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Orbit decomposition of Jordan matrix algebras of order three under the automorphism groups

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* **Abstract**. The orbit decomposition is given under the automorphism group on the real split Jordan algebra of all hermitian matrices of order three corresponding to any real split composition algebra, or the automorphism group on the complexification, explicitly, in terms of the cross product of H. Freudenthal and the characteristic polynomial.

0. Introduction.

Let $\mathcal{J}'$ be a split exceptional simple Jordan algebra over a field $\mathbb{F}$ of characteristic not two, that is, the set of all hermitian matrices of order three whose elements are split octonions over $\mathbb{F}$ with the Jordan product. And let G' be the automorphism group of $\mathcal{J}'$. N. Jacobson [16, p.389, Theorem 10] found that $X, Y \in \mathcal{J}'$ are in the same G'-orbit if and only if X, Y admit the same minimal polynomial and the same generic minimal polynomial, by imbedding a generating subalgebra with the identity element E in terms of the Jordan product into a special Jordan algebra. When $\mathbb{F} = \mathbb{R}$, the field of all real numbers, some elements of $\mathcal{J}'$ are not diagonalizable under the action of $G' = F_{4(4)}$, since $\mathcal{J}'$ admits a G'-invariant non-defnite $\mathbb{R}$-bilinear form such that the restriction to the subspace of all diagonal elements is positive-definite [19, Theorem 2], although every element of $\mathcal{J}'$ is diagonalizable under the action of a linear group $E_{6(6)}$ containing $F_{4(4)}$ on $\mathcal{J}'$ by [15] (cf. [17]) or under the action of the maximal compact subgroup $Sp(4)/\mathbb{Z}_2$ of $E_{6(6)}$ on $\mathcal{J}'$ given by [22].

This paper presents a concrete orbit decomposition under the automorphism group on a real split Jordan algebra of all hermitian matrices of order

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three corresponding to any real split composition algebra, or the complexification of it, that is special or exceptional as a Jordan algebra. As a result, $X, Y \in \mathcal{J}'$ are in the same G'-orbit if and only if X, Y admit the same dimension of the generating subspace with E by the cross product [8] and the same characteristic polynomial, which gives a simplification for N. Jacobson [16]'s polynomial invariants on G'-orbits when $\mathbb{F} = \mathbb{R}$ or the field of all complex numbers $\mathbb{C}$. To state the main results more precisely, let us give the precise notations:

Put $\mathbb{F} := \mathbb{R}$ or $\mathbb{C}$. Let V be an $\mathbb{F}$-linear space, and $\mathrm{End}_{\mathbb{F}}(V)$ (or $\mathrm{GL}_{\mathbb{F}}(V)$) denote the set of all $\mathbb{F}$-linear endomorphisms (resp. automorphims) on V. For a mapping $f : V \to V$ and $c \in \mathbb{F}$, put $V_{f,c} := \{v \in V |\ f(v) = cv\}$ and $V_{f,1} := V_f$. For a subgroup G of $\mathrm{GL}_{\mathbb{F}}(V)$, let G° be the identity connected component of G. For $v \in V$ and a mapping $\phi : V \to V$, put $\mathcal{O}_G(v) := \{\alpha(v)|\ \alpha \in G\}$, $G_v := \{\alpha \in G|\ \alpha(v) = v\}$ and $G^\phi := \{\alpha \in G|\ \phi \circ \alpha = \alpha \circ \phi\}$. For a subset W of V, put $G_W := \{\alpha \in G|\ \{\alpha w|\ w \in W\} = W\}$. For positive integers n, m, let $M(n, m; V)$ be the set of all $n\times m$-matrices with entries in V. Put $V^n := M(n, 1; V), V_m := M(1, m; V)$ and $M_n(V) := M(n, n; V)$. Since V can be considered as an $\mathbb{R}$-linear space, the complexification is defined as $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \sqrt{-1}V$ with an $\mathbb{R}$-linear conjugation: $\tau : V^{\mathbb{C}} \to V^{\mathbb{C}}; v_1 + \sqrt{-1}v_2 \mapsto v_1 - \sqrt{-1}v_2$ $(v_1, v_2 \in V)$. For any $\alpha \in \mathrm{End}_{\mathbb{R}}(V)$, put $\alpha^{\mathbb{C}} : V^{\mathbb{C}} \longrightarrow V^{\mathbb{C}}; v_1 + \sqrt{-1}v_2 \mapsto (\alpha v_1) + \sqrt{-1}(\alpha v_2)$ such that $\alpha^{\mathbb{C}}\tau = \tau\alpha^{\mathbb{C}}$, which is identified with $\alpha \in \mathrm{End}_{\mathbb{R}}(V)$: $\alpha = \alpha^{\mathbb{C}}$.

By W.R. Hamilton, *the quaternions* is defined as an $\mathbb{R}$-algebra $\mathbb{H} := \oplus_{i=0}^{3}\mathbb{R}e_i$ given as $e_0e_i = e_ie_0 = e_i$, $e_i^2 = -e_0$ $(i \in \{1, 2, 3\})$; $e_ke_{k+1} = -e_{k+1}e_k = e_{k+2}$ (where $k, k+1, k+2 \in \{1, 2, 3\}$ are counted modulo 3) with the unit element $1 := e_0$ and the conjugation $\overline{\sum_{i=0}^{3} x_ie_i} = x_0e_0 - \sum_{k=1}^{3} x_ke_k$, which contains *the complex numbers* $\mathbb{C} := \mathbb{R}e_0 \oplus \mathbb{R}e_1$ and the real numbers $\mathbb{R} := \mathbb{R}e_0$ as $\mathbb{R}$-subalgebras. By A. Cayley and J.T. Graves, *the octanions* is defined as a non-associative $\mathbb{R}$-algebra $\boldsymbol{O} := \mathbb{H} \oplus \mathbb{H}e_4$ given as follows [4]:

$$(x \oplus ye_4)(x' \oplus y'e_4) := (xx' - \overline{y'}y) \oplus (y\overline{x'} + y'x)e_4$$

with the $\mathbb{R}$-linear basis $\{e_i|\ i = 0, 1, 2, 3, 4, 5, 6, 7\}$, where the numbering is given as $e_5 := e_1e_4, e_6 := -e_2e_4, e_7 := e_3e_4$ after [26, p.127], [5, p.20] or [20]. Put $\boldsymbol{H} := \mathbb{C} \oplus \mathbb{C}e_4$ and $\boldsymbol{C} := \mathbb{R} \oplus \mathbb{R}e_4$. For $K := \boldsymbol{O}, \boldsymbol{H}, \boldsymbol{C}, \mathbb{R}$, put $d_K := \dim_{\mathbb{R}} K$. And put $\sqrt{-1} := e_0 \otimes e_1 \in K^{\mathbb{C}} := K \otimes_{\mathbb{R}} \mathbb{C}$ with the identification $K = K \otimes e_0 \subset K^{\mathbb{C}}$. Then $K^{\mathbb{C}} = K \oplus \sqrt{-1}K$ is *split* (i.e. non-division) as a $\mathbb{C}$-algebra with $\tau : K^{\mathbb{C}} \to K^{\mathbb{C}}; x+\sqrt{-1}y \mapsto x-\sqrt{-1}y$ $(x, y \in K)$

as the complex conjugation with respect to the real form K. Put

$$\gamma : \boldsymbol{O}^{\mathbb{C}} \longrightarrow \boldsymbol{O}^{\mathbb{C}}; \sum_{i=0}^{7} x_i e_i \mapsto \sum_{i=0}^{3} x_i e_i - \sum_{i=4}^{7} x_i e_i; \text{ and}$$
$$\epsilon : \boldsymbol{O}^{\mathbb{C}} \longrightarrow \boldsymbol{O}^{\mathbb{C}}; x := \sum_{i=0}^{7} x_i e_i \mapsto \bar{x} := x_0 - \sum_{i=1}^{7} x_i e_i$$

as $\mathbb{C}$-linear conjugations with respect to $\mathbb{H}^{\mathbb{C}}$ and $\mathbb{R}^{\mathbb{C}}$, respectively. And a $\mathbb{C}$-bilinear form are defined on $\boldsymbol{O}^{\mathbb{C}}$ as $(x|y) := (x\bar{y}+\overline{x\bar{y}})/2 = \sum_{i=0}^{7} x_i y_i \in \mathbb{C}$. The restrictions of γ, ϵ and $(x|y)$ on $K^{\mathbb{C}}$ are also well-defined and denoted by the same letters. Then $K^{\mathbb{C}}$ is a composition $\mathbb{C}$-algebra with respect to the norm form given by $N(x) := (x|x)$ [5, §I.3], because of $N((x \oplus ye_4)(x' \oplus y'e_4)) - N(x \oplus ye_4)N(x' \oplus y'e_4) = 2\{(y\overline{x'}|y'x) - (xx'|\overline{y'}y)\} = 2(\overline{y'}(y\overline{x'}) - (\overline{y'}y)\overline{x'}|\, x) = 0$ since $\mathbb{H}^{\mathbb{C}}$ is an associative composition algebra with respect to N [3, §6.4]. And $K = (K^{\mathbb{C}})_\tau$ is a division composition $\mathbb{R}$-algebra with the norm form $N(x)$ such that $a^{-1} = \bar{a}/N(a)$ for $a \neq 0$.

Put $K' := (K^{\mathbb{C}})_{\tau\gamma}$ as a composition $\mathbb{R}$-algebra with the norm form $N(x)$ such that $(K')_\gamma = K_\gamma = (K')_\tau = K' \cap K$. Precisely, $\boldsymbol{O}' = \{\sum_{i=0}^{3} x_i e_i + \sum_{i=4}^{7} x_i \sqrt{-1} e_i |\ x_i \in \mathbb{R}\}$ is the $\mathbb{R}$-algebra of *the split-octanions* containing the $\mathbb{R}$-subalgebra $\boldsymbol{H}' = \{\sum_{i=0}^{1} x_i e_i + \sum_{i=4}^{5} x_i \sqrt{-1} e_i |\ x_i \in \mathbb{R}\}$ of *the split-quaternions* and the $\mathbb{R}$-subalgebra $\boldsymbol{C}' = \{x_0 + x_4\sqrt{-1}e_4 |\ x_i \in \mathbb{R}\}$ of *the split-complex numbers* such that $\boldsymbol{O}' \cap \boldsymbol{O} = \mathbb{H}$, $\boldsymbol{H}' \cap \boldsymbol{H} = \mathbb{C}$ and $\boldsymbol{C}' \cap \boldsymbol{C} = \mathbb{R}$. Then $K'^{\mathbb{C}} = K' \oplus \sqrt{-1}K' = K^{\mathbb{C}}$ as a $\mathbb{C}$-subalgebra of $\boldsymbol{O}^{\mathbb{C}}$.

Put $\tilde{K} := K, K'$ (or $K'^{\mathbb{C}}$, $K^{\mathbb{C}}$) with $\mathbb{F} := \mathbb{R}$ (resp. $\mathbb{C}$) and $d_{\tilde{K}} := \dim_{\mathbb{F}} \tilde{K}$. For $A \in M_n(\tilde{K})$ with the (i,j)-entry $a_{ij} \in \tilde{K}$, let ${}^tA, \tau A, \epsilon A \in M_n(\tilde{K})$ be the transposed, τ-conjugate, ϵ-conjugate matrix of A such that the (i,j)-entry is equal to $a_{ji}, \tau(a_{ij}), \epsilon(a_{ij})$, respectively, with the trace $\mathrm{tr}(A) := \sum_{i=1}^{n} a_{ii} \in \mathbb{F}$, and the adjoint matrix $A^* := {}^t(\epsilon A) \in M_n(\tilde{K})$. Let denote the set of all hermitian matrices of order three corresponding to $\tilde{K}$ as follows:

$$\mathcal{J}_3(\tilde{K}) := \{X \in M_3(\tilde{K}) |\ X^* = X\}$$

with an $\mathbb{F}$-bilinear Jordan algebraic product $X \circ Y := \frac{1}{2}(XY + YX)$, the identity element $E := \mathrm{diag}(1,1,1)$ and an $\mathbb{F}$-bilinear symmetric form $(X|Y) := \mathrm{tr}(X \circ Y) \in \mathbb{F}$. After H. Freudenthal [8] (cf. [7, (7.5.1)], [25], [14], [16, p.232, (47)], [28]), *the cross product* on $\mathcal{J}_3(\tilde{K})$ is defined as follows:

$$X \times Y := X \circ Y - \frac{1}{2}(\mathrm{tr}(X)Y + \mathrm{tr}(Y)X - (\mathrm{tr}(X)\mathrm{tr}(Y) - (X|Y))E)$$

with $X^{\times 2} := X \times X$ as well as an $\mathbb{F}$-trilinear form $(X|Y|Z) := (X \times Y|Z)$ and the determinant $\det(X) := \frac{1}{3}(X|X|X) \in \mathbb{F}$ on $\mathcal{J}_3(\tilde{K})$ (cf. [9, p.163]). Put $E_i := \mathrm{diag}(\delta_{i1}, \delta_{i2}, \delta_{i3})$ for $i \in \{1, 2, 3\}$ with the Kronecker's delta δ_{ij}. For $x \in \tilde{K}$, put

$$F_1(x) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix}, \; F_2(x) := \begin{pmatrix} 0 & 0 & \overline{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \; F_3(x) := \begin{pmatrix} 0 & x & 0 \\ \overline{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $x \in K^{\mathbb{C}} = \mathbb{R}^{\mathbb{C}}, \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}$ or $\boldsymbol{O}^{\mathbb{C}}$, put $M_1(x), M_{23}(x) \in \mathcal{J}_3(K^{\mathbb{C}})$ such as

$$M_1(x) := (x|1)(E_2 - E_3) + F_1(\sqrt{-1}x), \quad M_{23}(x) := F_2(\sqrt{-1}\overline{x}) + F_3(x)$$

with $M_1 := M_1(1)$, $M_{23} := M_{23}(1)$. For $x \in K' = \boldsymbol{C}', \boldsymbol{H}'$ or $\boldsymbol{O}'$, put $M_{1'}(x), M_{2'3}(x) \in \mathcal{J}_3(K')$ such as

$$M_{1'}(x) := (x|1)(E_2 - E_3) + F_1(\sqrt{-1}e_4 x), \quad M_{2'3}(x) := F_2(-\sqrt{-1}e_4\overline{x}) + F_3(x)$$

with $M_{1'} := M_{1'}(1)$, $M_{2'3} := M_{2'3}(1)$. For $x \in \tilde{K}$, let denote

$$\begin{aligned}
&\tilde{M}_1(x) := M_1(x) \text{ (when } \tilde{K} = K^{\mathbb{C}}) \text{ or } M_{1'}(x) \text{ (when } \tilde{K} = K'), \\
&\tilde{M}_{23}(x) := M_{23}(x) \text{ (when } \tilde{K} = K^{\mathbb{C}}) \text{ or } M_{2'3}(x) \text{ (when } \tilde{K} = K'); \\
&\tilde{M}_1 := M_1 \text{ (when } \tilde{K} = K^{\mathbb{C}}) \text{ or } M_{1'} \text{ (when } \tilde{K} = K'), \\
&\tilde{M}_{23} := M_{23} \text{ (when } \tilde{K} = K^{\mathbb{C}}) \text{ or } M_{2'3} \text{ (when } \tilde{K} = K').
\end{aligned}$$

And denote

$$\begin{aligned}
&\mathcal{P}_2(\tilde{K}) := \{X \in \mathcal{J}_3(\tilde{K}) |\ X^{\times 2} = 0,\ \mathrm{tr}(X) = 1\}, \\
&\mathcal{J}_3(\tilde{K})_0 := \{X \in \mathcal{J}_3(\tilde{K}) |\ \mathrm{tr}(X) = 0\}, \\
&\mathcal{M}_1(\tilde{K}) := \{X \in \mathcal{J}_3(\tilde{K})_0 |\ X \neq 0,\ X^{\times 2} = 0\}, \\
&\mathcal{M}_{23}(\tilde{K}) := \{X \in \mathcal{J}_3(\tilde{K})_0 |\ X^{\times 2} \neq 0,\ \mathrm{tr}(X^{\times 2}) = \det(X) = 0\}.
\end{aligned}$$

When $\tilde{K} = K$, $\mathcal{P}_2(\tilde{K})$ has a structure of Moufang projective plane [9, p.162, 4.6, 4.7], the algebraization method of which motivates to define the cross product on $\mathcal{J}_3(\tilde{K})$ for any $\tilde{K}$. The automorphism group of $\mathcal{J}_3(\tilde{K})$ with respect to the $\mathbb{F}$-bilinear Jordan product $X \circ Y$ is denoted as follows:

$$G(\tilde{K}) := \mathrm{Aut}(\mathcal{J}_3(\tilde{K})) = \{\alpha \in GL_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))|\ \alpha(X \circ Y) = \alpha X \circ \alpha Y\},$$

which is a complex (resp. compact; real split) simple Lie group of type (F_4) (resp. $(F_{4(-52)})$; $(F_{4(4)})$) when $\tilde{K} = \boldsymbol{O'}^{\mathbb{C}} = \boldsymbol{O}^{\mathbb{C}}$ (resp. $\boldsymbol{O}$; $\boldsymbol{O'}$) by C. Chevalley and R.D. Schafer [2] (resp. [7], [9, p.161], [20, p.206, (2), (3)]; [29]). When $K = \mathbb{R}, \boldsymbol{C}$ or $\boldsymbol{H}$, the group $G(\tilde{K})$ is a simple Lie group of type $(A_1), (A_2)$ or (C_3), respectively (cf. [9, p.165]). Put $\gamma : \mathcal{J}_3(\tilde{K}) \longrightarrow \mathcal{J}_3(\tilde{K}); X \mapsto \gamma X$ such that $\gamma X := \sum_{i=1}^3 (\xi_i E_i + F_i(\gamma x_i))$ for $X = \sum_{i=1}^3 (\xi_i E_i + F_i(x_i)) \in \mathcal{J}_3(\tilde{K})$. Put $\tau : \mathcal{J}_3(\tilde{K}) \longrightarrow \mathcal{J}_3(\tilde{K}); X \mapsto \tau X$ such as $\tau X := \sum_{i=1}^3 ((\tau\xi_i)E_i + F_i(\tau x_i))$. Then $\tau \in GL_{\mathbb{R}}(\mathcal{J}_3(\tilde{K}))$ such that $\tau(X \circ Y) = (\tau X) \circ (\tau Y), \tau(X \times Y) = (\tau X) \times (\tau Y)$, $\mathrm{tr}(\tau X) = \tau(\mathrm{tr}(X))$, $(\tau X|\tau Y) = \tau(X|Y)$ and $\det(\tau X) = \tau(\det X)$, and that $\tau^2 = \mathrm{id}$, $\mathcal{J}_3(\tilde{K}) = \mathcal{J}_3(\tilde{K})_\tau \oplus \mathcal{J}_3(\tilde{K})_{-\tau}$ and $\mathcal{J}_3(K^{\mathbb{C}})_{-\tau} = \sqrt{-1}\mathcal{J}_3(K)$, so that $G(K) \equiv \{\alpha^{\mathbb{C}}|\ \alpha \in G(K)\} = G(K^{\mathbb{C}})^\tau$.

For $X \in \mathcal{J}_3(\tilde{K})$ and the indeterminate λ, put $\varphi_X(\lambda) := \lambda E - X$. Then *the characteristic polynomial of* X is defined as the polynomial $\Phi_X(\lambda) := \det(\varphi_X(\lambda))$ of λ with degree 3 and the derivative $\Phi'_X(\lambda)$ is $\frac{d}{d\lambda}\Phi_X(\lambda)$, so that $\Phi_X(\lambda) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ with some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$. In this case, the set $\{\lambda_1, \lambda_2, \lambda_3\}$ is said to be *the characteristic roots of* X. Put $\Lambda_X := \{\lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{C}$ with $\#\Lambda_X \in \{1, 2, 3\}$ and $V_X := \{aX^{\times 2} + bX + cE|\ a, b, c \in \mathbb{F}\}$ with $v_X := \dim V_X \in \{1, 2.3\}$.

PROPOSITION 0.1. *Let* $\tilde{K}$ *be* K, K' *or* $K^{\mathbb{C}}$ *with* $K = \mathbb{R}, \boldsymbol{C}, \boldsymbol{H}$ *or* $\boldsymbol{O}$.
(1) $G(\tilde{K}) \subsetneqq \{\alpha \in \mathrm{GL}_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))|\ \mathrm{tr}(\alpha X) = \mathrm{tr}(X),\ \alpha E = E\}$. *And*

$$\begin{aligned} G(\tilde{K}) &= \{\alpha \in \mathrm{GL}_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))|\ \det(\alpha X) = \det(X),\ \alpha E = E\} \\ &= \{\alpha \in \mathrm{GL}_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))|\ \Phi_{\alpha X}(\lambda) = \Phi_X(\lambda)\} \\ &= \{\alpha \in \mathrm{GL}_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))|\ \det(\alpha X) = \det(X),\ (\alpha X|\alpha Y) = (X|Y)\} \\ &= \{\alpha \in \mathrm{GL}_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))|\ \alpha(X \times Y) = (\alpha X) \times (\alpha Y)\}. \end{aligned}$$

Especially, $\Lambda_{\alpha X} = \Lambda_X$ *and* $v_{\alpha X} = v_X$ *for all* $X \in \mathcal{J}_3(\tilde{K})$ *and* $\alpha \in G(\tilde{K})$.
(2) $G(\tilde{K})^\tau$ *is a maximal compact subgroup of* $G(\tilde{K})$. *And* $\gamma \in G(\tilde{K})^\tau_{E_1,E_2,E_3}$.

(3) $\mathcal{P}_2(\tilde{K}) = \mathcal{O}_{G(\tilde{K})^\circ}(E_1)$,

(4) *Assume that* $\tilde{K} \neq K$, *i.e.,* $\tilde{K} = \boldsymbol{R}^{\mathbb{C}}, \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}, \boldsymbol{O}^{\mathbb{C}}; \boldsymbol{C}', \boldsymbol{H}'$ *or* $\boldsymbol{O}'$. *Then:*

(i) $\mathcal{M}_1(\tilde{K}) = \mathcal{O}_{G(\tilde{K})^\circ}(\tilde{M}_1)$,

(ii) $\mathcal{M}_{23}(\tilde{K}) = \mathcal{O}_{G(\tilde{K})^\circ}(\tilde{M}_{23})$.

Theorem 0.2. *Let* $K^{\mathbb{C}}$ *be* $\mathbb{R}^{\mathbb{C}}, \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}$ *or* $\boldsymbol{O}^{\mathbb{C}}$. *Then the orbit decomposition of* $\mathcal{J}_3(K^{\mathbb{C}})$ *over* $G(K^{\mathbb{C}})$ *or* $G(K^{\mathbb{C}})^\circ$ *is given as follows:*

(1) *Take* $X \in \mathcal{J}_3(K^{\mathbb{C}})$. *Then* $\#\Lambda_X = 3, 2$ *or* 1.

(i) *Assume that* $\#\Lambda_X = 3$ *with* $\Lambda_X = \{\lambda_1, \lambda_2, \lambda_3\}$. *Then* $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X)$ *with* $v_X = 3$.

(ii) *Assume that* $\#\Lambda_X = 2$ *with* $\Lambda_X = \{\lambda_1, \lambda_2\}$ *such that* $\Phi'_X(\lambda_2) = 0$. *Then* $v_X = 2$ *or* 3. *Moreover:*

(ii-1) $v_X = 2$ *iff* $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2) \in \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X)$*; and*

(ii-2) $v_X = 3$ *iff* $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2) + M_1 \in \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X)$.

(iii) *Assume that* $\#\Lambda_X = 1$ *with* $\Lambda_X = \{\lambda_1\}$. *Then:*

(iii-1) $v_X = 1$ *iff* $\lambda_1 E \in \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X)$*;*

(iii-2) $v_X = 2$ *iff* $\lambda_1 E + M_1 \in \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X)$*; and*

(iii-3) $v_X = 3$ *iff* $\lambda_1 E + M_{23} \in \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X)$.

(2) *For* $X, Y \in \mathcal{J}_3(K^{\mathbb{C}})$, $\mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X) = \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(Y)$ *iff* $\Lambda_X = \Lambda_Y$ *and* $v_X = v_Y$. *For any* $X \in \mathcal{J}_3(K^{\mathbb{C}})$, $\mathcal{O}(X) := \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X) = \mathcal{O}_{G(K^{\mathbb{C}})}(X)$ *and* $\mathcal{O}(X) \cap \mathcal{J}_3(\mathbb{R}^{\mathbb{C}}) \neq \emptyset$.

Theorem 0.3. *Let* K' *be* $\boldsymbol{C}', \boldsymbol{H}'$ *or* $\boldsymbol{O}'$. *Then the orbit decomposition of* $\mathcal{J}_3(K')$ *over* $G(K')$ *or* $G(K')^\circ$ *is given as follows:*

(1) *Take* $X \in \mathcal{J}_3(K')$. *Then* $\#\Lambda_X = 3, 2$ *or* 1.

(i) *Assume that* $\#\Lambda_X = 3$. *Then* $v_X = 3$. *And* $\Lambda_X = \{\lambda_1, \lambda_2, \lambda_3\}$ *for some* $\lambda_1 \in \mathbb{R}$ *and* $\lambda_2, \lambda_3 \in \mathbb{C}$ *such that* $\Lambda_X \subset \mathbb{R}$ *or* $\{\lambda_2, \lambda_3\} = \{p \pm q\sqrt{-1}\}$ *with some* $p \in \mathbb{R}$ *and* $q \in \mathbb{R}\backslash\{0\}$. *Moreover:*

(i-1) *If* $\Lambda_X \subset \mathbb{R}$ *with* $\lambda_1 > \lambda_2 > \lambda_3$, *then* $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{O}_{G(K')^\circ}(X)$*; and*

(i-2) *If* $\{\lambda_2, \lambda_3\} = \{p \pm q\sqrt{-1}\}$ *with some* $p, q \in \mathbb{R}$ *such that* $q > 0$, *then* $\mathrm{diag}(\lambda_1, p, p) + F_1(q\sqrt{-1}e_4) \in \mathcal{O}_{G(K')^\circ}(X)$.

(ii) *Assume that* $\#\Lambda_X = 2$ *with* $\Lambda_X = \{\lambda_1, \lambda_2\}$ *such that* $\Phi'_X(\lambda_2) = 0$. *Then* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $v_X = 2$ *or* 3. *Moreover:*

(ii-1) $v_X = 2$ *iff* $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2) \in \mathcal{O}_{G(K')^\circ}(X)$*; and*

(ii-2) $v_X = 3$ *iff* $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2) + M_{1'} \in \mathcal{O}_{G(K')^\circ}(X)$.

(iii) *Assume that* $\#\Lambda_X = 1$ *with* $\Lambda_X = \{\lambda_1\}$*. Then* $\lambda_1 \in \mathbb{R}$*. Moreover:*

(iii-1) $v_X = 1$ *iff* $\lambda_1 E \in \mathcal{O}_{G(K')^\circ}(X)$*;*

(iii-2) $v_X = 2$ *iff* $\lambda_1 E + M_{1'} \in \mathcal{O}_{G(K')^\circ}(X)$*; and*

(iii-3) $v_X = 3$ *iff* $\lambda_1 E + M_{2'3} \in \mathcal{O}_{G(K')^\circ}(X)$.

(2) *For* $X, Y \in \mathcal{J}_3(K')$*,* $\mathcal{O}_{G(K^{\mathbb{C}})^\circ}(X) = \mathcal{O}_{G(K^{\mathbb{C}})^\circ}(Y)$ *iff* $\Lambda_X = \Lambda_Y$ *and* $v_X = v_Y$*. For any* $X \in \mathcal{J}_3(K')$*,* $\mathcal{O}(X) := \mathcal{O}_{G(K')^\circ}(X) = \mathcal{O}_{G(K')}(X)$ *and* $\mathcal{O}(X) \cap \mathcal{J}_3(\boldsymbol{C}') \neq \emptyset$.

By Proposition 0.1 (1), Λ_X and v_X are invariants on $\mathcal{O}_{G(\tilde{K})}(X)$, so that the Theorems 0.2 (2) and 0.3 (2) follow from Theorems 0.2 (1) and 0.3 (1), respectively. Hence, this paper is concentrated in proving Theorems 0.2 (1) and 0.3 (1) with Proposition 0.1.

Note that the second equality of Propositoin 0.1 (1) was obtained by N. Jacobson [13, Lemma 1] in a more general setting (cf. [24, p.159, Proposition 5.9.4, §5.10]). In §1, by Lemma 1.2, it appears that the characteristic polynomial $\Phi_X(\lambda)$ of X equals the generic minimal polynomial of X defined by N. Jacobson [16, p.358 (5)]. By Lemma 1.6 (3), it appears that v_X equals the degree of N. Jacobson [16, p.389, Theorem 10]'s minimal polynomial for $X \in \mathcal{J}_3(\tilde{K})$ with respect to the Jordan product.

Contents

1. Preliminaries and Proposition 0.1 (1) and (2).

Let $i, i+1, i+2 \in \{1,2,3\}$ be the indices counted modulo 3. Then

$$\begin{cases} E_i \circ E_i = E_i, & E_i \circ E_{i+1} = 0, \\ E_i \circ F_i(x) = 0, & E_i \circ F_j(x) = \frac{1}{2} F_j(x) \ (i \neq j), \\ F_i(x) \circ F_i(y) = (x|y)(E_{i+1} + E_{i+2}), & F_i(x) \circ F_{i+1}(y) = \frac{1}{2} F_{i+2}(\overline{xy}); \end{cases}$$
$$\begin{cases} E_i \times E_i = 0, & E_i \times E_{i+1} = \frac{1}{2} E_{i+2}, \\ E_i \times F_i(x) = -\frac{1}{2} F_i(x), & E_i \times F_j(x) = 0 \ (i \neq j), \\ F_i(x) \times F_i(y) = -(x|y) E_i, & F_i(x) \times F_{i+1}(y) = \frac{1}{2} F_{i+2}(\overline{xy}) \end{cases}$$

for any $x, y \in \tilde{K}$. And

$$\begin{aligned} & M_1(x) \times M_1(y) = \sqrt{-1}\{(x|1)(y|1) - (x|y)\} E_1, \\ & M_1(x) \times M_{23}(y) = -\frac{1}{2}\{F_2(\sqrt{-1}(\overline{x} - (x|1))\overline{y}) + F_3(y(\overline{x} - (x|1)))\}, \\ & M_{23}(x) \times M_{23}(y) = (x|y) M_1, \ M_1 = M_{23}^{\times 2}; \text{ and} \\ & M_{1'}(x) \times M_{1'}(y) = \{(x|y) - (x|1)(y|1)\} E_1, \\ & M_{1'}(x) \times M_{2'3}(y) = \frac{1}{2}\{F_2(-(\overline{x}\sqrt{-1}e_4)\overline{y} + (x|1)\sqrt{-1}e_4\overline{y}) \\ & \qquad\qquad + F_3(-(y\sqrt{-1}e_4)(\overline{x}\sqrt{-1}e_4) + (x|1)y)\}, \\ & M_{23'}(x) \times M_{23'}(y) = (x|y) M_{1'}, \quad M_{1'} = M_{2'3}^{\times 2}. \end{aligned}$$

Let denote $\mathbb{X}(r;x) := \sum_{i=1}^{3} r_i E_i + \sum_{i=1}^{3} F_i(x_i)$ for any $r = (r_1, r_2, r_3) \in \mathbb{F}_3$ and $x = (x_1, x_2, x_3) \in \tilde{K}_3$. If $Y = \mathbb{X}(r;x) \in \mathcal{J}_3(\tilde{K})$, put $(Y)_{E_i} := (Y|E_i) = r_i$ and $(Y)_{F_i} := (Y|F_i(1))/2 = x_i$.

LEMMA 1.1. (1) *Let $i, i+1, i+2 \in \{1,2,3\}$ be counted modulo 3. Then*

$$\mathbb{X}(r;x)\times\mathbb{X}(s;y)=\frac{1}{2}\sum_{i=1}^{3}\{(r_{i+1}s_{i+2}+s_{i+1}r_{i+2}-2(x_i|y_i))E_i$$
$$+F_i(\overline{x_{i+1}y_{i+2}+y_{i+1}x_{i+2}}-r_iy_i-s_ix_i)\};$$
$$(\mathbb{X}(r;x)|\mathbb{X}(s;y))=\sum_{i=1}^{3}(r_is_i+2(x_i|y_i));$$
$$(\mathbb{X}(r;x)|\mathbb{X}(s;y)|\mathbb{X}(u;z))=(\mathbb{X}(r;x)|\mathbb{X}(s;y)\times\mathbb{X}(u;z))$$
$$=\sum_{i=1}^{3}\{\frac{r_i}{2}(s_{i+1}u_{i+2}+u_{i+1}s_{i+2})+(\overline{x_i}|y_{i+1}z_{i+2}+z_{i+1}y_{i+2})$$
$$-r_i(y_i|z_i)-s_i(z_i|x_i)-u_i(y_i|x_i)\};$$
$$\det(\mathbb{X}(r;x))=r_1r_2r_3+2(\overline{x_i}|x_{i+1}x_{i+2})-\sum_{j=1}^{3}r_jN(x_j)\text{ for }i\in\{1,2,3\}.$$

(2) *For* $X,Y,Z\in\mathcal{J}_3(\tilde{K})$*, all of* $X\circ Y$, $(X|Y)$, $X\times Y$, $(X\circ Y|Z)$ *and* $(X|Y|Z)$ *are symmetric. And* $(X|Y)$ *is non-degenerate.*

(3) $2E\times X=\mathrm{tr}(X)E-X=\varphi_X(\mathrm{tr}(X))$. *Especially,* $E^{\times 2}=E$ *and* $2E\times X^{\times 2}=\mathrm{tr}(X^{\times 2})E-X^{\times 2}=\varphi_{X^{\times 2}}(\mathrm{tr}(X^{\times 2}))$.

(4) $(X|Y|E)=\mathrm{tr}(X\times Y)=\frac{1}{2}(\mathrm{tr}(X)\mathrm{tr}(Y)-(X|Y))$.

Proof. (1) follows from the definitions except the 3rd equality, which is proved by [5, p.15, 3.5 (7)] as follows:

$$\begin{aligned}
&(\mathbb{X}(r;x)|\mathbb{X}(s;y)|\mathbb{X}(u;z))=\sum_{i=1}^{3}\{u_i(r_{i+1}s_{i+2}+s_{i+1}r_{i+2})/2\\
+\;&(\overline{z_i}|x_{i+1}y_{i+2}+y_{i+1}x_{i+2})-u_i(x_i|y_i)-r_i(y_i|z_i)-s_i(x_i|z_i)\}\\
=\;&\sum_{i=1}^{3}\{(u_{i+2}r_{i+3}s_{i+4}+u_{i+1}s_{i+2}r_{i+3})/2\\
+\;&(\overline{x_{i+3}}\;\overline{z_{i+2}}|y_{i+4})+(\overline{z_{i+1}}\;\overline{x_{i+3}}|y_{i+2})-u_i(x_i|y_i)-r_i(y_i|z_i)-s_i(x_i|z_i)\}\\
=\;&\sum_{i=1}^{3}\{r_i(s_{i+1}u_{i+2}+u_{i+1}s_{i+2})/2+(\overline{x_i}|y_{i+1}z_{i+2}+z_{i+1}y_{i+2})\\
-\;&r_i(y_i|z_i)-s_i(z_i|x_i)-u_i(y_i|x_i)\}\\
=\;&(\mathbb{X}(r;x)|\mathbb{X}(s;y)\times\mathbb{X}(u;z)).
\end{aligned}$$

(2) follows from the definitions or (1). (3) follows from direct computations. (4) follows from the definitions of $(X|Y|Z)$ and $X \times Y$. $\square$

For $X \in \mathcal{J}_3(\tilde{K})$, put $\Delta_X(\lambda) := -\frac{1}{2}\{3\lambda^2 - 2\mathrm{tr}(X)\lambda + \mathrm{tr}(X)^2 - 2(X|X)\}$, which values in $\mathbb{F}$ (or $\mathbb{C}$) if $\lambda \in \mathbb{F}$ (resp. $\mathbb{C}$).

Lemma 1.2. (1) $\Phi_X(\lambda) = \lambda^3 - \mathrm{tr}(X)\lambda^2 + \mathrm{tr}(X^{\times 2})\lambda - \det(X)$ *with* $\mathbb{F} \ni \mathrm{tr}(X) = \lambda_1 + \lambda_2 + \lambda_3$, $\mathrm{tr}(X^{\times 2}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_2$, $\det(X) = \lambda_1\lambda_2\lambda_3$ *if* $\Lambda_X = \{\lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{C}$ *for* $X \in \mathcal{J}_3(\tilde{K})$.

(2) $\Phi'_X(\lambda) = 3\lambda^2 - 2\mathrm{tr}(X)\lambda + \mathrm{tr}(X^{\times 2}) = \mathrm{tr}(\varphi_X(\lambda)^{\times 2}) = -2\Delta_X(\lambda) - \frac{1}{2}\{\mathrm{tr}(X)^2 - 3(X|X)\}$.

(3) *Put* $\mathcal{M}(\tilde{K}) := \{X \in \mathcal{J}_3(\tilde{K})_0 | \ X \neq 0, \ \Phi_X(\lambda) = \lambda^3\}$. *Then* $\mathcal{M}(\tilde{K}) = \{X \in \mathcal{J}_3(\tilde{K})_0 | \ X \neq 0, \mathrm{tr}(X^{\times 2}) = \det(X) = 0\} = \mathcal{M}_1(\tilde{K}) \cup \mathcal{M}_{23}(\tilde{K})$ *with* $\mathcal{M}_1(\tilde{K}) \cap \mathcal{M}_{23}(\tilde{K}) = \emptyset$. *And* $\{X \in \mathcal{J}_3(\tilde{K}) | \ \#\Lambda_X = 1\} = \mathbb{F}E \oplus (\{0\} \cup \mathcal{M}(\tilde{K}))$.

Proof. (1) $\Phi_X(\lambda) = \frac{1}{3}(\lambda E - X|\lambda E - X|\lambda E - X)$, which equals the required one by Lemma 1.1 (2, 3) and $\Phi_X(\lambda) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$.

(2) The first equality folllows from (1). By Lemma 1.1 (3), $\varphi_X(\lambda)^{\times 2} = (\lambda E - X)^{\times 2} = \lambda^2 E - (\mathrm{tr}(X)E - X)\lambda + X^{\times 2}$, so that $\mathrm{tr}(\varphi_X(\lambda)^{\times 2}) = 3\lambda^2 - 2\mathrm{tr}(X)\lambda + \mathrm{tr}(X^{\times 2})$. And $3\lambda^2 - 2\mathrm{tr}(X)\lambda + \mathrm{tr}(X^{\times 2}) = -2\Delta_X(\lambda) - \frac{1}{2}\{\mathrm{tr}(X)^2 - 3(X|X)\}$ by the second equality of Lemma 1.1 (4).

(3) The first claim follows from (1). For $X \in \mathcal{J}_3(\tilde{K})$, put $X_0 := X - \frac{1}{3}\mathrm{tr}(X)E \in \mathcal{J}_3(\tilde{K})_0$. Then $X = \frac{1}{3}\mathrm{tr}(X)E + X_0$, so that $\mathcal{J}_3(\tilde{K}) = \mathbb{F}E \oplus \mathcal{J}_3(\tilde{K})_0$. If $\Phi_X(\lambda) = \Pi_{i=1}^3(\lambda - \lambda_i)$, then $\Phi_{X_0}(\lambda) = \det((\lambda + \frac{1}{3}\mathrm{tr}(X))E - X) = \Pi_{i=1}^3(\lambda + \frac{1}{3}\mathrm{tr}(X) - \lambda_i)$, so that $\Phi_{X_0}(\lambda) = \lambda^3 \Leftrightarrow \frac{1}{3}\mathrm{tr}(X) - \lambda_i = 0 (i = 1, 2, 3) \Leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 \Leftrightarrow \#\Lambda_X = 1$, because of $\mathrm{tr}(X) = \sum_{i=1}^3 \lambda_i$ by (1). Hence, $\{X \in \mathcal{J}_3(\tilde{K}) | \ \#\Lambda_X = 1\} = \mathbb{F}E \oplus (\{0\} \cup \mathcal{M}(\tilde{K}))$. $\square$

Let V be an $\mathbb{F}$-algebra with the multiplication xy of $x, y \in V$. For $x \in V$, put an $\mathbb{F}$-linear endomorphism on V, $L_x : V \to V; y \mapsto xy$, as the left translation by x. And put the automorphism group of V as follows:

$$\mathrm{Aut}(V) := \{\alpha \in GL_{\mathbb{F}}(V) | \ \alpha(xy) = (\alpha x)(\alpha y); \ x, y \in V\}.$$

Lemma 1.3. (1) *Let* V *be an* $\mathbb{F}$*-algbra. Assume that* $\alpha \in \mathrm{Aut}(V)$*. Then* $\mathrm{trace}(L_{(\alpha x)}) = \mathrm{trace}(L_x)$, $\det(L_{(\alpha x)}) = \det(L_x)$ *for all* $x \in V$*. If moreover* V *admits the identity element* e*, then* $\alpha e = e$.

(2) *Let* L°_X *and* $L^{\times}_X$ *be the left translations by* $X \in \mathcal{J}_3(\tilde{K})$ *on* $\mathcal{J}_3(\tilde{K})$ *with respect to the product* $\circ$ *and the cross product* $\times$*, respectively. Then* $\mathrm{trace}(L^{\circ}_X) = (d_K + 1)\ \mathrm{tr}(X)$ *and* $\mathrm{trace}(L^{\times}_X) = \frac{-1}{2} d_K\ \mathrm{tr}(X)$.

Proof. (1) For $x, y \in V$ and $\alpha \in G(V)$, $L_{(\alpha x)} y = (\alpha x)y = \alpha(x(\alpha^{-1}y)) = (\alpha L_x \alpha^{-1})y$, i.e. $L_{(\alpha x)} = \alpha L_x \alpha^{-1}$, so that $\mathrm{trace}(L_{(\alpha x)}) = \mathrm{trace}(L_x)$ and $\det(L_{(\alpha x)}) = \det(L_x)$ as an $\mathbb{F}$-linear endomorphism on V. Assume that $ex = xe = x$ for any $x \in V$. Take $\alpha \in \mathrm{Aut}(V)$. Then $(\alpha e)(\alpha x) = (\alpha x)(\alpha e) = \alpha x$, so that $(\alpha e)y = y(\alpha e) = y$ for all $y \in V$. In particular, $\alpha e = (\alpha e)e = e$.

(2) $\{E_i, F_i(e_j/\sqrt{2}) |\ i = 1, 2, 3; j = 0, \cdots, d_K - 1\}$ forms an orthonormal basis of $(\mathcal{J}_3(K^{\mathbb{C}}), (*|*))$ by Lemma 1.1 (1). And L°_X and $L^{\times}_X$ can be identified with a $\mathbb{C}$-linear endomorphism on $\mathcal{J}_3(K^{\mathbb{C}}) = \mathcal{J}_3(\tilde{K})$ or $\mathbb{C} \otimes \mathcal{J}_3(\tilde{K})$. By Lemma 1.1 (1, 2), $\mathrm{trace}(L^{\circ}_X) = \sum_{i=1}^{3} \{(X \circ E_i | E_i) + \frac{1}{2} \sum_{j=0}^{d_K-1} (X \circ F_i(e_j) | F_i(e_j))\} = \sum_{i=1}^{3} \{(X | E_i \circ E_i) + \frac{1}{2} \sum_{j=0}^{d_K-1} (X | F_i(e_j) \circ F_i(e_j))\} = \sum_{i=1}^{3} \{(X|E_i) + \frac{1}{2} \sum_{j=0}^{d_K-1} (X | E_{i+1} + E_{i+2})\} = (d_K + 1)\ \mathrm{tr}(X)$; and $\mathrm{trace}(L^{\times}_X) = \sum_{i=1}^{3} \{(X \times E_i, E_i) + \frac{1}{2} \sum_{j=0}^{d_K-1} (X \times F_i(e_j) | F_i(e_j))\} = \sum_{i=1}^{3} \{(X | E_i \times E_i) + \frac{1}{2} \sum_{j=0}^{d_K-1} (X | F_i(e_j) \times F_i(e_j))\} = \sum_{i=1}^{3} \frac{1}{2} \sum_{j=0}^{d_K-1} (X | -E_i) = \frac{-1}{2} d_K \mathrm{tr}(X)$. $\square$

Proof of Proposition 0.1 (1). The first claim follows from Lemma 1.3 (1)(2). For the second claim, since $\det(X)$ is defined by $X \circ X$, $\mathrm{tr}(X)$ and E, the first equality is recognized as the inclusion $\subseteqq$. By $\Phi_X(\lambda) = \det(\lambda E - X)$, the 2nd equality is recognized as the inclusion $\subseteqq$. By Lemmas 1.1 (4) and 1.2 (1), the 3rd equality is recognized as the inclusion $\subseteqq$. By polarizing $3\det(X) = (X|X|X)$ with Lemma 1.1 (2), the 4th equality is recognized as the inclusion $\subseteqq$. Assume that $\alpha \in \mathrm{GL}_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))$ and $(\alpha X) \times (\alpha Y) = \alpha(X \times Y)$ for all $X, Y \in \mathcal{J}_3(\tilde{K})$. By Lemma 1.3, $\mathrm{tr}(\alpha X) = \mathrm{tr}(X)$. By Lemma 1.1 (4), $(X|Y) = \mathrm{tr}(X)\mathrm{tr}(Y) - 2\mathrm{tr}(X \times Y)$, so that $(\alpha X | \alpha Y) = (X|Y)$. By the definition of $\times$, $(X \circ Y | Z) = (X \times Y | Z) + (\mathrm{tr}(X)(Y|Z) + \mathrm{tr}(Y)(X|Z) - (\mathrm{tr}(X)\mathrm{tr}(Y) - (X|Y))\mathrm{tr}(Z))/2$, so that $((\alpha X) \circ (\alpha Y) | \alpha Z) = (X \circ Y | Z)$ for all $X, Y, Z \in \mathcal{J}_3(\tilde{K})$. By Lemma 1.1 (2), $\alpha^{-1}((\alpha X) \circ (\alpha Y)) = X \circ Y$, that is, $\alpha \in G(\tilde{K})$. Hence, all of the equations of the second claim follow. The last claim follows from these equations.

Proof of Proposition 0.1 (2). Note that $\tau\gamma = \gamma\tau$, $\gamma E_i = E_i$, $\gamma E = E$ and $\det(\gamma \mathbb{X}(r; x)) = \det(\mathbb{X}(r; x))$ by Lemma 1.1 (1), so that $\gamma \in G(\tilde{K})^{\tau}_{E_1, E_2, E_3}$. By Proposition 0.1 (1), the last claim follows. For the first claim, put $< X|Y > := (\tau X | Y) \in \mathbb{F}$ for $X, Y \in \mathcal{J}_3(\tilde{K})$, which defines a positive-definite symmetric (or hermitian) 2-form on $\mathcal{J}_3(K')$ (resp. $\mathcal{J}_3(K^{\mathbb{C}})$) over $\mathbb{R}$ (resp.

$\mathbb{C}$) by Lemma 1.1 (1). For $\alpha \in G(\tilde{K})$, $\alpha^* \in GL_F(\mathcal{J}_3(\tilde{K}))$ is defined such that $<\alpha X|Y>=<X|\alpha^* Y>$ for all $X, Y \in \mathcal{J}_3(\tilde{K})$. By (1), $<X|\alpha^* Y>= (\tau\alpha X|Y) = \tau(\alpha X|\tau Y) = \tau(X|\alpha^{-1}\tau Y) =< X|\tau\alpha^{-1}\tau Y>$, so that $\alpha^* = \tau\alpha^{-1}\tau \in G(\tilde{K})$ because of (1) by $\det(\alpha^* X) = \tau\det(\alpha^{-1}\tau X) = \tau^2\det(X) = \det(X)$ and $\alpha^* E = \tau\alpha^{-1}\tau E = E$. Then $G(\tilde{K}) \cong G(\tilde{K})^\tau \times \boldsymbol{R}$ as a polar decomposition of C. Chevalley [1, p.201] (resp. [12, p.450, Lemma 2.3]), so that $G(\tilde{K})^\tau$ is a maximal compact subgroup of $G(\tilde{K})$. □

For $i \in \{1, 2, 3\}$ and $a \in \tilde{K}$, put $B_i(a) : \mathcal{J}_3(\tilde{K}) \longrightarrow \mathcal{J}_3(\tilde{K}); \mathbb{X}(r; x) \mapsto \mathbb{X}(s; y)$ such that $s_i := 0, s_{i+1} := 2(a|x_i), s_{i+2} := -2(a|x_i), y_i := -(r_{i+1} - r_{i+2})a, y_{i+1} := -\overline{x_{i+2}a}, y_{i+2} := \overline{ax_{i+1}}$, where $i, i+1, i+2 \in \{1, 2, 3\}$ are counted modulo 3. Then $\exp(tB_i(a)) \in (G(\tilde{K})_{E_i})^\circ$ for $t \in \mathbb{F}$. In fact, $s_i = s_{i+1} = s_{i+2} = y_i = y_{i+1} = y_{i+2} = 0$ if $\mathbb{X}(r; x) = E_i$ or E. Put $X = \mathbb{X}(r; x)$. By Lemma 1.1 (1), $(B_i(a)X|X|X) = (B_i(a)X|X^{\times 2}) = 2\{(a|x_i)(r_{i+2}r_i - N(x_{i+1}) - r_i r_{i+1} + N(x_{i+2})) - (r_{i+1} - r_{i+2})(a|\overline{x_{i+1}x_{i+2}} - r_i x_i) - (\overline{x_{i+2}a}|\overline{x_{i+2}x_i} - r_{i+1}x_{i+1}) + (\overline{ax_{i+1}}|\overline{x_i x_{i+1}} - r_{i+2}x_{i+2})\} = 0$, so that $\exp(tB_i(a)) \in (G(\tilde{K})_{E_i})^\circ$ for all $t \in \mathbb{F}$ by Proposition 0.1 (1), as required. Note that $B_i(a)$ is nothing but $\tilde{A}^a_i$ given in H. Freudenthal [7, (5.1.1)].

For $\nu \in \{1, \sqrt{-1}\}$, put $C_\nu(t) := (e^{\nu t} + e^{-\nu t})/2$, $S_\nu(t) := (e^{\nu t} - e^{-\nu t})/(2\nu)$ as $\mathbb{F}$-valued functions of $t \in \mathbb{F}$. Then $(C_\nu(t), S_\nu(t)) = (\cosh(t), \sinh(t))$ or $(\cos(t), \sin(t))$ if $\nu = 1$ or $\sqrt{-1}$, respectively. Note that

$$
\begin{aligned}
&\tau C_\nu(t) = C_\nu(\tau t),\ \tau S_\nu(t) = S_\nu(\tau t),\\
&C_\nu(t_1)C_\nu(t_2) + \nu^2 S_\nu(t_1)S_\nu(t_2) = C_\nu(t_1 + t_2),\\
&C'_\nu(t) = \nu^2 S_\nu(t),\ S'_\nu(t) = C_\nu(t),\\
&S_\nu(t_1)C_\nu(t_2) + C_\nu(t_1)S_\nu(t_2) = S_\nu(t_1 + t_2),\\
&C'_\nu(0) = 0,\ S'_\nu(0) = 1,\ C_\nu(2t) = 1 + 2\nu^2 S^2_\nu(t).
\end{aligned}
$$

For $i \in \{1, 2, 3\}$, $t \in \mathbb{F}$, $a \in \tilde{K}$ and $\nu \in \{1, \sqrt{-1}\}$, put $\beta_i(t; a, \nu) : \mathcal{J}_3(\tilde{K}) \longrightarrow \mathcal{J}_3(\tilde{K}); \mathbb{X}(r; x) \mapsto \mathbb{X}(s; y)$ such that

$$
\begin{cases}
s_i &:= r_i,\\
s_{i+1} &:= \frac{r_{i+1}+r_{i+2}}{2} + \frac{r_{i+1}-r_{i+2}}{2}C_\nu(2t) + (a|x_i)S_\nu(2t),\\
s_{i+2} &:= \frac{r_{i+1}+r_{i+2}}{2} - \frac{r_{i+1}-r_{i+2}}{2}C_\nu(2t) - (a|x_i)S_\nu(2t),\\
y_i &:= x_i - a\frac{r_{i+1}-r_{i+2}}{2}S_\nu(2t) - 2a(a|x_i)S^2_\nu(t),\\
y_{i+1} &:= x_{i+1}C_\nu(t) - \overline{x_{i+2}a}S_\nu(t),\\
y_{i+2} &:= x_{i+2}C_\nu(t) + \overline{ax_{i+1}}S_\nu(t).
\end{cases}
$$

For $c \in \mathbb{F}$, put $\mathcal{S}_1(c, \tilde{K}) := \{x \in \tilde{K} |\ N(x) = c\}$, which is said to be *a generalized sphere* [11, p.42, (3.7)] *of first kind over* $\mathbb{F}$.

LEMMA 1.4. (1) (i) *Assume that* $i \in \{1,2,3\}$, $\nu \in \{1, \sqrt{-1}\}$ *and* $a \in \mathcal{S}_1(-\nu^2, \tilde{K})$. *Then* $\beta_i(t; a, \nu) = \exp(tB_i(a)) \in (G(\tilde{K})_{E_i})^\circ$ *for* $t \in \mathbb{F}$ *such that* $\beta_i(t; a, \nu)\tau = \tau\beta_i(\tau t; \tau a, \nu)$ *for all* $t \in \mathbb{F}$. *Especially,* $\sigma_i := \beta_i(\pi; 1, \sqrt{-1}) \in ((G(\tilde{K})^\tau_{E_i})^\circ)_{E_{i+1}, E_{i+2}}$.

(ii) *For* $i \in \{1,2,3\}$, *put* $\hat{\beta}_i := \beta_i(\frac{\pi}{2}; 1, \sqrt{-1})$. *Then* $\hat{\beta}_i \in (G(\tilde{K})^\tau_{E_i})^\circ$ *such that* $\hat{\beta}_i X = r_i E_i + r_{i+2}E_{i+1} + r_{i+1}E_{i+2} + F_i(-\overline{x_i}) + F_{i+1}(-\overline{x_{i+2}}) + F_{i+2}(\overline{x_{i+1}})$ *if* $X = \mathbb{X}(r; x) \in \mathcal{J}_3(\tilde{K})$. *Especially, for any permutation* $\mu = (\mu_1, \mu_2, \mu_3)$ *of the triplet* $(1,2,3)$, *there exists* $\hat{\beta} \in (G(\tilde{K})^\tau)^\circ$ *such that* $\hat{\beta}(\sum_{j=1}^3 r_j E_j) = \sum_{j=1}^3 r_{\mu_j} E_{\mu_j}$ *for all* $r_i \in \mathbb{F}$ $(i = 1,2,3)$.

(iii) *Put* $B_{23} := B_2(\sqrt{-1}) - B_3(1)$, $B_{23'} := B_2(1) - B_3(\sqrt{-1}e_4)$, $B_{2'3} := B_2(-\sqrt{-1}e_4) - B_3(1)$, $\beta_{23}(t) := \exp(tB_{23})$, $\beta_{23'}(t) := \exp(tB_{23'})$, $\beta_{2'3}(t) := \exp(tB_{2'3})$. *Then* $\beta_{23}(t) \in (G(K^{\mathbb{C}})^\circ)_{M_1}$ *and* $\beta_{23}(t)M_{23}(x) = 2t(x|1)M_1 + M_{23}(x)$ $(x \in K^{\mathbb{C}},\ t \in \mathbb{C})$. *And* $\beta_{23'}(t), \beta_{2'3}(t) \in (G(K')^\circ)_{M_{1'}}$ *such that* $\beta_{23'}(t)M_{2'3}(x) = 2t(\sqrt{-1}e_4|x)M_{1'} + M_{2'3}(x)$, $\beta_{2'3}(t)M_{2'3}(x) = 2t(1|x)M_{1'} + M_{2'3}(x)$ $(x \in K',\ t \in \mathbb{R})$.

(2) (i) *Let* $\mathcal{S}_1(1, \tilde{K})^\circ$ *be the connected component of* $\mathcal{S}_1(1, \tilde{K})$ *containing* $1 = e_0$ *in* $\tilde{K}$. *And* $O(\tilde{K}) := \{\alpha \in GL_F(\tilde{K}) |\ N(\alpha x) = N(x)\}$. *Then* $\mathcal{S}_1(1, \tilde{K}) = \mathcal{O}_{O(\tilde{K})}(e_0) = \mathcal{S}_1(1, \tilde{K})^\circ \cup (-\mathcal{S}_1(1, \tilde{K})^\circ)$. *Especially,* $\mathcal{S}_1(1, \tilde{K}) = \mathcal{S}_1(1, \tilde{K})^\circ = -\mathcal{S}_1(1, \tilde{K})^\circ$ *when* $\tilde{K} = \boldsymbol{H}', \boldsymbol{O}'; \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}, \boldsymbol{O}^{\mathbb{C}}$.

(ii) *For* $a \in \mathcal{S}_1(1, \tilde{K})$ *and* $i \in \{1,2,3\}$, *put* $\delta_i(a) \in \mathrm{End}_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))$ *with* $\mathbb{X}(s; y) := \delta_i(a)\mathbb{X}(r; x)$ *such that* $s_i := r_i, s_{i+1} := r_{i+1}, s_{i+2} := r_{i+2}, y_i := a x_i a, y_{i+1} := \overline{a} x_{i+1}, y_{i+2} := x_{i+2}\overline{a}$. *Then* $\delta_i(a) \in ((G(\tilde{K})_{E_i})^\circ)_{E_{i+1}, E_{i+2}}$ *such that* $\delta_i(a)\sigma_i = \sigma_i\delta_i(a) = \delta_i(-a)$ *and* $\delta_i(a)\tau = \tau\delta_i(\tau a)$. *Especially,* $\delta_i(a) \in (G(\tilde{K})_{E_1,E_2,E_3})^\circ$ *when* $\tilde{K} = \boldsymbol{H}', \boldsymbol{O}'; \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}, \boldsymbol{O}^{\mathbb{C}}$.

(iii) *Assume that* $d_{\tilde{K}} \leqq 4$. *For* $a \in \mathcal{S}_1(1, \tilde{K})$ *and* $i \in \{1,2,3\}$, *put* $\beta_i(a) \in \mathrm{End}_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))$ *with* $\mathbb{X}(s; y) := \beta_i(a)\mathbb{X}(r; x)$ *such that* $s_i := r_i, s_{i+1} := r_{i+1}, s_{i+2} := r_{i+2}, y_i := a x_i \overline{a}, y_{i+1} := a x_{i+1}, y_{i+2} := x_{i+2}\overline{a}$. *Then* $\beta_i(a) \in ((G(\tilde{K})_{E_i})^\circ)_{E_{i+1}, E_{i+2}, F_i(1)}$ *such that* $\beta_i(a)\sigma_i = \sigma_i\beta_i(a) = \beta_i(-a)$. *Especially,* $\beta_i(a) \in (G(\tilde{K})_{E_1,E_2,E_3,F_1(1)})^\circ$ *when* $\tilde{K} = \boldsymbol{H}'; \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}$.

Proof. (1) (i) Put $\mathbb{X}(u; z) := \frac{d}{dt}\beta_i(t; a, \nu)\mathbb{X}(r; x) - B_i(a)\mathbb{X}(r; x)$. Then $u_i = z_i = 0$, $u_{i+1} = (\nu^2 + N(a))((r_{i+1} - r_{i+2})S_\nu(2t) + 4(a, x_i)S_\nu^2(t)) = 0 = -u_{i+2}$, $z_{i+1} = (\nu^2 + N(a))x_{i+1}S_\nu(t) = 0$, $z_{i+2} = (\nu^2 + N(a))x_{i+2}S_\nu(t) = 0$, i.e. $\frac{d}{dt}\beta_i(t; a, \nu)\mathbb{X}(r; x) = B_i(a)\mathbb{X}(r; x)$ $(t \in \mathbb{F})$ with $\beta_i(0; a, \nu)\mathbb{X}(r; x) =$

$\mathbb{X}(r;x)$. Hence, $\beta_i(t;a,\nu)=\exp(tB_i(a))$, so that $\beta_i(t;a,\nu)\in(G(\tilde{K})_{E_i})^\circ$ and $\beta_i(t;a,\nu)\tau=\exp(tB_i(a))\tau=\tau\exp((\tau t)B_i(\tau a))=\tau\beta_i(t;a,\nu)$ for all $t\in\mathbb{F}$. Especially, $\sigma_i(t):=\beta_i(\pi t;1,\sqrt{-1})\in(G(\tilde{K})^\tau_{E_i})^\circ$ for all $t\in\mathbb{R}$ such that $\sigma_i=\sigma_i(1)$, $\sigma_iE_{i+1}=E_{i+1}$, $\sigma_iE_{i+2}=E_{i+2}$.

(ii) The first claim follows from (i), so that the second claim follows.

(iii) For $a,b\in K^{\mathbb{C}}$, $(B_2(a)-B_3(b))M_1=M_{23}(-\sqrt{-1}\overline{a}-b)$. Then $B_{23}M_1=0$ by $a=\sqrt{-1}$ and $b=1$. For $x\in K^{\mathbb{C}}$, $(B_2(a)-B_3(b))M_{23}(x)=\mathbb{X}(-2((a|\sqrt{-1}\overline{x})+(b|x)),2(b|x),2(a|\sqrt{-1}\overline{x});\overline{ax}+\sqrt{-1}\,\overline{b}x,0,0)$. In particular, $B_{23}M_{23}(x)=2(1|x)M_1$. Hence, $\beta_{23}(t)\in(G(K^{\mathbb{C}})^\circ)_{M_1}$ and $\beta_{23}(t)M_{23}(x)=2t(1|x)M_1+M_{23}(x)$ for $x\in K^{\mathbb{C}}$ and $t\in\mathbb{C}$. For $a,b\in K'$, $(B_2(a)-B_3(b))M_{1'}=M_{2'3}(\overline{a}\sqrt{-1}e_4-b)$. Then $B_{23'}M_{1'}=B_{2'3}M_{1'}=0$ by $b=\overline{a}\sqrt{-1}e_4$ with $(a,b)=(1,\sqrt{-1}e_4),(-\sqrt{-1}e_4,1)$. For $x\in K'$, $(B_2(a)-B_3(b))M_{2'3}(x)=\mathbb{X}(-2((a|-\sqrt{-1}e_4\overline{x})+(b|x)),2(b|x),2(a|-\sqrt{-1}e_4\overline{x});\overline{ax}-\overline{(\sqrt{-1}e_4\overline{x})b},0,0)$, so that $B_{23'}M_{2'3}(x)=2(\sqrt{-1}e_4|x)E_2-2(1|\sqrt{-1}e_4\overline{x})E_3+F_1(\overline{x}+\sqrt{-1}e_4(x\sqrt{-1}e_4))$ and $B_{2'3}M_{2'3}(x)=2(1|x)M_{1'}$. Put $x=p+q\sqrt{-1}e_4$ with $p,q\in\boldsymbol{H}$, so that $\overline{x}=\overline{p}-q\sqrt{-1}e_4$. Then $(\sqrt{-1}e_4)(x\sqrt{-1}e_4)=\overline{p}+\overline{q}\sqrt{-1}e_4$, $\overline{x}-\sqrt{-1}e_4(x\sqrt{-1}e_4)=-(q+\overline{q})\sqrt{-1}e_4=2(\sqrt{-1}e_4|x)\sqrt{-1}e_4$. And $B_{23'}M_{2'3}(x)=2(\sqrt{-1}e_4|x)M_{1'}$. Hence, $\beta_{23'}(t),\beta_{2'3}(t)\in(G(K^{\mathbb{C}})^\circ)_{M_{1'}}$ such that $\beta_{23'}(t)M_{2'3}(x)=2t(\sqrt{-1}e_4|x)M_{1'}+M_{2'3}(x)$ and $\beta_{2'3}(t)M_{2'3}(x)=2t(1|x)M_{1'}+M_{2'3}(x)$ for $x\in K'$ and $t\in\mathbb{R}$.

(2) (i) Since $\tilde{K}$ is a composition algebra, $L_a\in O(\tilde{K})$ for all $a\in\mathcal{S}_1(1,\tilde{K})$. Hence, $\mathcal{S}_1(1,\tilde{K})=\{L_a(e_0)|\ a\in\mathcal{S}_1(1,\tilde{K})\}=\mathcal{O}_{O(\tilde{K})}(e_0)$. Put $SO(\tilde{K}):=\{\alpha\in O(\tilde{K})|\ \det(\alpha)=1\}$. When $d_{\tilde{K}}=1$: $\mathcal{S}_1(1,\tilde{K})=\{\pm e_0\}$, $\mathcal{S}_1(1,\tilde{K})^\circ=\{e_0\}$, $\mathcal{S}_1(1,\tilde{K})=\mathcal{S}_1(1,\tilde{K})^\circ\cup(-\mathcal{S}_1(1,\tilde{K})^\circ)$. When $d_{\tilde{K}}=2,4,8$: $O(\tilde{K})=SO(\tilde{K})\cup SO(\tilde{K})\epsilon$ with $\epsilon(e_0)=e_0$, $SO(\tilde{K})=-SO(\tilde{K})$, so that $\mathcal{S}_1(1,\tilde{K})=\mathcal{O}_{SO(\tilde{K})}(e_0)=-\mathcal{O}_{SO(\tilde{K})}(e_0)=-\mathcal{S}_1(1,\tilde{K})$. Since $SO(K^{\mathbb{C}})$ is connected,

$$\mathcal{S}_1(1,K^{\mathbb{C}})^\circ=\mathcal{S}_1(1,K^{\mathbb{C}})=-\mathcal{S}_1(1,K^{\mathbb{C}})^\circ.$$

And $M_{d_{K'}}(\mathbb{R})\supsetneqq SO(K')\cong S(O(d_{K'}/2)\times O(d_{K'}/2))\times\mathbb{R}^{(d_{K'}/2)^2}$. Put

$$1_n:=\mathrm{diag}(1,\cdots,1),1'_n:=\mathrm{diag}(1_{n-1},-1)\in M_n(K').$$

When $d_{K'}/2=2,4$, $SO(K')$ admits just four connected components containing $\mathrm{id}_{K'}$, $\mathrm{diag}(1'_{d_{K'}/2},1_{d_{K'}/2})$, $\mathrm{diag}(1_{d_{K'}/2},1'_{d_{K'}/2})$, $\mathrm{diag}(1'_{d_{K'}/2},1'_{d_{K'}/2})$, so that $\mathcal{S}_1(1,K')=\mathcal{O}_{SO(K')}(e_0)=\mathcal{O}_{SO(K')^\circ}(e_0)=\mathcal{S}_1(1,K')^\circ$.

(ii) For $a\in\mathcal{S}_1(1,\tilde{K})$, $\delta_i(a)\in GL_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))_{E_1,E_2,E_3}$ with $\delta_i(a)^{-1}=\delta_i(\bar{a})$, $\delta_i(a)\tau=\tau\delta_i(\tau a)$ and $\delta_i(a)E=E$. By Lemma 1.1 (1), $\det(\delta_i(a)\mathbb{X}(r;x))=$

$r_1r_2r_3+2(\overline{ax_ia}|(\overline{a}x_{i+1})(x_{i+2}\overline{a}))-r_iN(ax_ia)-r_{i+1}N(\overline{a}x_{i+1})-r_{i+2}N(x_{i+2}\overline{a})=r_1r_2r_3+(2(\bar{x}_i|x_{i+1}x_{i+2})-r_iN(x_i))N(a)^2-(r_{i+1}N(x_{i+1})+r_{i+2}N(x_{i+2}))N(a)=\det(\mathbb{X}(r;x))$. By Proposition 0.1 (1), $\delta_i(a)\in(G(\tilde{K})_{E_1,E_2,E_3})^\circ$ for $a\in\mathcal{S}_1(1,\tilde{K})^\circ$. And $\delta_i(-a)=\sigma_i\delta_i(a)=\delta_i(a)\sigma_i$ with $\sigma_i\in((G(\tilde{K})_{E_i})^\circ)_{E_{i+1},E_{i+2}}$ by (1)(i). By (i), $\{\delta_i(a)|\ a\in\mathcal{S}_1(1,\tilde{K})\}=\{\delta_i(a),\delta_i(-a)|\ a\in\mathcal{S}_1(1,\tilde{K})^\circ\}=\{\delta_i(a),\delta_i(a)\sigma_i|\ a\in\mathcal{S}_1(1,\tilde{K})^\circ\}\subseteqq(G(\tilde{K})_{E_i})^\circ)_{E_{i+2},E_{i+3}}$. By the last claim of (i), $\{\delta_i(a)|\ a\in\mathcal{S}_1(1,\tilde{K})\}=\{\delta_i(a)|\ a\in\mathcal{S}_1(1,\tilde{K})^\circ\}\subseteqq(G(\tilde{K})_{E_1,E_2,E_3})^\circ$ when $\tilde{K}=\boldsymbol{H}',\boldsymbol{O}';\boldsymbol{C}^{\mathbb{C}},\boldsymbol{H}^{\mathbb{C}},\boldsymbol{O}^{\mathbb{C}}$.

(iii) $\tilde{K}$ is associative by $d_{\tilde{K}}\leqq 4$. Hence, $GL_{\mathbb{F}}(\mathcal{J}_3(\tilde{K}))_{E_1,E_2,E_3,F_i(1)}\ni\beta_i(a)$ is well-defined such that $\beta_i(a)^{-1}=\delta_i(\bar{a})$, $\beta_i(a)\tau=\tau\beta_i(\tau a)$, $\beta_i(a)E=E$. Then $\det(\beta_i(a)\mathbb{X}(r;x))=r_1r_2r_3+2(\overline{ax_i\overline{a}}|(ax_{i+1})(x_{i+2}\overline{a}))-r_iN(ax_i\overline{a})-r_{i+1}N(ax_{i+1})-r_{i+2}N(x_{i+2}\overline{a})=r_1r_2r_3+(2(\bar{x}_i|x_{i+1}x_{i+2})-r_iN(x_i))N(a)^2-(r_{i+1}N(x_{i+1})-r_{i+2}N(x_{i+2}))N(a)=\det(\mathbb{X}(r;x))$ by Lemma 1.1 (1). Because of Proposition 0.1 (1), $\beta_i(a)\in(G(\tilde{K})_{E_1,E_2,E_3,F_1(1)})^\circ$ for $a\in\mathcal{S}_1(1,\tilde{K})^\circ$. By (1) (i), $\sigma_i\in((G(\tilde{K})_{E_i})^\circ)_{E_{i+1},E_{i+2}}$ with $\beta_i(-a)=\sigma_i\beta_i(a)=\beta_i(a)\sigma_i$. By virtue of (i), $\{\beta_i(a)|\ a\in\mathcal{S}_1(1,\tilde{K})\}=\{\beta_i(a),\beta_i(-a)|\ a\in\mathcal{S}_1(1,\tilde{K})^\circ\}=\{\beta_i(a),\beta_i(a)\sigma_i|\ a\in\mathcal{S}_1(1,\tilde{K})^\circ\}$ is contained in $(G(\tilde{K})_{E_i})^\circ)_{E_{i+2},E_{i+3},F_i(1)}$. By the last claim of (i), $\{\beta_i(a)|\ a\in\mathcal{S}_1(1,\tilde{K})\}=\{\beta_i(a)|\ a\in\mathcal{S}_1(1,\tilde{K})^\circ\}$ is contained in $(G(\tilde{K})_{E_1,E_2,E_3,F_i(1)})^\circ$ when $\tilde{K}=\boldsymbol{H}';\boldsymbol{C}^{\mathbb{C}},\boldsymbol{H}^{\mathbb{C}}$ with $d_{\tilde{K}}\leqq 4$. □

Put $G_J(\tilde{K}_\tau):=\{\beta_j(t;a,\sqrt{-1})|\ j\in J,t\in\mathbb{R},a\in\tilde{K}_\tau,N(a)=1\}$ for any subset $J\subseteqq\{1,2,3\}$. By Lemma 1.4 (1) (i), $G_J(\tilde{K}_\tau)\subset(G(\tilde{K})^\tau)^\circ$. By Proposition 0.1 (2), $G(\tilde{K})^\tau$ and the identity connected component $(G(\tilde{K})^\tau)^\circ$ are compact.

LEMMA 1.5. (1) *For any $X\in\mathcal{J}_3(\tilde{K})_\tau$ and any closed subgroup H of $G(\tilde{K})^\tau$ such that $G_J(\tilde{K}_\tau)\subseteqq H$ with some $J\subseteqq\{1,2,3\}$,*

$$\mathcal{O}_H(X)\cap\{Y\in\mathcal{J}_3(\tilde{K})|\ (Y|F_j(x))=0\ (j\in J,\ x\in\tilde{K}_\tau)\}\neq\emptyset.$$

(2) $\mathcal{O}_{(G(\tilde{K})^\tau)^\circ}(X)\cap\{\mathrm{diag}(r_1,r_2,r_3)|\ r_i\in\mathbb{R}\ (i=1,2,3)\}\neq\emptyset$ *for any $X\in\mathcal{J}_3(\tilde{K})_\tau$, where $\{r_1,r_2,r_3\}=\Lambda_X$ iff* $\mathrm{diag}(r_1,r_2,r_3)\in\mathcal{O}_{(G(\tilde{K})^\tau)^\circ}(X)$.

(3) $\mathcal{O}_{G(K)^\circ}(X)\cap\{Y+\sqrt{-1}\mathrm{diag}(r_1,r_2,r_3)|\ Y\in\mathcal{J}_3(K),\ r_i\in\mathbb{R}\ (i=1,2,3)\}\neq\emptyset$ *for any $X\in\mathcal{J}_3(K^{\mathbb{C}})$.*

(4) $\mathcal{O}_{(G(K')^\tau)^\circ}(X)\cap\{\mathbb{X}(s;y)|\ s_i\in\mathbb{R},\ y_i=\sqrt{-1}p_ie_4,\ p_i\in K\cap K'\ (i=1,2,3)\}\neq\emptyset$ *for any $X\in\mathcal{J}_3(K')$.*

Proof. (1) (cf. [27, 3.3]): Since the closed subgroup H of the compact group $G(\tilde{K})^\tau$ is compact, the orbit $\mathcal{O}_H(X)$ is compact, which is contained in $\mathcal{J}_3(\tilde{K})_\tau$ if $X \in \mathcal{J}_3(\tilde{K})_\tau$. Put $\phi : \mathcal{J}_3(\tilde{K})_\tau \longrightarrow \mathbb{R}; \mathbb{X}(r;x) \mapsto \sum_{j=1}^3 r_j^2$, which is a continuous $\mathbb{R}$-valued function admitting a maximal point $\mathbb{X}(r;x) \in \mathcal{O}_H(X)$. Suppose that $(\mathbb{X}(r,x)|F_j(q)) \neq 0$ for some $j \in J$ and $q \in \tilde{K}_\tau$. By Lemma 1.1 (1), $2(x_j|q) \neq 0$. Since $(x|y)$ is non-degenerate on $\tilde{K}$ and $\tilde{K}_\tau$, $\tilde{K} = \tilde{K}_\tau \oplus \tilde{K}_\tau^\perp$ for $\tilde{K}_\tau^\perp := \{x \in \tilde{K}|\ (x|y) = 0\ (y \in \tilde{K}_\tau)\}$, so that $x_j = y_j + y_j^\perp$ for some $y_j \in \tilde{K}_\tau$ and $y_j^\perp \in \tilde{K}_\tau^\perp$. Then $(y_j|q) = (x_j|q) \neq 0$, so that $y_j \neq 0$ and $(y_j|y_j) = (\tau y_j|y_j) > 0$. Put $a := y_j/\sqrt{(y_j|y_j)} \in \tilde{K}_\tau$, so that $(a|a) = 1$. By Lemma 1.4 (1) (i), $\beta_j(t; a, \sqrt{-1}) \in G_J(\tilde{K}_\tau) \subsetneqq H$. Put $\varepsilon := (a|x_j) = (a|y_j) = \sqrt{(y_j|y_j)} > 0$, $s_j^\pm := (r_{j+1} \pm r_{j+2})/2 \in \mathbb{R}$ and $Y(t) := \beta_j(t; a, \sqrt{-1})\mathbb{X}(r;x) \in \mathcal{J}_3(\tilde{K})_\tau$. Then $\phi(Y(t)) = r_j^2 + \sum_\pm (s_j^+ \pm (s_j^- \cos(2t) + \varepsilon \sin(2t)))^2 = r_j^2 + 2(s_j^+)^2 + 2(s_j^- \cos(2t) + \varepsilon \sin(2t))^2 = r_j^2 + 2(s_j^+)^2 + 2((s_j^-)^2 + \varepsilon^2)\cos(2t + \theta)$ for some constant θ of t determined by s_j^- and $\varepsilon > 0$. Hence, $\phi(Y(\frac{-\theta}{2})) = r_j^2 + r_{j+1}^2 + r_{j+2}^2 + 2\varepsilon^2 = \phi(\mathbb{X}(r;x)) + 2\varepsilon^2 \geqq \phi(Y(\frac{-\theta}{2})) + 2\varepsilon^2$ by the maximality of $\phi(\mathbb{X}(r;x))$, which gives $\varepsilon = 0$, a contradiction.

(2) Take $X \in \mathcal{J}_3(\tilde{K})_\tau$. By (1) on $H := (G(\tilde{K})^\tau)^\circ \supsetneqq G_{\{1,2,3\}}(\tilde{K}_\tau)$, there exists $\beta \in H$ such that $(\beta X, F_i(x)) = 0$ $(x \in \tilde{K}_\tau; i = 1,2,3)$, so that $\beta X = \operatorname{diag}(r_1, r_2, r_3)$ for some $r_i \in \mathbb{R}$ $(i = 1,2,3)$. In this case, $\Phi_X(\lambda) = \Phi_{\operatorname{diag}(r_1,r_2,r_3)}(\lambda) = \Pi_{i=1}^3(\lambda - r_i)$, so that $\{r_1, r_2, r_3\} = \Lambda_X$. Conversely if $\{r_1, r_2, r_3\} = \Lambda_X$, then $\operatorname{diag}(s_1, s_2, s_3) \in \mathcal{O}_{(G(\tilde{K})^\tau)^\circ}(X)$ for some $\{s_1, s_2, s_3\} = \Lambda_X$, so that $\operatorname{diag}(r_1, r_2, r_3) \in \mathcal{O}_{(G(\tilde{K})^\tau)^\circ}(X)$ by Lemma 1.4 (1) (ii).

(3) Take $X \in \mathcal{J}_3(K^{\mathbb{C}})$. Then $X = X_1 + \sqrt{-1}X_2$ for some $X_i \in \mathcal{J}_3(K) = \mathcal{J}_3(K^{\mathbb{C}})_\tau$ $(i = 1,2)$. By (2), there exist $\beta \in (G(K^{\mathbb{C}})^\tau)^\circ = G(K)^\circ$ and $\{r_1, r_2, r_3\} \subset \mathbb{R}$ such that $\beta X_2 = \operatorname{diag}(r_1, r_2, r_3)$, so that $\beta X = \beta X_1 + \sqrt{-1}\beta X_2$ has the required form with $\beta X_1 \in \mathcal{J}_3(K)$.

(4) Take $X \in \mathcal{J}_3(K')$. Then $X = X_+ + X_-$ for some $X_\pm \in \mathcal{J}_3(K')_{\pm\tau}$. By (1) on $H := (G(K')^\tau)^\circ \supsetneqq G_{\{1,2,3\}}(\tilde{K}_\tau)$, there exists $\beta \in H$ such that $\beta X_+ = \operatorname{diag}(r_1, r_2, r_3)$ for some $r_i \in \mathbb{R}$. Then $\beta X = \beta X_+ + \beta X_-$ has the required form because of $\beta X_- \in \mathcal{J}_3(K')_{-\tau} = \{\mathbb{X}(0;y)|\ y_i = \sqrt{-1}p_i e_4;\ p_i \in K \cap K'\ (i = 1,2,3)\}$. $\square$

LEMMA 1.6. (1) *For a positive integer m, let $f(X_1, \cdots, X_m)$ be a $\mathcal{J}_3(\tilde{K})$-valued polynomial of E and $X_1, \cdots X_m \in \mathcal{J}_3(\tilde{K})$ with respect to $\circ$, $\times$ and the scalar multiples of* $\operatorname{tr}(X_i)$, $(X_i|X_j)$, $\det(X_i)$ *and* $(X_i|X_j|X_k)$ *for $i,j,k \in \{1, \cdots, m\}$. Assume that $f(X_1, X_2, \cdots, X_m) = 0$ for any $X_2, \cdots X_m \in$*

$\mathcal{J}_3(K)$ and all diagonal forms X_1 in $\mathcal{J}_3(\mathbb{R})$. Then $f(X_1, \cdots, X_m) = 0$ for all $X_1, \cdots X_m \in \mathcal{J}_3(K^{\mathbb{C}})$.

(2) *Assume that $X, Y \in \mathcal{J}_3(\tilde{K})$. Then:*

(i) $X \circ ((X \circ X) \circ Y) = (X \circ X) \circ (X \circ Y)$;

(ii) $X^{\times 2} \circ X = \det(X)E, \ \ (X^{\times 2})^{\times 2} = \det(X)X$;

(iii) $X^{\times 2} \times X = -\frac{1}{2}\{\mathrm{tr}(X)X^{\times 2} + \mathrm{tr}(X^{\times 2})X - (\mathrm{tr}(X^{\times 2})\mathrm{tr}(X) - \det(X))E\}$.

(3) *V_X is the minimal subspace over $\mathbb{F}$ generated by X and E under the cross product. Especially, $\varphi_X(\lambda)^{\times 2} \in V_X$ for all $\lambda \in \mathbb{F}$.*

(4) *$(\varphi_X(\lambda_1)^{\times 2})^{\times 2} = 0$ if $X \in \mathcal{J}_3(\tilde{K})$ and $\lambda_1 \in \mathbb{C}$ with $\Phi_X(\lambda_1) = 0$.*

(5) *$\mathcal{M}_{23}(\tilde{K}) = \{X \in \mathcal{J}_3(\tilde{K})_0 |\ X^{\times 2} \neq 0, \ \mathrm{tr}(X^{\times 2}) = 0, \ (X^{\times 2})^{\times 2} = 0\}$ and $\{X^{\times 2} |\ X \in \mathcal{M}_{23}(\tilde{K})\} \subsetneqq \mathcal{M}_1(\tilde{K})$.*

Proof. (1) (cf. [7, p.42], [28, p.74, $\ell\ell$.2–4], [6, p.91, Corollary V.2.6]): By Lemma 1.5 (2), any $X_1 \in \mathcal{J}_3(K)$ admits some $\beta \in G(K) = G(K^{\mathbb{C}})^{\tau}$ such that βX_1 is a diagonal form in $\mathcal{J}_3(\mathbb{R})$. Then $f(\beta X_1, X_2, \cdots, X_m) = 0$ for any $X_i \in \mathcal{J}_3(K)$ with $i \in \{2, \cdots, m\}$. By Proposition 0.1, β preserves $\circ$, $\times$, $\mathrm{tr}(*)$, $(*|*)$, $\det(*)$, $(*|*|*)$ and E, so that $f(X_1, \beta^{-1}X_2 \cdots, \beta^{-1}X_m) = 0$ for all $X_i \in \mathcal{J}_3(K)$ $(i = 2, \cdots, m)$. Hence, $f(X_1, \cdots, X_m) = 0$ for all $X_i \in \mathcal{J}_3(K)$ $(i = 1, \cdots, m)$. Since this formula consists of some polynomial equations on the $\mathbb{R}$-coefficients of each matrix entry of X_i's with respect to the $\mathbb{R}$-basis $\{e_j\}$ of K, the formula holds on $\mathcal{J}_3(K^{\mathbb{C}}) = \mathcal{J}_3(K) \otimes_{\mathbb{R}} \mathbb{C}$.

(2) The formulas in (i) and (ii) are polynomials of X, Y and E with respect to $\circ$, $\times$, $\mathrm{tr}(*)$, $(*|*)$, $\det(*)$, $(*|*|*)$. If X is a diagonal form in $\mathcal{J}_3(\mathbb{R})$, the formulas can be checked by Lemma 1.1 (1), easily. By (1), the formulas (i) and (ii) hold for any $X, Y \in \mathcal{J}_3(K^{\mathbb{C}})$. Hence, they hold for any $X, Y \in \mathcal{J}_3(\tilde{K}) \subsetneqq \mathcal{J}_3(K^{\mathbb{C}})$. The formula (iii) follows from the first formula of (ii) and the definition of cross product with $(X^{\times 2}|X) = 3\det(X)$.

(3) follows from the formulas in (ii), (iii) and Lemma 1.1 (3).

(4) $(\varphi_X(\lambda_1)^{\times 2})^{\times 2} = \det(\varphi_X(\lambda_1))\varphi_X(\lambda_1) = \Phi_X(\lambda_1)\varphi_X(\lambda_1) = 0$ by the second formula of (ii) in (2).

(5) By (2) (ii), $(X^{\times 2})^{\times 2} = \det(X)X$, so that $(X^{\times 2})^{\times 2} = 0$ if and only if $\det(X) = 0$, which gives the results. $\square$

The formula (i) of Lemma 1.6 (2) implies that $(\mathcal{J}_3(\tilde{K}), \circ)$ is a Jordan algebra over $\mathbb{F}$, which is also reduced simple in the sense of N. Jacobson [16, Chapters IV, IX], where $\mathcal{J}_3(\tilde{K})$ is called *split* iff $\tilde{K}$ is *split* (i.e. non-division), that is the case when $\tilde{K} = K'$ or $K'^{\mathbb{C}}$.

Proof of "Proposition 0.1 (3) when $\tilde{K} = K$" (cf, [7], [27, 4.1 Proposition]). Take any $X \in \mathcal{P}_2(K) \subset \mathcal{J}_3(K) = \mathcal{J}_3(K^{\mathbb{C}})^\tau$. By Lemma 1.5 (2) with $\tilde{K} = K^{\mathbb{C}}$, there exists $\alpha \in G(K)^\circ = (G(K^{\mathbb{C}})^\tau)^\circ$ such that $\alpha X = \mathrm{diag}(r_1, r_2, r_3)$ for some $r_1, r_2, r_3 \in \boldsymbol{R}$. By $\mathrm{tr}(\alpha X) = 1$ and $(\alpha X)^{\times 2} = 0$, $r_1 + r_2 + r_3 = 1$ and $r_2 r_3 = r_3 r_1 = r_1 r_2 = 0$, so that $(r_1, r_2, r_3) = (1,0,0), (0,1,0), (0,0,1)$. By Lemma 1.4 (1) (ii), there exists $\hat{\beta} \in (G(K^{\mathbb{C}})^\tau)^\circ = G(K)^\circ$ such that $\hat{\beta}(\alpha X) = E_1$. $\square$

H. Freudenthal [7, 5.1] gave the diagonalization theorem on $\mathcal{J}_3(\boldsymbol{O})$ with the action of $\{\alpha \in G(\boldsymbol{O}) |\ \mathrm{tr}(\alpha X) = \mathrm{tr}(X)\}$ (cf. [27, 3.3 Theorem], [20, p.206, Lemma 1], [23, Proposition 1.4], [6, p.90, Theorem V.2.5]), which is developing to Lemma 1.5 (2) for $\tilde{K} = \boldsymbol{O}^{\mathbb{C}}$ with $\mathcal{J}_3(\tilde{K})_\tau = \mathcal{J}_3(\boldsymbol{O}^{\mathbb{C}})_\tau = \mathcal{J}_3(\boldsymbol{O})$ under the action of $G(\tilde{K})^\tau = G(\boldsymbol{O}^{\mathbb{C}})^\tau \cong G(\boldsymbol{O}) =: F_4$. I. Yokota [27, 4.2 and 6.4 Theorems] proved the connectedness and the simply connectedness of F_4 by the diagonalization theorem of H. Freudenthal (cf. [18, Appendix], [20, p.210, Theorem 3], [10, p.175, Proposition 1.4]). O. Shukuzawa & I. Yokota [22, p.3, Remark] (cf. [29, p.63, Theorem 9; p.54, Remark]) proved the connectedness of $F_4' := G(\boldsymbol{O}')$ by showing the first formula of Proposition 0.1 (1) by virtue of Hamilton-Cayley formula on $\mathcal{J}_3(\boldsymbol{O}')$ given as the first formula of Lemma 1.6(2)(ii) (cf. [24, p.119, Proposition 5.1.5], [11, Lemma 14.96]). Because of $F_4^{\mathbb{C}} \cong (F_4^{\mathbb{C}})^\tau \times \mathbb{R}^{52}$ with $(F_4^{\mathbb{C}})^\tau = F_4$ [30, Theorem 2.2.2] (cf. Proposition 0.1 (2)), $F_4^{\mathbb{C}} := G(\boldsymbol{O}^{\mathbb{C}})$ is connected and simply connected, so that $F_4' = (F_4^{\mathbb{C}})^{\tau\gamma}$ is again proved to be connected by virtue of a theorem of P.K. Rasevskii [21].

2. Proposition 0.1 (3) and (4) (i).

Assume that $\tilde{K} = K'$ or $K^{\mathbb{C}}$ with $K' = \boldsymbol{C}', \boldsymbol{H}'$ or $\boldsymbol{O}'$; and $K^{\mathbb{C}} = \mathbb{R}^{\mathbb{C}}, \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}$ or $\boldsymbol{O}^{\mathbb{C}}$. And put $\sigma := \sigma_1$ defined in Lemma 1.4 (1) (i) such that $\sigma^2 = \mathrm{id}_{\mathcal{J}_3(\tilde{K})}$. Then $\mathcal{J}_3(\tilde{K}) = \mathcal{J}_3(\tilde{K})_\sigma \oplus \mathcal{J}_3(\tilde{K})_{-\sigma}$, $\mathcal{J}_3(\tilde{K})_\sigma = \{\sum_{i=1}^3 r_i E_i + F_1(x_1) |\ r_i \in \mathbb{F}, x_1 \in \tilde{K}\}$ such that $\mathcal{J}_3(\tilde{K})_{-\sigma} = \{F_2(x_2) + F_3(x_3) |\ x_2, x_3 \in \tilde{K}\} = \{X \in \mathcal{J}_3(\tilde{K}) |\ (X, Y) = 0\ (Y \in \mathcal{J}_3(\tilde{K})_\sigma)\}$. And $\mathcal{J}_3(\tilde{K})_\sigma = \mathbb{F}E_1 \oplus \mathcal{J}_2(\tilde{K})$ with $\mathcal{J}_2(\tilde{K}) := \{\sum_{i=2}^3 r_i E_i + F_1(x_1) |\ r_i \in \mathbb{F}, x_1 \in \tilde{K}\}$. By Lemma 1.1 (1), $\mathcal{J}_3(\tilde{K})_{L^\times_{2E_1}} = \{r(E_2 + E_3) |\ r \in \mathbb{F}\}$ and $\mathcal{J}_3(\tilde{K})_{-L^\times_{2E_1}} = \{r(E_2 - E_3) + F_1(x) |\ r \in \mathbb{F}, x \in \tilde{K}\}$, so that $\mathcal{J}_2(\tilde{K}) = \mathcal{J}_3(\tilde{K})_{L^\times_{2E_1}} \oplus \mathcal{J}_3(\tilde{K})_{-L^\times_{2E_1}}$.

Lemma 2.1. (1) $G(\tilde{K})_{E_1} = G(\tilde{K})_{E_1, E_2+E_3, \mathcal{J}_3(\tilde{K})_{\pm L^\times_{2E_1}}, \mathcal{J}_2(\tilde{K}), \mathcal{J}_3(\tilde{K})_{\pm\sigma}}$.

(2) (i) $\{\mathbb{X}((X|E_1), s_2, s_3; 0, 0, 0)|\ s_2, s_3 \in \mathbb{R}; s_2 \geqq s_3\} \cap \mathcal{O}_{(G(K)_{E_1})^\circ}(X) \neq \emptyset$ *and* $\{\mathbb{X}((X|E_1), t_2, t_3; u, 0, 0)|\ t_2, t_3, u \in \mathbb{R}; u \geqq 0\} \cap \mathcal{O}_{((G(K)_{E_3})^\circ)_{E_1,E_2}}(X) \neq \emptyset$ *if* $X \in \mathcal{J}_3(K)_\sigma$.

(ii) $\{\mathbb{X}((X|E_1), s_2, s_3; u\sqrt{-1}e_4, 0, 0)|\ s_2, s_3, u \in \mathbb{R}; u \geqq 0, s_2 \geqq s_3\} \cap \mathcal{O}_{(G(K')^\circ)^\tau_{E_1}}(X) \neq \emptyset$ *if* $X \in \mathcal{J}_3(K')_\sigma$.

(iii) $\{\mathbb{X}((X|E_1), t_2 + \sqrt{-1}s_2, t_3 + \sqrt{-1}s_3; u, 0, 0)|\ t_2, t_3, s_2, s_3, u \in \mathbb{R}; u \geqq 0, s_2 \geqq s_3\} \cap \mathcal{O}_{(G(K)^\circ)_{E_1}}(X) \neq \emptyset$ *if* $X \in \mathcal{J}_3(K^{\mathbb{C}})_\sigma$.

Proof. (1) For $\alpha \in G(\tilde{K})_{E_1}$, one has that $\alpha(E_2+E_3) = \alpha(E-E_1) = \alpha E - \alpha E_1 = E - E_1 = E_2 + E_3$ and $\alpha\mathcal{J}_3(\tilde{K})_{\pm L^\times_{2E_1}} = \mathcal{J}_3(\tilde{K})_{\pm L^\times_{2E_1}}$ since α preserves $\times$ by Proposition 0.1 (1), so that $\alpha\mathcal{J}_2(\tilde{K}) = \mathcal{J}_2(\tilde{K})$, $\alpha\mathcal{J}_3(\tilde{K})_\sigma = \mathcal{J}_3(\tilde{K})_\sigma$ and $\alpha\mathcal{J}_3(\tilde{K})_{-\sigma} = \mathcal{J}_3(\tilde{K})_{-\sigma}$ because of the orthogonal direct-sum decompositions of them and that α preserves $(*|*)$ on $\mathcal{J}_3(\tilde{K})$ by Proposition 0.1 (1).

(2) (i) Take any $X \in \mathcal{J}_3(K)_\sigma$. Then there exist $r_i \in \mathbb{R}$ and $x_1 \in K$ such that $X = \mathbb{X}(r_1, r_2, r_3; x_1, 0, 0)$. By Lemmas 1.4 (1) and 1.5 (1) with $\tilde{K} = K = K_\tau$, $F = \mathbb{R}$ and $H := (G(K)_{E_1})^\circ \supseteqq G_J(K)$ with $J = \{1\}$, there exists $\alpha \in H$ such that $(\alpha X)_{F_1} = 0$. By (1), $\alpha X \in \mathcal{J}_3(K)_\sigma$, so that αX is diagonal with $s_i := (\alpha X|E_i) \in \mathbb{R}$ $(i = 1, 2, 3)$ such that $s_1 = (\alpha X|\alpha E_1) = (X|E_1) = r_1$. If $s_2 \geqq s_3$, then αX gives an element of the left-handed set of the first formula. If $s_2 < s_3$, put $\alpha_1 := \hat{\beta}_1\alpha$ with $\hat{\beta}_1 \in (G(K)_{E_1})^\circ$ given in Lemma 1.4 (1) (ii), so that $\alpha_1 X$ gives an element of the left-handed set of the first formula. Hence, follows the first formula.

If $x_1 = 0$, then X gives an element of the left-handed side of the second formula with $u = 0 \in \mathbb{R}$. If $x_1 \neq 0$, put $a := x_1/\sqrt{(x_1|x_1)} \in \mathcal{S}_1(1, K)$, so that $\delta_3(a) \in ((G(K)_{E_3})^\circ)_{E_1,E_2}$ in Lemma 1.4 (2) (ii) such that $\delta_3(a)X = \mathbb{X}(r_1, r_2, r_3; u, 0, 0)$ with $u := \sqrt{(x_1|x_1)} > 0$, which gives an element of the left-handed side of the second formula. Hence, follows the second formula.

(ii) Take any $X \in \mathcal{J}_3(K')_\sigma$. By Lemmas 1.4 (1) and 1.5 (1) with $\tilde{K} = K'$ and $H := (G(K')^\tau_{E_1})^\circ \supseteqq G_J(K'_\tau)$ with $J = \{1\}$, there exists $\beta \in H$ such that $(\beta X|F_1(x)) = 0$ for all $x \in K'_\tau = K \cap K'$. By (1), $\beta X \in \mathcal{J}_3(K')_\sigma$. Hence, $\beta X = \mathbb{X}((X|E_1), s_2, s_3; \sqrt{-1}qe_4, 0, 0)$ for some $q \in K \cap K'$. Put $\alpha_1 := \beta$ (if $s_2 \geqq s_3$) or $\hat{\beta}_1\beta$ (if $s_2 < s_3$), so that $\alpha_1 \in H$ by Lemma 1.4 (1) (ii). Then $\alpha_1 X = \mathbb{X}((X|E_1), s_2, s_3; \sqrt{-1}qe_4, 0, 0)$ for some $q \in K \cap K'$, $s_2, s_3 \in \mathbb{R}$ with $s_2 \geqq s_3$. Put $\alpha := \alpha_1$ (if $q = 0$) or $\delta_3(a)\beta$ for $a := q/\sqrt{(q|q)} \in K'_\tau$ with $N(a) = 1$ (if $q \neq 0$), where $\delta_3(a) \in ((G(K')_{E_3})^\circ)^\tau_{E_1,E_2} \subseteqq (G(K')^\circ)^\tau_{E_1}$ by Lemma 1.4 (2) (ii). Then $\alpha X = \mathbb{X}((X|E_1), s_2, s_3; \sqrt{-1}ue_4, 0, 0)$ with $u := \sqrt{N(q)} \geqq 0$, which is an element of the left-handed set.

(iii) Take any $X \in \mathcal{J}_3(K^{\mathbb{C}})_\sigma$. Then $X = X_1 + \sqrt{-1}X_2$ for some $X_i \in \mathcal{J}_3(K)_\sigma$ $(i = 1, 2)$. By (i), there exist $\alpha_1 \in (G(K)_{E_1})^\circ$ such that $\alpha_1(X_2) = \mathbb{X}((X_2|E_1), s_2, s_3; 0, 0, 0)$ for some $s_2, s_3 \in \mathbb{R}$ with $s_2 \geqq s_3$. Because of $\mathcal{J}_3(K)_\sigma \ni \alpha_1(X_1) = \mathbb{X}((X_1|E_1), t_2, t_3; x, 0, 0)$ for some $t_2, t_3 \in \mathbb{R}$ and $x \in K$, so that $\alpha_1(X) = \mathbb{X}((X|E_1), t_2 + \sqrt{-1}s_2, t_3 + \sqrt{-1}s_3; x, 0, 0)$. Put $\alpha := \alpha_1$ (if $x = 0$) or $\delta_3(a)\alpha_1$ with $a := x/\sqrt{(x|x)} \in \mathcal{S}_1(1, K)$ (if $x \neq 0$), where $\delta_3(a) \in ((G(K)_{E_3})^\circ)_{E_1,E_2}$ by Lemma 1.4 (2) (ii). Then $\alpha \in (G(K)^\circ)_{E_1}$ and $\alpha X = \mathbb{X}((X|E_1), t_2 + \sqrt{-1}s_2, t_3 + \sqrt{-1}s_3; u, 0, 0)$ with $u := \sqrt{(x|x)} \geqq 0$, which is an element of the left-handed set. $\square$

For $c \in \mathbb{F}$, put $\mathcal{S}_2(c, \tilde{K}) := \{W \in \mathcal{J}_3(\tilde{K})_{-L^\times_{2E_1}} \mid (W|W) = c, W \neq 0\}$, which is said to be *a generalized sphere of second kind over* $\mathbb{F}$. Then $G(\tilde{K})_{E_1} = \cap_{c\in\mathbb{F}} G(\tilde{K})_{E_1,\mathcal{S}_2(c,\tilde{K})}$.

LEMMA 2.2. (1) $\mathcal{J}_3(K')_{-L^\times_{2E_1}} = (\cup_{c\in\mathbb{R}}\mathcal{S}_2(c, K')) \cup \{0\}$ *such that*

(i-1) $\mathcal{S}_2(c, K') = \mathcal{O}_{(G(K')^\circ)_{E_1}}(\sqrt{\frac{c}{2}}(E_2 - E_3))$ *for* $c > 0$*;*

(i-2) $\mathcal{S}_2(c, K') = \mathcal{O}_{(G(K')^\circ)_{E_1}}(\sqrt{\frac{-c}{2}}F_1(\sqrt{-1}e_4))$ *for* $c < 0$*;*

(ii) $\mathcal{S}_2(0, K') = \mathcal{O}_{(G(K')^\circ)_{E_1}}(M_{1'})$*; and*

(iii) $\{0\} = \mathcal{O}_{(G(K')^\circ)_{E_1}}(0)$.

(2) $\mathcal{J}_3(K^{\mathbb{C}})_{-L^\times_{2E_1}} = (\cup_{c\in\mathbb{C}}\mathcal{S}_2(c, K^{\mathbb{C}})) \cup \{0\}$ *such that*

(i) $\mathcal{S}_2(c, K^{\mathbb{C}}) = \mathcal{O}_{(G(K^{\mathbb{C}})^\circ)_{E_1}}(\sqrt{\frac{c}{2}}(E_2 - E_3))$ *for* $c \in \mathbb{C}\backslash\{0\}$*;*

(ii) $\mathcal{S}_2(0, K^{\mathbb{C}}) = \mathcal{O}_{(G(K^{\mathbb{C}})^\circ)_{E_1}}(M_1)$*; and*

(iii) $\{0\} = \mathcal{O}_{(G(K^{\mathbb{C}})^\circ)_{E_1}}(0)$.

Proof. (1) For $W \in \mathcal{J}_3(K')_{-L^\times_{2E_1}}$, put $c := (W|W) \in \mathbb{R}$. By Lemma 2.1 (1) and (2) (ii), $\alpha W = \mathbb{X}(0, s, -s; u\sqrt{-1}e_4, 0, 0)$ for some $s \geqq 0, u \geqq 0$ and $\alpha \in (G(K')^\circ)^\tau_{E_1}$. Then $c = (\alpha W|\alpha W) = 2(s^2 - u^2)$. For $t \in \mathbb{R}$, put $\mathbb{X}(r; x) := \beta_1(t; \sqrt{-1}e_4, 1)(\alpha W)$, so that $r_1 = x_2 = x_3 = 0$, $r_2 = -r_3 = \cosh(2t)(s - u\tanh(2t))$ and $x_1 = v\sqrt{-1}e_4$ with $v := \cosh(2t)(u - s\tanh(2t))$.

(i-1) If $c > 0$, then $s > u \geqq 0$ and $|u/s| < 1$, so that $\tanh(2t) = u/s$ for some $t \in \mathbb{R}$ such that $v = 0$ and $r_2 = \cosh(2t)(s^2 - u^2)/s > 0$. In this case, $\mathbb{X}(r; x) = r_2(E_2 - E_3)$ with $c = (W|W) = (\mathbb{X}(r; x)|\mathbb{X}(r; x)) = 2(r_2)^2$, so that $\mathbb{X}(r; x) = \sqrt{\frac{c}{2}}(E_2 - E_3) \in \mathcal{S}_2(c, K')$.

(i-2) If $c < 0$, then $u > s \geqq 0$ and $|s/u| < 1$, so that $\tanh(2t) = s/u$ for some $t \in \mathbb{R}$ such that $r_2 = 0$ and $v = \cosh(2t)(u^2 - s^2)/u > 0$. In this case,

$\mathbb{X}(r;x) = vF_1(\sqrt{-1}e_4)$ with $c = (W|W) = (\mathbb{X}(r;x)|\mathbb{X}(r;x)) = -2v^2$, so that $\mathbb{X}(r;x) = \sqrt{\frac{-c}{2}}F_1(\sqrt{-1}e_4) \in \mathcal{S}_2(c,K')$.

(ii, iii) If $c = 0$, then $s^2 - u^2 = c/2 = 0$, so that $s = u \geqq 0$ and $r_2 = v = ue^{-2t}$. When $u \neq 0$: $u > 0$ and $ue^{-2t} = 1$ for some $t \in \mathbb{R}$. In this case, $\mathbb{X}(r;x) = E_2 - E_3 + F_1(\sqrt{-1}e_4)) = M_{1'} \in \mathcal{S}_2(0,K')$. When $u = 0$: $r_2 = v = u = 0$ and $\mathbb{X}(r;x) = 0 \in \{0\}$.

(2) For $W \in \mathcal{J}_3(K^{\mathbb{C}})_{-L^{\times}_{2E_1}}$, put $c := (W|W) \in \mathbb{C}$. By Lemma 2.1 (1) and (2) (iii), $\alpha W = \mathbb{X}(0, t_2 + s_2\sqrt{-1}, -t_2 - s_2\sqrt{-1}; u, 0, 0)$ for some $t_2, s_2, u \in \mathbb{R}$ with $s_2, u \geqq 0$ and some $\alpha \in (G(K)_{E_1})^\circ \subsetneqq (G(K^{\mathbb{C}})_{E_1})^\circ$. Then $c = (\alpha W|\alpha W) = 2((t_2 + s_2\sqrt{-1})^2 + u^2)$. For $t \in \mathbb{R}$, put $\mathbb{X}(r;x) := \beta_1(t;1,\sqrt{-1})(\alpha W)$ with $\beta_1(t;1,\sqrt{-1}) \in (G(K)_{E_1})^\circ \subsetneqq (G(K^{\mathbb{C}})_{E_1})^\circ$, so that $r_1 = x_2 = x_3 = 0$, $r_2 = -r_3 = (t_2 + s_2\sqrt{-1})\cos(2t) + u\sin(2t)$ and $x_1 = u\cos(2t) - (t_2 + s_2\sqrt{-1})\sin(2t)$.

(i) If $c \neq 0$, then $(t_2 + (s_2+u)\sqrt{-1})(t_2 + (s_2-u)\sqrt{-1}) = c/2 \neq 0$, so that $e^{\sqrt{-1}4t} = (t_2 + (s_2+u)\sqrt{-1})/(t_2 + (s_2-u)\sqrt{-1}) \neq 0$ for some $t \in \mathbb{C}$, and that $x_1 = u(e^{\sqrt{-1}2t} + e^{-\sqrt{-1}2t})/2 - (t_2 + s_2\sqrt{-1})(e^{\sqrt{-1}2t} - e^{-\sqrt{-1}2t})/(2\sqrt{-1}) = \frac{\sqrt{-1}}{2}\{(t_2 + (s_2-u)\sqrt{-1})e^{\sqrt{-1}2t} - (t_2 + (s_2+u)\sqrt{-1})e^{-\sqrt{-1}2t}\} = 0$. In this case, $\mathbb{X}(r;x) = r_2(E_2 - E_3)$ with $c = (\mathbb{X}(r;x)|\mathbb{X}(r;x)) = 2(r_2)^2$, so that $\mathbb{X}(r;x) = \sqrt{\frac{c}{2}}(E_2 - E_3) \in \mathcal{S}_2(c,K^{\mathbb{C}})$.

(ii, iii) If $c = 0$, then $t_2^2 - s_2^2 + u^2 + 2t_2s_2\sqrt{-1} = c/2 = 0$, so that $t_2s_2 = 0$. When $s_2 = 0$: $t_2 = u = 0$, so that $\mathbb{X}(r;x) = 0 \in \{0\}$. When $s_2 \neq 0$: $t_2 = 0$, $u = s_2 > 0$, $r_2 = -r_3 = \sqrt{-1}ue^{-2t\sqrt{-1}}$, $x_1 = ue^{-2t\sqrt{-1}}$. There exists $t \in \mathbb{C}$ such that $\sqrt{-1}ue^{-2t\sqrt{-1}} = 1$, so that $\mathbb{X}(r;x) = E_2 - E_3 + F_1(\sqrt{-1}) = M_1 \in \mathcal{S}_2(0,K^{\mathbb{C}})$. $\square$

LEMMA 2.3. (1) *If* $Y = \mathbb{X}(r;x) \in \mathcal{J}_2(\tilde{K})$, *then* $\mathrm{tr}(Y) = r_2 + r_3$, $\det(E_1 + Y) = r_2r_3 - N(x_1)$ *and* $Y^{\times 2} = \det(E_1 + Y)E_1$.

(2) *For any* $X \in \mathcal{J}_3(\tilde{K})_\sigma$, *there exists* $Y \in \mathcal{J}_2(\tilde{K})$ *such that* $X = (X|E_1)E_1 + Y$ *and that* $Y = \frac{\mathrm{tr}(Y)}{2}(E_2 + E_3) + W$ *for some* $W \in \mathcal{J}_3(\tilde{K})_{-L^{\times}_{2E_1}}$ *such that* $(W,W) = \frac{1}{2}(\mathrm{tr}(Y)^2 - 4\det(E_1 + Y))$. *In this case, put* $\Psi_Y(\lambda) := \lambda^2 - \mathrm{tr}(Y)\lambda + \det(E_1 + Y) \equiv (\lambda - \lambda_2)(\lambda - \lambda_3)$ *with some* $\lambda_2, \lambda_3 \in \mathbb{C}$. *Then* $\Phi_X(\lambda) = (\lambda - (X|E_1))\Psi_Y(\lambda)$ *and* $2(W,W) = (\lambda_2 - \lambda_3)^2$.

(3) $\mathcal{O}_{(G(\tilde{K})^\tau)^\circ}(X) \cap \mathcal{J}_2(\tilde{K}) \neq \emptyset$ *if* $X \in \mathcal{J}_3(\tilde{K})$ *with* $X^{\times 2} = 0$.

Proof. (1) By Lemma 1.1, one has the first and the second equations. And $Y^{\times 2} = \frac{1}{2}(2r_2r_3 - 2N(x_1))E_1 = \det(E_1 + Y)E_1$.

(2) Take $X := \mathbb{X}(r_1, r_2, r_3; x_1, 0, 0) \in \mathcal{J}_3(\tilde{K})_\sigma$. Put

$$Y := \mathbb{X}(0, r_2, r_3; x_1, 0, 0),\ W := \frac{r_2 - r_3}{2}(E_2 - E_3) + F_1(x_1) \in \mathcal{J}_2(\tilde{K}).$$

Then $X = r_1E_1 + Y$, $Y = \frac{r_2+r_3}{2}(E_2+E_3)+W$; $\mathrm{tr}(Y) = r_2+r_3$, $\det(E_1+Y) = r_2r_3 - N(x_1)$ and $(W|W) = \frac{(r_2-r_3)^2}{2} + 2N(x_1) = \frac{1}{2}(\mathrm{tr}(Y)^2 - 4\det(E_1+Y))$. By Lemma 1.1 (1), $\varphi_X(\lambda) = (\lambda - r_1)(\lambda - r_2)(\lambda - r_3) - (\lambda - r_1)N(x_1) = (\lambda - r_1)(\lambda^2 - (r_2 + r_3)\lambda + (r_2r_3 - N(x_1))) = (\lambda - r_1)\Psi_Y(\lambda)$. Because of $\Psi_Y(\lambda) \equiv (\lambda - \lambda_2)(\lambda - \lambda_3)$, one has that $\mathrm{tr}(Y) = \lambda_2 + \lambda_3$, $\det(E_1 + Y) = \lambda_2\lambda_3$, so that $2(W, W) = (\lambda_2 + \lambda_3)^2 - 4\lambda_2\lambda_3 = (\lambda_2 - \lambda_3)^2$.

(3) Take $X \in \mathcal{J}_3(\tilde{K})$ with $X^{\times 2} = 0$. (i) When $\tilde{K} = K'$: By Lemma 1.5 (4), $\alpha X = \sum_{i=1}^{3}(s_iE_i + F_i(p_i\sqrt{-1}e_4))$ for some $p_i \in K \cap K'$, $s_i \in \mathbb{R}$ and $\alpha \in (G(K')^\tau)^\circ$, so that $0 = \alpha(X^{\times 2}) = (\alpha X)^{\times 2} = \sum_{i=1}^{3}(s_{i+1}s_{i+2}+2N(p_i))E_i + \sum_{i=1}^{3} F_i(\overline{p_{i+1}p_{i+2}} - s_ip_i\sqrt{-1}e_4)$, that is, $s_{i+1}s_{i+2}+2N(p_i) = \overline{p_{i+1}}p_{i+2} = s_ip_i = 0$ for all $i \in \{1, 2, 3\}$. (Case 1) When $p_i = 0$ for all i: $0 = s_2s_3 = s_3s_1 = s_1s_2$, so that $\alpha X = s_iE_i$ for some i. If $i = 2$ or 3, then $\alpha X \in \mathcal{J}_2(K')$. If $i = 1$, then $\hat{\beta}_3(\alpha X) = s_1E_2 \in \mathcal{J}_2(K')$ by $\hat{\beta}_3 \in (G(K')^\tau)^\circ$ defined in Lemma 1.4 (1) (ii). (Case 2) When $p_i \neq 0$ for some i: $p_{i+1} = p_{i+2} = 0$ and $s_i = 0$. If $i = 1$, then $\alpha X \in \mathcal{J}_2(K')$. If $i = 2$, then $\hat{\beta}_3(\alpha X) \in \mathcal{J}_2(K')$. If $i = 3$, then $\hat{\beta}_2(\alpha X) \in \mathcal{J}_2(K')$ by $\hat{\beta}_2 \in (G(K')^\tau)^\circ$ defined in Lemma 1.4 (1) (ii).

(ii) When $\tilde{K} = K^{\mathbb{C}}$: $\alpha X = Y + \sqrt{-1}\mathrm{diag}(r'_1, r'_2, r'_3)$ for some $r'_i \in \mathbb{R}$ ($i = 1, 2, 3$), $Y \in \mathcal{J}_3(K)$, and $\alpha \in G(K)^\circ = (G(K^{\mathbb{C}})^\tau)^\circ$ by Lemma 1.5 (3). Putting $Y = \mathbb{X}(r; x)$, $s_i := r_i + \sqrt{-1}r'_i \in \mathbb{C}$, one has $0 = (\alpha X)^{\times 2} = \sum_{i=1}^{3}\{(s_{i+1}s_{i+2} - 2N(x_i))E_i + F_i(\overline{x_{i+1}x_{i+2}} - s_ix_i)\}$, that is, $0 = r'_ix_i = \overline{x_{i+1}x_{i+2}} - r_ix_i = s_{i+1}s_{i+2} - 2N(x_i)$ for all $i \in \{1, 2, 3\}$. Then (Case 1) $x_i = 0$ for all i, (Case 2) $x_i \neq 0$, $x_{i+1} = x_{i+2} = 0$ for some i, (Case 3) $x_i \neq 0$, $x_{i+1} \neq 0$, $x_{i+2} = 0$ for some i; or (Case 4) $x_i \neq 0$ for all i. In (Case 1), $0 = s_{i+1}s_{i+2}$ for all i, so that $\alpha X = s_iE_i$ for some i. If $i = 2$ or 3, then $\alpha X \in \mathcal{J}_2(K^{\mathbb{C}})$. If $i = 1$, then $\hat{\beta}_3(\alpha X) = s_1E_2 \in \mathcal{J}_2(K^{\mathbb{C}})$ by $\hat{\beta}_3 \in (G(K^{\mathbb{C}})^\tau)^\circ$ defined in Lemma 1.4 (1) (ii). In (Case 2), $0 = r'_i = r_i$, so that $\alpha X = s_{i+1}E_{i+1} + s_{i+2}E_{i+2} + F_i(x_i)$ and that $\hat{\beta}_k(\alpha X) \in \mathcal{J}_2(K^{\mathbb{C}})$ for some $\hat{\beta}_k \in (G(K^{\mathbb{C}})^\tau)^\circ$ defined in Lemma 1.4 (1) (ii). In (Case 3), $0 = r'_i = r_i = r'_{i+1} = r_{i+1} = N(x_i) = N(x_{i+1})$, so that $\alpha X = s_{i+2}E_{i+2}$ and that $\hat{\beta}_k(\alpha X) \in \mathcal{J}_2(K^{\mathbb{C}})$ for some $\hat{\beta}_k \in (G(K^{\mathbb{C}})^\tau)^\circ$ defined in Lemma 1.4 (1) (ii). In (Case 4), $r'_i = 0$ for all i, so that $\alpha X \in \mathcal{J}_3(K)$ and that $\alpha_1(\alpha X)$ is diagonal for some $\alpha_1 \in G(K)^\circ$ by Lemma 1.5 (2). Then $\beta(\alpha_1(\alpha X))) \in \mathcal{J}_2(K^{\mathbb{C}})$ for some $\beta \in (G(K^{\mathbb{C}})^\tau)^\circ$ by the argument on (Case 1). $\square$

Proof of Proposition 0.1 (3) when $\tilde{K} \neq K$. Take any $X \in \mathcal{P}_2(\tilde{K})$. By (3), $\alpha_1 X \in \mathcal{J}_2(\tilde{K})$ for some $\alpha_1 \in (G(\tilde{K})^\tau)^\circ$. By (1), $\det(E_1 + \alpha_1 X)E_1 = (\alpha_1 X)^{\times 2} = \alpha(X^{\times 2}) = 0$, i.e. $\det(E_1 + \alpha_1 X) = 0$. By (2), $\alpha_1 X = \frac{\mathrm{tr}(\alpha_1 X)}{2}(E_2 + E_3) + W = \frac{1}{2}(E_2 + E_3) + W$ for some $W \in \mathcal{J}_3(\tilde{K})_{-L^\times_{2E_1}}$ such that $(W|W) = \frac{1}{2}(\mathrm{tr}(\alpha_1 X)^2 - 4\det(E_1 + \alpha_1 X)) = \frac{1}{2}$, so that $W \in \mathcal{S}_2(1/\sqrt{2}, \tilde{K})$. By Lemma 2.2 (1)(2), $\alpha_2 W = \frac{1}{2}(E_2 - E_3) \in \mathcal{S}_2(1/\sqrt{2}, \tilde{K})$ for some $\alpha_2 \in (G(\tilde{K})_{E_1})^\circ$. Then $\alpha_2(\alpha_1 X) = \frac{1}{2}(E_2 + E_3) + \frac{1}{2}(E_2 - E_3) = E_2$ by Lemma 2.1 (1). By $\hat{\beta}_3 \in (G(\tilde{K})^\tau_{E_3})^\circ$ defined in Lemma 1.4 (1) (ii), $\hat{\beta}_3(\alpha_2(\alpha_1 X)) = E_1$, where $\hat{\beta}_3 \alpha_2 \alpha_1 \in G(\tilde{K})^\circ$. □

Proof of Proposition 0.1 (4) (i). Take any $X \in \mathcal{M}_1(\tilde{K})$ defined in Lemma 1.6 (5). By (3), $0 \neq \alpha X \in \mathcal{J}_2(\tilde{K})$ for some $\alpha \in (G(\tilde{K})^\tau)^\circ$ with $\mathrm{tr}(\alpha X) = \mathrm{tr}(X) = 0$. In this case, by (2), $\alpha X = \frac{\mathrm{tr}(\alpha X)}{2}(E_2 + E_3) + W = W$ for some $W \in \mathcal{J}_3(\tilde{K})_{-L^\times_{2E_1}}$. And $(\alpha X|\alpha X) = (X|X) = (X \circ X|E) = -2(X \times X|E) = -2\mathrm{tr}(X^{\times 2}) = 0$ by Lemma 1.1 (4). Hence, $\alpha X \in \mathcal{S}_2(0, \tilde{K})$. By Lemma 2.2 (1) (ii) or (2) (ii), there exists $\beta \in (G(\tilde{K})^\circ)_{E_1}$ such that $\beta(\alpha X) = M_{1'}$ (when $\tilde{K} = K'$) or M_1 (when $\tilde{K} = K^{\mathbb{C}}$). □

3. Theorems 0.2 and 0.3 in (1) (i, ii).

Assume that $X \in \mathcal{J}_3(\tilde{K})$ admits a characteristic root $\lambda_1 \in \mathbb{F}$ of multiplicity 1. Then $0 \neq \Phi'_X(\lambda_1) = \mathrm{tr}(\varphi_X(\lambda_1)^{\times 2})$ by Lemma 1.2 (2), so that

$$E_{X,\lambda_1} := \frac{1}{\mathrm{tr}(\varphi_X(\lambda_1))}\varphi_X(\lambda_1)^{\times 2} \in V_X$$

is well-defined. Put $W_{X,\lambda_1} := X - \lambda_1 E_{X,\lambda_1} - \frac{\mathrm{tr}(X)-\lambda_1}{2}\varphi_{E_{X,\lambda_1}}(1) \in V_X$. Then

$$X = \lambda_1 E_{X,\lambda_1} + \frac{\mathrm{tr}(X) - \lambda_1}{2}\varphi_{E_{X,\lambda_1}}(1) + W_{X,\lambda_1}.$$

Lemma 3.1. *Assume that $X \in \mathcal{J}_3(\tilde{K})$ admits a characteristic root $\lambda_1 \in \mathbb{F}$ of multiplicity* 1. *Then:*

(1) $V_X \cap \mathcal{P}_2(\tilde{K}) \ni E_{X,\lambda_1} \neq 0$, $\varphi_{E_{X,\lambda_1}}(1) \neq 0$, $E^{\times 2}_{X,\lambda_1} = 0$, $2E_{X,\lambda_1} \times \varphi_{E_{X,\lambda_1}}(1) = \varphi_{E_{X,\lambda_1}}(1)$, $\varphi_{E_{X,\lambda_1}}(1)^{\times 2} = E_{X,\lambda_1}$, $2E_{X,\lambda_1} \times W_{X,\lambda_1} = -W_{X,\lambda_1}$*;*

(2) $V_X = \mathbb{F}E_{X,\lambda_1} \oplus \mathbb{F}\varphi_{E_{X,\lambda_1}}(1) \oplus \mathbb{F}W_{X,\lambda_1}$ *such that* $v_X = 2$ (*if* $W_{X,\lambda_1} = 0$) *or* $v_X = 3$ (*if* $W_{X,\lambda_1} \neq 0$) *with* $(E_{X,\lambda_1}|\varphi_{E_{X,\lambda_1}}(1)) = (E_{X,\lambda_1}|W_{X,\lambda_1}) = (\varphi_{E_{X,\lambda_1}}(1)|W_{X,\lambda_1}) = 0$, $(E_{X,\lambda_1}|E_{X,\lambda_1}) = 1$,
$(\varphi_{E_{X,\lambda_1}}(1)|\varphi_{E_{X,\lambda_1}}(1)) = 2$ *and* $(W_{X,\lambda_1}|W_{X,\lambda_1}) = \Delta_X(\lambda_1)$.

Proof. (1) Put $Z := \varphi_X(\lambda_1)$ and $Y := Z^{\times 2}$, so that $Y^{\times 2} = 0$ by Lemma 1.6 (4). Then $E_{X,\lambda_1} = \frac{1}{\mathrm{tr}(Y)}Y$, $\mathrm{tr}(E_{X,\lambda_1}) = 1$ and $E_{X,\lambda_1}^{\times 2} = 0$, so that $E_{X,\lambda_1} \in \mathcal{P}_2(\tilde{K}) \cap V_X$. Note that $\mathrm{tr}(\varphi_{E_{X,\lambda_1}}(1)) = \mathrm{tr}(E) - \mathrm{tr}(E_{X,\lambda_1}) = 3 - 1 = 2 \neq 0$, so that $\varphi_{E_{X,\lambda_1}}(1) \neq 0$. By Lemma 1.1 (3), $2E_{X,\lambda_1} \times \varphi_{E_{X,\lambda_1}}(1) = 2E_{X,\lambda_1} \times (E - E_{X,\lambda_1}) = 2E_{X,\lambda_1} \times E = \mathrm{tr}(E_{X,\lambda_1})E - E_{X,\lambda_1} = \varphi_{E_{X,\lambda_1}}(1)$ and $\varphi_{E_{X,\lambda_1}}(1)^{\times 2} = (E - E_{X,\lambda_1})^{\times 2} = E^{\times 2} - 2E \times E_{X,\lambda_1} = E - \varphi_{E_{X,\lambda_1}}(1) = E_{X,\lambda_1}$. By direct computations, $W_{X,\lambda_1} = \frac{\mathrm{tr}(Z)}{2}\varphi_{E_{X,\lambda_1}}(1) - Z$. By Lemma 1.6 (2) (iii) and $\det(Z) = 0$, $2E_{X,\lambda_1} \times Z = \frac{2}{\mathrm{tr}(Y)}Z^{\times 2} \times Z = \frac{2}{\mathrm{tr}(Y)}\frac{-1}{2}(\mathrm{tr}(Z)Y + \mathrm{tr}(Y)Z - \mathrm{tr}(Y)\mathrm{tr}(Z)E + \det(Z)E) = -\mathrm{tr}(Z)E_{X,\lambda_1} - Z + \mathrm{tr}(Z)E = -Z + \mathrm{tr}(Z)\varphi_{E_{X,\lambda_1}}(1)$. Hence, $2E_{X,\lambda_1} \times W_{\lambda_1} = \frac{\mathrm{tr}(Z)}{2}\varphi_{E_{X,\lambda_1}}(1) + Z - \mathrm{tr}(Z)\varphi_{E_{X,\lambda_1}}(1) = Z - \frac{\mathrm{tr}(Z)}{2}\varphi_{E_{X,\lambda_1}}(1) = -W_{X,\lambda_1}$.

(2) Since V_X is spanned by $E, X, X^{\times 2}$, $v_X := \dim_{\mathbb{F}} V_X \leqq 3$. If $W_{X,\lambda_1} \neq 0$, then E_{X,λ_1}, $\varphi_{E_{X,\lambda_1}}(1)$, W_{X,λ_1} are eigen-vectors of $L^{\times}_{2E_{X,\lambda_1}}$ with different eigen-values $0, 1, -1$, i.e. $v_X = 3$. If $W_{X,\lambda_1} = 0$, then $X = \lambda_1 E_{X,\lambda_1} + \frac{\mathrm{tr}(X)-\lambda_1}{2}\varphi_{E_{X,\lambda_1}}(1)$ and $X^{\times 2} = \frac{\lambda_1(\mathrm{tr}(X)-\lambda_1)}{2}\varphi_{E_{X,\lambda_1}}(1) + (\frac{\mathrm{tr}(X)-\lambda_1}{2})^2 E_{X,\lambda_1}$, so that V_X is spanned by E_{X,λ_1} and $\varphi_{E_{X,\lambda_1}}(1) = E - E_{X,\lambda_1}$, i.e. $v_X = 2$. By Lemmas 1.1 (2) and 1.6 (3), $L^{\times}_{2E_{X,\lambda_1}}$ is a symmetric $\mathbb{F}$-linear transformation on $(V_X, (*|*))$, so that $E_{X,\lambda_1}, \varphi_{E_{X,\lambda_1}}(1), W_{X,\lambda_1}$ are orthogonal as zero or eigen-vectors of $L^{\times}_{2E_{X,\lambda_1}}$ with the different eigen-values. By (1) and Lemma 1.1 (4), $0 = 2\mathrm{tr}(E_{X,\lambda_1}^{\times 2}) = \mathrm{tr}(E_{X,\lambda_1})^2 - (E_{X,\lambda_1}|E_{X,\lambda_1}) = 1 - (E_{X,\lambda_1}|E_{X,\lambda_1})$, so that $(E_{X,\lambda_1}|E_{X,\lambda_1}) = 1$ and $(\varphi_{E_{X,\lambda_1}}(1)|\varphi_{E_{X,\lambda_1}}(1)) = (E|E) - 2(E|E_{X,\lambda_1}) + (E_{X,\lambda_1}|E_{X,\lambda_1}) = 3 - 2\mathrm{tr}(E_{X,\lambda_1}) + 1 = 2$. Because of the orthogonality in (1), $(X|E_{X,\lambda_1}) = \lambda_1$, $(X|\varphi_{E_{X,\lambda_1}}(1)) = \mathrm{tr}(X) - \lambda_1$ and $(W_{X,\lambda_1}|W_{X,\lambda_1}) = (X|X) + \lambda_1^2 + \frac{(\mathrm{tr}(X)-\lambda_1)^2}{2} - 2\lambda_1(X|E_{X,\lambda_1}) - (\mathrm{tr}(X)-\lambda_1)(X|\varphi_{E_{X,\lambda_1}}(1)) = (X|X) - \frac{3}{2}\lambda_1^2 + \mathrm{tr}(X)\lambda_1 - \frac{1}{2}\mathrm{tr}(X)^2 = \Delta_X(\lambda_1)$. □

Note that $\Delta_X(\lambda_1) = -\frac{1}{2}\{3\lambda_1^2 - 2\mathrm{tr}(X)\lambda_1 + \mathrm{tr}(X)^2 - 2(X|X)\} \in \mathbb{F}$ is an invariant on $\mathcal{O}_{G(\tilde{K})}(X)$ if $\lambda_1 \in \mathbb{F}$ is a characteristic root of multiplicity 1 for $X \in \mathcal{J}_3(\tilde{K})$.

Lemma 3.2. *Assume that* $X \in \mathcal{J}_3(\tilde{K})$ *admits an eigen-value* $\lambda_1 \in \mathbb{F}$ *of multiplicity* 1. *Put* $\Phi_X(\lambda) \equiv \Pi_{i=1}^3(\lambda - \lambda_i)$ *for some* $\lambda_2, \lambda_3 \in \mathbb{C}$ *with* $\lambda_1 \neq$

λ_2, λ_3. *Then* $\mathcal{O}_{G(\tilde{K})^\circ} \ni \lambda_1 E_1 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_1) + W$ *for* $W \in \mathcal{J}_3(\tilde{K})_{-L^\times_{2E_1}}$ *such that* $(W|W) = \Delta_X(\lambda_1) = (\lambda_2 - \lambda_3)^2/2$ *given by* Λ_X *and* v_X *as follows:*

(1) *When* $\tilde{K} = K'$ *with* $\mathbb{F} = \mathbb{R}$*:*

(i-1) $W = \frac{\sqrt{\Delta_X(\lambda_1)}}{\sqrt{2}}(E_2 - E_3)$ *with* $\#\Lambda_X = v_X = 3$ *if* $\Delta_X(\lambda_1) > 0$*;*

(i-2) $W = \frac{\sqrt{-\Delta_X(\lambda_1)}}{\sqrt{2}} F_1(\sqrt{-1}e_4)$ *with* $\#\Lambda_X = v_X = 3$ *if* $\Delta_X(\lambda_1) < 0$*;*

(ii) $W = M_{1'}$ *if* $\Delta_X(\lambda_1) = 0$ *with* $v_X = 3$*;*

(iii) $W = 0$ *if* $\Delta_X(\lambda_1) = 0$ *with* $v_X = 2$*.*

(2) *When* $\tilde{K} = K^{\mathbb{C}}$ *with* $\mathbb{F} = \mathbb{C}$*:*

(i) $W = \frac{w}{\sqrt{2}}(E_2 - E_3)$ *for any* $w \in \mathbb{C}$ *such that* $w^2 = \Delta_X(\lambda_1)$ *with* $\#\Lambda_X = v_X = 3$ *if* $0 \neq \Delta_X(\lambda_1) \in \mathbb{C}$*;*

(ii) $W = M_1$ *if* $\Delta_X(\lambda_1) = 0$ *with* $v_X = 3$*;*

(iii) $W = 0$ *if* $\Delta_X(\lambda_1) = 0$ *with* $v_X = 2$*.*

Proof. By Lemma 3.1 (1) and Proposition 0.1 (3), $\alpha E_{X,\lambda_1} = E_1$ for some $\alpha \in G(\tilde{K})^\circ$, so that $\alpha\varphi_{E_{X,\lambda_1}}(1) = \alpha(E - E_{X,\lambda_1}) = E - E_1$. Put $W' := \alpha W_{X,\lambda_1}$. Then $\alpha X = \lambda_1 E_1 + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_1) + W'$ with $\Phi_{\alpha X}(\lambda) = \Pi_{i=1}^3(\lambda - \lambda_i)$. And $2E_1 \times W' = \alpha(2E_{X,\lambda_1} \times W_{X,\lambda_1}) = -\alpha W_{X,\lambda_1} = -W'$ by Lemma 3.1 (1), i.e. $W' \in \mathcal{J}_3(\tilde{K})_{-L^\times_{2E_1}} \subset \mathcal{J}_3(\tilde{K})_\sigma$. By Lemmas 3.1 (2) and 2.3 (2), $\Delta_X(\lambda_1) = (W_{X,\lambda_1}|W_{X,\lambda_1}) = (W'|W') = (\lambda_2 - \lambda_3)^2/2$, which is determined by Λ_X, so that $W' \in \mathcal{S}(\Delta_X(\lambda_1)) \cup \{0\}$. Note that $W' = 0$ (or $W' \neq 0$) iff $W_{X,\lambda_1} = 0$ (resp. $W_{X,\lambda_1} \neq 0$) iff $v_X = 2$ (resp. $v_X = 3$) by Lemma 3.1 (2). If $\Delta_X(\lambda_1) \neq 0$, then $(W'|W') \neq 0$, so that $W' \neq 0$ and $\lambda_2 \neq \lambda_3$, i.e. $v_X = \#\Lambda_X = 3$: By Lemma 2.2 (1) (i-1, 2) or (2) (i), $W := \beta W'$ is given as (1) (i-1, 2) or (2) (i) for some $\beta \in (G(\tilde{K})_{E_1})^\circ$, so that $\beta(\alpha X) = \lambda_1 \beta E_1 + \frac{\mathrm{tr}(X)-\lambda_1}{2}\beta(E - E_1) + W = \lambda_1 E_1 + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_1) + W$. If $\Delta_X(\lambda_1) = 0$, then $(W'|W') = 0$, so that $W' \in \mathcal{S}_2(0, \tilde{K}) \cup \{0\}$: By Lemma 2.2 (1) (ii, iii) or (2) (ii, iii), $W := \beta W'$ is given as (1) (ii, iii) or (2) (ii, iii) for some $\beta \in (G(\tilde{K})_{E_1})^\circ$, so that $\beta(\alpha X) = \lambda_1 \beta E_1 + \frac{\mathrm{tr}(X)-\lambda_1}{2}\beta(E - E_1) + W = \lambda_1 E_1 + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_1) + W$. □

Proof of Theorems 0.2 and 0.3 in (1) (i, ii). Let $X \in \mathcal{J}_3(\tilde{K})$ be such as $\#\Lambda_X \neq 1$, that is, X admits no characteristic root of multiplicity 3. Since the degree of $\Phi_X(\lambda)$ equals $3 = 1 + 1 + 1 = 1 + 2$, there exists a characteristic root $\mu_1 \in \mathbb{C}$ of multiplicity 1. If $\mathbb{F} \ni \mu_1$, put $\lambda_1 := \mu_1$. If $\mathbb{F} \not\ni \mu_1$, then

$\mathbb{F} = \mathbb{R} \not\ni \mu_1$, so that $\Phi_X(\lambda) = (\lambda - \mu_1)(\lambda - \overline{\mu_1})(\lambda - \nu_1)$ for some $\nu_1 \in \mathbb{R}$. In this case, put $\lambda_1 := \nu_1$. In all cases, put $\Lambda_X = \{\lambda_1, \lambda_2, \lambda_3\}$ with $\#\Lambda_X = 3$ or 2 such that $\Phi'_X(\lambda_1) \neq 0$ and $\mathrm{tr}(X) = \sum_{i=1}^{3} \lambda_i$, so that $\Delta_X(\lambda_1) = (\lambda_2 - \lambda_3)^2/2$ by Lemmas 3.1 (2) and 2.3 (2). By Lemma 3.2 (2) (if $\tilde{K} = K^{\mathbb{C}}$) or (1) (if $\tilde{K} = K'$), $\alpha X = \lambda_1 E_1 + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_1) + W$ for some $W \in \mathcal{J}_3(\tilde{K})_{-L^{\times}_{2E_1}}$ and $\alpha \in G(\tilde{K})^{\circ}$.

(0.2.1) *When* $\mathbb{F} = \mathbb{C}$*:* $\tilde{K} = K^{\mathbb{C}} = \mathbb{R}^{\mathbb{C}}, \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}$ or $\boldsymbol{O}^{\mathbb{C}}$.

(0.2.1.i) *The case of* $\#\Lambda_X = 3$*:* Put $w := (\lambda_2 - \lambda_3)/\sqrt{2}$. Then $\lambda_2 \neq \lambda_3$. And $\Delta_X(\lambda_1) = w^2 \neq 0$. By Lemma 3.2 (2) (i), $v_X = 3$ and $\alpha X = \lambda_1 E_1 + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_1) + \frac{w}{\sqrt{2}}(E_2 - E_3) = \lambda_1 E_1 + \frac{\lambda_2+\lambda_3}{2}(E_2 + E_3) + \frac{\lambda_2-\lambda_3}{2}(E_2 - E_3) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$.

(0.2.1.ii) *The case of* $\#\Lambda_X = 2$*:* $\lambda_2 = \lambda_3$ and $\Delta_X(\lambda_1) = 0$.

(0.2.1.ii-1) *When* $v_X = 2$*:* By Lemma 3.2 (2) (iii), $\alpha X = \lambda_1 E_1 + \lambda_2(E_2 + E_3) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)$.

(0.2.1.ii-2) *When* $v_X = 3$*:* By Lemma 3.2 (2) (ii), $\alpha X = \lambda_1 E_1 + \lambda_2(E_2 + E_3) + M_1 = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_2) + M_1$.

(0.3.1) *When* $\mathbb{F} = \mathbb{R}$*:* $\tilde{K} = K' = \boldsymbol{C}', \boldsymbol{H}'$ or $\boldsymbol{O}'$. And $\lambda_1 \in \mathbb{R}$, $\lambda_2, \lambda_3 \in \mathbb{C}$.

(0.3.1.i) *The case of* $\#\Lambda_X = 3$*:*

(0.3.1.i-1) *When* $\Lambda_X \subset \mathbb{R}$*:* It can be assumed that $\lambda_1 > \lambda_2 > \lambda_3$ by translation if necessary. Then $\Delta_X(\lambda_1) > 0$. By Lemma 3.2 (1) (i-1), $v_X = 3$ and $\alpha X = \lambda_1 E_1 + \frac{\lambda_2+\lambda_3}{2}(E_2 + E_3) + \frac{\lambda_2-\lambda_3}{2}(E_2 - E_3) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$.

(0.3.1.i-2) *When* $\Lambda_X \not\subset \mathbb{R}$*:* $\{\lambda_2, \lambda_3\} = \{p \pm q\sqrt{-1}\}$ for some $p, q \in \mathbb{R}$ with $q > 0$. And $\Delta_X(\lambda_1) = -2q^2 < 0$. By Lemma 3.2 (1) (i-2), $\alpha X = \lambda_1 E_1 + p(E_2 + E_3) + qF_1(\sqrt{-1}e_4) = \mathrm{diag}(\lambda_1, p, p) + F_1(q\sqrt{-1}e_4)$.

(0.3.1.ii) *The case of* $\#\Lambda_X = 2$*:* $\Lambda_X = \{\lambda_1, \lambda_2\}$ with $\Phi'_X(\lambda_2) = 0$. Then $\lambda_2 = \frac{1}{2}(\mathrm{tr}(X) - \lambda_1) \in \mathbb{R}$ and $\Delta_X(\lambda_1) = 0$.

(0.3.1.ii-1) *When* $v_X = 2$*:* By Lemma 3.2 (1) (iii), $\alpha X = \lambda_1 E_1 + \lambda_2(E_2 + E_3) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)$.

(0.3.1.ii-2) *When* $v_X = 3$*:* By Lemma 3.2 (1) (ii), $\alpha X = \lambda_1 E_1 + \lambda_2(E_2 + E_3) + M_{1'} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_2) + M_{1'}$. □

4. Proposition 0.1 (4) and Theorems 0.2 and 0.3 in (1) (iii).

Assume that $\tilde{K} \neq K$, i.e., $\tilde{K} = \boldsymbol{R}^{\mathbb{C}}, \boldsymbol{C}^{\mathbb{C}}, \boldsymbol{H}^{\mathbb{C}}, \boldsymbol{O}^{\mathbb{C}}; \boldsymbol{C}', \boldsymbol{H}'$ or $\boldsymbol{O}'$. Put $\mathcal{N}_1(\tilde{K}) := (\mathcal{J}_3(\tilde{K})_0)_{L_{\tilde{M}_1},0}$ and $\mathcal{N}_2(\tilde{K}) := \{X \in \mathcal{J}_3(\tilde{K})_0 |\ X^{\times 2} = \tilde{M}_1\}$.

LEMMA 4.1. (1) $\mathcal{N}_2(\tilde{K}) \subsetneqq \mathcal{N}_1(\tilde{K})$.

(2) $\mathcal{N}_1(\tilde{K}) = \{\tilde{M}_1(x) + \tilde{M}_{23}(y) |\ x, y \in \tilde{K}\}$;

(3) $\mathcal{N}_2(\tilde{K}) = \{s\tilde{M}_1 + \tilde{M}_{23}(y) |\ s \in \mathbb{F}, y \in \mathcal{S}_1(1, \tilde{K})\}$;

Proof. (1) Take $X \in \mathcal{N}_2(\tilde{K})$. Then $\mathrm{tr}(X) = 0$ and $\mathrm{tr}(X^{\times 2}) = \mathrm{tr}(\tilde{M}_1) = 0$. By Lemma 1.6 (2) (ii), $\det(X)X = (X^{\times 2})^{\times 2} = \tilde{M}_1^{\times 2} = 0$, so that $\det(X) = 0$. By Lemma 1.6 (2) (iii), $X \times \tilde{M}_1 = X \times (X^{\times 2}) = -\frac{1}{2}(\mathrm{tr}(X)X^{\times 2} + \mathrm{tr}(X^{\times 2})X - (\mathrm{tr}(X)\mathrm{tr}(X^{\times 2}) - \det(X))E) = 0$, as required.

(2) (i) Take $X = \mathbb{X}(r; x) \in \mathcal{N}_1(K^{\mathbb{C}})$. Then $r_1 + r_2 + r_3 = 0$ and that $0 = 2M_1 \times X = 2\mathbb{X}(0, 1, -1; \sqrt{-1}, 0, 0) \times \mathbb{X}(r; x) = (r_3 - r_2 - 2(\sqrt{-1}|x_1))E_1 - r_1E_2 + r_1E_3 + F_1(-\sqrt{-1}r_1) + F_2(\sqrt{-1}\overline{x_3} - x_2) + F_3(\sqrt{-1}\overline{x_2} + x_3)$ by Lemma 1.1 (1), that is, $x_2 = \sqrt{-1}\overline{x_3}$, $r_1 = 0$, $r_2 = -r_3 = -(\sqrt{-1}|x_1) = (1|-\sqrt{-1}x_1)$ with $x_1 = \sqrt{-1}(-\sqrt{-1}x_1)$, so that $X = M_1(-\sqrt{-1}x_1) + M_{23}(x_3)$.

(ii) Take $X = \mathbb{X}(r; x) \in \mathcal{N}_2(K')$. Then $r_1 + r_2 + r_3 = 0$ and that $0 = 2M_{1'} \times X = 2\mathbb{X}(0, 1, -1; \sqrt{-1}e_4, 0, 0) \times \mathbb{X}(r; x) = (r_3 - r_2 - 2(\sqrt{-1}e_4|x_1))E_1 - r_1E_2 + r_1E_3 + F_1(-r_1\sqrt{-1}e_4) + F_2(-\sqrt{-1}e_4\overline{x_3} - x_2) + F_3(-\overline{x_2}\sqrt{-1}e_4 + x_3)$ by Lemma 1.1 (1), that is, $x_2 = -\sqrt{-1}e_4\overline{x_3}$, $r_1 = 0$, $r_2 = -r_3 = -(\sqrt{-1}e_4|x_1) = (1|\sqrt{-1}e_4x_1)$ with $x_1 = (\sqrt{-1}e_4)^2x_1 = \sqrt{-1}e_4(\sqrt{-1}e_4x_1)$, so that $X = M_{1'}(\sqrt{-1}e_4x_1) + M_{2'3}(x_3)$.

(3) (i) Take $X \in \mathcal{N}_2(K^{\mathbb{C}})$. By (1), $X \in \mathcal{N}_1(K^{\mathbb{C}})$. By (2), $X = M_1(x_1) + M_{23}(x_3)$ for some $x_1, x_3 \in K^{\mathbb{C}}$, so that $X^{\times 2} = M_1(x_1)^{\times 2} + 2M_1(x_1) \times M_{23}(x_3) + M_{23}(x_3)^{\times 2} = \sqrt{-1}\{(x_1|1)^2 - N(x_1)\}E_1 - F_2(\sqrt{-1}(\overline{x_1} - (x_1|1))\overline{x_3}) - F_3(x_3(\overline{x_1} - (x_1|1))) + N(x_3)M_1$. Hence, $X^{\times 2} = M_1$ iff $N(x_3) = 1$ and $\overline{x_1} = (x_1|1) \in \mathbb{R}^{\mathbb{C}}$, i.e. $(x_1, x_3) = (s, x)$ for some $(s, x) \in \mathbb{R}^{\mathbb{C}} \times \mathcal{S}_1(1, K^{\mathbb{C}})$.

(ii) Take $X \in \mathcal{N}_2(K')$. By (1), $X \in \mathcal{N}_1(K')$. By (2), $X = M_{1'}(x_1) + M_{2'3}(x_3)$ for some $x_1, x_3 \in K'$, so that $X^{\times 2} = M_{1'}(x_1)^{\times 2} + 2M_{1'}(x_1) \times M_{2'3}(x_3) + M_{2'3}(x_3)^{\times 2} = (N(x_1) - (x_1|1)^2)E_1 - F_2(((\overline{x_1} - (x_1|1))\sqrt{-1}e_4)\overline{x_3}) + F_3(-(x_3\sqrt{-1}e_4)(\overline{x_1}\sqrt{-1}e_4) + (x_1|1)x_3) + N(x_3)M_{1'}$. Hence, $X^{\times 2} = M_{1'}$ iff $N(x_3) = 1$ and $x_1 \in \boldsymbol{R}$, as required. □

LEMMA 4.2. $\mathcal{N}_2(\tilde{K}) = \mathcal{O}_{(G(\tilde{K})^\circ)_{\tilde{M}_1}}(\tilde{M}_{23})$.

Proof. (1) Take any $X \in \mathcal{N}_2(K^{\mathbb{C}})$. By Lemma 4.1 (3), $X = sM_1 + M_{23}(x)$ for some $s \in \mathbb{R}^{\mathbb{C}}$ and $x \in \mathcal{S}_1(1, K^{\mathbb{C}})$. Put $x = \sum_{i=0}^{d_K-1} \xi_i e_i$ with $\xi_i \in \mathbb{R}^{\mathbb{C}}$ and $x_v := \sum_{i=1}^{d_K-1} \xi_i e_i$ such that $\overline{x_v} = -x_v$.

(Case 1) When $\xi_0 = 0$: By $x_v = x \in \mathcal{S}_1(1, K^{\mathbb{C}})$, $xx = -x\overline{x} = -1$. By Lemma 1.4 (2) (ii), $\delta_1(x)X = s(E_2 - E_3 - F_1(\sqrt{-1})) - F_2(\sqrt{-1}) + F_3(1)$, so that $\sigma_3\delta_1(x)X = sM_1 + M_{23}$. By Lemma 1.4 (1) (iii), $\beta_{23}(t)(\sigma_3\delta_1(x)X) =$

$(2t+s)M_1 + M_{23} = M_{23}$ if $t = -s/2$ with $\beta_{23}(-s/2) \in (G(K^{\mathbb{C}})^\circ)_{M_1}$ and $(\beta_{23}(-s/2)\sigma_3\delta_1(x))M_1 = \beta_{23}(-s/2)(M_1) = M_1$, as required.

(Case 2) When $\xi_0 \neq 0$: Put $Y := \beta_{23}(-s/(2\xi_0))X$ with $\beta_{23}(-s/(2\xi_0)) \in (G(K^{\mathbb{C}})^\circ)_{M_1}$. By Lemma 1.4 (1) (iii), $Y = M_{23}(x)$.

(i) When $x_v = 0$: $\xi_0^2 = (x|x) - (x_v|x_v) = 1 - 0 = 1$, $x = \xi_0 = \pm 1$ and $Y = M_{23}(\pm 1) = \pm M_{23}$, so that $Y = M_{23}$ or $\sigma_1 Y = M_{23}$ with $\sigma_1 \in (G(K^{\mathbb{C}})^\circ)_{M_1}$.

(ii) When $x_v \neq 0$ with $d_K \geqq 4$: Then $d_K - 1 \geqq 3$. If $\xi_1^2 + \xi_j^2 = 0$ for all $j \in \{2, \cdots, d_K - 1\}$ and $\xi_2^2 + \xi_3^2 = 0$, then $-\xi_1^2 = \xi_2^2 = \xi_3^2 = \cdots = \xi_{d_K-1}^2 = 0$, that is, $x_v = 0$, a contradiction. Hence, $\xi_i^2 + \xi_j^2 \neq 0$ for some $i, j \in \{1, \cdots, d_K - 1\}$ with $i \neq j$. Take $c \in \mathbb{R}^{\mathbb{C}}$ such that $c^2 = \xi_i^2 + \xi_j^2$. Put $a := (\xi_j e_i - \xi_i e_j)/c$, so that $-\overline{a} = a \in \mathcal{S}_1(1, K^{\mathbb{C}})$ and $(a|x) = 0$. Put $y := x\overline{a}$. Then $(y|1) = (a|x) = 0$ and $\sigma_3\delta_1^a Y = F_2(\sqrt{-1}\overline{y}) + F_3(y)$ with $\sigma_3\delta_1^a \in (G(K^{\mathbb{C}})^\circ)_{M_1}$. According to the (Case 1), $\beta(\sigma_3\delta_1^a Y) = M_{23}$ for some $\beta \in (G(K^{\mathbb{C}})^\circ)_{M_1}$.

(iii) When $x_v \neq 0$ with $d_K \leqq 4$: By Lemma 1.4 (2) (iii), $\beta_1(x)Y = M_{23}$ with $\beta_1(x) \in (G(K^{\mathbb{C}})^\circ)_{M_1}$.

(2) Take any $X \in \mathcal{N}_2(K')$. Then $X = sM_{1'} + M_{2'3}(x)$ for some $s \in \mathbb{R}$ and $x \in \mathcal{S}_1(1, K')$ by Lemma 4.1 (3). Take $p, q \in K'_\tau$ such that $x = p + q\sqrt{-1}e_4$, so that $\overline{p}p - \overline{q}q = N(x) = 1$. Note that $\sqrt{-1}e_4 x = \overline{q} + \overline{p}\sqrt{-1}e_4$, and that $x(\sqrt{-1}e_4 x) = (p + q\sqrt{-1}e_4)(\overline{q} + \overline{p}\sqrt{-1}e_4) = p(q + \overline{q}) + (q^2 + \overline{p}p)\sqrt{-1}e_4$.

(Case 1) When $(\sqrt{-1}e_4 x|1) = 0$: Then $q + \overline{q} = 2(\overline{q}|1) = 0$. By $q = -\overline{q}$, $q^2 + \overline{p}p = \overline{p}p - \overline{q}q = 1$, so that $x(\sqrt{-1}e_4 x) = \sqrt{-1}e_4$ and $\overline{x}\sqrt{-1}e_4\overline{x} = -\overline{x\sqrt{-1}e_4 x} = -\overline{\sqrt{-1}e_4} = \sqrt{-1}e_4$. By Lemma 1.4 (2) (ii), $\delta_1(x)M_{1'} = E_2 - E_3 + F_1(x\sqrt{-1}e_4 x) = M_{1'}$ and $\delta_1(x)X = sM_{1'} + F_2(-\overline{x}\sqrt{-1}e_4\overline{x}) + F_3(x\overline{x}) = sM_{1'} + M_{2'3}$. By Lemma 1.4 (1) (iii), $\beta_{2'3}(-s/2)(sM_{1'} + M_{2'3}) = M_{2'3}$ with $\beta_{23'}(t) \in (G(K')^\circ)_{M_{1'}}$.

(Case 2) When $(\sqrt{-1}e_4 x|1) \neq 0$: Then $(\sqrt{-1}e_4|x) = -(1|\sqrt{-1}e_4 x) \neq 0$. By Lemma 1.4 (1) (iii), $\beta_{23'}(-s/(2(\sqrt{-1}e_4|x)))X = M_{2'3}(x)$ with $\beta_{2'3}(t) \in (G(K')^\circ)_{M_{1'}}$. Note that $q + \overline{q} = 2(\sqrt{-1}e_4|x) \neq 0$, so that $q \in K'_\tau$ and $q \neq 0$.

(i) When $d_{K'} \geqq 4$: By $\dim_{\boldsymbol{R}} K'_\tau = d_{K'}/2 \geqq 2$, there exists $q_1 \in K'_\tau$ such that $(\overline{q}|q_1) = 0$, so that $(\sqrt{-1}e_4(x\overline{q_1})|1) = -(x\overline{q_1}|\sqrt{-1}e_4) = -(\overline{q_1}|\overline{x}\sqrt{-1}e_4) = (q_1|\sqrt{-1}e_4 x) = (q_1|\overline{q}) = 0$. Put $a := q_1/\sqrt{N(q_1)} \in \mathcal{S}_1(1, K')$. Because of $(\sqrt{-1}e_4 a|1) = 0$, $\delta_1(a)M_{1'} = M_{1'}$ as well as (Case 1), so that $\mathcal{N}_2(K') \ni \delta_1(a)M_{2'3}(x) = M_{2'3}(x\overline{a})$ with $x\overline{a} \in \mathcal{S}_1(1, K')$ such that $(\sqrt{-1}e_4(x\overline{a})|1) = 0$. Then $\delta_1(a)M_{1'} = M_{1'}$ and $\delta_1(x\overline{a})M_{2'3}(x\overline{a}) = M_{2'3}$ as well as (Case 1).

(ii) When $d_{K'} \leqq 2$: Then $K' = \boldsymbol{C}'$, so that $x \in \mathcal{S}_1(1, \boldsymbol{C}')$. By Lemma 1.4 (2) (iii), $\beta_1(x) \in (G(\boldsymbol{C}')^\circ)_{M_{1'}}$ such that $\beta_1(x)M_{2'3}(x) = M_{2'3}$. $\square$

Proof of Proposition 0.1 (4) (ii). Take any $X \in \mathcal{M}_{23}(\tilde{K})$. By Lemma 1.6 (5), $X^{\times 2} \in \mathcal{M}_1(\tilde{K})$. By Proposition 0.1 (4) (i), there exists $\alpha \in G(\tilde{K})^\circ$ such that $\tilde{M}_1 = \alpha(X^{\times 2}) = (\alpha X)^{\times 2}$, so that $\alpha X \in \mathcal{N}_2(\tilde{K})$. By Lemma 4.2, there exists $\beta \in (G(\tilde{K})^\circ)_{\tilde{M}_1}$ such that $\beta(\alpha X) = \tilde{M}_{23}$, as required. □

Proof of Theorems 0.2 and 0.3 in (1) (iii). Take any $X \in \mathcal{J}_3(\tilde{K})$ with $\#\Lambda_X = 1$ such as $\Lambda_X = \{\lambda_1\}$. By Lemmas 1.2 (1) and (3), $X = \lambda_1 E + X_0$ with some $X_0 \in \{0\} \cup \mathcal{M}_1(\tilde{K}) \cup \mathcal{M}_{23}(\tilde{K})$.

(iii-1) When $X_0 = 0$: $X = \lambda E$ and $v_X = \dim_{\mathbb{F}} V_X = \dim_{\mathbb{F}}\{aX^\times + bX + cE|\ a, b, c \in \mathbb{F}\} = \dim_{\mathbb{F}}\{cE|\ c \in \mathbb{F}\} = 1$.

(iii-2) When $X_0 \in \mathcal{M}_1(\tilde{K})$: By Proposition 0.1 (4) (i), there exists $\alpha \in G(\tilde{K})^\circ$ such that $\alpha X = \lambda_1 E + \tilde{M}_1$. By Lemma 1.1 (3) with $\tilde{M}_1^{\times 2} = \mathrm{tr}(\tilde{M}_1) = 0$, one has that $(\alpha X)^{\times 2} = \lambda_1^2 E - \lambda_1 \tilde{M}_1$, so that $v_X = \dim_{\mathbb{F}}\{a(\alpha X)^{\times 2} + b\alpha X + cE|\ a, b, c \in \mathbb{F}\} = \dim_{\mathbb{F}}\{(a\lambda_1^2 + b\lambda_1 + c)E + (b - a\lambda_1)\tilde{M}_1|\ a, b, c \in \mathbb{F}\} = 2$.

(iii-3) When $X_0 \in \mathcal{M}_{23}(\tilde{K})$: By Proposition 0.1 (4) (ii), there exists $\alpha \in G(\tilde{K})^\circ$ such that $\alpha X = \lambda_1 E + \tilde{M}_{23}$. By Lemma 1.1 (3) with $\tilde{M}_{23}^{\times 2} = \tilde{M}_1$ and $\mathrm{tr}(\tilde{M}_{23}) = 0$, one has that $(\alpha X)^{\times 2} = \lambda_1^2 E - \lambda_1 \tilde{M}_{23} + \tilde{M}_1$, so that $v_X = \dim_{\mathbb{F}}\{a(\alpha X)^{\times 2} + b\alpha X + cE|\ a, b, c \in \mathbb{F}\} = \dim_{\mathbb{F}}\{(a\lambda_1^2 + b\lambda_1 + c)E + (b - a\lambda_1)\tilde{M}_{23} + a\tilde{M}_1|\ a, b, c \in \mathbb{F}\} = 3$. □

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