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A test for the rank of the volatility process: the random perturbation approach

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Abstract

In this paper we present a test for the maximal rank of the matrix-valued volatility process in the continuous Itô semimartingale framework. Our idea is based upon a random perturbation of the original high frequency observations of an Itô semimartingale, which opens the way for rank testing. We develop the complete limit theory for the test statistic and apply it to various null and alternative hypotheses. Finally, we demonstrate a homoscedasticity test for the rank process.

Keywords: central limit theorem, high frequency data, homoscedasticity testing, Itô semimartingales, rank estimation, stable convergence.

AMS 2010 Subject Classification: 62M07, 60F05, 62E20, 60F17.

1 Introduction

In the last years asymptotic statistics for high frequency observations has received a lot of attention in the literature. This interest was mainly motivated by financial applications, where observations of stocks or currencies are available at very high frequencies. As under the no-arbitrage condition prices processes must be semimartingales (see e.g. [4]), a lot of research has been devoted to statistics of high frequency data of semimartingales. We refer to a recent book [10] for a comprehensive study of infill asymptotic for semimartingales.

This paper is devoted to testing for the maximal rank of the matrix-valued volatility process in the continuous Itô semimartingale framework, and more specifically for a d-dimensional continuous Itô semimartingale X which is observed at equidistant times over a fixed time interval $[0,T]$: we observe $(X_{i\Delta_n})_{0\leq i\leq [T/\Delta_n]}$, and the high-frequency approach consists in assuming $\Delta_n \to 0$.

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A continuous Itô semimartingale can be written as

$$dX_t = b_t dt + \sigma_t dW_t, \tag{1.1}$$

where W is a Brownian motion, and there are many representations of this form, with different Brownian motions W and, accordingly, different volatility processes σ. What is "intrinsic" is the drift coefficient b_t and the diffusion coefficient ("squared volatility") $c_t = \sigma_t \sigma_t^*$, in the sense that they are uniquely determined by X, up to a Lebesgue-null set of times (throughout the paper σ_t^* denotes the transpose of the matrix σ).

For modeling purposes and economical interpretation we would like to find, and often choose, the smallest possible dimension of the Brownian motion W in the representation (1.1). Assuming further that $t \mapsto c_t$ is continuous, this smallest possible dimension is the supremum in time of the rank of the $\mathbb{R}^{d\times d}$-valued process c over the time interval $[0, T]$. We are further interested in homoscedasticity testing for the rank process.

A partial answer to this question was given in [9]. The authors of this paper studied the problem of testing the null hypothesis $\sup_{t\in[0,T)} \operatorname{rank}(c_t) \geq r_0$ against $\sup_{t\in[0,T)} \operatorname{rank}(c_t) < r_0$ for a given number r_0. However, their method does not extend to testing null hypotheses of other types, e.g. $\sup_{t\in[0,T)} \operatorname{rank}(c_t) = r_0$ against $\sup_{t\in[0,T)} \operatorname{rank}(c_t) \neq r_0$ (which is much more useful). In the classical setting of i.i.d or weakly dependent data various estimation methods for the rank of an unknown covariance matrix (and related objects) have been proposed. We would like to mention Gaussian elimination method with complete pivoting of [3] and the test suggested in [13] among others. Unfortunately, these procedures can not be applied to our statistical problem as the probabilistic structure of the process X is more complex and the rank is time-varying.

Our method is based upon a random perturbation of the original data and determinant expansions. The main idea can be described as follows: if we compute $\det(c_t + he_t)$ for a positive definite $d \times d$ matrix e_t independent of c_t and $h \downarrow 0$, then, under appropriate conditions, its rate of decay to 0 depends on the unknown rank of c_t. Hence, the ratio $\det(c_t + 2he_t)/\det(c_t + he_t)$ asymptotically identifies the rank of c_t. Indeed, our main statistic is a partial sum of squared determinants of matrices build from d consecutive increments of the process X and the random perturbation is performed by a properly scaled Brownian motion W', which is independent of all ingredients of X. We remark that perturbation methods (and matrix expansions as well) find applications in various fields of mathematics; we refer for instance to [11] whose authors apply matrix perturbation methods to determine the number of components in a linear mixture model from high dimensional noisy samples. Furthermore, the methods of [2] also rely upon a generation of a new Brownian motion W' although in a completely different setting.

The paper is structured as follows. Section 2 is devoted to model assumptions, testing hypotheses and test statistics. We present the asymptotic theory for our estimators and apply it to maximal rank testing in section 3. In section 4 we develop a test for the null hypothesis of constant rank. All proofs are deferred to section 5.

2 Model, assumptions and a random perturbation

2.1 The setting and testing hypotheses

Our process of interest is a d-dimensional continuous Itô semimartingale X, given on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. In vector form, and with W denoting a q-dimensional Brownian motion, it can be written as

$$X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW_s, \tag{2.1}$$

where b_t is a d-dimensional drift process and σ_t is a $\mathbb{R}^{d\times q}$-valued volatility process, assumed to be continuous in time (and indeed much more, see Assumption (H) below). We set

$$c_t = \sigma_t \sigma_t^*, \qquad r_t = \mathrm{rank}(c_t), \qquad R_t = \sup_{s\in[0,t)} r_s. \tag{2.2}$$

We remark that the maximal rank R_T is not bigger than the rank of the integrated volatility $\int_0^T c_t dt$, but may be strictly smaller. As already mentioned, it is suitable to use the smallest possible dimension for W, on the time interval $[0,T]$. This is the $\mathbb{P}$-essential supremum of $\omega \mapsto R_T(\omega)$, but, since a single path $t \mapsto X_t(\omega)$ is (partially) observed, the only available information is R_T itself. So the problem really boils down to finding the behavior of the process r_t, and for this the choice of the dimension of W in (2.1) is irrelevant.

The rank r_t is the biggest integer $r \leq d$ such that the sum of the determinants of the matrices $(c_t^{ij})_{i,j\in J}$, where J runs through all subsets of $\{1,\cdots,d\}$ with r points, is positive (with the convention that a 0×0 matrix has determinant 1); see e.g. [9, Lemma 3]. Since c_t is continuous, this implies that for any r the random set $\{t : r_t(\omega) > r\}$ is open in $[0,T)$, so the mapping $t \mapsto r_t$ is lower semi-continuous. In particular, the set $\{t \in [0,T) : r_t(\omega) = R_T(\omega)\}$ is a non-empty open subset. These properties also yield that the process r_t is predictable and that the following subsets of Ω, which later will be the "testing hypotheses", are $\mathcal{F}_T$-measurable:

$$\begin{aligned} \Omega_T^r &= \{\omega :\ R_T(\omega) = r\} \\ \Omega_T^= &= \{\omega :\ r_t(\omega) = R_T(\omega) \text{ for all } t \in [0,T]\} \\ \Omega_T^{\neq} &= \{\omega :\ t \mapsto r_t(\omega) \text{ has finitely many discontinuities and is} \\ &\qquad\qquad \text{not Lebesgue-a.s. constant on } [0,T]\}. \end{aligned} \tag{2.3}$$

Notice that we impose that $r_T = R_T$ in $\Omega_T^=$, whereas the lower semi-continuity only implies in general that $r_T \leq R_T$. Observe also that *a priori* $t \mapsto r_t$ may be Lebesgue-a.s. constant and still have discontinuities (even infinitely many) on $[0,T]$. So, $\Omega_T^=$ and $\Omega_T^{\neq}$ are disjoint but $\Omega_T^= \cup \Omega_T^{\neq} \neq \Omega$ in general. The main aim of this paper is testing the null hypothesis Ω_T^r against $\Omega_T^{\neq r} = \cup_{r'\neq r, 0\leq r'\leq d} \Omega_T^{r'}$ (and related hypotheses) and testing the null hypothesis of $\Omega_T^=$ against $\Omega_T^{\neq}$.

2.2 Matrix perturbation

In order to explain the main idea of our method, we need to introduce some notation. Recall that d and q are the dimensions of X and W, respectively. Then $\mathcal{M}$ is the set of all $d \times d$ matrices, $\mathcal{M}_r$ for $r \in \{0, \cdots, d\}$ is the set of all matrices in $\mathcal{M}$ with rank r, and $\mathcal{M}'$ is the set of all $d \times q$ matrices. For any matrix A we denote by A_i the ith column of A; for any vectors $x_1, \ldots, x_d$ in $\mathbb{R}^d$, we write $\mathrm{mat}(x_1, \cdots, x_d)$ for the matrix in $\mathcal{M}$ whose ith column is the column vector x_i. For $r \in \{0, \cdots, d\}$ and $A, B \in \mathcal{M}$ we define

$$\mathcal{M}^r_{A,B} \;=\; \{G \in \mathcal{M} : \, G_i = A_i \text{ or } G_i = B_i \text{ with } \#\{i : \, G_i = A_i\} = r\}. \tag{2.4}$$

In other words, $\mathcal{M}^r_{A,B}$ is the set of all matrices $G \in \mathcal{M}$ with r columns equal to those of A (at the same places), and the remaining $d - r$ ones equal to those of B. Let us define

$$\gamma_r(A, B) = \sum_{G \in \mathcal{M}^r_{A,B}} \det(G). \tag{2.5}$$

We demonstrate the main ideas for a deterministic problem first. Let $A \in \mathcal{M}$ be an unknown matrix with rank r. Assume that, although A is unknown, we have a way of computing $\det(A + hB)$ for all $h > 0$ and some given matrix $B \in \mathcal{M}_d$. The multi-linearity property of the determinant implies the following asymptotic expansion

$$\det(A + hB) = h^{d-r} \gamma_r(A, B) + O(h^{d-r+1}), \tag{2.6}$$

which is the core of our method. Thus, if $\gamma_r(A, B) \neq 0$, we have

$$\frac{\det(A + 2hB)}{\det(A + hB)} \to 2^{d-r} \qquad \text{as} \quad h \downarrow 0. \tag{2.7}$$

and this convergence identifies the parameter r. However, it is impossible to choose a matrix $B \in \mathcal{M}$ which guarantees $\gamma_r(A, B) \neq 0$ for all $A \in \mathcal{M}_r$. To solve this problem we can use a random perturbation. As we will show later, for any $A \in \mathcal{M}_r$ we have $\gamma_r(A, B) \neq 0$ a.s. when B is the random matrix whose entries are independent standard normal. This idea will be the core of our testing procedure.

2.3 Assumptions and the test statistic

Before we proceed with the definition of the test statistic, we introduce the main assumptions. We need more structure than the mere Equation (2.1), namely that the processes b_t and σ_t, and also the volatility of σ_t, are continuous Itô semimartingales. In view of the previous discussion, it is no restriction to assume that all these are driven by the same q-dimensional Brownian motion, provided we take q large enough. This leads us to put

Assumption (H): The d-dimensional semimartingale X, defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, has the form

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s \\ \sigma_t &= \sigma_0 + \int_0^t a_s\,ds + \int_0^t v_s\,dW_s \\ b_t &= b_0 + \int_0^t a'_s\,ds + \int_0^t v'_s\,dW_s \\ v_t &= v_0 + \int_0^t a''_s\,ds + \int_0^t v''_s\,dW_s, \end{aligned} \tag{2.8}$$

where W is a q-dimensional Brownian motion, and b_t and a'_t are $\mathbb{R}^d$-valued, σ_t, a_t and v'_t are $\mathbb{R}^{d\times q}$-valued, v_t and a''_t are $\mathbb{R}^{d\times q\times q}$-valued, and v''_t is $\mathbb{R}^{d\times q\times q\times q}$-valued, all those processes being adapted. Finally, the processes a_t, v'_t, v''_t are càdlàg and the processes a'_t, a''_t are locally bounded. □

At this stage it is not quite clear why the full force of assumption (H) is required. In the standard limit theory for high frequency data of continuous Itô semimartingales, see e.g. [1, 7], only the first two representations of (2.8) are assumed. We will further explain condition (H) once we introduce the test statistic. When $b_t = g_1(X_t)$, $\sigma_t = g_2(X_t)$ with $g_1 \in C^2(\mathbb{R}^d)$ and $g_2 \in C^4(\mathbb{R}^d)$, then (H) is automatically satisfied, due to Itô's formula.

Remark 2.1 Since σ_t is not uniquely specified, whereas c_t is, and since we really are interested in specific properties of c_t, it would be much nicer to replace the structural assumption on σ_t (second equation in (2.8)) by a similar assumption on the process c_t itself.

This is of course a trivial matter when c_t is everywhere invertible: in this case c_t is a continuous Itô semimartingale if and only if σ_t is. But here we are precisely trying to describe the rank of the matrix c_t, so it is out of the question to assume that it is *a priori* invertible. Unfortunately, we were unable to replace the assumption on σ by a similar (and *de facto* weaker) assumption on c. □

Motivated by the matrix perturbation at (2.6), our tests will be based on statistics involving sums of (squared) determinants. The test function will be the nonnegative map f on $(\mathbb{R}^d)^d$ defined as

$$f(x_1,\dots,x_d) = \det(\mathrm{mat}(x_1,\cdots,x_d))^2. \tag{2.9}$$

The authors of [9] used the following statistics

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]-d+1} f\big(\Delta_i^n X/\sqrt{\Delta_n},\cdots,\Delta_{i+d-1}^n X/\sqrt{\Delta_n}\big), \qquad \Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}, \tag{2.10}$$

to test for the full rank, thus allowing for efficient testing of the null hypothesis Ω_T^d. On the sets Ω_T^r with $r < d$, however, it exhibits complex degeneracies and becomes difficult to study. In order to be able to analyze the asymptotic behavior of the preceding statistics, we introduce a random perturbation of the original process X as motivated at the end of Subsection 2.2 (a somewhat similar idea in a different context was applied in [2]). More specifically, we choose a non-random invertible $d \times d$ matrix $\widetilde{\sigma}$ and generate a new process

$$X'_t = \widetilde{\sigma} W'_t,$$

where W' is a d-dimensional Brownian motion independent of all processes in (2.8)(without loss of generality, for the mathematical treatment below we may assume that it is also defined on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$). Following the ideas of section 2.2, we add to X this new process X', with a multiplicative factor going to 0. As a matter of fact we introduce two such additions, and for $\kappa = 1$ or 2 we set

$$Z_t^{n,\kappa} = X_t + \sqrt{\kappa \Delta_n}\, X'_t. \tag{2.11}$$

Hence, with the notation of section 2.2, we use $h = \sqrt{\Delta_n}$, which leads later to the optimal rate of convergence.

Another problem arises, namely in (2.10) successive summands partly use the same increments of X, and this causes problems for the Central Limit Theorem. These problems can actually be overcome, at the expense of quite many additional technicalities, and with the advantage of a smaller asymptotic variance for our estimators below. However, in our case the crucial point is the choice of the tuning "parameter" $\widetilde{\sigma}$: this choice has an impact on the asymptotic variance as well, and since an "optimal" choice of $\widetilde{\sigma}$ seems out of reach, we will content ourselves with an arbitrary choice of $\widetilde{\sigma}$ and with a version of (2.10) with no overlapping of increments between the successive summands. This leads us to use the following two basic statistics:

$$S_t^{n,1} = 2d\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} f\left(\frac{Z^{n,1}_{(2id+1)\Delta_n} - Z^{n,1}_{2id\Delta_n}}{\sqrt{2\Delta_n}}, \cdots, \frac{Z^{n,1}_{(2id+d)\Delta_n} - Z^{n,1}_{(2id+d-1)\Delta_n}}{\sqrt{2\Delta_n}} \right) \tag{2.12}$$

$$S_t^{n,2} = 2d\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} f\left(\frac{Z^{n,1}_{(2id+2)\Delta_n} - Z^{n,1}_{(2id)\Delta_n}}{\sqrt{2\Delta_n}}, \cdots, \frac{Z^{n,1}_{(2id+2d)\Delta_n} - Z^{n,1}_{(2id+2d-2)\Delta_n}}{\sqrt{2\Delta_n}} \right).$$

Notice that the statistics $S_t^{n,1}$ and $S_t^{n,2}$ are essentially the same, except $S_t^{n,2}$ is computed using the frequency $2\Delta_n$. At stage n one observes the increments $\Delta_i^n X$ and simulates the increments $\Delta_i^n X'$ for $i \leq [t/\Delta_n]$, so one "observes" all variables incurring in the definition of these two statistics.

Remark 2.2 Now, let us explain why the assumption (H) and the random perturbation in (2.11) are required. A direct stochastic expansion of the increments $\Delta_i^n Z^n$ under assumption (H) implies the decomposition

$$\mathrm{mat}(\Delta_i^n Z^n/\sqrt{\Delta_n}, \ldots, \Delta_{i+d-1}^n Z^n/\sqrt{\Delta_n}) = \alpha_i^n + \sqrt{\Delta_n}(\beta_i^n(1) + \beta_i^n(2)) + O_{\mathbb{P}}(\Delta_n), \tag{2.13}$$

where the matrices $\alpha_i^n = \mathrm{mat}(\alpha_{i,1}^n, \ldots, \alpha_{i,d}^n)$, $\beta_i^n(k) = \mathrm{mat}(\beta_{i,1}^n(k), \ldots, \beta_{i,d}^n(k))$, $k = 1, 2$, in $\mathcal{M}$ are given by

$$\begin{aligned} \alpha_{i,j}^n &= \Delta_n^{-1/2} \sigma_{(i-1)\Delta_n} \Delta_{i+j-1}^n W, \\ \beta_{i,j}^n(1) &= b_{(i-1)\Delta_n} + \Delta_n^{-1}\, v_{(i-1)\Delta_n} \textstyle\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n} (W_s - W_{(i+j-1)\Delta_n}) dW_s \\ \beta_{i,j}^n(2) &= \Delta_n^{-1/2} \widetilde{\sigma} \Delta_{i+j-1}^n W'. \end{aligned} \tag{2.14}$$

We remark that the matrices $\alpha_i^n, \beta_i^n(1), \beta_i^n(2)$ are $O_{\mathbb{P}}(1)$. In the case $r_t \leq d-1$ for all t, the first order term α_i^n, which depends on the process σ_t, gives a degenerate limit when plugged in into the statistics (2.11) or (2.12). Hence the second order term $\sqrt{\Delta_n}(\beta_i^n(1)+\beta_i^n(2))$, which involves the processes b_t and v_t, becomes important. Indeed, we will see in section 3 that it affects the limits. Furthermore, it is important to control the error of the above decomposition, and this is done by using the last two equations in (2.8).

The asymptotic expansion in (2.13) is a stochastic analogue of the perturbation presented in (2.6) (up to an error term) with $A=\alpha_i^n$, $B=\beta_i^n(1)+\beta_i^n(2)$ and $h=\sqrt{\Delta_n}$. Under assumption (H) the term $\beta_i^n(1)$ already constitutes a random perturbation of the leading matrix α_i^n. However, this perturbation does not guarantee that the quantity $\gamma_r(\alpha_i^n,\beta_i^n(1))$ defined in (2.5) does not vanish when $\mathrm{rank}(\sigma_{(i-1)\Delta_n})=r$ (which is essential for our method). To illustrate this problem let us give a simple example. Let $d=3$, $q=1$ and define the processes

$$dX_t^j=\sigma_t^j dW_t, \qquad d\sigma_t^j=v_t^j dW_t, \qquad j=1,2,3.$$

(so W is a one-dimensional Brownian motion.) Then $\mathrm{rank}(\alpha_i^n)=1$, $\mathrm{rank}(\beta_i^n(1))=1$, and hence $\gamma_1(\alpha_i^n,\beta_i^n(1))=0$. The presence of the new independent process X', and thus of the term $\beta_i^n(2)$, regularizes the problem. Indeed, we will show that $\gamma_r(\alpha_i^n,\beta_i^n(1)+\beta_i^n(2))$ does not vanish whenever $\mathrm{rank}(\sigma_{(i-1)\Delta_n})=r$. Finally, the perturbation rate $h=\sqrt{\Delta_n}$ in front of the process X' is chosen to achieve the best rate of convergence for the normalized versions of the statistics $S_t^{n,1}, S_t^{n,2}$. □

Following the expansion (2.6) we know that that the order of $\det(\alpha_i^n+\sqrt{\Delta_n}(\beta_i^n(1)+\beta_i^n(2)))^2$ is increasing in $r=\mathrm{rank}(\sigma_{(i-1)\Delta_n})$. Consequently, as in (2.7), the ratio $S_T^{n,2}/S_T^{n,1}$ is expected to identify (asymptotically) the maximal rank R_T. The complete asymptotic theory is presented in the next section.

3 The asymptotic results and test for the maximal rank

3.1 Notation

In order to present the main asymptotic results we need to introduce a few more notation. We define the function F_r on $(\mathbb{R}^{2d})^d$ by

$$F_r(v_1,\dots,v_d)=\gamma_r(\mathrm{mat}(x_1,\cdots,x_d),\mathrm{mat}(y_1,\cdots,y_d))^2 \quad \text{if } v_j=\begin{pmatrix} x_j \\ y_j \end{pmatrix}\in\mathbb{R}^{2d}. \tag{3.1}$$

Next, let $\mathcal{U}=\mathcal{M}'\times\mathcal{M}\times\mathbb{R}^{dq^2}\times\mathbb{R}^d$, whose points are $\underline{u}=(\alpha,\beta,\gamma,a)$, where $\alpha\in\mathcal{M}'$ and $\beta\in\mathcal{M}$ and $\gamma\in\mathbb{R}^{dq^2}$ and $a\in\mathbb{R}^d$. Let us denote by $\overline{W}$ and $\overline{W}'$ two independent Brownian motions with respective dimensions q and d, defined on some space $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t),\overline{\mathbb{P}})$. If $\underline{u}\in\mathcal{U}$ and $\kappa=1,2$ and $i\geq 1$ we associate the $2d$-dimensional variables with the following components for $l=1,\cdots,d$:

$$\begin{aligned}\Psi(\underline{u},\kappa)_i^l &= \tfrac{1}{\sqrt{\kappa}}\textstyle\sum_{m=1}^q \alpha^{lm}(\overline{W}_{\kappa i}^m-\overline{W}_{\kappa(i-1)}^m)\\ \Psi(\underline{u},\kappa)_i^{d+l} &= a^l+\tfrac{1}{\sqrt{\kappa}}\textstyle\sum_{m=1}^d \beta^{lm}(\overline{W}'^m_{\kappa i}-\overline{W}'^m_{\kappa(i-1)})+\tfrac{1}{\kappa}\sum_{m,k=1}^q \gamma^{lmk}\int_{\kappa(i-1)}^{\kappa i}\overline{W}_s^k\, d\overline{W}_s^m.\end{aligned} \tag{3.2}$$

With the notation (3.1) we can then define the variables

$$\overline{F}_r(\underline{u},\kappa) = F_r\big(\Psi(\underline{u},\kappa)_1,\cdots,\Psi(\underline{u},\kappa)_d\big). \tag{3.3}$$

The two sequences $(\Psi(\underline{u},\kappa))_{i\geq 1}$ are not independent, but they have the same (global) law, for $\kappa=1,2$. Therefore if $\underline{u}=(\alpha,\beta,\gamma,a)$ we can set

$$\begin{aligned}
\Gamma_r(\underline{u}) &= \overline{\mathbb{E}}\big(\overline{F}_r(\underline{u},1)\big) = \overline{\mathbb{E}}\big(\overline{F}_r(\underline{u},2)\big)\\
\Gamma'_r(\underline{u}) &= \overline{\mathbb{E}}\big(\overline{F}_r(\underline{u},1)^2\big) - \Gamma_r(\underline{u})^2 = \overline{\mathbb{E}}\big(\overline{F}_r(\underline{u},2)^2\big) - \Gamma_r(\underline{u})^2\\
\Gamma''_r(\underline{u}) &= \overline{\mathbb{E}}\big(\overline{F}_r(\underline{u},1)\,\overline{F}_r(\underline{u},2)\big) - \Gamma_r(\underline{u})^2.
\end{aligned} \tag{3.4}$$

We then obtain the following crucial properties

Lemma 3.1 *Let $\underline{u}=(\alpha,\beta,\gamma,a)\in\mathcal{U}$ with $\beta\in\mathcal{M}_d$. Then if $r\in\{0,1,\cdots,d\}$,*

$$\begin{aligned}
\text{rank}(\alpha)=r &\implies \Gamma_r(\underline{u})>0,\quad \Gamma'_r(\underline{u})>\Gamma''_r(\underline{u})\\
\text{rank}(\alpha)<r &\implies \Gamma_r(\underline{u})=\Gamma'_r(\underline{u})=\Gamma''_r(\underline{u})=0.
\end{aligned} \tag{3.5}$$

3.2 The limiting results

The key result is the asymptotic behavior of the processes $S^{n,j}$ as $n\to\infty$. These processes enjoy a Law of Large Numbers and a Central Limit Theorem, the centering being around one of the following processes, where r is any (fixed) integer between 0 and r:

$$S(r)_t \;=\; \int_0^t \Gamma_r(\sigma_s,\widetilde{\sigma},v_s,b_s)\,ds\,. \tag{3.6}$$

We will in fact have a CLT for the two-dimensional processes $U(r)^n$ with components

$$U(r)^{n,\kappa} = \frac{1}{\sqrt{\Delta_n}}\Big(\frac{1}{(\kappa\Delta_n)^{d-r}}\,S^{n,\kappa} - S(r)\Big). \tag{3.7}$$

Of course, the centering process $S(r)$ depends on r, so one needs an additional assumption related with the particular value of r which is chosen below (in contrast, the centering term is the same for all components):

Theorem 3.2 *Assume (H), and also that $r_t(\omega)\leq r$ identically for some $r\in\{0,\cdots,d\}$. Then we have the stable (functional) convergence in law*

$$U(r)^n \overset{\mathcal{L}-s}{\Longrightarrow} \mathcal{U}(r), \tag{3.8}$$

where $\mathcal{U}(r)=(\mathcal{U}(r)^\kappa)_{\kappa=1,2}$ is defined on an extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq0},\widetilde{\mathbb{P}})$ of $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$ and is, conditionally on $\mathcal{F}$, a continuous centered Gaussian martingale with conditional covariance

$$\widetilde{\mathbb{E}}(\mathcal{U}(r)^\kappa_t\,\mathcal{U}(r)^{\kappa'}_t\mid\mathcal{F}) = V(r)^{\kappa\kappa'}_t := \begin{cases} 2d\int_0^t \Gamma'_r(\sigma_s,\widetilde{\sigma},v_s,b_s)\,ds & \text{if } \kappa=\kappa'\\ 2d\int_0^t \Gamma''_r(\sigma_s,\widetilde{\sigma},v_s,b_s)\,ds & \text{if } \kappa\neq\kappa'. \end{cases} \tag{3.9}$$

Note that in the above setting, if $r < r' \leq d$, we also have $r_t \leq r'$ and thus the results also hold with r' instead of r everywhere. This does not bring a contradiction because, by (3.5), in this case the processes $S(r')$ and $U(r')$ are identically vanishing.

Now, these processes $S^{n,j}$ are only tools, and at the end we will be interested, for any $T > 0$ fixed, in "estimators" for R_T, which are

$$\widehat{R}(n,T) = d - \frac{\log(S_T^{n,2}/S_T^{n,1})}{\log 2}. \tag{3.10}$$

The quantity is a transformed analogue of the term on the left side of (2.7). The following corollary is then a simple consequence of the previous theorem:

Corollary 3.3 *Assume (H), and let $r \in \{0, \cdots, d\}$ and $T > 0$. Then the following stable convergence in law holds:*

$$\frac{1}{\sqrt{\Delta_n}}\,(\widehat{R}(n,T) - r) \ \stackrel{\mathcal{L}-s}{\longrightarrow}\ \mathcal{S}(T) \qquad \text{on the set } \Omega_T^r, \tag{3.11}$$

where $\mathcal{S}(T)$ can be realized as $\mathcal{S}(T) = \frac{1}{\log 2}\,(\mathcal{U}(r)_T^1 - \mathcal{U}(r)_T^2)/S(r)_T$ and is thus defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and is, conditionally on $\mathcal{F}$, a centered Gaussian variable whose conditional variance is

$$\widetilde{\mathbb{E}}((\mathcal{S}(T))^2 \mid \mathcal{F}) = V(T),$$

where $V(T)$ is a.s. positive and given by

$$V(T) = \frac{1}{(\log 2)^2}\,\frac{V(r)_T^{1,1} + V(r)_T^{2,2} - 2V(r)_T^{1,2}}{(S(r)_T)^2} \qquad \text{on each set } \Omega_T^r. \tag{3.12}$$

In order to make this result feasible, we need consistent estimators for $V(T)$. For the denominator $S(r)_T^2$ we can of course take the square of $\Delta_n^{r-d}\,S_T^{n,1}$. As for the numerator, we need estimators for $V(r)_T^{\kappa,\kappa'}$. Up to normalization, natural ones are as follows:

$$\begin{aligned} V_t^{n,\kappa\kappa'} = 4d^2\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} & f\left(\frac{Z^{n,\kappa}_{(2id+\kappa)\Delta_n} - Z^{n,\kappa}_{2id\Delta_n}}{\sqrt{\kappa\Delta_n}}, \cdots, \frac{Z^{n,\kappa}_{(2id+\kappa d)\Delta_n} - Z^{n,\kappa}_{(2id+\kappa(d-1))\Delta_n}}{\sqrt{\kappa\Delta_n}}\right) \\ & \times f\left(\frac{Z^{n,\kappa'}_{(2id+\kappa')\Delta_n} - Z^{n,\kappa'}_{2id\Delta_n}}{\sqrt{\kappa'\Delta_n}}, \cdots, \frac{Z^{n,\kappa'}_{(2id+\kappa' d)\Delta_n} - Z^{n,\kappa'}_{(2id+\kappa'(d-1))\Delta_n}}{\sqrt{\kappa'\Delta_n}}\right). \end{aligned} \tag{3.13}$$

Proposition 3.4 *Assume (H).*

a) If $r_t(\omega) \leq r$ identically for some $r \in \{0, \cdots, d\}$, we have for $\kappa, \kappa' = 1, 2$:

$$\begin{aligned} &\tfrac{1}{(\kappa\kappa'\Delta_n^2)^{d-r}}\,V^{n,\kappa\kappa'} \ \stackrel{u.c.p.}{\Longrightarrow}\ 2d\int_0^{\cdot} \Theta_s^{r,\kappa,\kappa'}\,ds, \qquad \text{where} \\ &\Theta_s^{r,\kappa,\kappa'} = \begin{cases} \Gamma'_r(\sigma_s, \widetilde{\sigma}, v_s, b_s) + \Gamma_r(\sigma_s, \widetilde{\sigma}, v_s, b_s)^2 & \text{if } \kappa = \kappa' \\ \Gamma''_r(\sigma_s, \widetilde{\sigma}, v_s, b_s) + \Gamma_r(\sigma_s, \widetilde{\sigma}, v_s, b_s)^2 & \text{if } \kappa \neq \kappa'. \end{cases} \end{aligned} \tag{3.14}$$

b) We have

$$V(n,T) := \frac{V_T^{n,11} + 2^{2(\widehat{R}(n,T)-d)} V_T^{n,22} - 2^{1+\widehat{R}(n,T)-d} V_T^{n,12}}{(S_T^{n,1} \log 2)^2} \xrightarrow{\mathbb{P}} V(T). \tag{3.15}$$

Remark 3.5 The numerator of the right side of (3.12) is also $2(V(r)_T^{11} - V(r)_T^{12})$. Therefore we have

$$V'(n,T) = \frac{V_T^{n,11} - 2^{1+\widehat{R}(n,T)-d} V_T^{n,12}}{(S_T^{n,1})^2} \xrightarrow{\mathbb{P}} V(T) \quad \text{on the set } \Omega_T^r$$

as well. However, $V(n,T) \geq 0$ by construction (and it is even a.s. positive unless $r_t = 0$ identically on $[0,T]$), a property not shared by $V'(n,T)$. □

Now, by the delta-method for stable convergence in law, the two previous results immediately yield:

Corollary 3.6 *Under (H) and for any $T > 0$ we have*

$$\frac{\widehat{R}(n,T) - R_T}{\sqrt{\Delta_n V(n,T)}} \xrightarrow{\mathcal{L}-s} \Phi, \tag{3.16}$$

where $\Phi \sim \mathcal{N}(0,1)$ is defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and is independent of $\mathcal{F}$.

3.3 Tests for the maximal rank

So far, it seems that $\widehat{R}(n,T)$ are estimators for the maximal rank R_T, which equals r on the set Ω_T^r, and even feasible estimators if we use Corollary 3.6. In particular, this corollary seems to allow us to easily construct confidence intervals for R_T.

However, what precedes does not make much statistical sense: the parameter R_T to be estimated takes its values in $\{0, 1, \cdots, d\}$, whereas the estimators $\widehat{R}(n,T)$ are of course not integer-valued and can even be negative, or bigger than d. One could overcome this problem by taking the integer closest to $\widehat{R}(n,T)$, say $\widehat{R}'(n,T)$, and then use $\widehat{R}''(n,Y) = 0 \vee (\widehat{R}'(n,T) \wedge d)$ as the final estimator. Note that $\widehat{R}''(n,T)$ enjoys the same CLT as $\widehat{R}(n,T)$ does, on each Ω_T^r with $1 \leq r \leq d-1$, but of course not when $R_T = 0$ or $R_T = d$, in which cases the limiting law of the normalized error is "half Gaussian and half a Dirac mass at 0". Furthermore, confidence intervals have little meaning in this context, except perhaps when the dimension of X is very large.

So, it seems more appropriate here to do testing: we can test the null hypothesis that the path lies in Ω_T^r for some r, against the alternative that it is in $\Omega_T^{r'}$ for another specific $r' \neq r$, or for all $r' > r$ or all $r' < r$, or all $r' \neq r$. We may also use composite null hypotheses, such as being in Ω_T^r for some r smaller, or bigger, than a given value r_0.

We start with the problem of testing the null hypothesis Ω_T^r, against the alternative $\Omega_T^{\neq r} = \cup_{r'\neq r, 0\leq r'\leq d}\Omega_T^{r'}$. For any $\alpha \in (0,1)$, and with z_α being the symmetric α-quantile of $\mathcal{N}(0,1)$ defined by $\mathbb{P}(|\Phi| > z_\alpha) = \alpha$ when $\Phi \sim \mathcal{N}(0,1)$, we take the critical (rejection) region

$$\mathcal{C}(\alpha)_T^{n,=r} = \big\{\omega:\ |\widehat{R}(n,T) - r| > z_\alpha \sqrt{\Delta_n V(n,T)}\big\}. \tag{3.17}$$

Proposition 3.7 *Under (H), the tests (3.17) have the asymptotic level α for testing the null Ω_T^r, in the sense that*

$$A \subset \Omega_T^r,\ \mathbb{P}(A) > 0 \quad \Rightarrow \quad \mathbb{P}\big(\mathcal{C}(\alpha)_T^{n,=r} \mid A\big) \to \alpha \tag{3.18}$$

(above, $\mathbb{P}(. \mid A)$ is the usual conditional probability). They are also consistent for the alternative $\Omega_T^{\neq r}$, in the sense that

$$\mathbb{P}\big(\mathcal{C}(\alpha)_T^{n,=r} \cap \Omega_T^{\neq r}\big) \to \mathbb{P}(\Omega_T^{\neq r}). \tag{3.19}$$

One constructs one-sided tests in the same way. For example, if we want to test the null hypothesis $\Omega_T^{\leq r} = \cup_{r'\leq r}\Omega_T^{r'}$ against the alternative $\Omega_T^{> r} = \cup_{r'>r}\Omega_T^{r'}$, and if z'_α is the one-sided α-quantile defined by $\mathbb{P}(\Phi > z'_\alpha) = \alpha$, we take the critical region

$$\mathcal{C}(\alpha)_T^{n,\geq r} = \big\{\omega:\ \widehat{R}(n,T) > r + z'_\alpha \sqrt{\Delta_n V(n,T)}\big\}. \tag{3.20}$$

Exactly as above, one obtains the following proposition.

Proposition 3.8 *Under (H), the tests (3.20) have the asymptotic level at most α for testing the null $\Omega_T^{\leq r}$, and indeed satisfy*

$$A \subset \Omega_T^{\geq r},\ \mathbb{P}(A) > 0 \quad \Rightarrow \quad \mathbb{P}\big(\mathcal{C}(\alpha)_T^{n,=r} \mid A\big) \to \alpha\, \mathbb{P}(\Omega_T^r \mid A) \leq \alpha, \tag{3.21}$$

and are consistent for the alternative $\Omega_T^{> r}$.

The tests for the null $\Omega_T^{\geq r}$ against $\Omega_T^{< r}$ are obtained analogously.

Remark 3.9 Let us link our testing procedure with some other statistical problems:

a) In [5, 6] parametric estimation methods for the so called integrated diffusions have been developed. An integrated diffusion is a process that satisfies the first and the third equations of assumption (H) with $\sigma = 0$, i.e.

$$dX_t = b_t dt,$$

where b_t is a continuous Itô semimartingale. We refer to [5] for various applications of these models in natural sciences. Given high frequency observations of X, testing the null hypothesis of integrated diffusion versus the alternative of a diffusion with a present volatility part σ is equivalent to testing Ω_T^0 versus $\Omega_T^{\neq 0}$.

b) Another potential application of our method is a test for "perfect correlation" between the process X and the unobserved volatility σ. The problem can be formulated as follows: Let X and σ be two one-dimensional continuous Itô semimartingales of the form

$$dX_t = b_t dt + \sigma_t dW_t, \qquad d\sigma_t^2 = a_t dt + v_t dB_t,$$

where W and B are one-dimensional Brownian motions with the bracket process $[W,B]_t = \rho t$, $|\rho| \le 1$. For financial applications testing the hypothesis $|\rho| = 1$ versus $|\rho| < 1$ is of certain interest. Note that $|\rho| = 1$ appears in the SDE case, i.e. when $\sigma_t = g(X_t)$ with $g \in C^2(\mathbb{R})$. We refer to testing local volatility hypothesis in [12] for a more detailed discussion (see also [14] for related statistical problems). The aforementioned problem is equivalent to testing Ω_T^1 versus $\Omega_T^{\geq 1} = \Omega_T^2$ for the two-dimensional process (X, σ^2). Since the process σ^2 is unobserved, it has to be locally estimated from the high frequency observations of X first (see e.g. [12] for more details). □

4 A test for a constant rank

This section is devoted to a seemingly different topic, namely whether the *a priori* time-dependent rank is constant or not. Our test statistics will be based on a distance measure between the rank process r_t and the maximal rank R_T, which vanishes if and only if the rank is constant almost surely. For the formal testing procedure we will need some limiting results for the "spot estimators" of the rank. By this, we mean estimators for r_t, for any given t, at least under the assumption that r_s is equal to r_t for all s in some right or left neighborhood of t.

To describe these spot estimators we pick a sequence $k_n \geq 1$ of integers going to infinity, and such that $k_n\Delta_n \to 0$ (as for spot volatility estimators), and precise specifications for k_n will be given later, although we always assume $k_n \geq 4d$. For any integer $i \geq 1$ we set

$$\widehat{R}_i^n = d - \frac{\log \widehat{S}_i^n}{\log 2}, \qquad \widehat{S}_i^n = \frac{S^{n,2}_{2d(i+1)k_n\Delta_n} - S^{n,2}_{2dik_n\Delta_n}}{S^{n,1}_{2d(i+1)k_n\Delta_n} - S^{n,1}_{2dik_n\Delta_n}}. \tag{4.22}$$

Then $\widehat{R}_i^n$, more or less, plays the role of an estimator of the maximum of r_t over an interval of length $2dk_n\Delta_n$ around the time $2id\Delta_n$, and we set for any $p > 0$:

$$\begin{aligned} A(p)_t^n &= 2dk_n\Delta_n \textstyle\sum_{i=0}^{[t/2dk_n\Delta_n]-2} \{|\widehat{R}^n_{ik_n}|^p \wedge (d+1)^p\} \\ B(n,p,T) &= A(p)_T^n - a(n,T)(\widehat{R}(n,T))^p, \quad a(n,T) = 2dk_n\Delta_n\big([T/2dk_n\Delta_n] - 1\big). \end{aligned} \tag{4.23}$$

The asymptotic results for the quantity $B(n,p,T)$ are as follows.

Theorem 4.1 *Assume (H), and let $T > 0$, $p > 0$ and k_n be such that $k_n\Delta_n^{3/4} \to \infty$ and $k_n\Delta_n \to 0$.*

a) If $t \mapsto r_t(\omega)$ is continuous except at finitely many points on $[0,T]$, hence piecewise constant, we have

$$B(n,p,T) \ \stackrel{\mathbb{P}}{\longrightarrow}\ \int_0^T (r_s)^p\, ds - T(R_T)^p. \tag{4.24}$$

b) We have the stable convergence in law:

$$\frac{1}{\sqrt{\Delta_n}}\, B(n,p,T) \stackrel{\mathcal{L}-s}{\longrightarrow} \mathcal{B}(p,T) \quad \text{in restriction to the set } \Omega_T^{=} \cap \{R_T \geq 1\}, \tag{4.25}$$

where $\mathcal{B}(p,T)$ is defined on an extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}})$ of $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ and is, conditionally on $\mathcal{F}$, a centered Gaussian variable with conditional variance $\overline{V}(p,T) = \widetilde{\mathbb{E}}(\mathcal{B}(p,T)^2 \mid \mathcal{F})$ given on each set Ω_T^r by

$$\overline{V}(p,T) = \Big(\frac{pr^{p-1}}{\log 2}\Big)^2 \int_0^T \Big(\frac{1}{\Gamma_r(\sigma_s,\widetilde{\sigma},v_s,b_s)} - \frac{T}{S(r)_T}\Big)^2 \big(dV(r)_s^{11} + dV(r)_s^{22} - 2dV(r)_s^{12}\big), \tag{4.26}$$

with $V(r)^{\kappa\kappa'}$ being defined at (3.9).

Notice that the right side of (4.24) is 0 on the set $\Omega_T^{=}$, and strictly negative on $\Omega_T^{\neq}$.

Remark 4.2 The reader will notice that in the definition of $A(p)_t^n$ the summands are $|\widehat{R}_i^n|^p \wedge (d+1)^d$, instead of the more natural $|\widehat{R}_i^n|^p$. We could take this more natural form for (b) above, but it is useful (and innocuous from a practical viewpoint) to "bound" the summands, in order to obtain (a). We could bound them by d^p instead of $(d+1)^p$ and still have (4.24), but then (4.25) would then fail in case $r = d$ is the maximal rank: we would obtain a CLT with a non-Gaussian and non-centered limit. □

Remark 4.3 In the setting of (b) above, we will in fact prove a joint convergence for the variables $A(p)_T^n - a(n,T)r^p$ and $\widehat{R}(n,T) - r$, both normalized by $1/\sqrt{\Delta_n}$ (the second one being as in (3.11)), and from which (4.25) follows. Such a joint CLT even holds under the assumptions of (a), with a complicated limit, but this refinement is not useful for us in this paper.

Remark 4.4 One can also prove a joint convergence for the variables $A(p)_T^n - a(n,T)r^p$ with different values of p, and still normalized by $1/\sqrt{\Delta_n}$. However, when $p > p' > 0$ it turn out that the difference $\frac{1}{\sqrt{\Delta_n}}\,(A(p)_T^n - a(n,T)^{1-p'/p}\,(A(p')_T^n)^{p/p'})$ converges to 0, and no known normalization gives a proper CLT. □

As before, we need consistent estimators for the conditional variance $\overline{V}(p,T)$. Such estimators are constructed in a way analogous to (3.13). That is, we set with k_n as above:

$$\begin{aligned} \overline{V}_t^{n,\kappa\kappa'} &= 4d^2\Delta_n^{1+2d-2\widehat{R}(n,T)} \sum_{i=0}^{[t/2d\Delta_n]-k_n-1} \Big(\frac{2dk_n\Delta_n}{S^{n,1}_{2d(i+k_n)\Delta_n} - S^{n,1}_{2id\Delta_n}} - \frac{T}{S_T^{n,1}}\Big)^2 \\ &\times f\Big(\frac{Z^{n,\kappa}_{(2id+\kappa)\Delta_n} - Z^{n,\kappa}_{2id\Delta_n}}{\sqrt{\kappa\Delta_n}},\cdots,\frac{Z^{n,\kappa}_{(2id+\kappa d)\Delta_n} - Z^{n,\kappa}_{(2id+\kappa(d-1))\Delta_n}}{\sqrt{\kappa\Delta_n}}\Big) \\ &\times f\Big(\frac{Z^{n,\kappa'}_{(2id+\kappa')\Delta_n} - Z^{n,\kappa'}_{2id\Delta_n}}{\sqrt{\kappa'\Delta_n}},\cdots,\frac{Z^{n,\kappa'}_{(2id+\kappa'd)\Delta_n} - Z^{n,\kappa'}_{(2id+\kappa'(d-1))\Delta_n}}{\sqrt{\kappa'\Delta_n}}\Big). \end{aligned} \tag{4.27}$$

Theorem 4.5 *Assume (H), and let $T>0$, $p>0$ and k_n be such that $k_n\Delta_n^{3/4}\to\infty$ and $k_n\Delta_n\to 0$. Then we have*

$$\begin{aligned}\overline{V}(n,p,T) := \Big(\tfrac{p\widehat{R}(n,T)^{p-1}}{\log 2}\Big)^2 \big(\overline{V}^{n,11}_T + 2^{2(\widehat{R}(n,T)-d)}\overline{V}^{n,22}_T - 2^{1+\widehat{R}(n,T)-d}\overline{V}^{n,12}_T\big) \\ \stackrel{\mathbb{P}}{\longrightarrow}\ \overline{V}(p,T) \quad \text{on the set } \Omega^{=}_T.\end{aligned} \tag{4.28}$$

Moreover, the variables

$$Z(n,p,T) = \frac{B(n,p,T)}{\sqrt{\Delta_n\,(\overline{V}(n,p,T)\wedge(1/\sqrt{\Delta_n}\,))}} \tag{4.29}$$

have the following asymptotic behavior, where $\Phi\sim\mathcal{N}(0,1)$ is as in Corollary 3.6:

$$\begin{array}{llll} Z(n,p,T) & \stackrel{\mathcal{L}-s}{\longrightarrow} & \Phi & \text{in restriction to the set } \Omega^{=}_T\cap\{R_T\geq 1\} \\ Z(n,p,T) & \stackrel{\mathbb{P}}{\longrightarrow} & -\infty & \text{in restriction to the set } \Omega^{\neq}_T \end{array} \tag{4.30}$$

Having all instruments at hand we proceed with testing. What is easily available is a family of tests for the null $\Omega^{=}_T$, whereas the alternative is restricted to $\Omega^{\neq}_T$. One does not know how to test the null $\Omega^{\neq}_T$.

For this purpose we use the statistic $B(n,p,T)$. In fact, (4.30) gives us the behavior of this statistic on $\Omega^{=}_T\cap\{R_T\geq 1\}$, and this is the null which is tested below. Now, $\Omega^{=}_T$ is the union of $\Omega^{=}_T\cap\{R_T\geq 1\}$ and Ω^0_T, so if we are interested in testing the whole $\Omega^{=}_T$ one can do a double test, using what precedes and Proposition 3.8 with $r=0$.

We propose to use the following critical region, where $p>0$ is chosen arbitrarily and z'_α is again the one-sided α-quantile of $\mathcal{N}(0,1)$:

$$\mathcal{C}(\alpha)^{n,=}_T = \big\{\omega:\ B(n,p,T) < -z'_\alpha\sqrt{\Delta_n\,(\overline{V}(n,p,T)\wedge(1/\sqrt{\Delta_n}\,))}\big\}. \tag{4.31}$$

Exactly as in the previous section we obtain the following result.

Proposition 4.6 *Under (H), the tests (4.31) have the asymptotic level α for testing the null $\Omega^{=}_T\cap\{R_T\geq 1\}$, in the sense of (3.18), and are consistent for the alternative $\Omega^{\neq}_T$.*

5 Proofs

Before we start presenting the formal proofs, let us give the road map. Subsection 5.1 demonstrates some technical results on expansions of determinants. They are applied in Subsection 5.2 to prove Lemma 3.1. This Lemma implies that the process $S(r)_t$ defined at (3.6) is strictly positive on the set Ω^r_T, which is crucial for our method.

The first main result of our paper is Theorem 3.2 whose proof is rather involved. First, we will show that the standard localization procedure (see e.g. Section 3 in [1]) implies that all processes in (H) may be assumed to be bounded without loss of generality. This first step considerably simplifies the stochastic treatment of various quantities. A second crucial step is the stochastic expansion explained in Remark 2.2: we have (2.13) and (2.14). Subsection 5.3 deals with the formal justification of this expansion, for which we will use slightly different notation.

It turns out that the stochastic order of the error term related to the decomposition (2.13), namely $O_{\mathbb{P}}(\Delta_n)$, is not sufficient to show its asymptotic negligibility. However, we will prove that the error terms are martingale differences, so they will not affect the stable central limit theorem at (3.8). A similar treatment will be required for the error term connected with the stochastic version of the expansion (2.6).

The proof of Proposition 3.4 (consistent estimation of the asymptotic conditional covariance matrix) is somewhat easier. Corollary 3.3 follows essentially from Theorem 3.2 by the delta method for stable convergence. The proofs of these results are collected in Subsection 5.4. In particular, we apply a stable central limit theorem for semimartingales (see e.g. [8, Theorem IX.7.28]) to prove Theorem 3.2.

The proof of Theorems 4.1 and 4.5, which is presented in Subsection 5.5, is a bit more involved than one of Theorem 3.2, although the main techniques are similar. The additional difficulty comes from the fact that we need to use the stable convergence of Theorem 3.2, but for processes evaluated at random times. Corollary 3.6 and Propositions 3.7, 3.8 and 4.6 are straightforward consequences of the previous results.

5.1 Expansion of determinants.

We first prove some general and easy facts about determinants. Below $\|A\|$ denotes the Euclidean norm of a matrix $A \in \mathcal{M}$.

For $m \geq 1$ we call $\mathcal{P}_m$ the set of all multi-integers $\mathbf{p} = (p_1, \cdots, p_m)$ with $p_1 + \cdots + p_m = d$, and $\mathcal{I}_{\mathbf{p}}$ is the set of all partitions $\mathbf{I} = (I_1, \cdots, I_m)$ of $\{1, \cdots, d\}$ such that I_j contains exactly p_j points (so $I_j = \emptyset$ if $p_j = 0$). If $\mathbf{p} \in \mathcal{P}_m$ and $\mathbf{I} \in \mathcal{I}_{\mathbf{p}}$ and $A_1, \cdots, A_m \in \mathcal{M}$, we write $G^{\mathbf{I}}_{A_1,\cdots,A_m}$ for the matrix whose ith column is the ith column of A_j when $i \in I_j$. Letting $A, B, C \in \mathcal{M}$, we can rewrite (2.5) as

$$\gamma_r(A, B) = \sum_{\mathbf{I} \in \mathcal{I}_{(r,d-r)}} \det(G^{\mathbf{I}}_{A,B}), \tag{5.1}$$

and we set

$$\gamma'_r(A, B, C) = \sum_{\mathbf{I} \in \mathcal{I}_{(r,d-r-1,1)}} \det(G^{\mathbf{I}}_{A,B,C}). \tag{5.2}$$

In the following two lemmas we present some technical results on determinant expansions.

Lemma 5.1 *For any $m \geq 1$ and $A_1, \cdots, A_m \in \mathcal{M}$ we have*

$$\det(A_1 + \cdots + A_m) = \sum_{\mathbf{p} \in \mathcal{P}_m} \sum_{\mathbf{I} \in \mathcal{I}_{\mathbf{p}}} \det(G^{\mathbf{I}}_{A_1,\cdots,A_m}). \tag{5.3}$$

Proof. Letting $\mathcal{S}_d$ be the set of all permutations of $\{1,\cdots,d\}$ and $\mathrm{sign}(s)$ be the signature of $s\in\mathcal{S}$, we have

$$\begin{aligned}\det(A+B) &= \sum_{s\in\mathcal{S}_d}(-1)^{\mathrm{sign}(s)}\prod_{i=1}^d(a^{s(i),i}+b^{s(i),i})\\ &= \sum_{I\subset\{1,\cdots,d\}}\ \sum_{s\in\mathcal{S}_d}(-1)^{\mathrm{sign}(s)}\prod_{i\in I}a^{s(i),i}\prod_{i\notin I}b^{s(i),i} = \sum_{I\subset\{1,\cdots,d\}}\det(G_{A,B}^{(I,I^c)}).\end{aligned}$$

This readily implies that if (5.3) holds for some m, it also holds for $m+1$. Since (5.3) is obvious for $m=1$, the result follows by induction on m. □

Lemma 5.2 *There is a constant K such that, for all $r=0,\cdots,d$, all $h\in(0,1]$ and all $A,B,C,D\in\mathcal{M}$ with $\mathrm{rank}(A)\le r$ we have, with $\Lambda=\|A\|+\|B\|+\|C\|+\|D\|$ and with the convention $\gamma_{-1}(A,B)=0$:*

$$\begin{aligned}\big|\det(A+hB+h^2C+h^2D)-h^{d-r}\gamma_r(A,B)-h^{d-r+1}(\gamma_{r-1}(A,B)+\gamma'_r(A,B,C))\big|\\ \le Kh^{r-d+1}\Lambda^{d-1}(h\Lambda+\|D\|),\end{aligned}\tag{5.4}$$

$$\begin{aligned}\Big|\tfrac{1}{h^{2d-2r}}\ \det(A+hB+h^2C+h^2D)^2-\gamma_r(A,B)^2\\ -2h\,\gamma_r(A,B)(\gamma_{r-1}(A,B)+\gamma'_r(A,B,C))\Big|\le Kh\Lambda^{2d-1}(h\Lambda+\|D\|).\end{aligned}\tag{5.5}$$

Proof. Let $\mathbf{p}\in\mathcal{P}_4$ and $\mathbf{I}\in\mathcal{I}_{\mathbf{p}}$. Then $\det(G^{\mathbf{I}}_{A,hB,h^2C,h^2D})=h^{p_2+2p_3+2p_3}\det(G^{\mathbf{I}}_{A,B,C,D)})$ vanishes when $p_1>r$, and has absolute value smaller than $Kh^{p_2+2p_3+2p_4}\Lambda^{d-p_4}\|D\|^{p_4}$. Then (5.4) readily follows from (5.3), and by taking squares in (5.4) we deduce (5.5). □

With the same notation, and if further $A',B',C',D'\in\mathcal{M}$ with $\mathrm{rank}(A')\le r$ also and $\Lambda'=\|A'\|+\|B'\|+\|C'\|+\|D'\|$, and $h'\in(0,1]$, the same argument shows that

$$\begin{aligned}\Big|\tfrac{1}{(hh')^{2d-2r}}\ \det(A+hB+h^2C+h^2D)^2\ \det(A'+h'B'+h'^2C'+h'^2D')^2\\ -\gamma_r(A,B)^2\,\gamma_r(A',B')^2\Big|\le K(h+h')(\Lambda\,\Lambda')^{2d}.\end{aligned}\tag{5.6}$$

5.2 Proof of Lemma 3.1.

1) The results about $\Gamma_r(\underline{u})$. We write V_i and $\overline{V}_i$ for the d-dimensional variables whose components are respectively the d first and the d last components of $\Psi(\underline{u},1)_i$, for which we can take $\overline{W}=W$ and $\overline{W}'=W'$, and we set $A=\mathrm{mat}(V_1,\cdots,V_d)$ and $B=\mathrm{mat}(\overline{V}_1,\cdots,\overline{V}_d)$. If $\Delta_jW^{(\prime)}=W^{(\prime)}_j-W^{(\prime)}_{j-1}$, we have

$$V^l_i=\sum_{m=1}^q\alpha^{lm}\Delta_iW^m,\quad \overline{V}^l_i=a^l+\sum_{m=1}^d\beta^{lm}\Delta_iW'^m+\sum_{m,k=1}^q\gamma^{lkm}h_{i,km}(W),\tag{5.7}$$

where each $h_{i,lm}$ is a function of the path of W. Note also that $\overline{F}_r(\underline{u},1)=\gamma_r(A,B)^2$.

Assuming first that the rank of α is (strictly) smaller than r, we observe that the rank of A is also smaller than r, implying by (5.1) that $\gamma_r(A,B)=0$, hence $\Gamma_r(\underline{u})=0$.

Next we assume that the rank of α is r, and proceed to prove $\Gamma_r(\underline{u}) > 0$. We first simplify the problem as follows. The matrix β is invertible and the rank of $\beta^{-1}\alpha\alpha^*\beta^{-1,*}$ is r, so we can write $\beta^{-1}\alpha = \Pi\Lambda$, where $\Pi \in \mathcal{M}$ is an orthonormal matrix and $\Lambda \in \mathcal{M}$ is a diagonal matrix whose diagonal entries λ_j satisfy $\lambda_j \neq 0$ if $j \leq r$ and $\lambda_j = 0$ otherwise. Then, setting $V'_j = \Pi^*\beta^{-1}V_j$ and $\overline{V}'_j = \Pi^*\beta^{-1}\overline{V}_j$, the sequence $(V'_j, \overline{V}'_j)$ has the form (5.7), upon replacing W' by Π^*W' (another Brownian motion) and $\underline{u} = (\alpha, \beta, \gamma, a)$ by $\underline{u}' = (\alpha', J, \gamma', a')$, where J is the identity in $\mathcal{M}$ and $\alpha' = \Pi^*\beta^{-1}\alpha = \Lambda$ and $\gamma'^{ijl} = \sum_{m=1}^d (\Pi^*\beta^{-1})^{jm}\gamma^{mjl}$ and $a' = \Pi^*\beta^{-1}a$. Furthermore, $A' = \mathrm{mat}(V'_1, \cdots, V'_d) = \Pi^*\beta^{-1}A$ and $B' = \mathrm{mat}(\overline{V}'_1, \cdots, \overline{V}'_d) = \Pi^*\beta^{-1}B$, implying $\gamma_r(A', B') = \det(\Pi^*\beta^{-1})\gamma_r(A, B)$, which in turn yields $\Gamma_r(\underline{u}) = \frac{1}{\det(\beta)^2}\,\Gamma_r(\underline{u}')$, because $\det(\Pi) = 1$.

In other words, it is enough to prove the result when $\beta = J$ and $\alpha = \Lambda$ is diagonal as above, and below we assume this. The two matrices A and B can thus be realized as

$$A = \begin{pmatrix} \lambda_1\Phi_1^1 & \cdots & \lambda_1\Phi_d^1 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \lambda_d\Phi_1^d & \cdots & \lambda_d\Phi_d^d \end{pmatrix}, \qquad B = \begin{pmatrix} \Upsilon_1^1 + \Theta_1^1 & \cdots & \Upsilon_d^1 + \Theta_d^1 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \Upsilon_1^d + \Theta_1^d & \cdots & \Upsilon_d^d + \Theta_d^d \end{pmatrix}, \tag{5.8}$$

where all Φ_j^i and Υ_j^i are i.i.d. $\mathcal{N}(0,1)$ and the variables Θ_j^i are independent of the Υ_m^l's (note that we have incorporated the constant a^i in each variable Θ_j^i).

Let $\mathcal{J}_r$ be the class of all subsets of $\{1, \cdots, d\}$ with r points. Since $\lambda_j \neq 0$ if $j \leq r$ and $\lambda_j = 0$ otherwise, we see that, if $\mathbf{I} = (I, I^c)$ with $I = \{j_1 < \cdots < j_r\} \in \mathcal{J}_r$ and $I^c = \{j'_1 < \cdots < j'_{d-r}\}$, we have $\det(G_{A,B}^{\mathbf{I}}) = \varepsilon_I \det(A_I)\det(B_I)$, where A_I and B_I are the $r \times r$ and $(d-r) \times (d-r)$ matrices with entries $A_I^{l,m} = A^{j_l,m}$ and $B_I^{l,m} = B^{j'_l, r+l}$, and ε_I takes values in $\{-1, 1\}$. Thus

$$\gamma_r(A, B) = \sum_{I \in \mathcal{J}_r} \varepsilon_I \det(A_I)\,\det(B_I).$$

In this sum we single out the I's which contain d, and those which do not, and for the former ones the product $\det(A_I)\det(B_I)$ does not depend of the vector Υ_d. For those which do not contain d, we develop $\det(B_I)$ along the last column, which involves the determinants of the matrices $B_{I,i}$ which are the restrictions of B to the last $d-r$ lines except i, and to the column indexed by the complement I^c of I, except d. We thus get

$$\gamma_r(A, B) = Z + \sum_{i=1}^{d-r} (-1)^i \big(\Upsilon_d^{r+i} + \Theta_d^{r+i}\big) Z'_i, \quad Z'_i = \sum_{I \in \mathcal{J}_r,\, d \notin I} \varepsilon_I \det(A_I)\,\det(B_{I,i}), \tag{5.9}$$

where Z and all Z'_i and Θ_d^{r+i} are independent of the vector Υ_d. Since this random vector Υ_d has a density, it follows that the variable $\gamma_r(A, B)$ also has a density, provided $Z'_i \neq 0$ a.s. for at least one value of i.

At this stage, we observe that Z'_i has exactly the same structure as $\gamma_r(A, B)$, except that the dimension of each $B_{I,i}$ is $(d-r-1) \times (d-r-1)$ instead of $(d-r) \times (d-r)$, and that the last column of the original problem has totally disappeared. We can repeat the argument, to obtain that Z'_i has a density and is thus a.s. non-vanishing, as soon as some similar quantity (where the last two columns of the original problem no longer show up) is a.s. non-vanishing. Then, after an obvious induction, we deduce that $\gamma_r(A,B)$ has a density as soon as $\det(A_I)\neq 0$ a.s. for $I=\{1,\cdots,r\}$.

However, since the entries of this last A_I are $\lambda_i\Phi_j^i$ for $i,j=1,\cdots,r$, and all those λ_i are non zero, it is well known (and also a simple consequence of the previous proof, in which we develop $\det(A_I)$ according to its last column and perform the same induction procedure) that $\det(A_I)$ has a density. This indeed shows us that $\gamma_r(A,B)$ has a density, hence $\mathbb{E}(\gamma_r(A,B)^2)>0$ and the proof of the first part of (3.5) is complete. □

2) The results about $\Gamma'_r(\underline{u})$ and $\Gamma''_r(\underline{u})$. When the rank of α is smaller than r, we have seen that, with the previous notation, $\gamma_r(A,B)=0$, hence also $\Gamma'_r(\underline{u})=\Gamma''_r(\underline{u})=0$.

Next, we turn to the case when the rank of α is r. Exactly as in the previous proof, it suffices to show the result when $\underline{u}=(\Lambda,J,\gamma,a)$. Recalling that $\overline{F}_r(\underline{u},1)$ and $\overline{F}_r(\underline{u},2)$ have the same law, we have $\overline{\mathbb{E}}((\overline{F}_r(\underline{u},1)-\overline{F}_r(\underline{u},2))^2)=2(\Gamma'_r(\underline{u})-\Gamma''_r(\underline{u}))$ and thus the second part of (3.5) holds unless $\overline{F}_r(\underline{u},1)=\overline{F}_r(\underline{u},2)$ a.s.

With the previous notation, we have $\overline{F}_r(\underline{u},1)=\gamma_r(A,B)^2$, and also $\overline{F}_r(\underline{u},2)=\gamma_r(\overline{A},\overline{B})^2$, where $\overline{A}$ and $\overline{B}$ are given again by (5.8) with the same λ_j's, and random vectors $(\overline{\Phi}_j,\overline{\Upsilon}_j,\overline{\Theta}_j)$ having globally the same distribution as $(\Phi_j,\Upsilon_j,\Theta_j)$ (which may of course be defined for $j>d$): these two families of vector are not independent, and we have in fact

$$\overline{\Phi}_j=\frac{1}{\sqrt{2}}\,(\Phi_{2j-1}+\Phi_{2j}),\quad \overline{\Upsilon}_j=\frac{1}{\sqrt{2}}\,(\Upsilon_{2j-1}+\Upsilon_{2j}),\tag{5.10}$$

plus a more complicated relation relating $\overline{\Theta}_j$ with the $\Theta_{j'}$ for $j'\leq 2j$ and the vector a. What we need to prove is then $\mathbb{P}(|\gamma_r(A,B)|\neq|\gamma_r(\overline{A},\overline{B})|)>0$.

We have (5.9), and also, by the same argument,

$$\gamma_r(\overline{A},\overline{B})=\overline{Z}+\sum_{i=1}^{d-r}(-1)^i\Big(\frac{\Upsilon_{2d+1}^{r+i}+\Upsilon_{2d}^{r+i}}{\sqrt{2}}+\Theta_d^{r+i}\Big)\overline{Z}'_i,\quad \overline{Z}'_i=\sum_{I\in\mathcal{J}_r,\,d\notin I}\varepsilon_I\det(\overline{A}_I)\,\det(\overline{B}_{I,i}),$$

where we have also used the second part of (5.10). Here, the vector Υ_{2d} has a density and is independent of all other terms showing in the above expression, and also independent of $\gamma_r(A,B)$. Therefore, $|\gamma_r(A,B)|\neq|\gamma_r(\overline{A},\overline{B})|$ almost surely on the set $\{\overline{Z}'_i\neq 0\}$. Now, $\overline{Z}'_i$ is the same as Z'_i, upon replacing (A,B) by $(\overline{A},\overline{B})$ everywhere, hence the previous proof shows that indeed $\overline{Z}'_i\neq 0$ a.s. This shows that in fact $\mathbb{P}(|\gamma_r(A,B)|\neq|\gamma_r(\overline{A},\overline{B})|)=1$, thus ending the proof of the second part of (3.5). □

5.3 Some stochastic calculus preliminaries.

We assume (H) and, by localization (see e.g. section 3 in [1]), we may also assume that all processes X_t, σ_t, b_t, a_t, v_t, a'_t, v'_t, a''_t, v''_t are uniformly bounded in (ω,t). The constants are always written as K, or K_p if we want to stress the dependency on an additional parameter p, and never depend on t,i,n,j. For any process Y, we use the following simplifying notation:

$$\mathcal{F}_i^n = \mathcal{F}_{2id\Delta_n}, \qquad Y_i^n = Y_{2id\Delta_n}. \tag{5.11}$$

For all $p, t, s > 0$, we have by Burkholder-Gundy inequality

$$\mathbb{E}\big(\sup_{u\in[0,s]} |Y_{t+u} - Y_t|^p \mid \mathcal{F}_t\big) \leq K_q s^{q/2} \quad \text{if } Y = X, \sigma, b, v. \tag{5.12}$$

We set

$$\eta_{t,s} = \sup_{u\in[0,s],\ Y=a,v',v''} \|Y_{t+u} - Y_t\|^2, \qquad \eta_i^n = \sqrt{\mathbb{E}(\eta_{2id\Delta_n,2d\Delta_n} \mid \mathcal{F}_i^n)}. \tag{5.13}$$

Lemma 5.3 *For all $t > 0$ we have $\Delta_n \mathbb{E}\big(\sum_{i=0}^{[t/2d\Delta_n]-1} \eta_i^n\big) \to 0$.*

Proof. It suffices to prove the result separately when $Y = a$ or $Y = v'$ or $Y = v''$. Set $\gamma_t^n = \sup_{s\in(0,4d\Delta_n]} \|Y_{t+s} - Y_t\|^2$, so $\mathbb{E}((\eta_i^n)^2)$ is smaller than $\mathbb{E}(\gamma_0^n)$ when $i = 0$ and than $\frac{1}{2d\Delta_n}\int_{2(i-1)d\Delta_n}^{2id\Delta_n} \mathbb{E}(\gamma_s^n)\, ds$ when $i \geq 1$. Hence by the Cauchy-Schwarz inequality,

$$\Delta_n \mathbb{E}\Big(\sum_{i=1}^{[t/2d\Delta_n]-1} \eta_i^n\Big) \leq \frac{\sqrt{t}}{\sqrt{2d}} \Big(\mathbb{E}\Big(\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} (\eta_i^n)^2\Big)\Big)^{1/2} \leq K_t \Big(\mathbb{E}\Big(\gamma_0^n + \int_0^t \gamma_s^n\, ds\Big)\Big)^{1/2}.$$

We have $\gamma_s^n \leq K$, whereas the càdlàg property of Y yields that $\gamma_s^n(\omega) \to 0$ for all ω, and all s except for countably many strictly positive values (depending on ω). Then, the claim follows by the dominated convergence theorem. □

The proof of Theorem 3.2 is based on a decomposition of the increments $Z^n_{(2id+\kappa j)\Delta_n} - Z^n_{(2id+\kappa(j-1))\Delta_n}$. In order to understand better this decomposition, we first deduce from (2.8) that, for any $z \leq t \leq s$, and with vector notation,

$$\int_t^s b_u\, du = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \qquad \int_t^s \sigma_u\, dW_u = \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11},$$

where

$$\alpha_1 = b_z(s-t), \quad \alpha_2 = \int_t^s \Big(\int_z^u a'_w\, dw\Big) du \quad \alpha_3 = v'_z \int_t^s (W_u - W_z)\, du,$$
$$\alpha_4 = \int_t^s \Big(\int_z^u (v'_w - v'_z)\, dW_w\Big)\, du$$
$$\alpha_5 = \sigma_z(W_s - W_t), \quad \alpha_6 = a_z \int_t^s (u-z) dW_u, \quad \alpha_7 = \int_t^s \Big(\int_z^u (a_w - a_z)\, dw\Big)\, dW_u$$
$$\alpha_8 = v_z \int_t^s (W_u - W_z)\, dW_u, \quad \alpha_9 = \int_t^s \Big(\int_z^u \Big(\int_z^w a''_r\, dr\Big)\, dW_w\Big)\, dW_u$$
$$\alpha_{10} = v''_z \int_t^s \Big(\int_z^u (W_w - W_z)\, dW_w\Big)\, dW_u, \quad \alpha_{11} = \int_t^s \Big(\int_z^u \Big(\int_z^w (v''_r - v''_z)\, dW_r\Big)\, dW_w\Big)\, dW_u.$$

A repeated use of the Burkholder-Gundy and Hölder inequalities shows that, in view of our assumptions on the various coefficients, and for any $p \geq 1$:

$$\mathbb{E}(|\alpha_j|^p \mid \mathcal{F}_z) \leq \begin{cases} K_p(s-z)^{p/2} & \text{if } j = 5 \\ K_p(s-z)^p & \text{if } j = 1, 8 \\ K_p(s-z)^{3p/2} & \text{if } j = 3, 6, 10 \\ K_p(s-z)^{2p} & \text{if } j = 2, 9 \\ K_p(s-z)^{3p/2}\, \mathbb{E}(\eta^p_{z,s-z} \mid \mathcal{F}_z) & \text{if } j = 4, 7, 11. \end{cases}$$

We can then apply the previous decomposition with $z = 2id\Delta_n$, $t = (2id+\kappa(j-1))\Delta_n$ and $s = (2id+\kappa j)\Delta_n$, and add the increment of the process X', to obtain

$$\frac{Z^{n,\kappa}_{(2id+j\kappa)\Delta_n} - Z^{n,\kappa}_{(2id+(j-1)\kappa)\Delta_n}}{\sqrt{\kappa\Delta_n}} = \alpha^{n,\kappa}_{i,j} + \sqrt{\kappa\Delta_n}\,\beta^{n,\kappa}_{i,j} + \kappa\Delta_n\,\gamma^{n,\kappa}_{i,j} + \Delta_n\,\delta^{n,\kappa}_{i,j} \tag{5.14}$$

for $\kappa = 1,2$, and where (explicitly writing the components)

$$\begin{aligned}
\alpha^{n,\kappa,l}_{i,j} &= \tfrac{1}{\sqrt{\kappa\Delta_n}} \sum_{m=1}^q \sigma^{n,lm}_i (W^m_{(2id+\kappa j)\Delta_n} - W^m_{(2id+\kappa(j-1))\Delta_n}) \\
\beta^{n,\kappa,l}_{i,j} &= b^{n,l}_i + \tfrac{1}{\kappa\Delta_n} \sum_{m,k=1}^q v^{n,lmk}_i \int_{(2id+\kappa(j-1))\Delta_n}^{(2id+\kappa j)\Delta_n} (W^k_s - W^k_{2id\Delta_n})\, dW^m_s \\
&\quad + \tfrac{1}{\sqrt{\kappa\Delta_n}} \sum_{m=1}^d \widetilde{\sigma}^{lm} (W'^m_{(2id+\kappa j)\Delta_n} - W'^m_{(2id+\kappa(j-1))\Delta_n}) \\
\gamma^{n,\kappa,l}_{i,j} &= \tfrac{1}{(\kappa\Delta_n)^{3/2}} \Big(\sum_{m=1}^q a^{n,lm}_i \int_{(2id+\kappa(j-1))\Delta_n}^{(2id+\kappa j)\Delta_n} \big(s - 2id\Delta_n\big)\, dW^m_s \\
&\quad + \sum_{m=1}^q v'^{n,lm}_i \int_{(2id+\kappa(j-1))\Delta_n}^{(2id+\kappa j)\Delta_n} (W^m_s - W^m_{2id\Delta_n})\, ds \\
&\quad + \sum_{m,l,k=1}^q v''^{n,mlk}_i \int_{(2id+\kappa(j-1))\Delta_n}^{(2id+\kappa j)\Delta_n} \big(\int_{(2id+\kappa(j-1))\Delta_n}^s (W^k_u - W^k_{2id\Delta_n})\, dW^l_u \big) dW^m_s \Big)
\end{aligned}$$

and $\delta^{n,\kappa}_{i,j}$ is a remainder term, and for $p \geq 1$ we have the estimates when $j \leq 2d$ if $\kappa = 1$ and $j \leq d$ when $\kappa = 2$ (recalling $\eta_{t,s} \leq K$):

$$\begin{aligned}
&\mathbb{E}\big(\|\alpha^{n,\kappa}_{i,j}\|^p + \|\beta^{n,\kappa}_{i,j}\|^p + \|\gamma^{n,\kappa}_{i,j}\|^p \mid \mathcal{F}^n_i\big) \leq K_p \\
&\mathbb{E}\big(\|\delta^{n,\kappa}_{i,j}\|^p \mid \mathcal{F}^n_i\big) \leq K_p\,(\Delta_n^{p/2} + (\eta^n_i)^{2\wedge p}\big) \leq K_p.
\end{aligned} \tag{5.15}$$

We end these preliminaries with a lemma which compares $S^{n,\kappa}$ for $\kappa = 1,2$ with the following processes:

$$\begin{aligned}
&S(r)^{n,\kappa}_t = 2d\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \gamma_r(A^{n,\kappa}_i, B^{n,\kappa}_i)^2, \quad \text{where} \\
&A^{n,\kappa}_i = \mathrm{mat}(\alpha^{n,\kappa}_{i,1}, \cdots, \alpha^{n,\kappa}_{i,d}), \qquad B^{n,\kappa}_i = \mathrm{mat}(\beta^{n,\kappa}_{i,1}, \cdots, \beta^{n,\kappa}_{i,d}).
\end{aligned} \tag{5.16}$$

It also compares $V^{n,\kappa,\kappa'}$ of (3.13) with

$$V(r)^{n,\kappa,\kappa'}_t = 4d^2\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \gamma_r(A^{n,\kappa}_i, B^{n,\kappa}_i)^2\, \gamma_r(A^{n,\kappa'}_i, B^{n,\kappa'}_i)^2. \tag{5.17}$$

Lemma 5.4 *If $r_t(\omega) \leq r$ identically for some $r \in \{0, \cdots, d\}$, we have for $\kappa, \kappa' = 1,2$:*

$$\frac{1}{\sqrt{\Delta_n}} \Big(\frac{1}{(\kappa\Delta_n)^{d-r}}\, S^{n,\kappa} - S(r)^{n,\kappa} \Big) \stackrel{u.c.p.}{\Longrightarrow} 0 \tag{5.18}$$

and

$$\frac{1}{(\kappa\kappa'\,\Delta_n^2)^{d-r}}\, V^{n,\kappa,l} - V(r)^{n,\kappa,\kappa'} \stackrel{u.c.p.}{\Longrightarrow} 0 \tag{5.19}$$

Proof. We denote by $\xi_i^{n,\kappa}$ the ith summand in the definition (2.12) of $S_t^{n,\kappa}$. Besides the matrices in (5.16), we also define

$$C_i^{n,\kappa} = \mathrm{mat}(\gamma_{i,1}^{n,\kappa}, \cdots, \gamma_{i,d}^{n,\kappa}), \qquad D_i^{n,\kappa} = \mathrm{mat}(\delta_{i,1}^{n,\kappa}, \cdots, \delta_{i,d}^{n,\kappa}).$$

We start with (5.18). Applying (5.5) with $h = \sqrt{\kappa\Delta_n}$, the fact that each $A_i^{n,\kappa}$ has at most rank r (because $r_t \leq r$), and the estimates (5.15) plus the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\tfrac{1}{(\kappa\Delta_n)^{d-r}}\, \xi_i^{n,\kappa} = \gamma_r(A_i^{n,\kappa}, B_i^{n,\kappa})^2 + 2\sqrt{\kappa\Delta_n}\, \zeta_i^{n,\kappa} + \widetilde{\zeta}_i^{n,\kappa}, \quad \text{where} \\ &\quad \zeta_i^{n,\kappa} = \gamma_r(A_i^{n,\kappa}, B_i^{n,\kappa})(\gamma_{r-1}(A_i^{n,\kappa}, B_i^{n,\kappa}) + \gamma'(A_i^{n,\kappa}, B_i^{n,\kappa}, C_i^{n,\kappa})) \\ &\quad \mathbb{E}(|\widetilde{\zeta}_i^{n,\kappa}|) \leq K\Delta_n + K\sqrt{\Delta_n}\, \mathbb{E}(\eta_i^n). \end{aligned}$$

In view of Lemma 5.3, $\sqrt{\Delta_n} \sum_{i=0}^{[t/2d\Delta_n]-1} \widetilde{\zeta}_i^{n,\kappa} \stackrel{\text{u.c.p.}}{\Longrightarrow} 0$. Since $S_t^{n,\kappa} = 2d\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \xi_i^{n,\kappa}$, it remains to prove that $\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \zeta_i^{n,\kappa} \stackrel{\text{u.c.p.}}{\Longrightarrow} 0$.

For this purpose we use the decomposition $\zeta_i^{n,\kappa} = \zeta_i'^{n,\kappa} + \zeta_i''^{n,\kappa}$, where $\zeta_i'^{n,\kappa} = \mathbb{E}(\zeta_i^{n,\kappa} \mid \mathcal{F}_i^n)$. By Doob's inequality, (5.15) and the fact that $\zeta_i''^n$ is $\mathcal{F}_{i+1}^n$-measurable, we have

$$\mathbb{E}\Big(\sup_{s\leq t}\Big(\sum_{i=0}^{[s/2d\Delta_n]-1} \zeta_i''^{n,\kappa}\Big)^2\Big) \leq 2^{d+1}\,\mathbb{E}\Big(\sum_{i=0}^{[t/2d\Delta_n]-1} |\zeta_i^n|^2\Big) \leq \frac{Kt}{\Delta_n}.$$

Thus $\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \zeta_i''^{n,\kappa} \stackrel{\text{u.c.p.}}{\Longrightarrow} 0$, and the result will hold if we can prove that $\zeta_i'^{n,\kappa} = 0$. We even prove the stronger statement that $\mathbb{E}(\zeta_i^{n,\kappa} \mid \mathcal{G}^{W'} \vee \mathcal{F}_i^n) = 0$, where $\mathcal{G}^{W'}$ is the σ-field generated by the whole process W', and this is implied by

$$\begin{aligned} &\mathbf{I} \in \mathcal{I}_{(r,d-r)}, \quad \mathbf{I}' \in \mathcal{I}_{(r-1,d-r+1)}, \quad \mathbf{I}'' \in \mathcal{I}_{(r,d-r-1,1)} \quad \Longrightarrow \\ &\quad \mathbb{E}\big(\det(G^{\mathbf{I}}_{A_i^{n,\kappa},B_i^{n,\kappa}})\,\det(G^{\mathbf{I}'}_{A_i^{n,\kappa},B_i^{n,\kappa}}) \mid \mathcal{G}^{W'} \vee \mathcal{F}_i^n\big) = 0 \\ &\quad \mathbb{E}\big(\det(G^{\mathbf{I}}_{A_i^{n,\kappa},B_i^{n,\kappa}})\,\det(G^{\mathbf{I}''}_{A_i^{n,\kappa},B_i^{n,\kappa},C_i^{n,\kappa}}) \mid \mathcal{G}^{W'} \vee \mathcal{F}_i^n\big) = 0. \end{aligned} \tag{5.20}$$

The variables $\alpha_{i,j}^{n,\kappa,l}$, $\beta_{i,j}^{n,\kappa,l}$ and $\gamma_{i,j}^{n,\kappa,l}$ have the form $\Phi(\omega, (W(\omega)_{2id\Delta_n+t} - W(\omega)_{2id\Delta_n})_{t\geq 0})$, with Φ a $(\mathcal{G}^{W'} \vee \mathcal{F}_i^n) \otimes \mathcal{C}^d$-measurable function on $\Omega \times C(\mathbb{R}_+, \mathbb{R}^d)$, where $C(\mathbb{R}_+, \mathbb{R}^d)$ is the set of all continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$ and $\mathcal{C}^d$ is its Borel σ-field for the local uniform topology. When $\Phi = \alpha_{i,j}^{n,\kappa,l}$ or $\Phi = \gamma_{i,j}^{n,\kappa,l}$, the map $x \mapsto \Phi(\omega, x)$ is odd, in the sense that $\Phi(\omega, -x) = \Phi(\omega, x)$, and it is even when $\Phi = \beta_{i,j}^{n,\kappa,l}$.

In (5.20), the three variables $\det(G^{\mathbf{I}}_{A_i^{n,\kappa},B_i^{n,\kappa}})$, $\det(G^{\mathbf{I}'}_{A_i^{n,\kappa},B_i^{n,\kappa}})$, $\det(G^{\mathbf{I}''}_{A_i^{n,\kappa},B_i^{n,\kappa},C_i^{n,\kappa}})$ are associated with three functions Ψ, Ψ', Ψ'' of the same type. What precedes yields that Ψ is even (resp. odd) if r is even (resp. odd), and both Ψ' and Ψ'' are even (resp. odd) if r is odd (resp. even). Consequently, the products $\Psi\Psi'$ and $\Psi\Psi''$ are odd in all cases. Since the $\mathcal{G}^{W'} \vee \mathcal{F}_i^n$-conditional law of $(W_{2id\Delta_n+t} - W_{2id\Delta_n})_{t\geq 0}$ is invariant by the map $x \mapsto -x$ on $C(\mathbb{R}_+, \mathbb{R}^d)$, we deduce (5.20), hence (5.18) holds.

Finally, we turn to (5.19). Let θ_i^n be the ith summand in the right side of (3.13), for κ,κ' fixed. We can apply (5.6) with $h=\sqrt{\kappa\Delta_n}$ and $h=\sqrt{\kappa'\Delta_n}$, and (5.14) and (5.15) again, to get

$$\mathbb{E}\Big(\Big|\frac{1}{(\kappa\kappa'\Delta_n^2)^{d-r}}\,\theta_i^n-\gamma_r(A_i^{n,\kappa},B_i^{n,\kappa})^2\,\gamma_r(A_i^{n,\kappa'},B_i^{n,\kappa'})^2\Big|\Big)\leq K\sqrt{\Delta_n}.$$

(5.19) follows, and the proof is complete. □

5.4 Proof of Theorem 3.2, Corollary 3.3, and Proposition 3.4.

1) Observe that, with the notation (3.2) and (3.3), and upon taking

$$\underline{u}_i^n=(\sigma_i^n,\widetilde{\sigma},v_i^n,b_i^n),\quad \overline{W}_t=\frac{W_{(2id+t)\Delta_n}-W_{2id\Delta_n}}{\sqrt{\Delta_n}},\quad \overline{W}'_t=\frac{W'_{(2id+t)\Delta_n}-W'_{2id\Delta_n}}{\sqrt{\Delta_n}},\tag{5.21}$$

we have $\gamma_r(A_i^{n,\kappa},B_i^{n,\kappa})^2=\overline{F}(\underline{u}_i^n,\kappa)$. We consider the two-dimensional variables ξ_i^n with components

$$\xi_i^{n,\kappa}=2d\sqrt{\Delta_n}\,\big(\gamma_r(A_i^{n,\kappa},B_i^{n,\kappa})^2-\Gamma_r(u_i^n)\big),\qquad \kappa=1,2.\tag{5.22}$$

Since $\underline{u}_i^n$ is $\mathcal{F}_i^n$-measurable, whereas the processes $\overline{W}$ and $\overline{W}'$ above are independent of $\mathcal{F}_i^n$, we deduce from (3.4), and from (5.15) for the estimate below, that

$$\begin{array}{l}\mathbb{E}(\xi_i^{n,\kappa}\mid\mathcal{F}_i^n)=0,\qquad \mathbb{E}(\|\xi_i^n\|^4\mid\mathcal{F}_i^n)\leq K\Delta_n^2\\ \mathbb{E}(\xi_i^{n,\kappa}\,\xi_i^{n,\kappa'}\mid\mathcal{F}_i^n)=\left\{\begin{array}{ll}4d^2\Delta_n\,\Gamma'_r(\underline{u}_i^n) & \text{if } \kappa=\kappa'\\ 4d^2\Delta_n\,\Gamma''_r(\underline{u}_i^n) & \text{if } \kappa\neq\kappa'\end{array}\right.\end{array}\tag{5.23}$$

□

2) By (5.18), for Theorem 3.2 it is enough to prove the stable convergence $U'(r)^n\stackrel{\mathcal{L}-s}{\Longrightarrow}\mathcal{U}(r)$, where $U'(r)^n$ is the two-dimensional process with components $U'(r)^{n,\kappa}=\frac{1}{\sqrt{\Delta_n}}\,(S(r)^{n,\kappa}-S(r))$ and the quantity $S(r)^{n,\kappa}$ is defined in (5.16). We have $U'(r)^n=Y^n+Y'^n$, where

$$\begin{array}{l}Y_t^n=\sum_{i=0}^{[t/2d\Delta_n]-1}\xi_i^n\\ Y_t'^n=\frac{1}{\sqrt{\Delta_n}}\Big(2d\Delta_n\sum_{i=0}^{[t/2d\Delta_n]-1}\Gamma_r(\sigma_i^n,\widetilde{\sigma},v_i^n,b_i^n)-\int_0^t\Gamma_r(\sigma_s,\widetilde{\sigma},v_s,b_s)\,ds\Big),\end{array}$$

and ξ_i^n is given in (5.22). Since the three processes σ,v,b are Itô semimartingales, whereas Γ_r is a C^∞ function, it is well known that $Y'^n\stackrel{\text{u.c.p.}}{\Longrightarrow}0$ (see e.g. section 8 in [1]). We are thus left to prove that

$$Y^n\stackrel{\mathcal{L}-s}{\Longrightarrow}\mathcal{U}(r).\tag{5.24}$$

By virtue of the first two parts of (5.23), a standard CLT for triangular arrays of martingale (see [8, Theorem IX.7.28]) increments shows that, for (5.24) to hold, it suffices to show the next two properties:

$$\sum_{i=0}^{[t/2d\Delta_n]-1}\mathbb{E}(\xi_i^{n,\kappa}\,\xi_i^{n,\kappa'}\mid\mathcal{F}_i^n)\ \stackrel{\mathbb{P}}{\longrightarrow}\ V(r)_t^{\kappa\kappa'}\tag{5.25}$$

$$\sum_{i=0}^{[t/2d\Delta_n]-1} \mathbb{E}\big(\xi_i^n \left(N_{(2(i+1)d\Delta_n} - N_{2id\Delta_n}\right) \mid \mathcal{F}_i^n\big) \stackrel{\mathbb{P}}{\longrightarrow} 0 \tag{5.26}$$

for all $t > 0$ and for any bounded martingale N orthogonal to (W, W') and also for $N = W^m$ or $N = W'^m$ for any m.

The last part of (5.23) and the càdlàg property of σ, v, b, plus the fact that Γ'_r and Γ''_r are polynomials, immediately gives us (5.25) by Riemann integration.

The proof of (5.26) is also standard: By construction, ξ_i^n is a two-dimensional variable of the form $\Phi(\omega, (W_{2id\Delta_n+t}(\omega) - W_{2id\Delta_n}(\omega))_{t\geq 0}, (W'_{2id\Delta_n+t}(\omega) - W'_{2id\Delta_n}(\omega))_{t\geq 0})$ similar to the functions occurring in Lemma 5.4, and since in the definition of ξ_i^n one takes squared determinants, all these functions Φ are globally even in the sense that $\Phi(\omega, x, y) = \Phi(\omega, -x, -y)$ for any two d-dimensional functions x, y on $\mathbb{R}_+$. So, on the one hand, after multiplying the function Φ corresponding to ξ_i^n by $x^m(2d\Delta_n)$ or $y^m(2d\Delta_n)$, one gets an odd function, and (5.26) when $N = W^m$ or $N = W'^m$ follows. On the other hand, by the representation theorem one can write ξ_i^n as the sum of two integrals over $(2id\Delta_n, 2(i+1)d\Delta_n]$ with respect to W and W', for suitable predictable integrands; thus when N is orthogonal to W and W', the increment $N_{(2(i+1)d\Delta_n} - N_{2id\Delta_n}$ has $\mathcal{F}_i^n$-conditional correlation 0 with both those integrals, thus yielding (5.26) again.

Therefore, the proof of Theorem 3.2 is complete. □

3) A simple calculation shows that

$$\widehat{R}(n,T) - r = \frac{1}{\log 2} \log \frac{1 + \sqrt{\Delta_n}\, U(r)_T^{n,1}/S(r)_T}{1 + \sqrt{\Delta_n}\, U(r)_T^{n,2}/S(r)_T} \quad \text{if} \quad S(r)_T > 0,$$

hence on the set Ω_T^r. Since the sequence $U(r)_T^n$ is tight, it follows from a Taylor expansion that

$$\frac{1}{\sqrt{\Delta_n}} \big(\widehat{R}(n,T) - r\big) - \frac{1}{S(r)_T \log 2} \big(U(r)_T^{n,1} - U(r)_T^{n,2}\big) \stackrel{\mathbb{P}}{\longrightarrow} 0 \tag{5.27}$$

on Ω_T^r again. Then Corollary 3.3 follows from Theorem 3.2, upon observing that the $\mathcal{F}$-conditional variance of $U(r)_T^1 - U(r)_T^2$ is the numerator of the right side if (3.12). □

4) Now we turn to the proof of (3.14), and by (5.19) it suffices to prove the convergence of $V(r)^{n,\kappa,\kappa'}$. We suppose that $\kappa = \kappa'$, the proof in the case $\kappa \neq \kappa'$ being analogous. We set

$$\eta_i^n = \gamma_r(A_i^{n,\kappa}, B_i^{n,\kappa})^4, \qquad \eta_i'^n = \mathbb{E}(\eta_i^n \mid \mathcal{F}_i^n), \qquad \eta_i''^n = \eta_i^n - \eta_i'^n.$$

As for (5.23), we deduce from (3.4) and (5.15) that

$$\eta_i'^n = 2d\big(\Gamma'_r(\underline{u}_i^n) - \Gamma_r(\underline{u}_i^n)^2\big), \qquad \mathbb{E}(|\eta_i''^n|^2) \leq K.$$

On the one hand, the same argument as for proving (5.25) shows that $4d^2\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \eta_i'^n$ converges in the u.c.p. sense to the right side of (3.14) (for $\kappa = \kappa'$). On the other hand, since $\eta_i''^n$ is a martingale increment relative to the filtration $(\mathcal{F}_i^n)_{i\geq 0}$, we deduce from Doob's inequality that $4d^2\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \eta_i''^n \stackrel{\text{u.c.p.}}{\Longrightarrow} 0$. We then deduce (3.14). □

5) Finally, for (3.15) it is enough to show the convergence in probability in restriction to each set Ω_T^r, for $r = 0, \cdots, d$. For this we use the following convergence properties, which readily follow from (3.8), (3.11), in restriction to the set Ω_T^r:

$$\frac{1}{\Delta_n^{d-r}}\, S^{n,1} \stackrel{\mathbb{P}}{\longrightarrow} S(r)_T > 0, \qquad \widehat{R}(n,T) \stackrel{\mathbb{P}}{\longrightarrow} r,$$

together with (3.14) applied at time T, which also holds on Ω_T^r. Then (3.15) follows after a (slightly tedious) calculation, in view of the form (3.12) of $V(T)$ on Ω_T^r: the proof of Proposition 3.4 is complete. □

5.5 Proof of Theorems 4.1 and 4.5.

We begin with a lemma. Its setting apparently extends the setting of the theorem to be proved, but this will be useful for the proof itself. The extension concerns the fact that we replace the non-random terminal time T by a stopping time, still denoted by T, which is positive and bounded. In this case, the notation (2.3) still makes sense, as well as $A(p)_T^n$ and $a(n,T)$ and $\overline{V}_T^{n,\kappa\kappa'}$, as given by (4.23) and (4.27).

Lemma 5.5 *Assume (H) and $r_t = r$ for all $t \leq T$ with T a positive finite stopping time and $r \in \{0, \cdots, d\}$. Then for all $p > 0$, $\kappa, \kappa' \in \{1,2\}$ and $\Theta_s^{r,\kappa\kappa'}$ as in (3.14) we have*

$$A(p)_T^n \stackrel{\mathbb{P}}{\longrightarrow} T\, r^p \tag{5.28}$$

$$\frac{1}{(\kappa\kappa'\Delta_n^2)^{d-r}}\, \overline{V}_T^{n,\kappa\kappa'} \stackrel{\mathbb{P}}{\longrightarrow} 2d \int_0^T \Big(\frac{1}{\Gamma_r(\sigma_s, \widetilde{\sigma}, v_s, b_s)} - \frac{T}{S(r)_T}\Big)^2\, \Theta_s^{r,\kappa\kappa'}\, ds. \tag{5.29}$$

Moreover, if $r \geq 1$, the following stable convergence in law holds, where $\mathcal{U}(r)$ is defined in Theorem 3.2:

$$\begin{gathered}\Big(U(r)_T^n, \tfrac{1}{\sqrt{\Delta_n}}\,(A(p)_T^n - a(n,T)\, r^p)\Big) \stackrel{\mathcal{L}-s}{\longrightarrow} \\ \Big(\mathcal{U}(r)_T, \tfrac{pr^{p-1}}{\log 2} \int_0^T \tfrac{1}{\Gamma_r(\sigma_s, \widetilde{\sigma}, v_s, b_s)}\, (d\mathcal{U}(r)_s^1 - d\mathcal{U}(r)_s^2)\Big).\end{gathered} \tag{5.30}$$

Proof. 1) Let $\gamma_t = \Gamma_r(\sigma_t, \widetilde{\sigma}, v_t, b_t)$, which is a continuous process, positive on $[0,T]$ by Lemma 3.1. Thus $T_m = m \wedge T \wedge \inf(t : \gamma_t < 1/m)$ satisfies $\mathbb{P}(T_m = T) \to 1$ as $m \to \infty$ and, if any one of the claimed convergence holds for each T_m (instead of T), it also holds for T. In other words, we can assume $T \leq A$ and $1/\gamma_t \leq A$ for some constant A and all $t \in [0,T]$. Moreover, Γ_r is a polynomial, so the process γ_t is a continuous Itô semimartingale, and by localization again one can assume that for some other constant A',

$$\mathbb{E}(|\gamma_{t+s} - \gamma_t|^2) \leq A'\, s. \tag{5.31}$$

The sequence $U(r)^n$ converges in law toward a continuous process, so the moduli of continuity $\rho(n,x) = \sup(\|U(r)_{t+s}^n - U(r)_t^n\| : \ t \leq A', \ |s| \leq x)$ satisfy $\lim_{x\downarrow 0} \limsup_n \mathbb{P}(\rho(n,x) > 1) = 0$, and thus with the simplifying notation $w_n = 2dk_n\Delta_n$ we have

$$\mathbb{P}(\Omega_n) \to 1, \qquad \text{where} \ \ \Omega_n = \big\{\|U(r)_{t+s}^n - U(r)_t^n\| \leq 1 \ \forall\, t \leq A', \ s \leq w_n\big\}. \tag{5.32}$$

2) Observe that

$$\widehat{R}_i^n = r + \tfrac{1}{\log 2}\,\log\tfrac{\zeta_i^n+\sqrt{\Delta_n}\,\eta_i^{n,1}}{\zeta_i^n+\sqrt{\Delta_n}\,\eta_i^{n,2}}, \quad \text{where}$$
$$\zeta_i^n = S(r)_{2id\Delta_n+w_n} - S(r)_{2id\Delta_n}, \quad \eta_i^{n,k} = U(r)^{n,k}_{2id\Delta_n+w_n} - U(r)^{n,k}_{2id\Delta_n}.$$

Recalling $1/\gamma_t \leq A$, and since $\zeta_i^n = w_n(\gamma_{2id\Delta_n} + \rho_i^n)$, where $\mathbb{E}(|\rho_i^n|^2) \leq A'' w_n$ by (5.31), one has

$$0 \leq \frac{w_n}{\zeta_i^n} \leq A, \qquad \mathbb{E}\Big(\Big|\frac{w_n}{\zeta_i^n} - \frac{1}{\gamma_{2id\Delta_n}}\Big|^2\Big) \leq A^2 A'' w_n. \tag{5.33}$$

Moreover, take $\alpha \in (0,1/2)$ such that $\frac{2|\log(1-\alpha)|}{\log 2} \leq 1/2$. For n large enough we have $A\sqrt{\Delta_n}\,/w_n \leq \alpha$ because $k_n\Delta_n^{3/4} \to \infty$. In this case, in restriction to the set Ω_n, for all $i \leq [T/2d\Delta_n] - k_n - 1$ we have with a constant K (varying from place to place below):

$$\begin{array}{ll} \Big|\frac{\sqrt{\Delta_n}\,\eta_i^{n,k}}{\zeta_i^n}\Big| \leq \frac{A\sqrt{\Delta_n}}{w_n} \leq \frac12, & \Big|\widehat{R}_i^n - r - \frac{\sqrt{\Delta_n}}{\log 2}\,\frac{\eta_i^{n,1}-\eta_i^{n,2}}{\zeta_i^n}\Big| \leq K\,\frac{\Delta_n}{w_n^2} \\ r \geq 1 \ \Rightarrow\ \Big|\frac{\widehat{R}_i^n}{r} - 1\Big| \leq \frac{K\sqrt{\Delta_n}}{w_n} \bigwedge \frac12, & r = 0 \ \Rightarrow\ |\widehat{R}_i^n| \leq \frac{K\sqrt{\Delta_n}}{w_n} \bigwedge \frac12. \end{array} \tag{5.34}$$

3) Recalling (4.23) and $T \leq A$, when $r = 0$ the last estimate above yields

$$\mathbb{E}(A(p)_T^n\,1_{\Omega_n}) \leq KA\,\frac{\Delta_n^{p/2}}{w_n^p},$$

which goes to 0 because $k_n\sqrt{\Delta_n} \to \infty$. Thus in view of (5.32) one gets (5.28) when $r = 0$.

4) At this stage, we start proving (5.30), and thus assume $r \geq 1$. We observe that

$$Y_n := \frac{1}{\sqrt{\Delta_n}}\,(A(p)_T^n - a(n,T)\,r^p) = w_n \sum_{i=0}^{[T/w_n]-2} \xi_i^n, \ \text{ where } \xi_i^n = \frac{1}{\sqrt{\Delta_n}}\,\Big(|\widehat{R}_{ik_n}^n|^p \wedge (d+1)^p - r^p\Big).$$

(5.34) implies that for n large enough, $|\widehat{R}_i^n| \leq d+1$ (recall $r \leq d$), hence a Taylor expansion of the function $x \mapsto |r+x|^p - r^p$ imply, again for n large enough:

$$\Big|\xi_i^n - \frac{pr^{p-1}}{\log 2}\,\frac{\eta_{ik_n}^{n,1} - \eta_{ik_n}^{n,2}}{\zeta_{ik_n}^n}\Big| \leq K\frac{\sqrt{\Delta_n}}{w_n^2} \quad \text{on } \Omega_n \text{ and for } i \leq [T/w_n] - 2.$$

Upon using (5.33), and by the Cauchy-Schwarz inequality, it follows that

$$\mathbb{E}\Big(\Big|\xi_i^n - \frac{pr^{p-1}}{\log 2}\,\frac{\eta_{ik_n}^{n,1} - \eta_{ik_n}^{n,2}}{w_n\,\gamma_{(i-1)w_n}}\Big|\,1_{\Omega_n}\Big) \leq K\frac{\sqrt{\Delta_n}}{w_n^2} + K\sqrt{w_n},$$

hence

$$\mathbb{E}\big(|Y_n - Y'_n|\,1_{\Omega_n}\big) \to 0, \quad \text{where } \ Y'_n = \frac{pr^{p-1}}{\log 2}\sum_{i=0}^{[T/w_n]-2} \frac{\eta_{ik_n}^{n,1} - \eta_{ik_n}^{n,2}}{\gamma_{(i-1)w_n}},$$

because $k_n\Delta_n^{3/4}\to\infty$. Recall also (5.27). Then, by virtue of (5.32), the convergence (5.30) will follow from

$$\big(U(r)_T^n, Y_n'\big) \stackrel{\mathcal{L}-s}{\longrightarrow} (\mathcal{U}(r)_T,\mathcal{Y}), \quad \mathcal{Y}=\frac{pr^{p-1}}{\log 2}\int_0^T \frac{1}{\Gamma_r(\sigma_s,\widetilde{\sigma},v_s,b_s)}\,(d\mathcal{U}(r)_s^1-d\mathcal{U}(r)_s^2). \tag{5.35}$$

5) By Theorem VI.6.15 of [8] it follows from (5.23) and (5.25) that not only does the sequence of processes $U(r)^n$ converge in law, but it also enjoys the so-called P-UT property (predictable uniform tightness). By a trivial extension of Theorem VI.6.22 in [8], this implies that if a sequence H^n of adapted càdlàg two-dimensional processes on Ω is such that the pair $(U(r)^n,H^n)\stackrel{\mathcal{L}-s}{\Longrightarrow}(\mathcal{U}(r),H)$ (functional convergence for the Skorokhod topology), the bi-dimensional processes $\big(U(r)^n,\int_0^t H^n_{s-}\,dU(r)^n_s\big)$ converge stably in law to $\big(\mathcal{U}(r),\int_0^t H_{s-}\,d\mathcal{U}(r)_s\big)$, and since $\mathcal{U}(r)$ is continuous and T is $\mathcal{F}$-measurable, this in turn implies the stable convergence of the variables $\big(U(r)^n_T,\int_0^T H^n_{s-}\,dU(r)^n_s\big)\stackrel{\mathcal{L}-s}{\longrightarrow}\big(\mathcal{U}(r)_T,\int_0^T H_{s-}\,d\mathcal{U}(r)_s\big)$. At this point, (5.35) follows, by taking the processes H^n and H with components

$$H_t^{n,1}=-H_t^{n,2}=\frac{pr^{p-1}}{\gamma_{(i-1)w_n}\log 2}\ \text{ if } t\in((i-1)w_n,(iw_n)\wedge T], \quad H_t^1=-H_t^2=\frac{pr^{p-1}}{\gamma_t\log 2}\,1_{\{t\le T\}}.$$

(Note that the joint stable convergence $(U(r)^n,H^n)\stackrel{\mathcal{L}-s}{\Longrightarrow}(\mathcal{U}(r),H)$ holds because $1/\gamma_t$ is continuous.) This ends the proof of (5.30).

6) Since (5.30) implies (5.28) when $r\ge 1$, we are left to prove (5.29). We fix κ,κ'. Our first observation is that, since $U(r)^n_T\stackrel{\mathcal{L}-s}{\longrightarrow}\mathcal{U}(r)_T$ follows from (3.8) as seen before, the proof of (3.11) carries over to the case T is a stopping time. Therefore $(\widehat{R}(n,T)-r)\,\log\Delta_n\stackrel{\mathbb{P}}{\longrightarrow}0$ because here $\Omega_T^r=\Omega$, and thus $\Delta_n^{\widehat{R}(n,T)-r}\stackrel{\mathbb{P}}{\longrightarrow}1$. It follows that (5.29) amounts to proving the same result for the variable $\widetilde{V}_T^{n,\kappa\kappa'}$ which is the same as $\overline{V}_T^{n,\kappa\kappa'}$ except that in front of the sum we substitute $\Delta_n^{1+2d-2\widehat{R}(n,T)}$ with $\Delta_n^{1+2d-2r}$.

With θ_i^n being as for Proposition 3.4, the ith summand in the right side of (3.13), we have

$$\begin{aligned}&\widetilde{V}_T^{n,\kappa\kappa'}=\sum_{j=0}^2\Big(\tfrac{T\Delta_n^{d-r}}{S_T^{n,1}}\Big)^j B(j)_T^n, \quad \text{where } B(j)_T^n=4d^2\Delta_n\sum_{i=0}^{[T/2d\Delta_n]-k_n-1}\upsilon(j)_i^n\theta_i^n\\ &\upsilon(0)_i^n=\Big(\tfrac{w_n}{\zeta_i^n+\sqrt{\Delta_n}\,\eta_i^{n,1}}\Big)^2, \qquad \upsilon(1)_i^n=-2\,\tfrac{w_n}{\zeta_i^n+\sqrt{\Delta_n}\,\eta_i^{n,1}}, \qquad \upsilon(2)_i^n=1.\end{aligned}\tag{5.36}$$

Combining (5.33) and (5.34), we obtain for $i\le[T/2d\Delta_n]-k_n-1$ and all n large enough:

$$\mathbb{E}\Big(\Big|\upsilon(0)_i^n-\frac{1}{(\gamma_{2id\Delta_n})^2}\Big|^2 1_{\Omega_n}\Big)+\mathbb{E}\Big(\Big|\upsilon(1)_i^n+\frac{2}{\gamma_{2id\Delta_n}}\Big|^2 1_{\Omega_n}\Big)\le K\Big(\frac{\Delta_n}{w_n^2}+w_n\Big).$$

Since by localization we may assume that the processes σ_t,v_t,b_t are bounded, we may also assume $\theta_i^n\le K$, and upon using (5.32) once more, we then deduce that $B(j)_T^n$ as the same asymptotic behavior as $B'(j)_T^n$ which is given by the same formula, with $\upsilon(j)_i^n$ substituted with the following variables $\upsilon'(j)_i^n$:

$$\upsilon'(0)_i^n = \frac{1}{(\gamma_{2id\Delta_n})^2}, \qquad \upsilon'(1)_i^n = -2\,\frac{1}{\gamma_{2id\Delta_n}}, \qquad \upsilon'(2)_i^n = 1.$$

This allows us to get, with $\Upsilon(0) = \frac{1}{\gamma^2}$, $\Upsilon(1) = \frac{1}{\gamma}$ and $\Upsilon(2) = 1$:

$$\frac{1}{(\kappa\kappa'\Delta_n^2)^{d-r}}\, B(j)_T^n \xrightarrow{\mathbb{P}} 2d \int_0^T \Upsilon(j)_s \Theta_s^{r,\kappa\kappa'}\, ds \tag{5.37}$$

(indeed, the case $j = 2$ is (3.14), and the other two cases follow from a standard argument, similar to Step 5 above, but simpler because the integrand is a càdlàg bounded process not depending on n, and $\theta_i^n \geq 0$). Using further $T\Delta_n^{d-r}/S_T^{n,1} \xrightarrow{\mathbb{P}} T/S(r)_T$ and recalling (5.36), and upon expanding the square in the right side of (5.29), we obtain this convergence and the lemma is proved. □

Proof of (a) of Theorem 4.1. Since $\widehat{R}(n,T) \xrightarrow{\mathbb{P}} R_T$ by (3.11), it suffices to prove that

$$A(p)_T^n \ \xrightarrow{\mathbb{P}}\ A(p)_T := \int_0^T (r_s)^p\, ds. \tag{5.38}$$

The assumption implies the existence of a sequence of stopping times τ_j increasing to infinity, such that $\tau_0 = 0$ and $\tau_j < \tau_{j+1}$ if $\tau_j < \infty$, and such that the process r_t takes a constant (random) value $\rho(j)$ on the time interval $J_j = (\tau_{j-1}, \tau_j)$, with $\rho(j) \neq \rho(j+1)$ if $0 < \tau_j < \infty$. In view of the discussion preceding (2.3), the values r_{τ_j} is necessarily smaller than or equal to $\rho(j) \wedge \rho(j+1)$, but is irrelevant to our discussion. We also denote by N_T the biggest j such that $\tau_j \leq T$.

With an empty sum being set to 0, we have

$$A(p)_T^n = \sum_{j=1}^{N_T} Y(j)_n + Z_n, \qquad Y(j)_n = w_n \sum_{i=[\tau_{j-1}/w_n]+1}^{[(\tau_j \wedge T)/w_n]-2} |\widehat{R}_i^n|^p,$$

and where Z_n is the sum of at most $3N_T$ terms of the form $w_n(|\widehat{R}_i^n|^p \wedge (d+1)^p)$. Since N_T is finite and $w_n \to 0$, we have $Z_t^n \to 0$ (pointwise), and it suffices to show that for each $j \geq 1$ we have

$$Y(j)_n \ \xrightarrow{\mathbb{P}}\ Y(j) := \big((T \wedge \tau_j) - (T \wedge \tau_{j-1})\big)\rho(j)^p. \tag{5.39}$$

We then fix j. The variable $Y(j)_n$ is the process $A(p)^n$ evaluated at time $T_j = T \wedge \tau_j - \tau_{j-1}$ relative to the underlying process $X(j)_t = X_{\tau_{j-1}+t}$, up to at most two border terms. We thus might be tempted to apply (5.28) right away, and indeed $X(j)$ satisfies (H) for the filtration $\mathcal{F}(j)_t = \mathcal{F}_{\tau_{j-1}+t}$, relative to which T_j is a positive bounded stopping time. There are, however, a few problems to overcome:

1. The rank $r_t(X(j))$ associated with $X(j)$ is equal to $\rho(j)$ for all $t \in (0, T_j)$, but not necessarily for $t = 0$, nor for $t = T_j$;

2. This rank $\rho(j)$ is random, albeit $\mathcal{F}(j)_0$-measurable;

To solve these problems we fix $\varepsilon > 0$ and consider the process $X(j,\varepsilon)_t = X_{\tau_{j-1}+\varepsilon+t}$, satisfying (H) for the filtration $\mathcal{F}(j,\varepsilon)_t = \mathcal{F}_{\tau_{j-1}+\varepsilon+t}$, and the $\mathcal{F}(j,\varepsilon)_t$-stopping time $T(j,\varepsilon) = T \wedge \tau_j - T \wedge \tau_{j-1} - 2\varepsilon$. The associated rank is thus $\rho(j)$ for all $t \in [0, T(j,\varepsilon)]$, and we will show that if $A(p,j,\varepsilon)$ is associated with $X(j,\varepsilon)$ by (4.23), we have

$$A(p,j,\varepsilon)^n_{T(j,\varepsilon)} \stackrel{\mathbb{P}}{\longrightarrow} T(j,\varepsilon)\,\rho(j)^p \;=\; (T \wedge \tau_j - T \wedge \tau_{j-1} - 2\varepsilon)\,\rho(j)^p. \tag{5.40}$$

Indeed, it suffices to prove this in restriction to each set $\Omega'_r = \{\rho(j) = r\}$ satisfying $\mathbb{P}(\Omega'_r) > 0$. If $\mathbb{P}_r$ denotes the (usual) conditional probability $\mathbb{P}(\cdot \mid \Omega'_r)$, the process $X(j,\varepsilon)$, on the space $(\Omega'_r, \mathcal{F} \cap \Omega'_r, (\mathcal{F}(j,\varepsilon)_t \cap \Omega'_r), \mathbb{P}_r)$, still satisfies (H) and the associated rank is now r on the time interval $[0, T(j,\varepsilon)]$. Then Lemma 5.5 yields the convergence (5.40) under $\mathbb{P}_r$, hence under $\mathbb{P}$ in restriction to each Ω'_r, hence under $\mathbb{P}$ on Ω itself.

Finally, the difference $Y(j)_n - A(p,j,\varepsilon)^n_{T(j,\varepsilon)}$ is a sum of at most $2[\varepsilon/w_n]$ terms, each one smaller than $w_n(d+1)^p$, so this difference is smaller than $K\varepsilon$, as is the difference between the two right sides of (5.39) and (5.40). Hence (5.39) follows from (5.40), by taking first $n \to \infty$ and then $\varepsilon \to 0$. This completes the proof. □

Proof of (b) of Theorem 4.1. Exactly as in the previous proof, it is enough to prove the result when $r_t = r \geq 1$ identically, for some non-random $r \in \{1, \cdots, d\}$. By a standard localization procedure we can assume that $\Gamma_r(\sigma_t, \widetilde{\sigma}, v_t, b_t)$, which is positive everywhere, is bounded from below by a constant $1/A$ with $A > 0$, so the assumptions of Lemma 5.5 are satisfied. Therefore, (5.27) and (5.30) yield that, with Y_n and $\mathcal{Y}$ as in the proof of Lemma 5.5 and with $Z_n = \frac{1}{\sqrt{\Delta_n}}\,(\widehat{R}(n,T) - r)$ and $\mathcal{Z} = \frac{1}{S(r)_T \log 2}\,(\mathcal{U}(r)^1_T - \mathcal{U}(r)^2_T)$, we have

$$(Y_n, Z_n) \stackrel{\mathcal{L}-s}{\longrightarrow} (\mathcal{Y}, \mathcal{Z}). \tag{5.41}$$

Then we obtain

$$\frac{1}{\sqrt{\Delta_n}}\,B(n,p,T) = Y_n + \frac{a(n,T)}{\sqrt{\Delta_n}}\,\big(r^p - \big|r + \sqrt{\Delta_n}\,Z_n\big|^p\big).$$

On the one hand, $a(n,T) \to T$. On the other hand, since Z_n converges in law and $r \geq 1$, we have by the mean value theorem

$$\frac{1}{\sqrt{\Delta_n}}\,\big(r^p - \big|r + \sqrt{\Delta_n}\,Z_n\big|^p\big) + p r^{p-1}\,Z_n \stackrel{\mathbb{P}}{\longrightarrow} 0.$$

Hence (5.41) yields

$$\frac{1}{\sqrt{\Delta_n}}\,B(n,p,T) \stackrel{\mathcal{L}-s}{\longrightarrow} \mathcal{B}(p,T) = \mathcal{Y} - T p r^{p-1} \mathcal{Z}.$$

The pair $(\mathcal{Y}, \mathcal{Z})$ being $\mathcal{F}$-conditionally centered Gaussian, the same is true of $\mathcal{B}(p,T)$, and the form (4.26) of its conditional variance is easily checked, by virtue of (3.9). □

Proof of Theorem 4.5. It is easy to construct a process X' which satisfies the assumptions of (a) of Theorem 4.1 and such that $X'_t = X_t$ for all $t \leq T$ on the set $\Omega_T^{\neq}$. Then on this set $B(n,p,T)$ is the same, when constructed upon X or upon X', and thus it converges in probability on this set to a strictly negative variable. On the other hand, $Z(n,p,T)$ is $B(n,p,T)$ divided by a quantity which by construction is smaller than $\Delta_n^{1/4}$. Then the convergence $Z(n,p,T) \to -\infty$ on $\Omega_T^{\neq}$ is clear.

It suffices to prove (4.28) on the set $\Omega_T^{=} \cap \Omega_T^r$ for any $r \in \{0, \cdots, d\}$ such that $\mathbb{P}(\Omega_T^r) > 0$. For this we can argue under the conditional probability $\mathbb{P}_r = \mathbb{P}(\cdot \mid \{r_0 = r\})$, or equivalently suppose that we have in fact $r_0 = r$. As above, one can construct a process X' which satisfies the assumptions of Lemma 5.5 for some stopping time T' which satisfies $T' \geq T$ on the set $\Omega_T^{=}$, and we can apply (5.29) to X' and the stopping time $T' \wedge T$. This gives us (5.29) for X, in restriction to the set $\Omega_T^{=}$.

At this point, (4.28) follows from (5.29) by exactly the same calculations as (3.15) follows from (3.14).

Finally, since $\overline{V}(n,p,T) \stackrel{\mathbb{P}}{\longrightarrow} \overline{V}(p,T)$ on $\Omega_T^{=}$, we have $Z(n,p,T) = B(n,p,T)/\sqrt{\Delta_n BV(n,p,T)}$ on a set Ω''_n whose probability goes to 1. The first part of (4.30) than follows from (4.25) and (4.28) by delta method for stable convergence. □

5.6 Proof of Corollary 3.6.

The same stopping argument as in Step 2 of the previous proof allows us to show that, without assumptions on the rank process r_t, the stable convergence in law (3.8) holds in restriction to the set Ω_T^r, as soon as we restrict our attention to the time interval $[0,T]$.

At this stage, the claim of Corollary 3.6 follows from (3.11), an application of the delta method, (3.15) and classical properties of stable convergence in law. □

5.7 Proof of Propositions 3.7, 3.8 and 4.6.

(3.18) is an obvious consequence of the stable convergence (3.16). For the alternative-consistency, it suffices to prove that for any $r' \neq r$ we have

$$\mathbb{P}\big(\mathcal{C}(\alpha)_T^{n,=r} \cap \Omega_T^{r'}\big) \to \mathbb{P}(\Omega_T^{r'}). \tag{5.42}$$

On the set $\Omega_T^{r'}$ we have $S(n,T) \stackrel{\mathbb{P}}{\longrightarrow} 2^{d-r'}$, and by (3.15) the variables $V(n,T)$ converge in probability to a limit which is $[0,\infty)$-valued (actually, it is a.s. positive, but we do not use this fact here), so that $\Delta_n\, V(n,T) \stackrel{\mathbb{P}}{\longrightarrow} 0$. Since $r' \neq r$, (5.42) readily follows from the definition of $\mathcal{C}(\alpha)_T^{n,=r}$.

Propositions 3.8 and 4.6 are proved analogously, the alternative-consistency in the latter case following from the second part of (4.30). □

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