
where \( A = m_2 \left( \int_{a}^{b} \left| K [s, t, 0] \right| dt \right) \), and

\[
m_2 \left( \int_{a}^{b} K [s, t, \phi] dt \right) \leq B \cdot m_2 \left( \int_{a}^{b} K [s, t, \phi] dt \right)
\]

where \( B \) is only dependent on \( k \), because \( m_2 \) is the upper bounded modular. Therefore, for a partition \( \{ J_0, J_1, \ldots, J_j \} \) of \( J \), which satisfy \( \text{mes} (J_i) < \delta \) for \( i = 1, 2, \ldots, j \) where \( j = \lceil 1/\delta \rceil + 1 \), we have

\[
m_2 \left( \frac{1}{j} \int_{a}^{b} K [s, t, \phi] dt \right) \leq \sum_{i=1}^{j} \frac{1}{j} m_2 \left( \int_{a}^{b} K [s, t, \phi] dt \right) \leq B (1 + A),
\]

and hence it follows that, by the upper boundedness of \( m_2 \),

\[
m_2 \left( \int_{a}^{b} K [s, t, \phi] dt \right) \leq C \cdot m_2 \left( \frac{1}{j} \int_{a}^{b} K [s, t, \phi] dt \right) \leq CB (1 + A)
\]

where \( C \) is only dependent on \( j \). Thus, it is shown that \( A \phi (s) \in L_{\mathcal{E}} \) and further \( \| \phi \|_{\mathcal{E}} \leq \tau \) implies \( m_2 (A \phi) < CB (1 + A) \), that is, \( \| A \phi \|_{\mathcal{E}} < 2 CB (1 + A)^* \), where \( k = \lceil \tau/\delta \rceil + 1 \).

Next, we prove the continuity of the operator \( A \phi \).

If \( \lim_{k \to \infty} \| \phi_k - \phi \|_{\mathcal{E}} = 0 \) (\( \phi_k, \phi \in L_{\mathcal{E}} \)) then \( \{ \| \phi_k - \phi \|_{\mathcal{E}} \} \) converge to 0 weakly and hence \( \lim_{k \to \infty} \int_{a}^{b} |\phi_k (t) - \phi (t)| dt = 0 \). Accordingly, we can select a subsequence \( \{ \phi_n (t) \} \) converging to \( \phi_n (t) \) for almost all \( t \).

Since \( \phi_n (t) \) is almost all finite on \( J \), for any natural number \( \nu \) there exist \( M_\nu > 0 \) and a subset \( E_\nu \subset J \) such that \( \text{mes} (E_\nu) \geq 1 - 1/\nu \) and \( |\phi_n (t)| \leq M_\nu \) for all \( t \in E_\nu \).

Furthermore, by Egoroff's theorem, for any \( \varepsilon > 0 \) there exists a subset \( X \subset J \) such that

\[
\text{mes} (J - X) < \varepsilon \quad \text{and} \quad \{ \phi_n \} \quad \text{converge to} \quad \phi \quad \text{uniformly on} \quad X.
\]

Putting \( X = X \cap E_\nu \), we have \( \text{mes} (J - X) \leq \varepsilon + 1/\nu \) and for all of sufficiently large \( n \),

\[
\left| \int_{x_\nu} K [s, t, \phi_n (t)] dt \right| \leq \int_{x_\nu \setminus [M_\nu - \varepsilon]} |K [s, t, u]| dt < +\infty
\]

for almost all \( s \), because, by Lemma 1, the assumption a) implies

\[
m_2 \left( \int_{a}^{b} \sup_{|M_\nu - \varepsilon| \leq \xi} |K [s, t, u]| dt \right) < +\infty,
\]

and hence

\[\text{This is obtained from the fact that } m_2 (x) \leq 1 \text{ implies } \| x \|_{\mathcal{E}} \leq 1.\]

\[\text{The step function } f (t) = 1 \text{ on } J \text{ belongs to the conjugate space } L_{\mathcal{F}} \text{ of } L_{\mathcal{F}}, \text{ because}
\]

\[
\int_{J} \Phi_1 (\phi_1 [1, x], x) dx + \int_{J} \Phi_1 (1, x) dx = \int_{J} \phi_1 (1, x) dx < +\infty.
\]

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\[ \int_{\mathbb{R}} \sup_{|y_{t,n}| \leq \varepsilon} |K[s, t, u]| \, dt \in L_{\|\cdot\|}. \]

Therefore, by Lebesgue's theorem, we have
\[ \lim_{n \to \infty} \int_{X_n} K[s, t, \phi_n] \, dt = \int_{X_n} K[s, t, \phi_0] \, dt \]
for all \( u \) and almost all \( s \), since \( K[s, t, \phi_n] \) converge to \( K[s, t, \phi_0] \) for almost all \( t, s \), and consequently it follows that
\[ \lim_{n \to \infty} \Phi_2 \left( \left\{ \int_{X_n} \left( K[s, t, \phi_n] - K[s, t, \phi_0] \right) \, dt \right\}, s \right) = 0 \]
for almost all \( s \).

And, we have for all of sufficiently large \( n \)
\[ \Phi_2 \left( \left\{ \frac{1}{2} \left| \int_{X_n} \left( K[s, t, \phi_n] - K[s, t, \phi_0] \right) \, dt \right|, s \right\} \right) \]
\[ \leq 2 \Phi_2 \left( \left\{ \int_{X_n} K[s, t, \phi_n] \, dt \right\}, s \right) + 2 \Phi_2 \left( \left\{ \int_{X_n} K[s, t, \phi_0] \, dt \right\}, s \right) \]
\[ \leq \Phi_2 \left( \int_{X_n} \sup_{|y_{t,n}| \leq \varepsilon} |K[s, t, u]| \, dt, s \right), \]
and the last term is integrable by a) and Lemma 1, so that, by Lebesgue's theorem,
\[ \lim_{n \to \infty} \int_{X_n} \Phi_2 \left( \left\{ \int_{X_n} \left( K[s, t, \phi_n] - K[s, t, \phi_0] \right) \, dt \right\}, s \right) \, ds = 0, \]

because \( m_n \) is upper bounded.

Now, for any \( \varepsilon > 0 \), when we select \( \nu, \varepsilon, \) in the above as which satisfy \( \varepsilon + 1/\nu < \delta \) where \( \delta = \delta(\varepsilon) \) is the number in the assumption b), there exists \( n_c = n_c(\varepsilon) \) such that
\[ \|\phi_n - \phi_0\|_{\|\cdot\|} < \delta \]
and
\[ \int_{\mathbb{R}} \Phi_2 \left( \left\{ \int_{X_n} \left( K[s, t, \phi_n] - K[s, t, \phi_0] \right) \, dt \right\}, s \right) \, ds < \varepsilon, \]
and consequently, it follows that, by the convexity and upper boundedness of \( m_n \),
\[ m_n \left( A\phi_n(s) - A\phi_0(s) \right) < N \cdot \varepsilon \]
where \( N \) is a constant for which satisfies
\[ \Phi_2 (2\xi, s) \leq N \cdot \Phi_2 (\xi, s) \quad \text{for all } \xi \geq 0 \quad \text{and } s. \]

This shows that \( \{A\phi_n\} \) converges to \( A\phi_0 \) by the modular and hence it follows that
\[ \lim_{n \to \infty} \|A\phi_n - A\phi_0\|_{\|\cdot\|} = 0. \]

If we suppose that \( \lim_{n \to \infty} \|\phi_n - \phi_0\|_{\|\cdot\|} = 0 \) and

* If a modular \( m \) is upper bounded, then \( \lim_{n \to \infty} \|\phi_n - x\|_{\|\cdot\|} = 0 \) for all \( \xi \geq 0 \) is equivalent to \( \lim_{n \to \infty} m(\xi(\phi_n - x)) = 0 \), and that the modular convergence coincides with the norm convergence. (cf. H. Nakano\(^2\))

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\[ \| A\phi_k - A\phi_n \|_{L^\varphi} \geq \varepsilon \quad \text{for some } \varepsilon > 0 \quad \text{and } k = 1, 2, \ldots, \]
then we can find a subsequence \( \{ \phi_n(t) \} \) converging to \( \phi_k(t) \) in almost all \( t \in \mathcal{D} \) and hence it follows, as is shown above, that
\[
\lim_{n \to \infty} \| A\phi_n - A\phi_k \|_{L^\varphi} = 0.
\]
This is contradiction to (21). Thus the operator is continuous.

**Remark.** In the operator of Hammerstein type, i.e.
\[
H\phi(s) = \int_{\mathcal{D}} K(s, t) f(t, \phi(t)) \, dt
\]
it is known that the operator \( H\phi \) is continuous (moreover, it is compact) in Orlicz space \( L^\varphi \) if it satisfies the following conditions:
\begin{enumerate}
\item \[
\int_\mathcal{D} \varphi \left( \int_{\mathcal{D}} \varphi \left( \left| K(s, t) \right| \right) \, ds \right) \, ds > +\infty;
\]
\item \[
|f(t, u)| \leq a(t) + \Phi^{-1} [b\varphi(|u|)]
\]
\end{enumerate}
where \( a(t) \in L^\varphi, \ b > 0 \) and \( \Phi, \varphi, \) and their complementary Young’s functions \( \Psi, \varphi, \)
satisfy the \( \{\mathcal{D}\}\)-condition.5,11,13

Those conditions satisfy the conditions in Theorem 1, because the condition 2) implies the boundedness of the operator \( f : L^\varphi \to L^\varphi, \)
and also the bounded set \( \Omega \) in \( L^\varphi, \) is the absolutely equi-continuous integrals9, since
\[
\int_\mathcal{D} f(x) \cdot \psi_i (f(x)) \, dx \leq M < +\infty \quad \text{for all } f(x) \in \Omega,
\]
where \( \psi_i \) is a left-derivative of \( \varphi_i, \) consequently, the condition b) is satisfied.

4. In this section, we will consider the compactness of Uryson’s operator.
L. A. Ladyzhenskii\( ^{7} \) given a sufficient condition of the compactness of the operator acting in the space \( C, \) which it is proved by use of Ascoli-Arzela’s theorem. V. V. Nemyckii\( ^{9} \) shown a sufficient condition of the compactness of the operator in the space \( C \) and his proof is placed on the basis of Kolmogoroff’s criteria concerning for a compactness of a set. Those conditions have been established under the assumption that \( \mathcal{D} \) is bounded closed set in \( n \)-dimensional Euclidian space \( \mathbb{R}^n \) with Lebesgue measure.

We will give a theorem concerning for the compactness of the operator which acts in modlared function spaces defining on a bounded set in \( \mathbb{R}^n. \)

**Theorem 2.** Let the operator \( A\phi \) be the bounded operator which acts from the unit sphere \( S_1 \) of \( L^\varphi \) into \( L^\varphi. \) Further, if it satisfies the condition
\[ (\#) \quad \int_{\mathcal{D}} \left| K [x, t, \phi(t)] - K [s, t, \phi(t)] \right| \, dt \leq f(s) \cdot p(h) \quad (\phi \in S_1) \]

* \( L^\varphi \) means the Orlicz space satisfying \( \{\mathcal{D}\}\)-condition. cf. A. C. Zaanen\( ^{10} \)
for \(|x-s| \leq h\) (\(|x|\) is the usual norm in \(\mathbb{R}^n\), where \(f(s) \in L_{s_x}\) and \(p(h)\) is some real function tending to zero as \(h \to 0\), then \(A\phi\) is the compact operator from \(S\) into \(L_{s_x}\).

**Proof.** Putting

\[
(A\phi(s))^s = \frac{1}{V(\delta)} \int_{U(s, \delta)} A\phi(x) \, dx
\]

where \(V(\delta)\) is the volume of \(U(s, \delta)\) which is the sphere with the center \(s\) and the radius \(\delta\), we have, by (92),

\[
\Phi_2(|A\phi-(A\phi)^s|, s) \\
\leq \Phi_2\left(\frac{1}{V(\delta)} \int_{U(\delta)} \left\{ K[s, t, \phi] - K[x, t, \phi] \right\} dt \, dx, s \right) \\
\leq \Phi_2(f(s) p(\delta), s) \text{ for almost all } s \in J \text{ and all } \phi \in S_1,
\]

and the last term is integrable on \(J\).

On the other hand, we have obviously

\[
\lim_{s \to 0} \Phi_2(f(s) p(\delta), s) = 0 \quad \text{for almost all } s \in J.
\]

Therefore, we have

\[
\lim_{s \to 0} m_2(A\phi-(A\phi)^s) = 0 \quad \text{uniformly on } S_1,
\]

i.e., for any \(\varepsilon > 0\) there exists \(\delta = \delta(\varepsilon) > 0\) such that

\[
\|A\phi-(A\phi)^s\|_{s_x} < \varepsilon \quad \text{for all } \phi \in S_1.
\]

Accordingly, if it is shown that \(\{(A\phi)^s\} \ (\phi \in S_1)\) is compact set in \(L_{s_x}\) then the compactness of the operator \(A\phi\) is obvious.

Since \(L_{s_x}\) is reflexive as Banach space, the boundedness of \(\{A\phi\} \ (\phi \in S_1)\) implies the weak compactness of \(\{A\phi\} \ (\phi \in S_1)\). Therefore, for any infinite sequence in \(\{A\phi\} \ (\phi \in S_1)\) we can find a subsequence, such that for every \(\delta > 0\)

\[
\lim_{n \to \infty} (A\phi_n(s))^s = (\phi_0(s))^s \quad \text{for almost all } s \in J,
\]

where \(\phi_0(s), (\phi_0(s))^s \in L_{s_x}\). Also, we have

\[
\| (A\phi_n(s))^s - (\phi_0(s))^s \| \leq \frac{1}{V(\delta)} \| X_2 \|_{s_x} \cdot \| A\phi_n - \phi_0 \|_{s_x} \\
\leq \frac{1}{V(\delta)} \| X_2 \|_{s_x} \{ M + \| \phi_0 \|_{s_x} \}
\]

for almost all \(s \in J\), where \(M = \sup_{\phi \in S_1} \| A\phi \|_{s_x}^s\).

* Weakly convergent sequence \(\{A\phi_n\}\) is the requirement, because all step functions belong to \(L_{s_x}\).

** It follows that \(\int_{U(s, \delta)} \phi_0(x) \, dx \leq \| X_2 \|_{s_x} \| \phi_0 \|_{s_x}\), and all step functions on \(J\) belong to the spaces \(L_{s_x}\) and \(L_{s_{x_x}}\).
Therefore, we have
\[ \lim_{n \to \infty} \| (A_{\phi_n} (s))' - (\phi_n (s))' \|_{L_{\phi}} = 0, \]
namely, \( \{ (A_{\phi})' \} (\phi \in S) \) is compact in \( L_{\phi} \) by the definition.

**Theorem 3.** When \( L_{\phi} (i=1, 2) \) are Orlicz spaces satisfying the \( (\Delta_2) \)-condition, we can replace the condition \( (\#) \) by the weaker conditions:
for almost all \( s \)
\[ (\exists_\phi) \lim_{s \to \infty} s \int_s^\infty |K [x, t, \phi] - K [s, t, \phi]| \, dt = 0 \]
uniformly on \( S \);

\[ (\exists_\phi) \sup_{\phi \in S} |A_{\phi} (s)| = f(s) \in L_{\phi}. \]

**Proof.** Since we know easily that
\[ \Phi_2 (|A_{\phi} - (A_{\phi})'|) \leq C \left\{ \Phi_2 (|A_{\phi}| + \Phi_2 \left( \frac{1}{V(\phi)} \int_{\delta} \int_{\gamma(s, \phi)} K [x, t, \phi] \, dx \, dt \right) \right\} \]
\[ \leq C \left\{ \Phi_2 (|f(s)|) + \Phi_2 \left( \frac{1}{V(\phi)} \| Z_{\phi} \|_{s} \cdot \| A_{\phi} \|_{s} \right) \right\} \]
\[ \leq C \left\{ \Phi_2 (|f(s)|) + \Phi_2 (\alpha \cdot M) \right\} \in L_{\phi}, \]
where \( \alpha = \lim_{s \to \infty} \frac{s}{V(\phi)} < + \infty \), \( M = \sup_{\phi \in S} \| A_{\phi} \|_{s} \) and \( C \) is some constant, the theorem is proved by the same method as the proof of Theorem 2.

**Lemma 2.** If \( L_{\phi} (\Delta, \mu) \) is a modulared function space, defining on a bounded set \( \Delta \) in \( \mathbb{R}_n \), with the upper bounded modular, then for any \( \varphi \in L_{\phi} \), we have
\[ \lim_{|x| \to 0} \| \varphi (x+h) - \varphi (x) \|_{\phi} = 0 \]
where \( \varphi (x+h) = 0 \) if \( x+h \in \Delta \) and \( \| h \| \) is the usual norm in \( \mathbb{R}_n \).

**Proof.** For any \( \varepsilon > 0 \) and \( \varphi \in L_{\phi} \), there exists a closed subset \( G \) of \( \Delta \) such that \( \| \varphi - \varphi_0 \| < \varepsilon \) where
\[ \varphi_0 (t) = \left\{ \begin{array}{ll}
\varphi (t) & \text{if } t \in G \\
0 & \text{if } t \notin G
\end{array} \right. \]
and \( \varphi_0 \in L_{\phi} \).

Therefore, we will prove the lemma for a function on \( G \).

(i) Putting, for \( x \in G \)
\[ \varphi_n (x) = \left\{ \begin{array}{ll}
n & \text{if } \varphi (x) \geq n \\
\varphi (x) & \text{if } -n \leq \varphi (x) \leq n \\
n & \text{if } \varphi (x) \leq -n,
\end{array} \right. \]
we have \( \lim_{n \to \infty} |\varphi_n (x) - \varphi (x)| = 0 \) for almost all \( x \in G \) and \( |\varphi_n (x) - \varphi (x)| \leq 2 |\varphi (x)| \),

\[ \lim_{s \to \infty} \frac{s}{V(\phi)} < + \infty \text{ is equivalent to } \lim_{s \to \infty} \frac{s}{V(\phi)} = 0 \text{ and } \| x \|_{s} \leq 1/V(\phi). \]

(226)
therefore, it follows that
\[
\lim_{n \to \infty} \| \varphi_n - \varphi \|_\sigma = 0, \text{ i.e., for any } \varepsilon > 0, \text{ there exists } n_\varepsilon = n_\varepsilon(\varepsilon) > 0 \text{ such that }
\| \varphi_n - \varphi \|_\sigma < \varepsilon \text{ and } |\varphi_n(x)| \text{ is bounded on } G.
\]

(ii) Let \( f(x) \) is bounded on \( G \), i.e., \( |f(x)| \leq M \) on \( G \). For any \( \varepsilon > 0 \) and \( \sigma > 0 \), there exists a continuous function \( g(x) \) on \( G \) such that \( |g(x)| \leq M \) on \( G \) and
\[
\mu(\{x; |f(x) - g(x)| \geq \sigma\} < \varepsilon.
\]

This statement is proved by the same method as the proof of Borel’s theorem which is stated for \( G = [0, 1] \) (cf. I. P. Natanson’).

Namely, for such natural number \( l \) as \( M/l < \sigma \), putting
\[
E_i = \{x; (i-1)M/l \leq f(x) \leq iM/l \text{ and } x \in G\} \quad (i = 1-l, 2-l, \ldots, l-1)
\]
and
\[
E_i = \{x; (l-1)M/l \leq f(x) \leq M\},
\]
we get a partition \( \{E_i\} \) \( (i = 1-l, 2-l, \ldots, l) \) of \( G \).

Since \( E_i \) are Lebesgue measurable sets, we can select closed sets \( F_i \) such that
\[
\mu(F_i) > \mu(E_i) - \varepsilon / 2l \quad \text{and} \quad F_i \subset E_i.
\]

Defining a continuous function \( g_i(x) \) on \( F = \bigcup_{i=1-l}^l F_i \) such as
\[
g_i(x) = iM/l \quad \text{if} \quad x \in F_i \quad (i = 1-l, 2-l, \ldots, l),
\]
we have \( |f(x) - g_i(x)| \leq M/l < \sigma \) for \( x \in F \).

Further, we get a continuous function \( g(x) \) on \( G \) such that it is an extension of \( g_i(x) \) on \( G \), for which satisfies
\[
|g(x)| \leq M \quad \text{and} \quad g(x) = 0 \quad \text{if} \quad x \in G - F.
\]

The function \( g(x) \) is the requirement.

(iii) By (ii), there exists a sequence \( \{g_n(x)\} \) of continuous functions on \( G \) such that it converges in measure on \( G \). Therefore we have, by Lebesgue’s theorem,
\[
\lim_{k \to \infty} \int \Phi_k(\|f(x) - g_{n_k}(x)\|, x) \, dx = 0
\]
for some subsequence of \( \{g_n(x)\} \). Accordingly, we have
\[
\lim_{k \to \infty} \|f - g_{n_k}\|_\sigma = 0,
\]
and hence there exists a continuous function \( g(x) \) on \( G \) such that
\[
\|f - g\|_\sigma < \varepsilon.
\]

(iv) If we assume that \( f(x + h) - g(x + h) = 0 \) for \( x + h \in G \), then we have, for enough small \( \|h\| \), \( \|g(x + h) - g(x)\|_\sigma < \varepsilon \) and \( \|f(x + h) - g(x + h)\|_\sigma < \varepsilon \), which implies the required fact, i.e.,
\[
\|\varphi(x + h) - \varphi(x)\|_\sigma < 5\varepsilon.
\]
**Remark.** Suppose $\mathcal{A}$ is a bounded set in $\mathbb{R}^n$. Let $\Phi_i, \Psi_i$ $(i=1, 2)$ be Young's functions satisfying the $(L_i)$-condition.

If

$$\int_\mathcal{A} \int_\mathcal{B} \Psi (R(s,t)) ds \, dt < \infty$$

where $\Psi \equiv \Phi_i [\Psi_i]$, then the linear operator $\int_\mathcal{A} R(s,t) \Phi_i(t) \, dt$ satisfies the conditions in Theorem 3, and hence the operator is a compact operator from $S_i \subset L^+_\Phi$ into $L^+_\Phi$. (cf. A. C. Zaanen\textsuperscript{10}, Krasnoselskii and Ya. B. Rutitskii; Dokl. Akad. Nauk SSSR (n. s) 85 (1952), 33-36. Russian)

Because, by Lemma 2, we have

$$\lim_{||h|| \to 0} \int_\mathcal{A} \int_\mathcal{B} \Psi (|R(s+h,t+h)-R(s,t)|) \, ds \, dt = 0$$

and hence

$$\lim_{||h|| \to 0} \int_\mathcal{A} \int_\mathcal{B} \Psi (|R(s+h,t)-R(s,t)|) \, ds \, dt = 0 \quad \text{for almost all } s \in \mathcal{A}.$$

And, we have also

$$\int_\mathcal{A} |R(s+h,t)-R(s,t)| \cdot |\Phi_i(t)| \, dt \leq \|R(s+h,t)-R(s,t)\|_{\Psi_i} \quad \text{for } \Phi \in S.$$

Namely, the assumptions of Theorem 3 are satisfied.

5. Combined the results in the section 3 with those in the section 4, we get the conditions of the complete continuity of the operator.

**Theorem 4.** Let $L_{\Phi_i}$ $(i=1, 2)$ are modular function spaces with the bounded modulars, defining on a bounded set $\mathcal{A}$ in $\mathbb{R}^n$. Let $K[s,t,u]$ be continuous in $u (-\infty < u < +\infty)$ for fixed $(s,t)$ and measurable on $\mathcal{A} \times \mathcal{A}$ for fixed $u$ satisfying the following conditions:

a) for every bounded measurable function $h(s,t)$ on $\mathcal{A} \times \mathcal{A}$

$$m_2 \left( \int_\mathcal{A} K[s,t,h(s,t)] \, ds \right) < +\infty;$$

b) for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi - \varphi\|_{\Phi_i} < \delta$ implies

$$m_2 \left( \int_\mathcal{A} K[s,t,\varphi] - K[s,t,\varphi] \, ds \right) < \varepsilon$$

for $\text{mes}(F)^2 \leq \delta (F \subset \mathcal{A});$

c) for $\|x-s\| < h$, where $f(s) \in L_{\Phi}$ and $p(h)$ tends to zero as $h \to 0$, then the operator $A \phi(s)$ acts from $S_i \subset L_{\Phi}$ into $L_{\Phi}$ and is completely continuous.

**Theorem 5.** In Theorem 4, if $L_{\Phi_i}$ are Orlicz spaces, the condition c) is replaced by the following condition:

c') for any bounded set $\mathcal{A}$ in $L_{\Phi}$ and almost all $s \in \mathcal{A},$
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\[
\lim_{z \to x} \int_{\Omega} \left| K(x, t, \phi(t)) - K(s, t, \phi(t)) \right| dt = 0 \text{ uniformly on } \Omega
\]

and

\[
\sup_{s \in \Omega} |A \phi(s)| = f(s) \in L_\omega. 
\]

**Remark.** Under the assumptions in the remark of the section 3, we obtain that the operator of Hammerstein type \( H \phi(s) \) acts in Orlicz space \( L_\omega^\ast \) and is completely continuous in the unit sphere \( S_i \) of \( L_\omega^\ast \). Since it is shown that the operator acts in \( L_\omega^\ast \) and is continuous it is sufficient to show the compactness of the operator.

Putting

\[
K[s, t, u] = R(s, t)f(t, u)
\]

we have for any \( \phi \in S_i \)

\[
\int \left| K(x, t, \phi) - K(s, t, \phi) \right| dt = \int \left| R(x, t) - R(s, t) \right| \left| f(t, \phi) \right| dt \leq \| R(x, t) - R(s, t) \|_{\omega} \| f(t, \phi) \|_{\omega} \leq \| R(x, t) - R(s, t) \|_{\omega} M
\]

and

\[
\| R(s, t) \|_{\omega} \in L_\omega^\ast \text{ from the assumption 1), where } \sup_{s \in \Omega} \| f(t, \phi) \|_{\omega} = M < \infty,
\]

because the operator \( f : L_\omega^\ast \in \phi(t) \to f(t, \phi) \in L_\omega^\ast \) is bounded\(^{(2)}\). Therefore, on the assumptions 1), 2) and Lemma 2, we will know that the conditions (\( L_\omega^\ast \)) and (\( L_\omega^\ast \)) in Theorem 3 are satisfied, namely the operator \( H \phi(s) \) is compact.

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