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Improving the Lagrangian perturbative solution for a cosmic fluid: Applying Shanks transformation

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We study the behavior of Lagrangian perturbative solutions. For a spherical void model, the higher order the Lagrangian perturbation we consider, the worse the approximation becomes in late-time evolution. In particular, if we stop to improve until an even order is reached, the perturbative solution describes the contraction of the void. To solve this problem, we consider improving the perturbative solution using Shanks transformation, which accelerates the convergence of the sequence. After the transformation, we find that the accuracy of higher-order perturbation is recovered and the perturbative solution is refined well. Then we show that this improvement method can apply for a ΛCDM model and improved the power spectrum of the density field.

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I. INTRODUCTION

There are various structures in the universe which are gravitationally bounded; for example, galaxies, groups of galaxies, clusters of galaxies, voids, large-scale structure, and so on. These structures have evolved spontaneously from a primordial density fluctuation.

The scenario for the growth of density perturbation is analyzed by several methods. When we do not consider the superhorizon scale or an extremely dense region like a supermassive black hole, the motion of the cosmological fluid can be described by Newtonian cosmology. Further, the Lagrangian description for the cosmological fluid can be usefully applied to the structure formation scenario. This description provides a relatively accurate model even in a quasilinear regime. Zel'dovich [1] proposed a linear Lagrangian approximation for dust fluid. This approximation is called the Zel'dovich approximation (ZA) [1–9]. After that, higher-order approximation for the Lagrangian description was proposed [10–17].

How accurate is the Lagrangian perturbation? To verify its accuracy, we often use simple models to compare exact and perturbative solutions. One of the simplest models is the ''top-hat'' spherical symmetric model, which has a constant density. For this model, we have obtained an exact solution. Therefore, to estimate validity in some approximated model, we often use the top-hat model. According to recent analyses of several symmetric models [18], the spherical symmetric model has a little difficulty accurately describing evolution with the Lagrangian perturbation.

Munshi, Sahni, and Starobinsky [19] derived up to the third-order perturbative solution. In addition to these, Sahni and Shandarin [20] obtained up to the fifth-order perturbative solution. We [8] have derived up to the 11th-order solution.

If the density fluctuation is positive, the model will collapse. After a caustic formation at the center of the model, the equation of motion cannot describe the evolution. In past analyses with the Lagrangian perturbation, if we consider higher perturbative solutions, the approximation is improved all the more. On the other hand, if the density fluctuation is negative, the spherical void expands. In this case, ZA remains the best approximation to apply to the late-time evolution of voids. Especially if we stop expansion until an even order (2nd, 4th, 6th, $\cdots$) is reached, the perturbative solution describes the contraction of a void. From the viewpoint of the convergence of the series, we conjectured that the higher the order of perturbation we consider, the worse the approximation becomes in late-time evolution.

In this paper, we apply Shanks transformation, which accelerates the convergence of the sequence [21]. When we regard the perturbative solution as partial sums of infinite sequence, we must consider sequence convergence if description accuracy is to be discussed. Applying Shanks transformation, we can converge the sequence with a few terms. Therefore, we can improve the perturbative solutions. Here we consider the spherical void evolution using Shanks transformation. As a result, the higher-order perturbative solution recovers its accuracy and describes late-time evolution well. Because this method can apply not only to a spherical model but also in a more generic case, we can think that a new perturbative approach for Lagrangian description has been found. From comparison of Shanks transformation and Padé approximation, which is another improvement method for the sequence, we show several merits of Shanks transformation for the perturbative approach.

To generalize our approach, we apply Shanks transformation for a ΛCDM model. From the power spectrum of the density field, we show that Shanks transformation also recovers the accuracy of Lagrangian perturbation in a ΛCDM model.

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In Sec. II, we briefly show the evolution equation for a spherical symmetric model and derive exact solutions. Then we introduce Lagrangian perturbative solutions (Sec. III). In Sec. IV, we describe Shanks transformation, the important method we have applied. Using this transformation, we indicate the accuracy of new perturbative solutions and show that Shanks transformation improves their solutions (Sec. V). For comparison, we also compute Padé approximation and show its behavior (Sec. VI). In Sec. VII, we consider the generic case and the evolution of a ΛCDM model with N-body simulation and Lagrangian approximations. Then we compare the power spectrum of the density field between the simulation and the approximations. Finally we offer our summary and conclusion (Sec. VIII).

II. EVOLUTION EQUATION AND EXACT SOLUTIONS

We consider the "top-hat" spherical symmetric model, which has a constant density. In the E-dS Universe model, the equation of motion of a spherical shell is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2 \frac{\mathrm{d}x}{\mathrm{d}t}\right) = -\frac{2a^2 x}{9t^2}\left[\left(\frac{x_0}{x}\right)^3 - 1\right], \tag{1}$$

where x is a comoving radial coordinate and $x_0 = x(t_0)$ [19]. Under the initial condition $|\delta| = a$ for $a \to 0$, Eq. (1) can be integrated.

$$\left(\frac{\mathrm{d}R}{\mathrm{d}a}\right)^2 = a\left(\frac{1}{R} - \frac{3}{5}\right), \tag{2}$$

where $R(\theta) = a(t)x/x_0$ is a physical particle trajectory. The exact solution for the spherical collapse (Eq. (2)) can be parameterized as follows:

$$R_+(\theta) = \tfrac{3}{10}(1 - \cos\theta), \tag{3}$$

$$a(\theta) = \tfrac{3}{5}\left[\tfrac{3}{4}(\theta - \sin\theta)\right]^{2/3}. \tag{4}$$

Similarly, the exact solution for the expansion of a top-hat void (Eq. (2)) can be parameterized as follows:

$$R_-(\theta) = \tfrac{3}{10}(\cosh\theta - 1), \tag{5}$$

$$a(\theta) = \tfrac{3}{5}\left[\tfrac{3}{4}(\sinh\theta - \theta)\right]^{2/3}. \tag{6}$$

From these equations, we can obtain density fluctuation.

$$\delta(x) = (\tfrac{x_0}{x})^3 - 1, \tag{7}$$

$$\delta_+(x) = \frac{9(\theta - \sin\theta)^2}{2(1 - \cos\theta)^3} - 1, \tag{8}$$

$$\delta_-(x) = \frac{9(\theta - \sinh\theta)^2}{2(\cosh\theta - 1)^3} - 1, \tag{9}$$

TABLE I. The perturbative coefficients in Lagrangian description.

k	C_k
1	1/3
2	1/21
3	23/1701
4	1894/392931
5	3293/1702701
6	2418902/2896294401
7	55964945/147711014451
8	611605097/3430178002251
9	4529700278678/52512595036460559
10	2008868248800940/4710379774705121423
11	29117328566723/1356899523596443827

where subscript + and − denote the case of spherical collapse and of void expansion, respectively.

III. LAGRANGIAN PERTURBATION

In the Lagrangian description, the inhomogeneity of mass distribution is described by the displacement from homogeneous distribution. The Lagrangian perturbative solution for spherical symmetric models in the E-dS Universe model is given by

$$R_\pm(t) = R_0\left[1 - \sum_{k=1}^{n} (\pm 1)^k C_k a^k\right], \tag{10}$$

where C_k are Lagrangian perturbative coefficients. The sign in coefficients corresponds to positive and negative density fluctuation, respectively. Substituting Eq. (10) to (2), we derive the coefficients C_k.

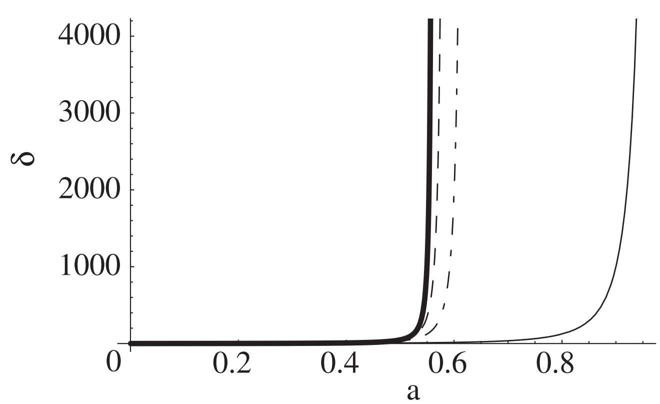

FIG. 1. The evolution of the spherical model for positive fluctuation. The thick solid line shows evolution by exact solution. The fine solid line, the dashed-dotted line, and the dashed line show evolution by first-, fifth-, and 11th-order Lagrangian perturbation, respectively. In this case, the higher-order Lagrangian approximation gives an accurate description. In this figure, the scale factor is normalized by the time of a caustic formation in a first-order perturbation.

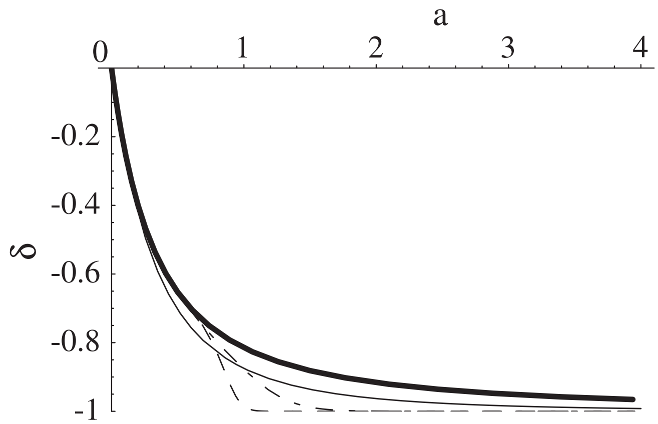

FIG. 2. The evolution of the spherical model for negative fluctuation. In this figure, the scale factor is normalized as in Fig. 1. The thick solid line shows the evolution by the exact solution. The fine solid line, the dashed-dotted line, and the dashed line shows the evolution by first-, fifth-, and 11th-order Lagrangian perturbation, respectively. In this case, the higher-order Lagrangian approximation deviates from the exact solution at a late time. In other words, ZA gives the best description for the late-time evolution of a void.

Munshi, Sahni, and Starobinsky [19] derived up to the third-order perturbative solution (C_1, C_2, C_3). In addition to these, Sahni and Shandarin [20] obtained C_4 and C_5. Furthermore we [8] derived $C_6, \cdots, C_{11}$. The coefficients C_k are shown in Table I.

The Lagrangian perturbation causes a serious problem. If the density fluctuation is positive, the spherical fluctuation collapses. The higher-order Lagrangian approximation gives accurate description (Fig. 1). On the other hand, if the density fluctuation is negative, the spherical void expands. In this case, ZA remains the best approximation to apply to the late-time evolution of voids. Especially, if we stop to improve until an even order (2nd, 4th, 6th, $\cdots$), the perturbative solution describes the contraction of a void. In other words, ZA gives the best description for the late-time evolution of voids (Fig. 2).

IV. SHANKS TRANSFORMATION

The Lagrangian perturbation causes a serious problem at late-time evolution for the spherical void model. How do we improve the description for spherical void evolution? From the viewpoint of series, the higher order the perturbation we consider, the narrower the radius of the convergence becomes. How do we improve the convergence of the perturbation?

For the contradiction in the Lagrangian approximation, we consider to improve the convergence rate of a sequence of partial sums. As a good way to speed up the convergence of a slowly converging series, Shanks transformation had been proposed [21]. First we consider a simple example. Suppose the n-th term in the sequence takes this form:

$$A_n = A + \alpha q^n (|q| < 1). \tag{11}$$

The sequence converges $A_n \to A$ as $n \to \infty$. To obtain the limit of a sequence A, we solve algebraic equations with A_{n-1}, A_n, and A_{n+1}.

$$A = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}. \tag{12}$$

This formula is exact only if the sequence A_n is described by the form in (11). For the generic case, we consider the nth term in the sequence takes the form:

$$A_n = A(n) + \alpha q^n, \tag{13}$$

where for large n, $A(n)$ is a more slowly varying function of n than A_n. Let us suppose that $A(n)$ varies sufficiently slowly so that $A(n-1)$, $A(n)$, and $A(n+1)$ are all approximately equal. Then the above discussion motivates the nonlinear transformation

$$S(A_n) = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}. \tag{14}$$

This transformation is called Shanks transformation, creating a new sequence $S(A_n)$ which often converges more rapidly than the old sequence A_n. The sequence $S^2(A_n) = S[S(A_n)]$ and $S^3(A_n) = S[S[S(A_n)]]$ may be even more rapidly convergent.

Damour, Jaranowski, and Schäfer applied Shanks transformation to the post-Newtonian approximation of general relativity [22]. They improved the analytical determination of various last stable orbits in circular general relativistic orbits of two point masses. In the next section, we apply the transformation to improve of the Lagrangian perturbation.

V. IMPROVEMENT OF THE LAGRANGIAN PERTURBATIVE SOLUTION

Here we apply Shanks transformation for the Lagrangian perturbation. We consider the spherical void case and adopt the 11th-order solution [8]. From the Lagrangian perturbation (Eq. (10)), we obtain a new solution via Shanks transformation.

$$\tilde{R}_n = \frac{R_{n+1}R_{n-1} - R_n^2}{R_{n+1} + R_{n-1} - 2R_n}, \tag{15}$$

$$R_n \equiv \left[1 + \sum_{k=1}^{n} (-1)^{k+1} C_k a^k\right]. \tag{16}$$

From the new sequence or perturbative solution $\tilde{R}_n$, we can derive a more refined solution. Figure 3 shows the evolution of the spherical void using exact and the Lagrangian perturbative solutions. We apply Shanks transformation once, twice, and 3 times for the perturbation. After transformation, the solution is refined and recovered its accuracy. In ordinary Lagrangian perturbation, we cannot improve the perturbative solution for late-time evolution.

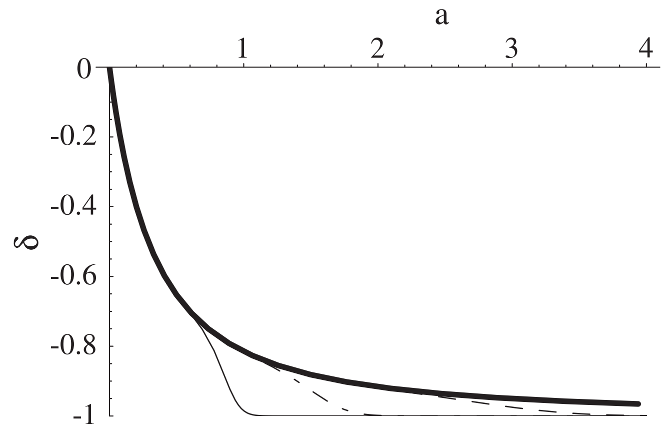

FIG. 3. The evolution of the spherical model for negative fluctuation. In this figure, the scale factor is normalized as in Fig. 1. We apply Shanks transformation to the Lagrangian perturbative solution and improve its accuracy. The thick solid line shows the evolution by the exact solution. The fine solid line shows the behavior of original 11th-order perturbative solution. The dashed-dotted line, and the dashed line shows the evolution by once transformed and twice transformed perturbative solutions, respectively. After 3 times transformation, because the difference between exact solution and perturbative solution becomes quite small. Therefore, if we also draw the curve of 3 times transformed perturbative solutions, we cannot find the difference on the figure. In this case, by application of Shanks transformation, the Lagrangian approximation improves its accuracy. In other words, we can describes late-time evolution of voids well.

Using Shanks transformation, we can obtain a well-refined perturbative solution.

We have shown one of the simplest cases. This improvement method is not limited to a special case. It can be applied in generic cases. Suppose the third-order perturbative solution takes this form:

$$S = D_+(t)S^{(1)} + D_+(t)^2 S^{(2)} + D_+(t)^3 S^{(3)}, \tag{17}$$

where D_+ is a linear growing factor. In the E-dS model, when we consider only the primordial growing mode in the longitudinal mode, the perturbative solution can be described by this form exactly [12–14].

$$\nabla^2 S^{(2)} = -\tfrac{3}{7}[(\nabla^2 S^{(1)})^2 - S^{(1)}_{,ij} S^{(1)}_{,ji}], \tag{18}$$

$$\nabla^2 S^{(3)} = -\tfrac{1}{3}\det(S^{(1)}_{,ij}) + \tfrac{10}{21}[\nabla^2 S^{(1)} \nabla^2 S^{(2)} - S^{(1)}_{,jk} S^{(2)}_{,kj}], \tag{19}$$

where ∇ and subscript denote the Lagrangian spacial derivative. For other universe models, the perturbative solution can be approximated by this form in the matter-dominant era. Applying Shanks transformation (14), the perturbative solution is transformed to

$$\tilde{S} = D_+ S^{(1)} + \frac{D_+^2 (S^{(2)})^2}{D_+ S^{(3)} - S^{(2)}}. \tag{20}$$

In Sec. VII, we will treat a generic case with Shanks transformation.

VI. COMPARISON WITH PADÉ APPROXIMATION

For a convergence of series, there are other methods. One of these methods is known as Padé approximation [21]. Padé approximation seems to be a generalization of Taylor expansion. For a given function $f(t)$, Padé approximation is written as the ratio of two polynomials

$$f(t) \simeq \frac{\sum_{k=0}^{M} \alpha_k t^k}{1 + \sum_{k=1}^{N} \beta_k t^k}, \tag{21}$$

where α_k and β_k are constant coefficients. Assume we already know the coefficient γ_l $(0 \le l \le M + N)$ of the Taylor expansion around $x = 0$. Then,

$$f(t) = \sum_{l=0}^{M+N} \gamma_l t^l + o(t^{M+N+1}). \tag{22}$$

Comparing the coefficients α_k, β_k, and γ_k, we determine α_k and β_k.

$$\alpha_0 = \gamma_0, \tag{23}$$

$$\alpha_k = \sum_{m=1}^{N} \beta_m \gamma_{k-m} \qquad (k = 1, \cdots, N), \tag{24}$$

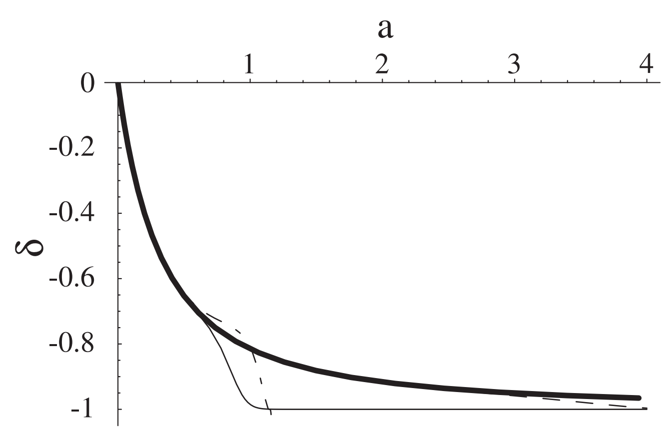

FIG. 4. The evolution of the spherical model for negative fluctuation. In this figure, the scale factor is normalized as in Fig. 1. We apply Padé approximation to the Lagrangian perturbative solution and improve its accuracy. The thick solid line shows the evolution by the exact solution. The fine solid line shows the behavior of original 11th-order perturbative solution. The dashed-dotted line, and the dashed line shows the evolution by the case of $(M, N) = (1, 10)$ and (3,8), respectively. When N is greatly different from M, the approximation is not improved well. When we choose $(M, N) = (5, 6)$, the difference between exact solution and perturbative solution becomes quite small. Using Padé approximation, we can improve the perturbative solution well.

$$\sum_{m=1}^{N} \beta_m \gamma_{N-m+k} = -\gamma_{N+k} \qquad (k = 1, \cdots, N). \tag{25}$$

The advantage of Padé approximation is that even if we consider a same-order expansion, Padé approximation describes original function rather better than Taylor expansion does.

Yoshisato, Matsubara, and Morikawa [23] have proposed an application of Padé approximation for Eulerian perturbative solutions. Furthermore, Matsubara, Yoshisato, and Morikawa [24] have applied Padé approximation for the Lagrangian description. They also showed that Padé approximation can improve the Lagrangian perturbative solution.

Here we apply Padé approximation to the spherical void. In Padé approximation, it is quite important to choose the numbers of terms M and N. When N is greatly different from M, the approximation is not improved well. Figure 4 shows the evolution of the spherical void using exact and the Lagrangian perturbative solutions. We apply Padé approximation with several cases. Here we show the case of $(M, N) = (1, 10)$, (3,8) and (5,6). It is important to choose the parameter M and N. When N is greatly different from M, not only the approximation is not improved well, but also the solution will diverges. For late-time evolution, although the original 11-order perturbative solution converges to $\delta \rightarrow -1$, the Padé approximation with $(M, N) = (1, 10)$, (3,8) diverges. On the other hand, when we choose $(M, N) = (5, 6)$, the perturbative solution can approximate the exact solution at late time. We can improve the perturbative solution with Padé approximation, too.

VII. GENERIC CASE: THE ΛCDM MODEL

We showed the evolution of a homogeneous spherical void and noted the improvement for Lagrangian approximation. Because we know the exact solutions for the spherical collapse and void evolution, we do not know whether or not the Shanks transformation is useful for the generic case. Therefore we must apply the transformation for generic models.

Here we consider a ΛCDM (Low density Cold Dark Matter) model. The cosmological parameter at the present time ($z = 0$, here we define $a \equiv 1$ at the present time) is given by a WMAP 3rd-year result [25]

$$\Omega_m = 0.28, \tag{26}$$

$$\Omega_{\rm DE} = 0.72, \tag{27}$$

$$H_0 = 74\ [\text{km/s/Mpc}], \tag{28}$$

$$\sigma_8 = 0.74. \tag{29}$$

The Gaussian density field is generated by COSMICS [26]. We set up the initial condition at decoupling time ($a = 10^{-3}$). The initial peculiar velocity and the density fluctuation are adjusted by the growing solution of ZA.

For time evolution, we consider Lagrangian third-order approximation, Shanks transformation, and N-body simulation. For computation of the Lagrangian perturbations, we set the parameters as follows:

$$\text{Number of grids: } N = 128^3,$$
$$\text{Box size: } L = 128h^{-1}\ \text{Mpc} \qquad (\text{at } a = 1).$$

The N-body simulation is applied by a particle-particle particle-mesh (P^3M) method [27] whose code was originally written by Bertschinger.

For N-body simulations, we set the parameters as follows:

$$\text{Number of particles: } N = 128^3,$$
$$\text{Box size: } L = 128h^{-1}\ \text{Mpc} \qquad (\text{at } a = 1),$$
$$\text{Softening length: } \varepsilon = 50h^{-1}\ \text{kpc}(\text{at } a = 1).$$

Then we impose a periodic boundary condition.

The Lagrangian approximation in ΛCDM is expanded as

$$S = h_1(t)S^{(1)} + h_2(t)S^{(2)} + h_3(t)S^{(3)}, \tag{30}$$

where $h_n(t)$ is the growing factor for nth-order approximation. The spacial parts are given by Eqs. (18) and (19). The growing factors are derived with a numerical method [13]. Applying Shanks transformation (Eq. (14)), the perturbative solution is transformed to

$$\tilde{S} = h_1 S^{(1)} + \frac{h_2^2 (S^{(2)})^2}{h_3 S^{(3)} - h_2 S^{(2)}}. \tag{31}$$

In order to avoid the divergence of the density fluctuation, we need to consider a smoothed density field over the scale R. This density field is related to the unsmoothed density field $\rho(\boldsymbol{x})$ as

$$\rho(\boldsymbol{x}; R) = \int d^3\boldsymbol{y} W(|\boldsymbol{x} - \boldsymbol{y}|; R)\rho(\boldsymbol{y}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \tilde{W}(kR)\tilde{\rho}(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{32}$$

where W denotes the window function and $\tilde{W}$ and $\tilde{\rho}$ represent the Fourier transforms of the corresponding quantities. In this paper, we adopt the top-hat window function,

$$\tilde{W} = \frac{3(\sin x - x\cos x)}{x^3}. \tag{33}$$

Then, the density fluctuation $\delta(\boldsymbol{x}; R)$ at the position $\boldsymbol{x}$ smoothed over the scale R can be constructed in terms of $\rho(\boldsymbol{x}; R)$.

Here we choose the smoothing scale $R = 1h^{-1}$ Mpc. Then we calculate the power spectrum of density fields. In

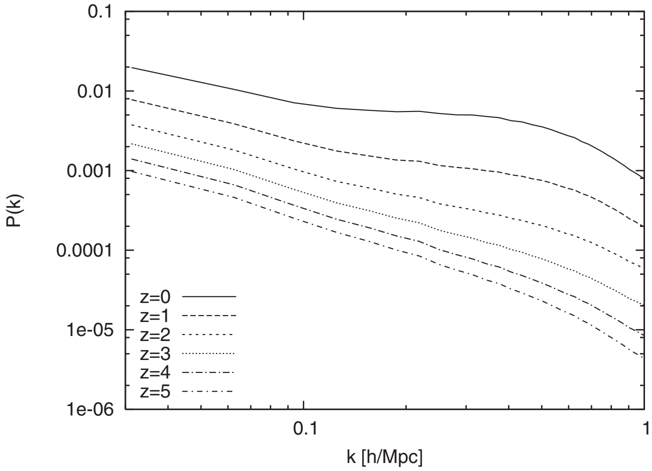

FIG. 5. The power spectrum of the density field in N-body simulation. Because of a nonlinear effect in small structures, the large-k components in the power spectrum grow remarkably.

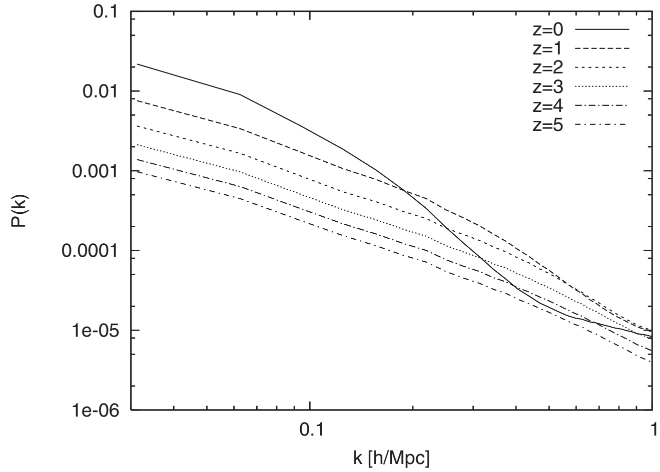

FIG. 7. The power spectrum of the density field in Shanks transformation. Because of the improvement of the nonlinear effect, the distortion of the spectrum is improved well.

order to obtain the power spectrum, we generate 50 samples for the primordial density fluctuations. Then we pick up snapshots at $z = 5, 4, 3, 2, 1, 0$.

Figure 5 shows the power spectrum of the density field in N-body simulation. During evolution, because a strongly nonlinear region promotes the growth of the fluctuation, the large-k components in the power spectrum grows remarkably.

On the other hand, the power spectrum in the Lagrangian third-order spectrum sinks (Fig. 6). At the high-z era, shell-crossing occurs in small structures. After shell-crossing, the cluster it forms spreads eternally. Then the negative influence of shell-crossing affects the large-scale structure. Therefore the spectrum sinks from the small scale gradually. At $z = 0$, the spectrum differs greatly from that in N-body simulation.

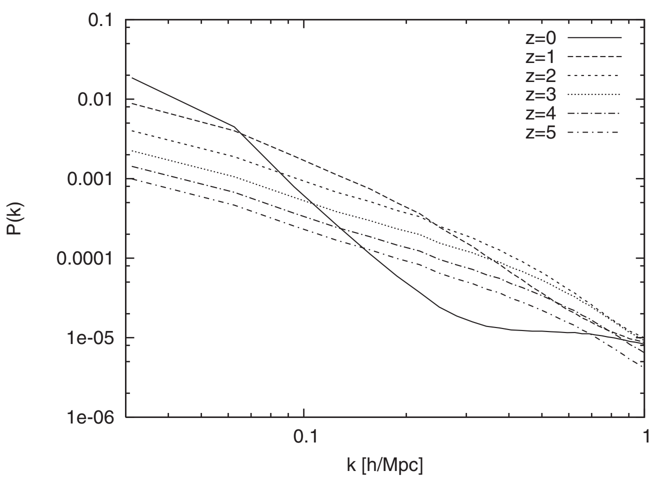

FIG. 6. The power spectrum of the density field in Lagrangian third-order approximation. Because of the shell-crossing, the spectrum distorted from the large-k components to the small-k components gradually. At $z = 0$, the spectrum is really differs greatly that in N-body simulation.

When we apply Shanks transformation, although a strongly nonlinear effect in the large-k components is not realized, the distortion of the spectrum is well improved (Fig. 7). Even if we consider a low-z era, Shanks transformation can realize the spectrum, except for the small scale, well.

VIII. SUMMARY

We have discussed the evolution of the spherical void in the framework of the Lagrangian perturbation. Using ordinary Lagrangian perturbation, the higher-order Lagrangian approximation deviates from the exact solution at late time. In other words, ZA gives the best description for the late-time evolution of voids. Then we generalized Shanks transformation for the Lagrangian perturbation, i.e., we analyzed time evolution for a ΛCDM model.

We apply Shanks transformation, which accelerates the convergence of series for the Lagrangian perturbation. Although the transformation in valid within a convergent radius of the series, the transformation creates a new sequence which often converges more rapidly than the old sequence. In the spherical void model, the transformation is valid for a long time. Then we can improve the accuracy of the Lagrangian description. Using this method, we can solve the problem whereby the higher order the perturbation we consider, the worse the approximation becomes in late-time evolution.

In this paper, we also compare the accuracy of our improved methods, between Shanks transformation and Padé approximation, that is. In the comparison, we found that Shanks transformation has several merits: both Shanks transformation and Padé approximation can be derived with algebraic procedures. In Padé approximation, although we can solve algebraic equations and find unique solutions, the equation is quite complicated. Furthermore, according to our analyses, if the difference between two parameters M and N in Padé approximation is large, the perturbative solution will diverge. On the other hand, Shanks transformation does not diverge.

However, in several points, Shanks transformation shows its weakness. To apply Padé approximation, we need to obtain second-order perturbation. On the other hand, when we consider the Shanks transformation, we must obtain at least third-order perturbation. Using the Shanks transformation, we obtain a new perturbative solution $\tilde{R}_n$ from R_{n-1}, R_n, R_{n+1}. Then, to repeatedly apply Shanks transformation, we require $R_{n-2}, \cdots, R_{n+2}$. In general, when we apply n times transformation, at least we must know $2n + 1$th-order perturbative solutions. To improve the perturbation well, we must repeat the transformation several times.

From the viewpoint of algebraic procedures, Shanks transformation has an advantage. In Padé approximation, we must solve nonlinear simultaneous equations. Then the solution is extremely complicated in a higher-order case. For example, when we improve an 11th-order solution with $(M, N) = (5, 6)$, we derived about 50-digit coefficients.

For a ΛCDM model, Shanks transformation recovers accuracy for the description of the density field, too. We showed that the power spectrum in Shanks transformation becomes better than that of ordinary Lagrangian approximations. Even if we consider the spectrum at $z = 0$, the Shanks transformation can describe quite well, except for the small scale.

However, in a generic case, the critical problem in Shanks transformation appeared. When we continue to apply Shanks transformation, the divergence of the perturbation occurs. For example, when we apply the transformation with third-order approximation, the perturbation is written as Eq. (31). If the higher-order perturbation is smaller than the lower-order perturbation, we consider that the perturbative method is valid. From Eq. (31), when second-order and third-order perturbation become

$$h_2 S^{(2)} \simeq h_3 S^{(3)}, \tag{34}$$

the perturbation diverges. In other words, because the Lagrangian perturbation is excluded from the convergence series during evolution, the perturbative method becomes invalid. Therefore when we apply Shanks transformation, we notice the validity of the perturbative expansion. In the spherical collapse case, the density fluctuation diverges before divergence of the Lagrangian perturbation. After the shell-crossing, the divergence of the perturbation is settled. Then the perturbation describes the displacement of the fluid.

Even if we improve the Lagrangian perturbation, we cannot avoid the problem of shell-crossing. Because of shell-crossing, the approximation becomes worse at late time. To solve this problem, Scoccimarro and Sheth proposed an extrapolation method [28]. First, they considered structure formation with second-order Lagrangian approximation. After that, they extrapolated the density distribution in the high-density region from NFW profile. This hybrid method realizes the density distribution well. Although it is very complicated, this method is effective for higher-order perturbation. We can then expect that this hybrid method will realize the density distribution with high precision by combining it with third-order Lagrangian approximation or Shanks transformation.

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