SEARCH-THEORETIC MONEY, CAPITAL AND INTERNATIONAL EXCHANGE RATE FLUCTUATIONS★

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ABSTRACT

In this paper we develop a two-country global monetary economy where a monetary equilibrium exists because of fundamental decentralized trade frictions – a Lagos-Wright search and matching friction. In the decentralized markets (DM), the terms of trade can be determined either by bargaining or by competitive price taking (baseline model). We show that the baseline model is capable of generating quite realistic real and nominal exchange rate volatility observed in the data, without relying on more ad-hoc sticky price assumptions commonly used in the international macroeconomics literature. The key mechanism lies in the role of search and matching frictions and a primitive technological assumption – that capital is also a complementary input to production in the DM. This creates an internal propagation mechanism by modifying asset-pricing relations and relative price dynamics in the model.

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1. Introduction

It is well known that the real and nominal exchange rate of the world's largest economies are very volatile and persistent. The seminal work of Chari, Kehoe, and McGrattan [2002] explored whether these features of the data could be understood in the context of a standard two country real business cycle model with sticky prices. They concluded that such models can explain the volatility of the real exchange rate but that they can not match its persistence. In a nutshell, ad-hoc sticky price models are able to generate volatile real and nominal exchange rate processes, because, by assumption prices are made to not adjust too quickly to aggregate shocks. In an open economy, the nominal exchange rate and therefore, the real exchange rate, have to overreact. This is a manifestation of the textbook Dornbusch [1976] exchange rate overshooting hypothesis.

In this paper, we revisit this literature and ask if a perfectly flexible price, two-country, search theoretic model of money along the lines of Lagos and Wright [2005] and Aruoba, Waller, and Wright [2008], is capable of explaining real and nominal exchange rate dynamics. This question is necessarily quantitative. We then explore how the underlying search and matching friction, and, a capital complementarity effect in the decentralized market production, help to generate the dynamic features of the exchange rates in the model. These aspects of the model are arguably more primitive than assuming exogenous sticky prices. The former (search and matching friction) relates to an inherent feature of market incompleteness in the sense of missing Walrasian markets and a resultant spatial-temporal friction, whilst the latter pertains to a standard description of primitive technology– i.e. capital being used in a two sector-economy. In the search and money literature very little focus has been placed on the quantitative and therefore potential policy relevance of the theories put forward. A model must be empirical plausible first, before it can be of policy relevance. Exceptions are Aruoba, Waller, and Wright [2008] (long run empirical plausibility) and Aruoba [2010] (closed-economy business-cycles), where respectively, the task has been to investigate the quantitative relevance of the most recent generation of search and money models. We think our task, one of quantitatively validating the search-theoretical foundation for international monetary economics, is an important one to undertake, given the nature of openness of large economies to trade and capital flows today.

By attempting to capture a slightly deeper friction – as a stand-in for anonymity (or absence of contractual enforcement) and lack of double coincidence of wants – that generate a role for money, we can study the impact of different roles for money on the evolution on the exchange rate. In particular, we consider an environment where agents participate in sequential markets, first in a decentralized market (DM) where they trade specialized consumption goods bilaterally and then in a centralized Walrasian market (CM) where they trade general consumption goods, which require domestic and foreign inputs, and assets, capital and bonds. Physical capital is a factor of production in both markets. Demand for money arises because the particular frictions in the decentralized markets require a medium of exchange. Moreover, money in this type of environment also provides a store of value and precautionary function since agents face individual uncertainty before trading in the DM.

Quantitatively, we calibrate our models to quarterly data representing the U.S. and the rest of the world. We find that, without even resorting to exogenous sticky-price assumptions, the baseline model is capable of generating very volatile but moderately persistent real and nominal exchange rates. Empirically, Steinsson [2008] showed that the real exchange rate has hump-shaped dynamics. He then showed that a standard sticky price model and its variations cannot replicate this fact, without introducing exogenous real shocks such as a Phillips curve shock. In constrast, our baseline search model is capable of producing hump-shaped dynamics in the real exchange rate, without relying on additional exogenous real shocks. This strong internal propagating mechanism of our model contributes to the equilibrium persistence of the real exchange rate. The models are also capable of generating the observation that the real exchange rate is more volatile than U.S. GDP.

We show in this paper that a key mechanism that results in the volatility and persistence of the exchange rates is the basic Lagos and Wright [2005]-type search friction, indexed by a single parameter σ. This feature introduces an explicit value for money as medium of exchange, and, as a corrollary, it affects the precautionary aspect of holding money. Using comparative impulse response analyses, we provide an explanation of how this return to money modifies asset pricing relations in an otherwise standard real-business-cycle model, and thus the exchange rates. We also conduct a sensitivity analysis on the role of capital complementarity in DM production in the model. This parameter is also key in controlling the volatility of the exchange rate outcomes, as it restricts the asset-pricing relations in the model by modifying the return on capital investment made in each CM.

Another interesting puzzle in equilibrium international business cycle models is the perfectly positive correlation between relative consumption and the real exchange rate. In the data, this statistic tends to be slightly negative or zero. The puzzle is attributed to the fact that with complete markets in equilibrium business cycle models, there is a perfect link between a marginal rate of substitution transform of relative consumption and the real exchange rate. However, even in the sticky-price monetary models with incomplete markets of Chari, Kehoe, and McGrattan [2002], this correlation remains at unity. However, in our model, the correlation is no longer perfect. The reason behind this is the nature of the DM in our model, which is akin to a nontraded goods sector.

The paper is organized as follows. In section 2, we outline the details and assumption of the baseline quantitative-theoretical model. We then work through the equilibrium constructs and implications of the model in Section 3. Next, in Section 4, we provide some insight into the key mechanisms in the model. We then take the theory to the data in Section 5. We discuss the model's business cycle features relative to the data and other existing models in Section 6. We then explain how the mechanisms interact to produce the business cycle features, by using the partial tool of impulse response analysis, in Section 7. We conclude in Section 8.

2. Baseline Environment: Complete Asset Markets in the CM

Consider a two-country model, each referred to as Home and Foreign. Variables and parameters without an asterisk (or with a subscript h) will refer to the Home country, and those without an asterisk (or with a subscript f), will refer to the Foreign country. Time is denumerable, and a time period is denoted by $t \in \mathbb{N} := \{0, 1, 2, ...\}$. Agents exist on a continuum $[0, 1]$ and have a common discount factor $\beta \in (0, 1)$. Each $t \in \mathbb{N}$ is composed of two arbitrary sub-periods, night and day. In the night, agents trade anonymously in bilateral random matches, in decentralized markets (DM). In the day, agents trade in centralized markets (CM). The nature of consumption, production and trade in each market will be explained in detail in sections 2.5 and 2.6.

2.1. **Preferences and DM technology.** Let $q^b \in \mathbb{R}_+$ be an agent's consumption (if the agent is a buyer) and $q^s \in \mathbb{R}_+$ be an agent's output (if the agent is a seller) of a "specialized", or, agent-specific and non-storable good in the DM. Similar to Lagos and Wright [2005], an agent who is a producer of a q^s is assumed to not value it, but may, with probability $\sigma \leq 1/2$ exchange it (for money) with another agent who, with symmetric probability σ, wishes to consume it (a buyer) – *i.e.* it is a q^b to this buyer. Thus, with probability $1 - 2\sigma$, an agent will leave a DM with no exchange. For simplicity, assume that "doble-coincidence-of-wants" events, where buyers and sellers in the DM are able to barter and money is inessential, occur with probability zero.

Let $X \in \mathbb{R}_+$, $k \in \mathbb{R}_+$ and $H \in [0, \overline{H}]$, where $\overline{H} < +\infty$, denote consumption, individual capital stock and labor in the CM, respectively. Agents' per-period preferences are given by

$$(q, X, H) \mapsto \begin{cases} u(q) + U(X) + h(H) & \text{if buyer in current DM} \\ -c(q,k) + U(X) + h(H) & \text{if seller in current DM} \end{cases}$$

where $u(q)$ is the per-period payoff from q if the agent is a buyer, $c(q,k)$ is the cost of producing q with fixed within-period k determined in the previous CM. $U(X)$ is the immediate payoff from consuming X in the CM, and $-h(H)$ is the disutility of work effort in the CM. We make the following assumptions.

Assumption 1. *The functions $u, U, h : \mathbb{R}_+ \to \mathbb{R}$ and $c : \mathbb{R}^2_+ \to \mathbb{R}$ have the following properties:*

(i) *First and second derivatives exist everywhere: $u, U \in \mathbf{C}^2(\mathbb{R}_+)$ and $c \in \mathbf{C}^2(\mathbb{R}^2_+)$;*
(ii) *$u_q > 0$, $c_q > 0$, $c_k < 0$, $U_X > 0$ and $h_H > 0$;*
(iii) *$u_{qq} < 0$, $c_{qq} \geq 0$, $c_{qk} < 0$, $U_{XX} \leq 0$ and $h_{HH} = 0$;*
(iv) *$u(0) = c(0,0) = 0$; and*
(v) *$u(q) > c(q,k)$ for every (q,k).*

2.2. **CM Technology.** In the CM the final good in the Home country is produced according to a constant returns technology, $(y_h, y_f) \mapsto G(y_h, y_f)$, where y_h denotes the input demand for an intermediate good produced in the home country, and, y_f represents the demand of a substitutable input produced in the foreign country. Similarly, the foreign final good is given by, $(y^*_f, y^*_h) \mapsto G(y^*_f, y^*_h)$. Assume that $G \in \mathbf{C}^2(\mathbb{R}^2_+)$ and that $G_i > 0, G_j > 0, G_{ii} < 0, G_{jj<0}$, and, $F(i,0) = F(0,j) = 0$, for some inputs i, j.

Let K denote an aggregate capital stock in each home country. The production of the different intermediate goods are given by another constant returns technology, $(K, H) \mapsto zF(K, H)$ which is subject to a stochastic productivity shock, z. Assume $(z_t)_{t \in \mathbb{N}}$ is a strictly positive and bounded stochastic process. Assume that $F \in \mathbf{C}^2(\mathbb{R}^2_+)$ and that $F_K > 0, F_H > 0, F_{KK} < 0, F_{HH<0}$, and, $F(K,0) = F(0,H) = 0$.

2.3. **State variables.** Let $m \in \mathbb{R}_+$ be the stock of an agent's local nominal money holding in the Home country. [1] Denote b as the current stock of an internationally

[1] It will be immediately apparent to the serious monetary theorist that we are placing an *ad hoc* restriction that agents can only use the local currency to buy local goods, especially in the DM. We

traded complete state-contingent money claim, held by an agent in the Home country. Each b is denominated in the Home currency. Since these complete contingent claims require knowledge of traders' histories, it is natural that they are not issued or traded in the DM with anonymous randomly matched trades. They are traded only during each CM subperiod.

Suppose we relax the last assumption a little bit. Following Aruoba, Waller, and Wright [2008], suppose that conditional on being a buyer or seller, the exogenous probability that a buyer or seller would engage in an exchange where record keeping is possible is $(1-\kappa) \in [0,1]$. That is, the event that a buyer *or* a seller can buy or sell a good in the DM using credit occurs with the discrete probability measure $\sigma(1-\kappa)$. Since credit is assumed to be enforceable in such an event, a buyer is willing to take (and a seller is willing to give) out the nominal loan l in exchange for a good, say $\check{q}$. This loan is required to be repaid in full in the following CM. Thus there is no complication with discounting given the timing of the sub-markets. Then we let q denote a DM specialized good that is exchanged for money in events where exchange occurs with measure $\sigma\kappa$ for a buyer *or* seller.

Denote the vector of exogenous shocks as $\mathbf{z} \in Z$. For example, we consider Home and Foreign, technology (z) and money growth (ψ), shocks. Thus $\mathbf{z} := (z, z^*, \psi, \psi^*)$, and $Z \subset \mathbb{R}^4$. Let the time-t *aggregate* (global) CM state vector relevant to an agent in country $i \in \{h, f\}$ be $\mathbf{s} := (M, M^*, B, B^*, K, K^*, \phi, \phi^*, e, \mu_h, \mu_f, \mathbf{z})$, consisting of, respectively, the global/aggregate Home and Foreign capital stocks, the global Home and Foreign DM-specific capital stocks, the total Home and Foreign holding of the state-contingent claims, Home and Foreign nominal money stock, the value of money in the Home CM ($\phi := 1/p_X$), the value of money in the Foreign CM ($\phi^* := 1/p_X^*$), the nominal exchange rate in Home CM currency terms (e), and, $\mu_i(\cdot, \mathbf{z}) : \mathcal{B}_i(\mathbf{z}) \to [0,1]$ which is the time-t probability measure on the Borel σ-field $\mathcal{B}_i(\mathbf{z}) := \mathcal{B}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ generated by the product state space

appeal to observed facts that this is indeed what we see–one does not pay for a haircut in the United States using the South Korean Won. Of course, what we expect to see in the model ought to be the result of a possible equilibrium in an environment where agents are not *a priori* restricted to hold a particular currency. One possible microfoundation lies in sellers' unwillingless to accept a currency they do not recognize [see e.g. Lester, Postlewaite, and Wright, 2008]. However, these explorations are beyond the scope of this paper. We have nevertheless tried a version of the model where we do not restrict trades of local goods to local currencies. In this case, the composition of each agent's currency portfolio, and therefore, the nominal exchange rate, will be indeterminate–i.e. we have a stochastic version of Kareken and Wallace [1981]. Unfortunately, this less stringent version of the model does not admit any stable rational expectations equilibrium, given a data consistent calibration of the model.

containing (m, b, k, l), for each vector of exogenous state variables, $\mathbf{z}$.[2] Also, let the space of all such distributions be $\mathcal{P}_{\mathbf{z}}(\mathcal{B}_i)$, for each fixed $\mathbf{z}$.

At the beginning of the time-t DM, the *aggregate* (global) state vector for an agent in country $i \in \{h, f\}$ is $\hat{\mathbf{s}} := (M, M^*, B, B^*, K, K^*, \phi, \phi^*, e, \nu_h, \nu_f, \mathbf{z})$. The explicit switch in notation from ν_i to μ_i takes into account that, in general, the distribution of assets upon the economy i entering each period's DM, ν_i, may be different to the distribution μ_i upon its leaving the DM, and into the CM, in the same period.[3]

2.4. **Timing.** Figure 1 depicts the sequence of events within each $t \in \mathbb{N}$. The relevant aggregate state vector $\mathbf{s}$ is realized at the beginning of each t. This is public information for all agents. An agent in the Home country, first entering the DM with assets (m, b, k) respectively, money, bonds, and capital, given $\hat{\mathbf{s}}$, is publicly known by the *individual* state $(\mathbf{a}, \hat{\mathbf{s}}) := (m, b, k, 0, \hat{\mathbf{s}})$. For simplicity, we make the restriction that each country i agent does not hold another country's currency as asset.[4] Since bilateral matches in the DM are random, agents within each country i only know the state of their trade partners *ex post*. *Ex ante* they only know the probability distribution of traders in the DM, which is $(\sigma, \sigma, 1 - 2\sigma)$ with support $\{Buyer, Seller, Neither\}$. Conditional on either events $\{Buyer\}$ or $\{Seller\}$, there is an identical distribution $\{\kappa, 1 - \kappa\}$ faced by the agent of a trade being either anonymous (monetary) or monitored (credit).

Upon leaving the DM, an agent's *individual* state changes to

$$(\mathbf{a}', \mathbf{s}) := \begin{cases} (m', b, k, 0, \mathbf{s}) & \text{w.p. } 2\sigma\kappa \\ (m, b, k, l, \mathbf{s}) & \text{w.p. } 2\sigma(1-\kappa) \end{cases}$$

reflecting the possibility that money had changed hands as a result of the agent being a buyer or seller. As a result of that, the distribution of assets (namely money) would also have changed from $\nu_i \in \hat{\mathbf{s}}$ to $\mu_i \in \mathbf{s}$. The components (b, k) have not changed since they are predetermined at the beginning of t. Thus, within t, the agent enters the CM with possible state $(\mathbf{a}', \mathbf{s})$. Agents do not discount payoffs within each period t.

[2]Note that if $Z = \emptyset$, *i.e.* in the absence of aggregate exogenous shocks, then the solution of the Markov equilibrium is characterized by a deterministic difference equation system, as in Lagos and Wright [2005]. Also, note that the aggregate prices (ϕ, ϕ^*, e) are explicitly included as (auxiliary) state variables, following Duffie, Geanakoplos, Mas-Colell, and McLennan [1994], so that we can restrict our characterization of equilibria to stationary Markov equilibria.

[3]It is straightforward to prove that the probability measures ν_i for each $i \in \{h, f\}$, is degenerate in any equilibrium, as a stochastic extension to the original proof in Lagos and Wright [2004]. This affords us plenty of tractability and ease of computation later.

[4]See Head and Shi [2003] for the environment where agents trade currency internationally.

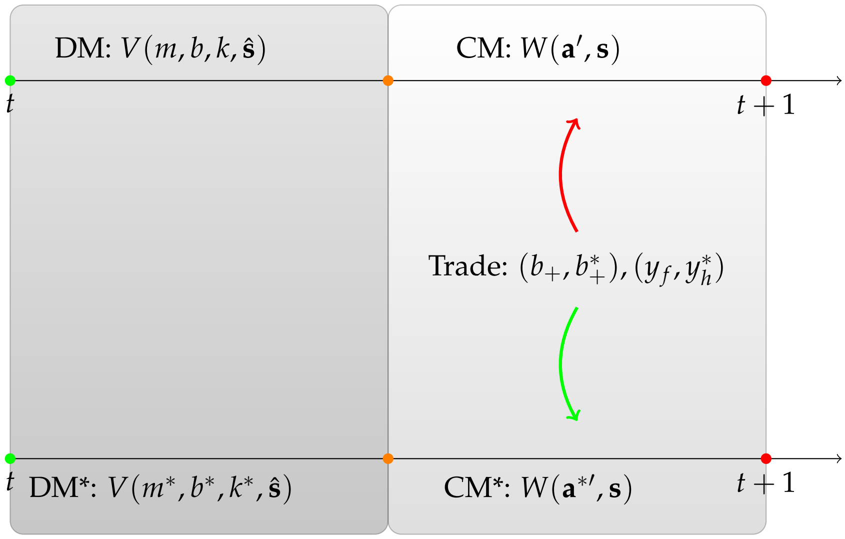

FIGURE 1. Agents engage in domestic decentralized trades (DM) over *special goods* during the night using money. During the day in the centralized market (CM), they trade domestically a *general good* which is not exported, and borrow or lend using nominal claims. Note: (i) In the "complete markets" version of the CM, we have $(b_+, b^*_+) := \{b(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s}), b^*(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s})\}$ denoting a complete set of money claims contingent on all measurable events $\{\mathbf{a}_+, \mathbf{s}_+\}$, reachable from $(\mathbf{a}, \mathbf{s})$; and (ii) In the "incomplete markets" version of the CM, we have $(b_+, b^*_+) := (b(\mathbf{s}), b^*(\mathbf{s}))$ denoting single non-contingent money claims payable in all continuation states $(\mathbf{a}_+, \mathbf{s}_+)$.

In the next two sections we describe in detail the sub-period problems, DM and CM, in a backward fashion. To economize on notation, we use the following convention. A variable or vector with a "$+$" subscript will denote its time $t+1$ contingent outcome. A state with a "$-$" subscript will denote its time $t-1$ realization. However, in some cases, variables with a "$+$" subscript, such as capital and bonds, are predetermined at the beginning of time $t+1$. In such cases, these are decision or control variables which will be made obvious in the problems below. The same variable without the "$+$" subscript denotes its current or time-t realization.

2.5. **Centralized markets.** In the CM, an agent consumes a general good $X \in \mathbb{R}_+$ which is produced using CM-specific labor $H \in \mathbb{R}_+$ and capital k. In contrast to Lagos and Wright [2005], we model a set of nominally complete state-contingent claims issued by both countries. Agents in each country's CM who consume more (less) than their total wealth can also trade in these securities. Note that we do not model international trade in final consumption goods or intermediate inputs. This keeps the model manageable and more importantly, allows us to focus on the effect of money and capital on the only channel for international linkage.

Let $A > 0$ be the constant marginal disutility of work effort. Let $\delta \in [0, 1]$ be the depreciation rate of capital and τ_K a proportional tax rate on capital income. Denote $r := r(\mathbf{s}) \equiv (1 - \tau_K)(\tilde{r}(\mathbf{s}) - \delta)$ and $r^* := r^*(\mathbf{s}) \equiv (1 - \tau_K)(\tilde{r}^*(\mathbf{s}) - \delta)$ be, respectively, the after-tax competitive rate of return to physical capital, $\tilde{r}$ net of depreciation, at Home and in the Foreign country. Similar, denote $w(\mathbf{s}) := (1 - \tau_H)\tilde{w}(\mathbf{s})$ as the after-tax real wage rate. Finally, denote τ_X as the proportional tax rate on CM consumption X.

Let $m_+ := m(\mathbf{a}, \mathbf{s})$, $k_+ := k(\mathbf{a}, \mathbf{s})$, and $b_+ := b(\mathbf{a}, \mathbf{s})$, so that $\mathbf{a}_+ = (m_+, b_+, k_+, 0)$. $Q(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s})$ is the domestic price of one unit of the state-contingent claim with nominal $(\mathbf{a}_{i+}, \mathbf{s}_+)$-contingent payoffs, *i.e.* $b(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s})$. Let $\phi := \phi(\mathbf{s}) = 1/p_X(\mathbf{s})$ and $\phi := \phi^*(\mathbf{s}) = 1/p^*_X(\mathbf{s})$ be the inverses of the prices of X (i.e. the value of a unit of each currency), in the respective Home and Foreign countries.

At each $t \in \mathbb{N}$, a price-taking agent (at the beginning of the CM sub-period in the Home country) named $(m, b, k, l, \mathbf{s})$ solves the recursive problem of:

$$W(m, b, k, l, \mathbf{s}) = \max_{X, H, m_+, k_+, b_+} \Big\{ U(X) - AH + \beta \int V(m_+, b_+, k_+, \mathbf{s}_+)\lambda(\mathbf{s}, d\hat{\mathbf{s}}_+) \Big\} \tag{1}$$

subject to

$$\mathbf{s}_+ = G(\mathbf{s}, \mathbf{v}_+), \qquad \mathbf{v} \overset{\text{i.i.d.}}{\sim} \varphi \tag{2}$$

$$\begin{aligned} &(1 + \tau_X)X(\mathbf{a}, \mathbf{s}) + k(\mathbf{a}, \mathbf{s}) - k - \phi(\mathbf{s})b + T(\mathbf{s}) \\ &\quad = \phi(\mathbf{s})\,[m - m(\mathbf{a}, \mathbf{s}) - l] + w(\mathbf{s})H(\mathbf{a}, \mathbf{s}) + r(\mathbf{s})k \\ &\quad - \phi(\mathbf{s}) \iint_{\mathbf{s}_+, \mathbf{a}_+} b(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s})Q(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s})\mu_h(\mathbf{s}_+, d\mathbf{a}_+)\lambda(\mathbf{s}, d\mathbf{s}_+) \end{aligned} \tag{3}$$

where $\lambda(\mathbf{s}, \cdot)$, for each given $\mathbf{s}$, is induced by $\mathcal{G} \circ \varphi$, and defines an equilibrium product probability measure over Borel-subsets containing $\hat{\mathbf{s}}_+$. Constraint (2) describes a transition law, where the mapping $\mathcal{G} = \mathcal{G}_{\{\mathbf{s}\}\setminus\{\mathbf{z}\}} \circ \mathcal{G}_{\{\mathbf{z}\}}$, with component $\mathcal{G}_{\{\mathbf{s}\}\setminus\{\mathbf{z}\}}$ inducing the $\mathbf{z}$-dependent stochastic process for endogenous aggregate states, $\{\mathbf{s}\} \setminus \{\mathbf{z}\}$, is to be pinned down in equilibrium, and $(\mathbf{z}, \mathbf{v}_+) \mapsto \mathcal{G}_{\{\mathbf{z}\}}(\mathbf{z}, \mathbf{v}_+)$ is an exogenous map for the aggregate shocks. Implicit in the constraint (2) is the equilibrium transition of the distribution of *individual* states from the period-t CM, to the period-$(t+1)$ DM:

$$\nu_h(\hat{\mathbf{s}}_+, \cdot) = \mathcal{G}_\nu \left[\mu_h(\mathbf{s}, \cdot), \mathbf{z}_+\right]. \tag{4}$$

such that the relevant conditional distribution of assets at the beginning of the time-$(t+1)$ CM subperiod is given by:

$$\mu_h(\mathbf{s}_+, \cdot) = \mathcal{G}_\mu \left[\nu_h(\hat{\mathbf{s}}_+, \cdot), \mathbf{z}_+\right] \equiv \mathcal{G}_\mu \circ \mathcal{G}_\nu(\mathbf{s}, \mathbf{z}_+). \tag{5}$$

where $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$ are components of $\mathcal{G}_{\{\mathbf{s}\}\backslash\{\mathbf{z}\}}$.

The sequence of state-contingent one-period budget constraints given by (3) say the following: For each given state $(m, b, k, \mathbf{s})$, consumption of the general good X is to be financed by the variation in real money holdings, by real labor income wH, net of investment flows to physical capital made in the CM, net of contingent claims in real terms, and net of lump-sum government taxes, T.

2.5.1. *Optimal individuals' decisions in the CM.* Eliminating H in (1) using the budget constraint (3), the optimal decision rules satisfy the following conditions for every state $(\mathbf{a}, \mathbf{s})$ and every measurable continuation state $(\mathbf{a}_+, \hat{\mathbf{s}}_+)$.

The optimal trade-off between current CM consumption X and leisure $-H$, given the competitive real wage $w := w(\mathbf{s})$, is

$$X: \qquad U_X\left[X(\mathbf{a}, \mathbf{s})\right] = \frac{A(1+\tau_X)}{w(\mathbf{s})}, \tag{6}$$

where the marginal utility of leisure is a constant $A > 0$. In what follows, note that the derivative functions of V is determined in general equilibrium. The optimal trade-off between a current increase in marginal utility of X in the CM and the present-value expected marginal value of entering the next-period DM with a marginal increment of money holdings is

$$m_+: \qquad \frac{A\phi(\mathbf{s})}{w(\mathbf{s})} = \beta \int V_{m_+}(m_+, b_+, k_+, 0, \hat{\mathbf{s}}_+)\lambda(\mathbf{s}, d\hat{\mathbf{s}}_+). \tag{7}$$

Similar to condition (7), conditions (8)-(9) below provide the optimal trade-offs between the current utility of consumption of X and the expected discounted marginal value of entering the DM with more assets. Specifically, the optimal choice of the complete state-contingent money claims, or bonds, is given by

$$b_+(\cdot;\mathbf{s}): \qquad \frac{A\phi(\mathbf{s})}{w(\mathbf{s})}\left[Q(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s})\mu_h(\mathbf{s}_+, d\mathbf{a}_+)\right]\lambda(\mathbf{s}, d\hat{\mathbf{s}}_+)$$
$$= \beta V_{b_+}(m_+, b_+, k_+, 0, \hat{\mathbf{s}}_+), \tag{8}$$

which holds for every $\mathbf{s}$, every $\hat{\mathbf{s}}_+$, and implicitly, every $\mathbf{s}_+$.

The optimal choice of the Home-produced capital stock available for production in the next period satisfies

$$k_+ : \qquad \frac{A}{w(\mathbf{s})} = \beta \int V_{k_+}(m_+, b_+, k_+, 0, \hat{\mathbf{s}}_+)\lambda(\mathbf{s}, d\hat{\mathbf{s}}_+). \tag{9}$$

2.5.2. *Envelope conditions for W in the CM.* At an optimum, the envelope conditions for the agent's CM decision problem are as follows. The marginal value of money holdings upon entering the CM is

$$W_m(m, b, k, l, \mathbf{s}) = \frac{A\phi(\mathbf{s})}{w(\mathbf{s})}, \tag{10}$$

the marginal value of holding bonds upon entering the CM, respectively, are

$$W_b(m, b, k, l, \mathbf{s}) = \frac{A\phi(\mathbf{s})}{w(\mathbf{s})}, \tag{11}$$

and the marginal value of holding the each of the four types of capital stocks at the beginning of the CM are as follows. With respect to a Home agent's holding of capital stock in the Home country, the marginal CM value is:

$$W_k(m, b, k, l, \mathbf{s}) = \frac{A}{w(\mathbf{s})}\left[1 + r(\mathbf{s})\right]. \tag{12}$$

With respect to a Home agent's holding of credit in the Home country, the marginal CM value is:

$$W_l(m, b, k, l, \mathbf{s}) = -\frac{A\phi(\mathbf{s})}{w(\mathbf{s})}. \tag{13}$$

The envelope conditions (10)-(13) imply that, W is linear in (m, b, k, l), for each fixed aggregate state $\mathbf{s}$. So we can write W as

$$W(m, b, k, l, \mathbf{s}) = W(0, 0, \mathbf{0}, \mathbf{s}) + \frac{A}{w}\Big[\phi(m + b) + (1 + r)k\Big]. \tag{14}$$

2.5.3. *Firms.* Let P_h be the Home currency price of the Home produced intermediate good, and P_y be that of the Foreign produced intermediate good use by the Home final-good firm. The Home final-good firm solves:

$$\max_{y_h, y_f} \left\{ \frac{G[y_h(\mathbf{s}), y_f(\mathbf{s})]}{\phi(\mathbf{s})} - P_h(\mathbf{s})y_h(\mathbf{s}) - P_f(\mathbf{s})y_f(\mathbf{s}) \right\}.$$

The profit-maximizing conditions are:

$$\phi(\mathbf{s})P_h(\mathbf{s}) = G_{y_h}[y_h(\mathbf{s}), y_f(\mathbf{s})], \tag{15}$$

and

$$\phi(\mathbf{s})P_f(\mathbf{s}) = G_{y_f}[y_h(\mathbf{s}), y_f(\mathbf{s})]. \tag{16}$$

The Home intermediate goods producer solves

$$\max_{H,K} \left\{ P_{y_h}(\mathbf{s}) \cdot zF_k[K(\mathbf{s}_-), H(\mathbf{s})] - \frac{[w(\mathbf{s})H(\mathbf{s}) + r(\mathbf{s})K(\mathbf{s}_-)]}{\phi(\mathbf{s})} \right\}.$$

where the market for inputs to F is perfectly competitive. Profit maximization is characterized by the usual first order conditions where capital and labor are paid a respective rental rate which equals their marginal products in every aggregate state $\mathbf{s}$:

$$\tilde{r}(\mathbf{s}) = \phi(\mathbf{s})P_h(\mathbf{s}) \cdot zF_k[K(\mathbf{s}_-), H(\mathbf{s})], \tag{17}$$

and

$$\tilde{w}(\mathbf{s}) = \phi(\mathbf{s})P_h(\mathbf{s}) \cdot zF_H[K(\mathbf{s}_-), H(\mathbf{s})], \tag{18}$$

where

$$H(\mathbf{s}) = \int_{\mathbf{a}} H(\mathbf{a}, \mathbf{s})\mu_h(\mathbf{s}, d\mathbf{a})$$

is aggregate labor supply in the Home CM.

Without loss of generality for the rest of the model, we shall assume that $(z_t)_{t\in\mathbb{N}}$ is induced by some Markov process to be described in the application later on. A foreign country's CM agent named $(m^*, b^*, k^*, l^*, \mathbf{s})$ and its firm have a symmetric problem to (1)-(3), (15)-(16), and (17)-(18).

2.6. **Decentralized markets.** At the beginning of the CM at each $t \in \mathbb{N}$, an agent named $(m, b, k, 0, \hat{\mathbf{s}})$ enters the DM.[5] With a fixed probability σ this agent is the buyer of the special good that some other agent produces, q^b, where the other agent (seller) is indexed by the state $(\tilde{\mathbf{a}}, \hat{\mathbf{s}}) := (\tilde{m}, \tilde{b}, \tilde{k}, \hat{\mathbf{s}})$, but not vice-versa. With probability $\sigma\kappa$, the buyer parts with d^b "dollars" and realizes a payoff of $u(q^b) \in \mathbb{R}$. The buyer then enters the day CM with a value of $W\left(m - d^b, b, k, 0, \mathbf{s}\right)$. With probability $\sigma(1-\kappa)$, the buyer does not use money, but takes out a nominal loan l, from the seller he meets, and realizes a payoff of $u(\breve{q}^b) \in \mathbb{R}$. The buyer then enters the day CM with a value of $W(m, b, k, l, \mathbf{s})$.

[5]Note that m implicitly includes any aggregate monetary transfer or injection from the government, which we denote later as $\iota(\hat{\mathbf{s}})$, so then, $m(\hat{\mathbf{s}}) = m(\mathbf{s}_-) + \iota(\hat{\mathbf{s}})$.

Symmetrically, with probability $\sigma\kappa$, agent $(m, b, k, \hat{\mathbf{s}})$ has a special good q^s which other buyers want to buy, but not vice-versa. This agent receives d^s dollars in exchange for exerting a production cost of $c(q^s, k) \in \mathbb{R}_+$. Notice that capital obtained from the previous period's CM, k, accrues a return in the DM in the form of the marginal benefit to producing q (q^s or $\breve{q}^s$), i.e. $c_k(q, k)$.[6] This seller then enters the day CM with a value of $W(m + d^s, b, k, \mathbf{s})$. With probability $\sigma(1-\kappa)$, a seller may sell $\breve{q}^s$ by extending a loan l to a matched buyer.

These four events described above are known as single-coincidence-of-wants meetings, where money is a portable medium of exchange in events that occur with probability $2\sigma\kappa$, and where credit l is the medium of exchange in events with probability $2\sigma(1-\kappa)$. With probability $1 - 2\sigma$, agent $(m, b, k, \hat{\mathbf{s}})$ leaves the DM and enters the day with his assets intact, and begins his activity in the CM with value $W(m, b, k, 0, \mathbf{s})$. For simplicity, we assume the probability of a "double-coincidence" meeting, and hence the occurrence of pure barter, is zero.

Formally, an agent named $(m, b, k, 0, \hat{\mathbf{s}})$ has a value $V(m, b, k, 0, \hat{\mathbf{s}})$ at the beginning of the DM that satisfies the following problem:

$$\begin{aligned} V(m, b, k, 0, \hat{\mathbf{s}}) = \sigma V^b(m, b, k, 0, \hat{\mathbf{s}}) \\ + \sigma V^s(m, b, k, 0) + (1-2\sigma) W(m, b, k, \mathbf{s}). \end{aligned} \tag{19}$$

where, in general:

$$\begin{aligned} V^b(m, b, k, 0) =& \kappa \int \left[u(q^b) + W\left(m - d^b, b, k, 0, \mathbf{s}\right)\right] \nu_h(d\tilde{\mathbf{a}}, \hat{\mathbf{s}}) \\ &+ (1-\kappa) \int \left[u(\breve{q}^b) + W\left(m, b, k, l^b, \mathbf{s}\right)\right] \nu_h(d\tilde{\mathbf{a}}, \hat{\mathbf{s}}), \end{aligned}$$

and,

$$\begin{aligned} V^s(m, b, k, 0) =& \kappa \int \left[-c(q^s, k) + W\left(m + d^s, b, k, 0, \mathbf{s}\right)\right] \nu_h(d\tilde{\mathbf{a}}, \hat{\mathbf{s}}) \\ &+ (1-\kappa) \int \left[-c(\breve{q}^s, k) + W\left(m, b, k, -l^s, \mathbf{s}\right)\right] \nu_h(d\tilde{\mathbf{a}}, \hat{\mathbf{s}}). \end{aligned}$$

are the value functions of ex-post buyer and sellers respectively.

[6] This feature was first introduced by Aruoba, Waller, and Wright [2008, Appendix A.1]. The authors showed that whether there exist two kinds of capital goods, for use in the DM and in the CM production, respectively, is of negligible quantitative consequence in their model.

2.6.1. *Walrasian price taking.* Consider a version where $(q^b, q^s, d^b, d^s, \breve{q}^b, \breve{q}^s, l^b, l^s)$ are determined by Walrasian price taking. Then, we have:

$$\begin{aligned} V^b(m,b,k,0) =& \kappa \max_{q^b \in [0, m/\tilde{p}]} \left[u(q^b) + W\left(m - \tilde{p}q^b, b, k, 0, \mathbf{s}\right)\right] \\ &+ (1-\kappa) \max_{\breve{q} \in [0, l^b/\breve{p}]} \left[u(\breve{q}^b) + W\left(m, b, k, l^b, \mathbf{s}\right)\right], \end{aligned}$$

where $d^b = \tilde{p}q^b$, and,

$$\begin{aligned} V^s(m,b,k,0) =& \kappa \max_{q^s} \left[-c(q^s,k) + W\left(m + d^s, b, k, 0, \mathbf{s}\right)\right] \\ &+ (1-\kappa) \max_{\breve{q}^s} \left[-c(\breve{q}^s,k) + W\left(m, b, k, -l^s, \mathbf{s}\right)\right], \end{aligned}$$

where $\tilde{p}$ and $\breve{p}$ are the respective prices of the special good, taken as given by all buyers and sellers.

2.7. **Monetary policy.** New money supply is injected at the end of the period in the CM.[7] Specifically, the monetary authority follows a monetary supply rule:

$$M(\mathbf{s}) = \exp(\psi) M(\mathbf{s}_-), \tag{20}$$

where $\exp\{\psi\} - 1$ is the one-period money supply growth rate between time t and $t+1$. Assume that $(\exp(\psi_t))_{t \in \mathbb{N}}$ follows a Markov process that lives in the compact set $[1, N]$, with $N < +\infty$. We define this process later.

The monetary authority's nominal budget constraint for each $\mathbf{s}$ is:

$$M(\mathbf{s}) = \iota(\mathbf{s}) + M(\mathbf{s}_-). \tag{21}$$

Thus, aggregate money injection $\iota(\mathbf{s})$ is given by:

$$\iota(\mathbf{s}) = [\exp(\psi) - 1]\, M(\mathbf{s}_-).$$

Also, note that if we set $\exp(\psi) = 1$ for all periods, then money supply remains constant forever. In this case, $\iota(\mathbf{s}) = 0$ for all $\mathbf{s}$ so no new money is injected into the economy. Again, the Foreign country would have a symmetric description of its government policy.

3. Stationary Markov Monetary Equilibrium

In this section, we state a key result which is just an extension of Lagos and Wright [2005] to environments with aggregate uncertainty. We claim here that in an equilibrium, the endogenous distribution of agents' asset holdings is degenerate at

[7] This is merely for mathematical convenience, so that within each DM, agents do not have to deal with a stochastic total payoff function, W.

the start of each period (and hence DM), such that all agents in each country choose the same allocations that depend only on the global state. We further characterize the equilibrium conditions in the DM and list the conditions for market clearing in the CM. We then define the elements that constitute a *stationary Markov monetary equilibrium*, which includes non-Walrasian or decentralized bilateral random matches with alternative pricing mechanisms: Walrasian price-taking, generalized Nash bargaining, and proportional bargaining.

In general, because of the memory-less random matching process in the DM, we will need to track the history of aggregate distribution of assets held by agents in any equilibrium where money has value. However, because of the quasi-linear assumption on each agent's per-period payoff function, it can be shown that in equilibrium asset holdings at the beginning of each $t \in \mathbb{N}$ are identical across all agents within each country i, so that,

$$\begin{aligned}(m,b,k)(\mathbf{s}) &= \int (m,b,k)\nu_i(\hat{\mathbf{s}}, dm, db, dk) \\ &=: (M,B,K)(\hat{\mathbf{s}}) \\ &=: (M,B,K)(\mathbf{z}). \end{aligned} \tag{22}$$

for each $i \in \{h,f\}$, for all $\hat{\mathbf{s}}$. This implies that we can explicitly write $\nu(\hat{\mathbf{s}},\cdot)$ as $\nu(\mathbf{z},\cdot)$, and furthermore, for every $\mathbf{z}$, and every $A \in \mathcal{B}_i(\mathbf{z})$,

$$\nu_i(\mathbf{z}, A) = \begin{cases} 1 & \text{if } (m,b,k) = (M,B,K) \in A \\ 0 & \text{otherwise} \end{cases}.$$

However, we can see that even if $\nu_i(\mathbf{z},\cdot)$ is degenerate at the end of the CM, $\mu_i(\mathbf{z},\cdot)$ is not. Thus, explicitly, agents at the beginning of each CM will still face an aggregate state variable $\mathbf{s}$ that contains a non-degenerate distribution of individual states. Specifically, the non-degeneracy is along the dimension of money holdings out of the DM.

3.1. **Walrasian Price Taking and equilibrium decisions.** In equilibrium, the constraints $d \leq m$, and $l \leq \check{p}\check{q}$ bind, and $q^b = q^s = q$. Thus for the $\sigma\kappa$ proportion of agents who are sellers that meet buyers and they trade with money, we have the equilibrium condition that the relative price of the special good in terms of the CM final good, is equal to the marginal cost of the DM seller:

$$\frac{A\phi}{w} M = c_q(q,K)q \equiv g(q,K). \tag{23}$$

Note that $\tilde{p} = M/q$ in equilibrium.

For the $\sigma(1-\kappa)$ proportion of buyers and sellers, we have:

$$\frac{A\phi}{w}l = c_q(\breve{q}, K)\breve{q} \equiv g(\breve{q}, K). \tag{24}$$

Since by assumption contracts are enforceable for these agents, then credit attains the first best allocation in terms of $\breve{q}$ satisfying

$$u_q(\breve{q}) = c_q(\breve{q}, K). \tag{25}$$

Therefore we can substitute out credit in the equilibrium conditions later, using

$$l = \frac{w u_q(\breve{q})\breve{q}}{A\phi}. \tag{26}$$

3.2. **Envelope conditions for V in the DM.** At an interior optimum consistent with equilibrium, we have the following envelope conditions. Utilizing the linearity of W, the marginal value of money at the beginning of the DM is

$$V_M(M, B, K, 0, \hat{\mathbf{s}}) = \frac{A\phi}{w}\left[(1-\sigma\kappa) + \sigma\kappa\frac{u_q(q)}{g_q(q, K)}\right] > 0. \tag{27}$$

The marginal value of the state-contingent money claims at the beginning of the DM is

$$V_B(M, B, K, 0, \hat{\mathbf{s}}) = W_b(M, B, K, 0, \mathbf{s}) = \frac{A\phi}{w}. \tag{28}$$

The DM marginal value of the capital stock, is

$$V_K(M, B, K, 0, \hat{\mathbf{s}}) = \frac{A\phi}{w}(1+r) - \sigma\kappa\gamma(q, K) - \sigma(1-\kappa)\gamma(\breve{q}, K) > 0, \tag{29}$$

where

$$\gamma(q, K) = c_K(q, K) < 0. \tag{30}$$

The function γ is strictly negative due to two effects that capture the reduction in marginal cost of production in the DM. The first term on the right of (30) is the indirect effect on marginal cost through the effect of an additional capital stock on the terms of trade q.

This reflects the fact that parties to a monetary trade in the DM are not price takers. The terms of trade determined by the bargaining solution thus takes into account the seller's stock of capital in the DM. The second term is the capital-effort complementarity effect in the production of q in the sense that marginally more capital stock reduces the cost of producing q.

3.3. **Market clearing in the CM.** In an equilibrium, since agents within each country choose the same asset holdings, i.e. $(m, b, k) = (M, B, K)$, then they do not borrow from, or, lend to each other, only countries lend to each other. Therefore, in the global equilibrium, state-contingent money claims by Home and Foreign have zero excess demand:

$$B(\mathbf{s}) + B^*(\mathbf{s}) = 0. \tag{31}$$

in every state $\mathbf{s}$. The Home resource constraint is given by

$$G[y_h(\mathbf{s}), y_f(\mathbf{s})] = X(\mathbf{s}) + I(\mathbf{s}) + G^d(\mathbf{s}), \tag{32}$$

where $I(\mathbf{s}) = K(\mathbf{s}) - (1-\delta)K(\mathbf{s}_-)$ is domestic capital investment, and,

$$G^d(\mathbf{s}) = [T(\mathbf{s}) + (M(\mathbf{s}) - M(\mathbf{s}_-))\phi(\mathbf{s})] \\ + \tau_X X(\mathbf{s}) + \tau_H H(\mathbf{s}) + \tau_K(\tilde{r}(\mathbf{s}) - \delta)K(\mathbf{s}_-).$$

The Foreign resource constraint is given by

$$G[y_f^*(\mathbf{s}), y_h^*(\mathbf{s})] = X^*(\mathbf{s}) + I^*(\mathbf{s}) + G^{d*}(\mathbf{s}), \tag{33}$$

where $I^*(\mathbf{s}) = K^*(\mathbf{s}) - (1-\delta)K^*(\mathbf{s}_-)$ is the Foreign country's investment in its own capital stock, and,

$$G^{d*}(\mathbf{s}) = [T^*(\mathbf{s}) + (M^*(\mathbf{s}) - M^*(\mathbf{s}_-))\phi^*(\mathbf{s})] \\ + \tau_X X^*(\mathbf{s}) + \tau_H H^*(\mathbf{s}) + \tau_K(\tilde{r}^*(\mathbf{s}) - \delta)K^*(\mathbf{s}_-).$$

Market clearing for the intermediate goods must hold:

$$zF[K(\mathbf{s}_-), H(\mathbf{s})] = y_h(\mathbf{s}) + y_h^*(\mathbf{s}) \tag{34}$$

$$z^*F[K^*(\mathbf{s}_-), H^*(\mathbf{s})] = y_f^*(\mathbf{s}) + y_f(\mathbf{s}) \tag{35}$$

Definition 1. *A monetary stationary Markov equilibrium (SME), given any feasible monetary policy rule (ψ, ψ^*), is a set of time-invariant maps consisting of*

E1. *strictly positive pricing functions (ϕ, ϕ^*, e) and (w, r, w^*, r^*, Q),*
E2. *transition laws $(\mathcal{G}, \varphi)$ and $(\mathcal{G}^*, \varphi^*)$,*
E3. *value functions V, W and V^*, W^*,*
E4. *CM decision rules $(X, X^*, m, m^*, b, k, b^*, k^*)$, and*
E5. *DM terms of trade (decision rules), $(d, q, \check{q})$ and $(d^*, q^*, \check{q}^*)$,*

such that:

(1) *given prices (E1), the value functions V and W satisfy the functional equations (1), (2), (3), and (19) and symmetrically V^*, W^* solve the Foreign country counterpart problems;*
(2) *given the value functions V and W, and prices (E1), the decision rules E4 solve (1), (2), (3) in the CM, for the Home country and symmetrically for the Foreign country, given V^* and W^*;*
(3) *Firms optimize: (17) and (18);*
(4) *given the value functions W and V, the decision rules E5 solve and (23), (25), and (26) in the DM, and symmetrically for the Foreign country, given W^*;*
(5) *The government budget constraint (21) is satisfied for Home and symmetrically for Foreign.*
(6) *Markets clear in the CM and CM*: (31), (32) and (33), where $m = M$, $b = B$ and $k = K$, and $m^* = M^*$, $b^* = B^*$ and $k^* = K^*$.*

4. Equilibrium Asset-pricing Properties

We are now in a position to gain further insights into the international asset pricing properties of the model. The insight will be driven by the endogenous complementarity effect of capital k on the production cost in the DM, measured by the endogenous function γ in equation (30). This γ acts as an endogenous wedge in equilibrium asset pricing conditions in the model.

For ease of notation and exposition, and without loss of generality, we set $\kappa = 1$ for now and $\tau_X = \tau_H = \tau_K = 0$. Using the first-order conditions in the CM and DM, the corresponding envelope conditions, and imposing equilibrium, we can derive a set of stochastic Euler functional equations necessary for characterizing a *stationary Markovian equilibrium* (SME). We can write the SME conditions as ones that characterize the solutions as $\mathbf{s}$-dependent processes. Recall that in any equilibrium, agents end up choosing the same asset allocations regardless of their personal state. Thus, with a slight abuse of notation, we drop the dependency on aggregate state variables such as $\mu_i(\mathbf{s}, \cdot)$, $i \in \{h, f\}$, from the definition of $\mathbf{s}$ in equilibrium. In other words, the Euler equations below will have the appearance as though they were—and indeed they are—characterizing equilibrium of some representative agent model.

First, from (6), we can easily deduce that in equilibrium, $X(\mathbf{a}, \mathbf{s}) = X(\mathbf{s})$ and $X^*(\mathbf{a}_f, \mathbf{s}) = X^*(\mathbf{s})$, for all $\mathbf{s}$. Also, we have, in equilibrium, $q(m, \tilde{k}, \mathbf{s}) = q(M, K, \mathbf{s}) \equiv q(\mathbf{s})$ and $q^*(m^*, \tilde{k}^*, \mathbf{s}) = q^*(M^*, K^*, \mathbf{s}) \equiv q^*(\mathbf{s})$. Together with (7) and (27), we have the SME version of the Euler functional equation for optimal money holdings in the Home country:

$$U_X[X(\mathbf{s})] = \beta \mathbb{E}_\lambda \left\{ U_X[X(\mathbf{s}_+)] \frac{\phi(\mathbf{s}_+)}{\phi(\mathbf{s})} \left[(1-\sigma) + \sigma \frac{u_q[q(\mathbf{s}_+)]}{g_q[q(\mathbf{s}_+), K(\mathbf{s})]} \right] \right\}, \tag{36}$$

where, $\mathbb{E}_\lambda$ denotes the expectation operator with respect to the conditional distribution $\lambda(\mathbf{s}, \cdot)$, and, the term in the square brackets is the state-contingent one-period nominal gross return on money holding. This return is made up of two terms: (i) With measure $1-\sigma$, the positive one-for-one effect on the value of entering the CM with more money holding in the case that the agent does not trade in the DM net of the negative one-for-one effect on the value of entering the CM with less money holding in the case that the agent spent his money holding as buyer; and (ii) the positive marginal effect of money holding on utility of consumption via the specific good q, conditional of the agent being a buyer with probability σ. Alternatively, we can write (36) as an integral-functional equation solely in terms of the functions q and K:

$$\frac{g[q(\mathbf{s}), K(\mathbf{s}_-)]}{M(\mathbf{s}_-)} = \beta \mathbb{E}_\lambda \left\{ \frac{g[q(\mathbf{s}_+), K(\mathbf{s})]}{M(\mathbf{s})} \left[(1-\sigma) + \sigma \frac{u_q[q(\mathbf{s}_+)]}{g_q[q(\mathbf{s}_+), K(\mathbf{s})]} \right] \right\}. \tag{37}$$

A parallel of this equation for the Foreign country characterizing equilibrium q^* is:

$$\frac{g[q^*(\mathbf{s}), K^*(\mathbf{s}_-)]}{M^*(\mathbf{s}_-)} = \beta \mathbb{E}_\lambda \left\{ \frac{g[q^*(\mathbf{s}_+),_+ K^*(\mathbf{s})]}{M^*(\mathbf{s})} \left[(1-\sigma) + \sigma \frac{u_q[q^*(\mathbf{s}_+)]}{g_q[q^*(\mathbf{s}_+),_+ K^*(\mathbf{s})]} \right] \right\}. \tag{38}$$

Second, since in equilibrium, $X(\mathbf{a}, \mathbf{s}) = X(\mathbf{s})$ for all $\mathbf{s}$, along with (8) and (28), we then have an Euler equation for optimal Home bond holdings:

$$\begin{aligned} Q(\mathbf{s}_+|\mathbf{s}) &:= \left[\int_{\mathbf{a}_+} Q(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s}) \mu_h(\mathbf{s}_+, d\mathbf{a}_+) \right] \lambda(\mathbf{s}, d\mathbf{s}_+) \\ &= \beta \frac{U_X[X(\mathbf{s}_+)]}{U_X[X(\mathbf{s})]} \frac{\phi(\mathbf{s}_+)}{\phi(\mathbf{s})} \lambda(\mathbf{s}, d\mathbf{s}_+), \qquad \forall \mathbf{s}, \mathbf{s}_+. \end{aligned} \tag{39}$$

Third, Foreign agents would also have a first order condition for bonds similar to (39), which, in Home currency terms is:

$$\begin{aligned} Q(\mathbf{s}_+|\mathbf{s}) &:= \left[\int_{\mathbf{a}^*_+} Q(\mathbf{a}^*_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s}) \mu_f(\mathbf{s}_+, d\mathbf{a}^*_+) \right] \lambda(\mathbf{s}, d\mathbf{s}_+) \\ &= \beta \frac{U_X[X^*(\mathbf{s}_+)]}{U_X[X^*(\mathbf{s})]} \frac{\phi^*(\mathbf{s}_+)}{\phi^*(\mathbf{s})} \frac{e(\mathbf{s})}{e(\mathbf{s}_+)} \lambda(\mathbf{s}, d\mathbf{s}_+), \qquad \forall \mathbf{s}, \mathbf{s}_+. \end{aligned} \tag{40}$$

From (6), (9) and knowing V_K, we have an Euler equation for optimal Home capital holdings:

$$U_X[X(\mathbf{s})] = \beta\mathbb{E}_\lambda\left\{U_X[X(\mathbf{s}_+)]\left[(1+r(\mathbf{s}_+)-\delta)-\sigma\frac{\gamma[q(\mathbf{s}_+),K(\mathbf{s})]}{U_X[X(\mathbf{s}_+)]}\right]\right\}. \tag{41}$$

The equation characterizing optimal holding of capital by Foreign agents is:

$$U_X[X^*(\mathbf{s})] = \beta\mathbb{E}_\lambda\left\{U_X[X^*(\mathbf{s}_+)]\left[(1+r^*(\mathbf{s}_+)-\delta)-\sigma\frac{\gamma[q^*(\mathbf{s}_+),K^*(\mathbf{s})]}{U_X[X^*(\mathbf{s}_+)]}\right]\right\}. \tag{42}$$

4.1. **Inspecting the mechanism.** Equating the pricing kernel from (39) and (40) and iterating, we have

$$\frac{U_X[X(\mathbf{s})]}{U_X[X(\mathbf{s}_0)]}\frac{\phi(\mathbf{s})}{\phi(\mathbf{s}_0)} = \frac{U_X[X^*(\mathbf{s})]}{U_X[X^*(\mathbf{s}_0)]}\frac{e(\mathbf{s}_0)}{e(\mathbf{s})}\frac{\phi^*(\mathbf{s})}{\phi^*(\mathbf{s}_0)} \tag{43}$$

where $\mathbf{s}_0$ is the initial aggregate state. Assume that the initial condition, given by

$$\kappa_0 := \frac{e(\mathbf{s}_0)U_X[X(\mathbf{s}_0)]\phi(\mathbf{s}_0)}{U_X[X^*(\mathbf{s}_0)]\phi^*(\mathbf{s}_0)}$$

is fixed. we can re-write the above expression in (43) as the equilibrium determination of the nominal exchange rate:

$$e(\mathbf{s}) = \kappa_0\frac{U_X[X^*(\mathbf{s})]}{U_X[X(\mathbf{s})]}\frac{\phi^*(\mathbf{s})}{\phi(\mathbf{s})}. \tag{44}$$

This warrants some remark. Up to this point, in terms of equilibrium complete state-contingent money claims, we have derived a standard complete markets result for the nominal exchange rate [see e.g. Chari, Kehoe, and McGrattan, 2002]. Specifically, what equation (44) says is that the nominal exchange rate, at each state of the world, is proportional to the within-period the relative value of the marginal rate of substitution of the general good between Home and Foreign consumers. Intuitively, and quite obviously, in the absence of frictions, we will obtain similar dynamics for the nominal exchange rate, x, as in the standard international real business cycle model with money [see Schlagenhauf and Wrase, 1995].

However, we can show that in our model with monetary search friction and capital complementarity in DM production, there exists an additional function that acts as a connection between outcomes in the nontraded DM to the marginal utility of X expressions, and thus, the nominal exchange rate x. To see this, we can insert the conditions (41) and (42) into (44) to obtain a relationship between the nominal exchange rate, e, and DM activity:

$$e(\mathbf{s}) \propto \frac{\phi^*(\mathbf{s})}{\phi(\mathbf{s})} \frac{\mathbb{E}_\lambda \left\{ U_X[X(\mathbf{s}_+)] \left[(1 + z_+ F_k[K(\mathbf{s}), H(\mathbf{s}_+)] - \delta) - \sigma \frac{\gamma[q(\mathbf{s}_+), K(\mathbf{s})]}{U_X[X(\mathbf{s}_+)]} \right] \right\}}{\mathbb{E}_\lambda \left\{ U_X[X^*(\mathbf{s}_+)] \left[(1 + z^*_+ F_k[K^*(\mathbf{s}), H^*(\mathbf{s}_+)] - \delta) - \sigma \frac{\gamma[q^*(\mathbf{s}_+), K^*(\mathbf{s})]}{U_X[X^*(\mathbf{s}_+)]} \right] \right\}}. \tag{45}$$

More generally, the key mechanisms that makes the difference in our model are as follows:

(1) The equilibrium price processes $(\phi^{-1}, (\phi^*)^{-1})$ that depend on the interaction between DM and CM outcomes;
(2) The link between DM and CM which shows up as the additional return to capital (encoded in the function γ) and money (implicitly in the fact that $c_{qK} < 0$);
(3) and the interaction of these with DM market incompleteness (individual-level uncertainty), indexed by the probabilities $\{\sigma\kappa, \sigma\kappa, \sigma(1-\kappa), \sigma(1-\kappa), 1-2\sigma\}$ on the support $\{buyer-money, seller-money, buyer-credit, seller-credit, none\}$.

These are well-understood in the search and money literature, but their implications for the international exchange rate business cycle dynamics have not been explored. To understand what the model is capable of, we will next resort to a calibration exercise to discipline the choice of parameterization.

4.2. **Other variable definitions.** Since the model features a DM sector that is akin to a nontraded goods sector, we will need to define a relevant price index in order to define a real exchange rate. Empirically, the relevant exchange rate will have to a some measure of a broad index, such as The U.S. Federal Reserve Board of Governor's Broad Index used by Heathcote and Perri [2002]. First we define a DM price index as the convex combination of the pricing outcome in monetary and credit trades:

$$p_{DM} := \kappa\tilde{p} + (1-\kappa)\check{p}.$$

The foreign counterpart will be p^*_{DM}. Denote the aggregate DM consumption as

$$q_{DM} := \kappa q + (1-\kappa)\check{q}.$$

Now we can define our measure of consumer price index as

$$P_{CPI} = \zeta\phi^{-1} + (1-\zeta)p_{DM},$$

where

$$\zeta = \frac{X}{X + \sigma q_{DM}},$$

is the CM consumption share in total domestic consumption. Note that this share is time-varying in the sense that it is dependent on the aggregate state **s**. The foreign CPI is defined analogously as P^*_{CPI}. Now we define the real exchange rate as

$$RER(\mathbf{s}) := \frac{e(\mathbf{s})P^*_{CPI}(\mathbf{s})}{P_{CPI}(\mathbf{s})}.$$

5. Computational Exercise

As a first exercise, we consider a simpler pricing mechanism in the DM – Walrasian price taking. We can show analytically that in this case, the SME nests a unique non-stochastic steady state equilibrium.[8] For our numerical experiments, we consider the following specific functions to represent the model primitives. In the CM, preferences and technology are described by

$$U(X) = B\frac{X^{1-\gamma} - 1}{1-\gamma}, \qquad zF(K, H) = zK^{\alpha}H^{1-\alpha},$$

respectively, where $B > 0$, $\gamma > 0$, and $\alpha \in (0, 1)$. The symmetric description holds for the Foreign country. Note however, the notation for the final goods production function G is such that

$$G(y_h, y_f) = \left[\vartheta(y_h)^{\frac{1}{\epsilon}} + (1-\vartheta)(y_f)^{\frac{1}{\epsilon}}\right]^{\epsilon},$$

for the Home country, and,

$$G(y^*_f, y^*_h) = \left[\vartheta(y^*_f)^{\frac{1}{\epsilon}} + (1-\vartheta)(y^*_h)^{\frac{1}{\epsilon}}\right]^{\epsilon},$$

for the Foreign country, where $\vartheta \in (0, 1)$ and $-\infty \leq 1/\epsilon \leq 1$. The elasticity of substitution between the inputs to G is given by $\sigma_\epsilon = \epsilon/(\epsilon - 1)$. These functional forms are quite standard in models with international trade in intermediate goods [see e.g. Heathcote and Perri, 2002; Chari, Kehoe, and McGrattan, 2002].

In the DM, preferences and technology are given respectively by

$$u(q) = C\frac{(q + \underline{q})^{1-\eta} - b^{1-\eta}}{1-\eta}, \qquad c(q, K) = q^{\varpi}(K)^{1-\varpi}$$

where $C = 1$ without loss of generality, $\underline{q} \searrow 0$ ($b = 0$ if DM trade is determined by Walrasian price taking), $\eta > 0$ and $\varpi \geq 1$.

[8] Later, in the bargaining case, we have to rely on numerical methods to find the non-stochastic steady state solutions.

Given these function, we can characterize the SME with either (i) DM bargaining or (ii) DM Walrasian price taking assumptions on the pricing mechanism in DM trades. As a first pass, we will consider (ii) DM Walrasian price taking, since this avoid additional sources of inefficiencies arising from bargaining (i.e. the holdup problems on money and capital). More important, with DM Walrasian price taking, we can first focus on the role that capital complementarity plays in DM production and its pure effect on the SME and real exchange rate dynamics. (Later, when we incorporate the bargaining assumption, we can also see the additional effect of bargaining holdups on capital and its resulting effect on the SME real exchange rate dynamics.)

TABLE 1. Baseline parameterization.

Parameter	Values
β	.99
$\underline{q}$	0 (0.0001 if bargaining)
$\eta = \gamma$	1
δ	0.025
α	0.3
A	0.256
ϖ	1.373
σ	0.065
B	0.075
ϑ	0.9397

5.1. **Baseline model calibration.** Table 1 summarizes the baseline parameter values for the model. To avoid arbitrary parameterization and therefore outcomes for the model's behavior, we calibrate the model to match long run stylized facts. First, we discuss parameters that can be easily estimated or fixed indepedently. Similar to Aruoba, Waller, and Wright [2008], we calibrate α to match the target of labor share in output, which is about 0.7 in the data [see also Aruoba, 2010]. We fix $\delta = 0.1$ as estimated in Heathcote and Perri [2002] for a two country model. Following Aruoba, Waller, and Wright [2008] and Aruoba [2010], we set $\sigma = 0.065$ to match the long-run money demand semi-elasticity with respect to the nominal interest rate, where money is defined by M1 for the U.S.. The risk aversion parameters η and γ imply that both U and u are natural log functions of X and q, respectively. This restriction is required for the baseline model to have a balanced growth path, since the per-period utility function is linearly separable in consumption and leisure [see Waller, 2010]. The constant marginal taxes on capital, labor and CM-consumption, $(\tau_K, \tau_H, \tau_X) = (0.548, 0.242, 0.069)$, are chosen as in Aruoba, Waller, and Wright [2008]. The estimate of ω is from Backus, Kehoe, and Kydland [1994].

Second, we calibrate simultaneously the remaining parameters (A, B, ϖ) to match the targets of proportion of hours worked, H, a measure of non-traded consumption goods share in total consumption, NTS, and the long run capital output ratio, K/Y. The value of H is roughly 0.33, which is standard. This value can be thought of as pinning down the marginal utility of labour parameter A. B is calibrated, in this model, to match a DM consumption (interpreted as a nontradable good in this model) share of total consumption to be close to 0.50 for the U.S., a share estimated by Stockman and Tesar [1995]. This is in contrast to the closed-economy models in Aruoba, Waller, and Wright [2008] and Aruoba [2010], where intuitively, B is calibrated to match the velocity of money. The target capital-output ratio, K/Y, is 2.23 in annual terms. Given other parameters, this ratio can be thought of as pinning down the calibration for ϖ from the Euler equation characterizing equilibrium capital accumulation along the steady state path. The calibrated value of $\varpi > 1$, implies that the more capital is installed for use in the DM production, the lower the cost of producing a unit of DM output q. By duality, this implies that capital is a complementary input to labor effort in DM production.

In the baseline model, we assume that all the TFP levels (and their shocks), in both CM and DM, are uncorrelated with each other [see also Chari, Kehoe, and McGrattan, 2002]. In parameterizing the exogenous TFP autocorrelation parameters (ρ_Z, ρ_{Z^*}) we borrow values from Heathcote and Perri [2002]. The money supply growth stochastic processes are the estimates from Schlagenhauf and Wrase [1995].

6. International business cycle features

In this section, we discuss the business cycle dynamics of the calibrated baseline model. We report the quantitative predictions of our benchmark model relative to a business cycle model with sticky prices [Chari, Kehoe, and McGrattan, 2002, labelled CKM in the tables], and a real business cycle model of **?** (HP in the tables).

As we can see from Table 2, the benchmark model can account for the volatilities of the key business cycle data for the U.S. quite well. In particular, the model can account for up to 90% of the consumption volatility, 80% of the volatility in domestic investment, and about 90% of labor volatility. The model over-predicts the nominal exchange rate volatility.

Overall, in terms of the nominal and real exchange rate volatilities, the model is able to reproduce qualitatively the observation that both exchange rates are much

TABLE 2. Percentage standard deviation (relative to U.S. GDP).

	Data	Benchmark	CKM	HP
e	4.67	8.92	4.14-4.66	n.a.
RER	2.23-4.36	2.82	4.09-4.98	2.23
X	0.81-0.83	0.70	0.83-0.92	0.51-0.53
I	2.78-2.84	2.23	1.32-1.70	2.04-2.74
H	0.66-0.90	0.80	0.48-0.63	0.28-0.32

more volatile than U.S. GDP. As opposed to Chari, Kehoe, and McGrattan [2002] and Heathcote and Perri [2002], our benchmark model does not rely on large relative risk aversion parameters, sticky prices nor imperfections in international risk sharing to generate volatility.[9] Furthermore, in contrast to standard flexible price two-country CIA models [see Schlagenhauf and Wrase, 1995], which is the limit of our baseline model when $\sigma = 0$ (and this is different to the limiting economy where $\sigma \searrow 0$), the CIA models are unable to reproduce any realistic volatilities in the real and nominal exchange rates.

TABLE 3. First-order autocorrelations.

	Data	Benchmark	CKM	HP
e	0.86	0.66	0.46-0.69	n.a.
RER	0.83	0.68	0.48-0.69	n.a.
X	0.89	0.99	0.48-0.61	n.a.
I	0.91	0.92	0.47-0.60	n.a.
H	0.90	0.90	0.48-0.69	n.a.
Y	0.88	0.94	0.49-0.70	n.a.

In term of endogenous persistence in the model, we consider the first order autocorrelation coefficients of the equilibrium processes in Table 3. In terms of consumption in the traditional Walrasian (or RBC) sector of the model, investment, labor allocation, and GDP, the model matches the empirical persistence in the data very well. However, in terms of the real and nominal exchange rates, the model under predicts the persistence observed in the data. However, the baseline model is able to do just as well as the models of Chari, Kehoe, and McGrattan [2002], without requiring any exogenous sticky-price assumption.

[9] On the other hand, the competitive equilibrium in our model features incomplete markets as a result of idiosyncratic shocks to agent types each period as they enter the DM. Since there is a link between the DM and CM outcomes via capital, not all consumption risk can be fully insured.

TABLE 4. Contemporaneous cross-correlations with GDP.

	Data	Benchmark	CKM	HP
RER	0.13	0.08	0.17-0.52	0.65
X	0.86	0.80	n.a.	0.92-0.96
I	0.95	0.98	n.a.	0.96-0.99
H	0.87	0.84	n.a.	0.97-0.99

As another test of the model's empirical plausibility, we consider the relevant business cycle co-movements in Table 4 (co-movement with U.S. GDP) and Table 4 (various other correlations).

The model predicts correctly a procyclical real exchange rate, consumption, investment and hours data. The strength of the correlations are very to that in the data as well, except for the real exchange rate. In this case, the model can only explain more than 50% of the correlation.

TABLE 5. Other contemporaneous correlations.

	Data	Benchmark	CKM	HP
(Y, Y^*)	0.60	0.26	0.43-0.58	0.17-0.24
(X, X^*)	0.38	0.29	0.48-0.50	0.65-0.85
$(x, X/X^*)$	-0.35	0.33	1.00	n.a.
(RER, e)	0.99	0.96	0.75-0.88	n.a.
(RER, NX)	0.14	0.38	0.75-0.88	n.a.

In terms of the other correlations in the data, Table 4 shows that the model is able to generate realistic cross-country output, and consumption correlations, that are weakly positive. Moreover, the model is able to generate an real-nominal exchange rate correlation that is very close to the data. The model is able to explain just under 50% of the mild positive correlation between the real exchange rate and net exports in the data. What stands out is that the model breaks away from the Backus and Smith [1993] puzzle–that relative consumption across countries is counterfactually perfectly correlated with the real exchange rate. This is not surprising since now, since our DM sector where the special good is assumed to be produced and traded locally, behaves exactly a nontraded goods sector. This essentially breaks the perfect link between the real exchange rate and the real terms of trade induced by the marginal rate of subsitution of consumption across countries in the traded goods sector (even with complete international asset markets).

7. Inspecting the Mechanism

Now we will explain the mechanism in this model that contributes to the business cycle properties of the benchmark model reported in Tables 2-5 and discussed in Section 6.

There are two key assumptions in this model that depart from a frictionless monetary business cycle model. One is indexed by the underlying DM matching friction, σ, and the other, conditional on $\sigma \in (0, 1/2)$, indexes capital complementarity in DM production, ϖ. These, individually, are crucial to contributing to the dynamics of the international real and nominal exchange rates, respectively, in the face of total factor productivity and money supply growth differentials across countries. We analyze each feature of the model separately, beginning with σ.

7.1. **Random matching trade friction, σ.** Consider the following broad-brush experiment where all shocks to (z, z^*, ψ, ψ^*) are present. In Table 6, we take σ from the benchmark calibration of $\sigma = 0.065$, holding all other calibrations constant, to $\sigma = 0.26$. We can see that the volatility of each key variable increases as σ becomes smaller (calibrated benchmark). In particular, we can almost double the volatility on real (RER) and nominal (e) exchange rates, and also increase the persistence of the variables. From this we make the following observation on the benchmark model.

Result 1. *Greater individual uncertainty in the DM, σ, induces greater volatility in all allocations and prices, namely the real and nominal exchange rates, in the DM and CM. Moreover, it results in more persistence in the real and nominal exchange rates.*

We can dissect this result into the contribution of σ to the dynamics, in the face of each TFP and money supply growth shocks. This exercise is reported in terms of comparative impulse response functions in Figure 2 for a shock to Home TFP z, and in Figure 3 for a shock to Home money supply growth ψ.

7.1.1. *TFP shock and σ.* As in standard real-business-cycle models, here, a one-percent positive shock to the Home TFP, given by z, has the following effects: (i) It raises CM output directly and therefore agents income from renting out labor and capital in the CM: $zF(K, H)$; and (ii) It also raises the marginal products of labor, $zF_H(K, H)$, and capital, $zF_k(K, H)$, directly.

Since there is no wealth effect on leisure (by quasilinearity of preferences), there is only a substitution effect on leisure. Thus employment H increases. CM Consumption X is a normal good, so it must increase with an increase in income. In

TABLE 6. Effects of random matching (σ).

Statistic	$\sigma = 0.065$ (Benchmark)	$\sigma = 0.26$
Rel. Std.		
RER	2.82	2.59
e	8.92	8.17
I	2.23	1.51
X	0.70	0.68
q	1.99	1.71
H	0.80	0.57
ϕ	4.73	4.49
Autocorrelation		
RER	0.68	0.65
e	0.66	0.65
I	0.93	0.84
X	0.99	0.98
q	0.61	0.60
H	0.88	0.80
ϕ	0.61	0.60

order to smooth consumption across date (via capital accumulation K) and state-and-date events (via international contingent money claims, $B(\mathbf{s}_+|\mathbf{s})$), there will be respectively, an increase in physical capital investment (on the final goods demand side) and demand for intermediate goods (on the final goods supply side). The latter must be consistent with movements in the terms of trade P_{y_f}/P_{y_h}, such that any changes in net exports will be equal to the negative of the current account, which measures the net real flow of contingent claims between the countries. In equilibrium, a relatively higher TFP at Home, creates a depreciation in the real exchange rate (i.e. an increase in real purchasing power of Home residents) and this is consistent with a fall in the CM final goods (normalized) price level $\hat{p} := 1/\hat{\phi}$.

We can now explain how σ contributes to the dynamics. We know from Lagos and Wright [2005], that random matching frictions will induce individual uncertainty *ex-ante* in the DM. This creates stochastic opportunities for traders (*i.e.* the total measure of $2\sigma\kappa$ anonymous buyer and sellers) where they get to trade using either money or credit. Hence this would result in a distribution of heterogenous agents' assets at the end of each DM. A smaller σ, all else constant, induces greater uncertainty for both buyers and sellers, and this results in larger responses in the DM outcome q and price level $\hat{\phi}$, and in H as $\sigma \searrow 0$, as seen in Figure 2. Intuitively, *ex-post* DM buyers have to now work harder in each period to rebalance their money holdings. Given this larger risk to individual DM trading opportunities, capital investment, I, in each CM is now more responsive to a TFP shock, since with greater probability, the additional return on a given amount of investment in each current CM may not be realized in subsequent DM monetary or credit trades with total measure $2\sigma(1-\kappa)$. However, an increased H and I response, also results in less volatile CM consumption (X) response. All else constant, the latter, together with more volatile $\hat{\phi}$, would mean a less volatile nominal exchange rate (e). This is evident from the global risk sharing condition in equation (50). Recall that this is only part of the mechanism.

7.1.2. *Monetary shock and σ.* Consider now a shock to Home money supply growth ψ. This would raise q (not shown), and reduce the value of money $\hat{\phi} := 1/\hat{p}$ (see Figure 3). All else constant, this tends to create a nominal depreciation of the Home currency e, inducing the Home CM final good producer to substitute away from the relatively more expensive intermediate good y_h, to increasing its demand for the relatively cheaper y_f.

However, now consider in equilibrium that the DM pricing outcome (51) must also hold. Since the monetary shock in the DM affects the equilibrium pricing condition (51) by raising the amount of money individuals bring into the DM, the marginal utility value of a $\exp\{\psi\}$-unit of Home currency in the left-hand-side of (51) – *i.e.* $A\hat{\phi}/w(1-\tau_H)$ must fall. We observe that employment H falls (so that real wage would rise) to consistently offset the fall in CM marginal utility $U_X(X)$ of consumption (X) – i.e. also satisfying equation (46). This results in less current production in the CM intermediate goods sector and may have an effect on the terms of trade opposite to the direct effect of a depreciation of e.

Under our baseline calibration, the equilibrium response is such that y_h decreases as Home the intermediate goods producer substitutes towards the imported intermediate good component (see Figure 3).

Again, how does σ contribute to the dynamics in the face of a money supply growth shock? The effect on labor is opposite to that of a TFP shock. With money balances at the start of each DM higher over time, individual (monetary trade) buyers now have less need to rebalance their money holdings via labor (see Figure 3). This has then a smaller supply side effect on the intermediate goods production channel outline previously. On the other hand, a smaller σ also means that there is a smaller probability that money has value as a medium of exchange in future DM's. In order to maintain some smoothing of consumption X across dates and states, and given the relatively smaller return to money as a medium of exchange (*viz.* also as a means of precautionary saving), then the value of money must fall by more, *i.e.* $\hat{p} := 1/\hat{\phi}$ rises by more (see Figure 3). The effect on the nominal exchange rate of σ is almost negligible.

7.1.3. *All together now (σ reprise).* Now combining the analysis in the last two sections, and recalling, that a similar analysis is to be done for shock to Foreign TFP z^* and money supply growth ψ^*, we obtain the net result in Table 6.

7.2. **DM Capital complementarity, ϖ.** Conditional on $\sigma \in (0, 1/2]$, we now elucidate on the role the second feature of the model – capital complementary in DM production. This is indexed by the parameter ϖ. As before, we first report an overall result of a experiment where we consider taking $\varpi = 1.373$ in the baseline calibration to $\varpi = 1$ holding all else in the model constant. The experiment thus looks at the limit in which capital is not employed in the DM. Again all primitive shocks are present in this overall result, reported in Table 7.

TABLE 7. Effects of capital complementarity (ϖ).

Statistic	$\varpi = 1.373$ (Benchmark)	$\varpi \searrow 1.00$
Rel. Std.		
RER	2.82	1.04
e	8.92	7.11
I	2.23	2.54
X	0.70	0.65
q	1.99	1.47
H	0.80	0.84
ϕ	4.73	3.82
Autocorrelation		
RER	0.68	0.77
e	0.66	0.67
I	0.93	0.96
X	0.99	0.99
q	0.61	0.61
H	0.88	0.93
ϕ	0.61	0.62

Result 2. *Conditional on the baseline calibration of σ, greater capital complementarity in the DM seller's technology, ϖ, induces greater volatility in the real and nominal exchange rates. However, it results in lower persistence in the exchange rates.*

7.2.1. *TFP shock and ϖ.* Consider first a Home TFP shock, shown in Figure 4. The mechanism in the baseline is the same as in the baseline case in the first experiment on σ.

With a lower (or without) capital complementarity in the DM production, expected return on holding money (and investing in capital) in the CM is lower because $c_{qK}(q_+, K_+) < 0$ (and $\sigma c_K(q_+, K_+)$) for all measurable future realizations of (q_+, K_+). From the equilibrium asset pricing relation, for example equation (41), in terms of optimal capital accumulation, we can deduce that on average, the level of K is lower. By the concavity of the production technologies, the expected marginal product of capital will be more variable. Hence domestic investment will be more variable and persistent. Then labor and prices will be more variable and persistent, although relative cross-country consumption is less variable, but more persistent. From equilibrium nominal exchange determination, it turns out that the nominal exchange rate is more variable and but less persistent. The real exchange rate turns out to be less volatile and persistent.

7.2.2. *Monetary shock and ϖ.* Consider now money supply growth shocks, shown in Figure 5. How does ϖ contribute to the dynamics in the face of a money supply growth shock? Consider first the equilibrium condition on money demand, given by the Euler equation (37). Now, a smaller ϖ also means that money has lower value as a medium of exchange in future DM's, since $g_{qK} = c_{qK} < 0$, so that the DM value of money as medium of exchange, given by the term $\sigma \kappa u_q / g_q$ will be smaller on average. Given an increase in money supply growth, inflation tax on money holdings is larger. Given uncertainty of realizing the payoff from holding money in the DM, σ, *ex-ante* in the CM, there will be more volatility in the pricing outcome in the DM. This is observed in Figure 5 as a larger response of the price levels and the CPI index. Then, consistent with the asset pricing relation (37), there more smoothing of CM consumption relative to foreign consumption.

Note that as $\varpi = 1$, we effectively have a dichotomy in the determination of real and monetary outcomes in the model [see Aruoba and Wright, 2003]. This explains why the impulse responses to a money supply growth shock has virtually no real effects in Figure 5. That shows up as less variability in RER although, and the effect of the change in ϖ is almost negligible in the nominal exchange rate, suggesting that most of the nominal exchange rate dynamics still comes from domestic real and external shocks.

7.2.3. *All together now (ϖ reprise).* Again, combining the analysis in the last two sections, and recalling that a similar analysis is to be done for shock to Foreign TFP z^* and money supply growth ψ^*, we obtain the net result in Table 7.

8. Conclusion

In this paper we develop a two-country global monetary economy where a monetary equilibrium exists because of fundamental decentralized trade frictions – a Lagos-Wright search and matching friction. In the decentralized markets (DM), the terms of trade can be determined either by bargaining or by competitive price taking. We show that the baseline model (with DM price taking) is capable of generating quite realistic the real exchange rate volatility observed in the data. In constrast to flexible price CIA models [see e.g. Schlagenhauf and Wrase, 1995] which fail at matching volatility and persistence of the exchange rates, our baseline model is able to explain about 7/8 of the persistence in the exchange rates without requiring ad-hoc assumptions of sticky prices nor additional real shocks as in Steinsson [2008]. The key mechanisms lie in the underlying DM trading friction and a primitive technological assumption – that capital is also a complementary input to production in the DM. This creates an internal propagation mechanism by modifying asset-pricing relations and relative price dynamics in the model. In continuing work, we also explore how alternative DM pricing mechanisms alter the exchange rate dynamics.

Appendix A. SME Characterization

Consider a simplification of the model with $\kappa = 1$ and $\tau_K = \tau_X = \tau_H = 0$. Since the processes (ψ) and (ψ^*) are bounded below by zero, this implies that nominal variables, namely M, M^*, ϕ and ϕ^* will grow unboundedly. We can perform a change of variables in the equilibrium conditions for nominal variables as follows. We normalize Home and Foreign nominal variables by $M(\mathbf{s}_-)$ and $M^*(\mathbf{s}_-)$, respectively, such that

$$\hat{\imath}(\mathbf{s}) := \frac{\iota(\mathbf{s})}{M(\mathbf{s}_-)}, \qquad \hat{\imath}^*(\mathbf{s}) := \frac{\iota^*(\mathbf{s})}{M^*(\mathbf{s}_-)}, \qquad \hat{\phi}(\mathbf{s}) := \phi(\mathbf{s})M(\mathbf{s}_-),$$
$$\hat{\phi}^*(\mathbf{s}) := \phi^*(\mathbf{s})M^*(\mathbf{s}_-), \qquad \hat{e}(\mathbf{s}) := \frac{e(\mathbf{s})M^*(\mathbf{s}_-)}{M(\mathbf{s}_-)},$$
$$\hat{P}_h(\mathbf{s}) = P_h(\mathbf{s})/M(\mathbf{s}_-), \qquad \hat{P}_f(\mathbf{s}) = P_f(\mathbf{s})/M(\mathbf{s}_-).$$

Then our SME conditions can be equivalently written as follows. The Home and Foreign money supply injection are respectively given as:

$$\hat{\imath}(\mathbf{s}) = \exp(\psi) - 1$$
$$\hat{\imath}^*(\mathbf{s}) = \exp(\psi^*) - 1$$

Labor market clearing in the CM in Home and Foreign, respectively, are

$$U_X[X(\mathbf{s})] = \frac{A}{zF_H\left[K(\mathbf{s}_-), H(\mathbf{s})\right]} \tag{46}$$
$$U_X[X^*(\mathbf{s})] = \frac{A}{z^*F_H\left[K^*(\mathbf{s}_-), H^*(\mathbf{s})\right]} \tag{47}$$

The Home resource constraint in equilibrium is given by

$$G(y_h(\mathbf{s}), y_f(\mathbf{s})) = X(\mathbf{s}) + K(\mathbf{s}) - (1-\delta)K(\mathbf{s}_-). \tag{48}$$

The Foreign resource constraint is given by

$$G(y_f^*(\mathbf{s}), y_h^*(\mathbf{s})) = X^*(\mathbf{s}) + K^*(\mathbf{s}) - (1-\delta)K^*(\mathbf{s}_-). \tag{49}$$

Complete international risk sharing entails

$$\frac{\hat{e}(\mathbf{s})\hat{\phi}(\mathbf{s})}{\hat{\phi}^*(\mathbf{s})} = \kappa_0 \frac{U_X[X^*(\mathbf{s})]}{U_X[X(\mathbf{s})]}. \tag{50}$$

where $\kappa_0 = 1$, implying a symmetric initial steady state, without loss of generality.

Aggregate general-good price levels in Home and Foreign, respectively, are pinned down by

$$\frac{A\hat{\phi}(\mathbf{s})}{w(\mathbf{s})} \exp\{\psi\} = g[q(\mathbf{s}), K(\mathbf{s}_-)], \tag{51}$$

and

$$\frac{A\hat{\phi}^*(\mathbf{s})}{w^*(\mathbf{s})} \exp\{\psi^*\} = g[q^*(\mathbf{s}), K^*(\mathbf{s}_-)]. \tag{52}$$

The equilibrium Euler equations for Home are:

$$g[q(\mathbf{s}), K(\mathbf{s}_-)] = \beta\mathbb{E}_\lambda\left\{ g[q(\mathbf{s}_+), K(\mathbf{s})] \exp\{-\psi\} \left[(1-\sigma) + \sigma \frac{u_q[q(\mathbf{s}_+)]}{g_q[q(\mathbf{s}_+), K(\mathbf{s})]}\right]\right\} \tag{53}$$

$$U_X[X(\mathbf{s})] = \beta\mathbb{E}_\lambda\left\{ U_X[X(\mathbf{s}_+)] \left[(1 + r(\mathbf{s}_+) - \delta) - \sigma \frac{\gamma[q(\mathbf{s}_+), K(\mathbf{s})]}{U_X[X(\mathbf{s}_+)]}\right]\right\}. \tag{54}$$

The equilibrium Euler equations for Foreign are:

$$g[q^*(\mathbf{s}), K^*(\mathbf{s}_-)] = \beta \mathbb{E}_\lambda \Bigg\{ g[q^*(\mathbf{s}_+), K^*(\mathbf{s})] \exp\{-\psi^*\} \times \left[(1-\sigma) + \sigma \frac{u_q[q^*(\mathbf{s}_+)]}{g_q[q^*(\mathbf{s}_+), K^*(\mathbf{s})]} \right] \Bigg\} \tag{55}$$

$$U_X[X^*(\mathbf{s})] = \beta \mathbb{E}_\lambda \Bigg\{ U_X[X^*(\mathbf{s}_+)] \Bigg[(1 + r^*(\mathbf{s}_+) - \delta) - \sigma \frac{\gamma[q^*(\mathbf{s}_+), K^*(\mathbf{s})]}{U_X[X^*(\mathbf{s}_+)]} \Bigg] \Bigg\}. \tag{56}$$

Note that capital and labor rental pricing functions are given by:

$$r(\mathbf{s}) = \hat{\phi}(\mathbf{s}) \hat{P}_h(\mathbf{s}) \cdot z F_k[K(\mathbf{s}_-), H(\mathbf{s})], \tag{57}$$

and

$$w(\mathbf{s}) = \hat{\phi}(\mathbf{s}) \hat{P}_h(\mathbf{s}) \cdot z F_H[K(\mathbf{s}_-), H(\mathbf{s})], \tag{58}$$

for Home, and

$$r^*(\mathbf{s}) = \frac{\hat{\phi}^*(\mathbf{s}) \hat{P}_f(\mathbf{s})}{e(\mathbf{s})} \cdot z^* F_k[K^*(\mathbf{s}_-), H^*(\mathbf{s})], \tag{59}$$

and

$$w^*(\mathbf{s}) = \frac{\hat{\phi}^*(\mathbf{s}) \hat{P}_f(\mathbf{s})}{e(\mathbf{s})} \cdot z^* F_H[K^*(\mathbf{s}_-), H^*(\mathbf{s})], \tag{60}$$

for Foreign, where we have made use of the law of one price for intermediate goods.

Intermediate goods trade and market clearing are given by:

$$\hat{\phi}(\mathbf{s}) \hat{P}_h(\mathbf{s}) = G_{y_h}[y_h(\mathbf{s}), y_f(\mathbf{s})], \tag{61}$$

and

$$\hat{\phi}(\mathbf{s}) \hat{P}_f(\mathbf{s}) = G_{y_f}[y_h(\mathbf{s}), y_f(\mathbf{s})]. \tag{62}$$

for Home, and

$$\frac{\hat{\phi}^*(\mathbf{s}) \hat{P}_f(\mathbf{s})}{e(\mathbf{s})} = G_{y_f^*}[y_f^*(\mathbf{s}), y_h^*(\mathbf{s})], \tag{63}$$

and

$$\frac{\hat{\phi}^*(\mathbf{s}) \hat{P}_h(\mathbf{s})}{e(\mathbf{s})} = G_{y_h^*}[y_f^*(\mathbf{s}), y_h^*(\mathbf{s})]. \tag{64}$$

for Foreign, where we have again made use of the law of one price for intermediate goods.

Market clearing for intermediate goods are:

$$zF[K(\mathbf{s}_-), H(\mathbf{s})] = y_h(\mathbf{s}) + y_h^*(\mathbf{s}) \tag{65}$$

$$z^*F[K^*(\mathbf{s}_-), H^*(\mathbf{s})] = y_f^*(\mathbf{s}) + y_f(\mathbf{s}). \tag{66}$$

Definition 2. *A stationary Markov monetary equilibrium (with decentralized bargaining) is given by time-invariant functions of* $\mathbf{s}$*, i.e.*

(1) *Consumption functions* $(X, X^*, H, H^*, q, q^*, y_h, y_f, y_f^*, y_h^*)$,
(2) *Savings functions* (K, K^*), *and,*
(3) *Pricing functions* $(w, w^*, r, r^*, \hat{e}, \hat{\phi}, \hat{\phi}^*, \hat{P}_h, \hat{P}_y)$,

that induce bounded stochastic processes satisfying the recursions (46)-(66), for given policies $(\hat{\iota}(\mathbf{s}), \hat{\iota}^*(\mathbf{s}), G(\mathbf{s}), G^*(\mathbf{s}))$.

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FIGURE 2. Experiment 1: The role of σ. Impulse responses of variables to 1% Home TFP z increase: Experiment (- ○ -), Benchmark (- ◇ -)

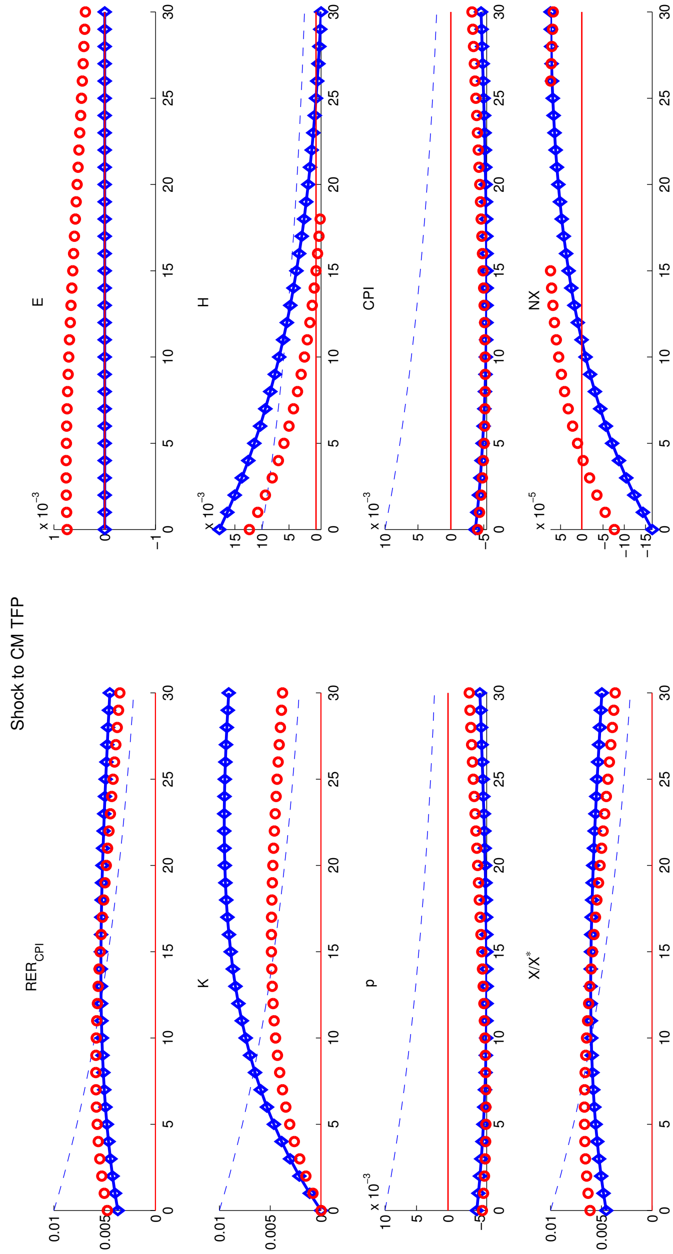

FIGURE 3. Experiment 1: The role of σ. Impulse responses of variables to 1% Home money-supply growth ψ increase: Experiment (- ○ -), Benchmark (- ◇ -)

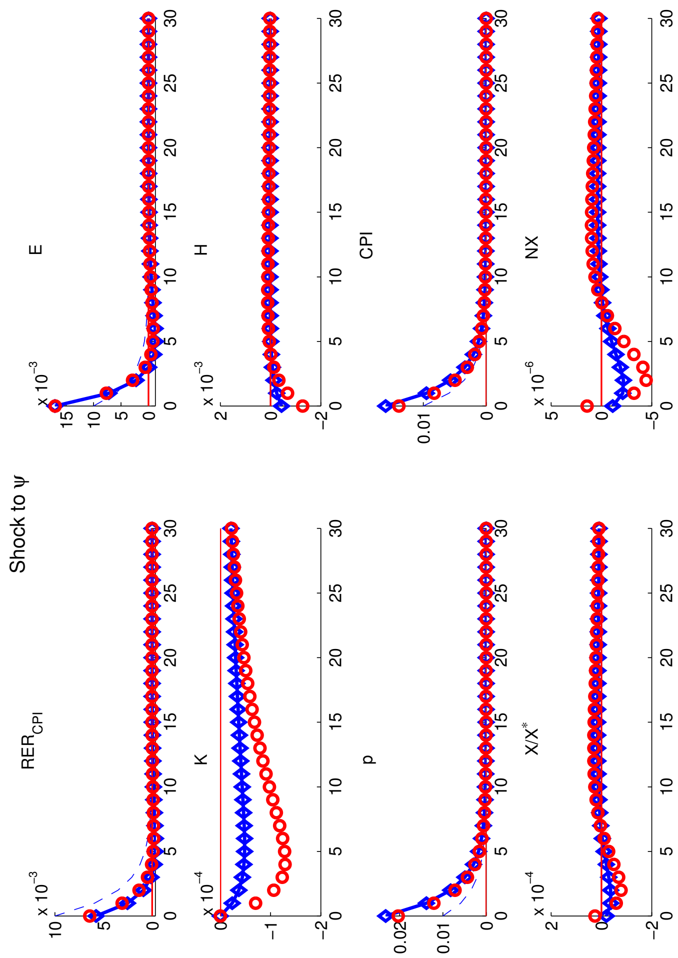

FIGURE 4. Experiment 2: The role of ϖ. Impulse responses of variables to 1% Home TFP z increase: Experiment (- ○ -), Benchmark (- ◇ -)

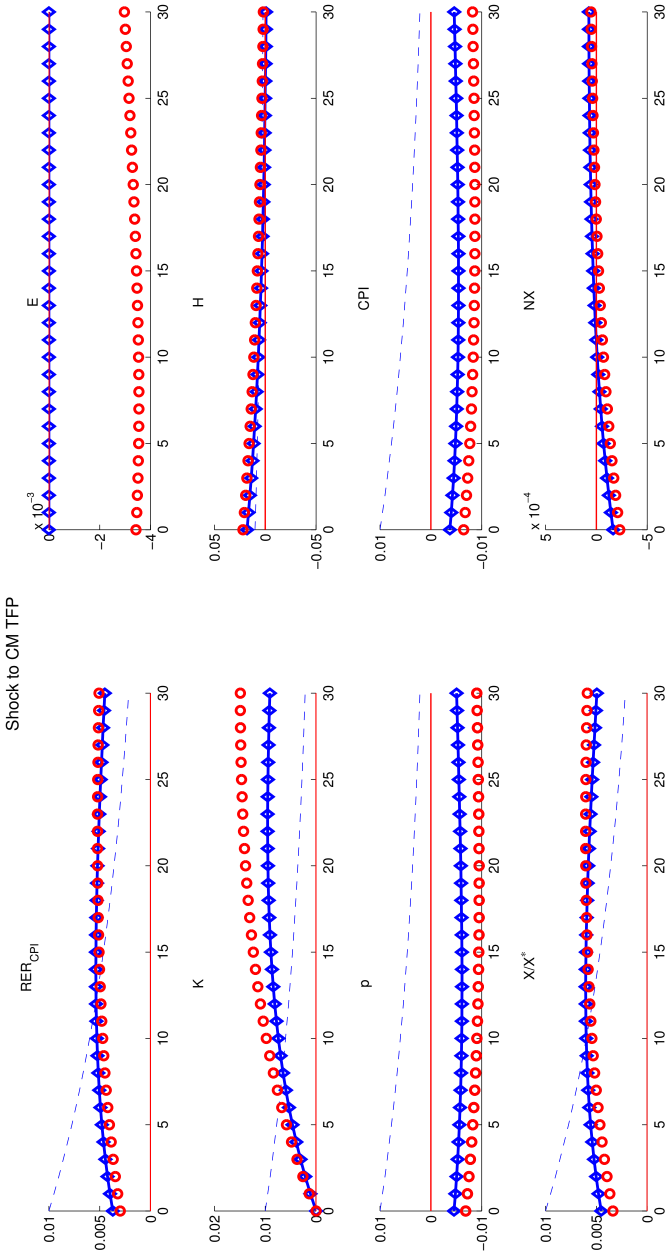

FIGURE 5. Experiment 2: The role of ϖ. Impulse responses of variables to 1% Home money-supply growth ψ increase: Experiment (- ○ -), Benchmark (- ◇ -)