Note on Chords of Regions Mapped by Multivalent Functions.

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Let $f(z)$ ($f(0)=0, f'(0)=1$) be any function regular and univalent for $|z|<1$.

G. Szegö has proved that, for the length l of any Hauptsehne of the region into which the unit circle is mapped by $f(z)$,

$$l \geqq 1.$$

And that, if $f(z)$ is bounded ($|f(z)|<M$), this inequality can be replaced by

$$l \geqq 2M\,(M - \sqrt{M^2-1})$$

has been given by the auther [1] [2]。

In this paper we shall generalize these theorems, in slightly different forms, to the case of multivalent functions of order p which are regular or meromorphic for $|z|<1$.

We denote by $\mathfrak{R}$ or $\mathfrak{M}$ respectively the class of p-valent functions which are regular or meromorphic for $|z|<1$ and of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

and by $\mathfrak{R}_M$ the subclass of $f(z)$ such that functions are bounded, *i.e.*

$$|f(z)| < M \ (M \geqq 1).$$

Then the following lemma given by M. Biernacki is a generalization of Koebe's theorem for univalent functions [3] [4].

Lemma. *Let $f(z)$ be any function $\in \mathfrak{R}$, then $w=f(z)$ takes every value in $|w|<1/4$, and $1/4$ cannot be replaced by any greater number.*

Now we suppose that $f(z) \in \mathfrak{M}$ and γ_1, γ_2 are two boundary points of the region mapped by $f(z)$ such that

$$\arg \gamma_2 = \arg\gamma_1 + \theta \qquad (\,0 < \theta < 2\pi\,).$$

Then the function

$$\phi(z) = \frac{\gamma_1 f(z)}{\gamma_1 - f(z)} \tag{1}$$

is regular, p—valent and can be expanded as follows:

$$\phi(z) = z^p + \cdots\cdots\cdots.$$

Therefore $\phi(z) \in \mathfrak{R}$ and

$$\gamma_1\gamma_2/(\gamma_1 - \gamma_2)$$

is a boundary point of the region mapped by $\phi(z)$.

Hence we have by the lemma given above,

(2) $$\left|\frac{\gamma_1 \gamma_2}{\gamma_1 - \gamma_2}\right| \geqq 1/4.$$

On the other hand we have

(3) $$|\gamma_1 - \gamma_2| \geqq l sin\theta/2$$

(4) $$|\gamma_1 \gamma_2| \leqq l^2/4$$

where $l = |\gamma_1| + |\gamma_2|$.

Consequently we get by (2),

(5) $$l \geqq sin\theta/2$$

and by (3),

(6) $$|\gamma_1 - \gamma_2| \geqq sin^2 \theta/2.$$

The equality signs in (5) and (6) hold only when $|\gamma_1| = |\gamma_2|$ and $w = \phi(z)$ has a boundary point on $|w| = 1/4$.

We define $f(z)$ by

(7) $$f(z) = \frac{z^p}{1 - 2icot\theta/2 \cdot z^p + z^{2p}} = z^p + \cdots\cdots,$$

then $f(z)$ is regular, p—valent for $|z| < 1$ with the exception of poles at $z = \sqrt[p]{-tan\theta/4}$ $(0<\theta<\pi)$ or at $z = \sqrt[p]{cot\theta/4}$ $(\pi<\theta<2\pi)$, and consequently $f(z) \in \mathfrak{M}$. Furthermore $w = f(z)$ mapps $|z| < 1$ into the full w—plane that is furnished with a circular slit, and, for the two end points

$$\gamma_1 = \frac{i}{2} sin \frac{\theta}{2} e^{-i\theta/2}, \quad \gamma_2 = \frac{i}{2} sin \frac{\theta}{2} e^{i\theta/2}$$

of this slit, the equality signs in (5) and (6) can be attained.

Thus we can establish the following theorem.

Theorem 1. *Let $f(z)$ be any function $\in \mathfrak{M}$ and $\overline{PQ}$ be any chord of the mapped region of the unit circle by $f(z)$ such that*

$$P\hat{O}Q = \theta \ (0<\theta<2\pi, \ \theta \neq \pi, \ O: Origin),$$

then $$\overline{PQ} \geqq Sin^2\theta/2.$$

And this lower bound for $\overline{PQ}$ is the best possible one.

〔N. B.〕 The above arguments hold also true for $\theta = \pi$, but the extremal function (7) belongs to $\mathfrak{R}$ in this case and hence Therom 1 is reduced to a generalization of the theorem for univalent functions due to G. Szegö to multivalent functions $\in \mathfrak{R}$.

Next we suppose that $f(z)$ be any function $\in \mathfrak{R}_M$. Then we can prove the following theorem analogously by using

$$\phi(z) = \frac{M^2\gamma_1 f(z)}{[\gamma_1 - f(z)] \ [M^2 + |\gamma_1 \gamma_2| / \gamma_2 \cdot f(z)]} \in \mathfrak{R}$$

instead of (1).

Theorem 2. *Let $f(z)$ be any function $\in \mathfrak{R}_M$ and $\overline{PQ}$ be any chord of the mapped region of the unit circle by $f(z)$ such that*

$$P\hat{O}Q = \theta \ (0<\theta<2\pi, \ O: Origin),$$

then

$$\overline{PQ} \geqq 2M\,(M - \sqrt{M^2 - sin^2\theta/2})$$

$$\overline{OP} + \overline{OQ} \geqq 2M\,(M - \sqrt{M^2 - sin^2\theta/2}\,)/sin\theta/2.$$

〔N. B.〕 Let $f(z) \in \mathfrak{K}_M$, then it is known that, for any boundary point r of $f(z)$,

$$|r| \geqq M\,(2M - 1 - 2\sqrt{M^2 - M}\,).$$

Therefore, for

$$2M\,(2M - 1 - 2\sqrt{M^2 - M}\,) \geqq 2M\,(M - \sqrt{M^2 - sin^2\theta/2})/sin\theta/2$$

i.e.

$$M/(2M - 1) \geqq sin\theta/2,$$

the above theorom becomes trivial.

Finally putting $\theta = \pi$ in Theorem 2 gives us the relation

(8) $$\overline{PQ} \geqq 2M\,(M - \sqrt{M^2 - 1}\,).$$

It can be proved as follows that this inequality is sharp. We define $f(z) \in \mathfrak{K}_M$ by

(9) $$\frac{M^2\rho f(z)}{[M\rho + f(z)]\,[M + \rho f(z)]} = \frac{z^p}{(1+z^p)^2}$$

where $\rho = M - \sqrt{M^2 - 1}$, then the two boundary points γ_1, γ_2 of the mapped region which correspond to $z^p = 1$ and $z^p = -1$ are given by

$$\gamma_1 = M\rho, \qquad \gamma_2 = -M\rho.$$

Hence we obtain

$$arg\,\gamma_2 = arg\gamma_1 + \pi, \quad |\gamma_1 - \gamma_2| = 2M\rho.$$

So that we have the conclusion as follows.

Theorem 3. *Let $f(z)$ be any function $\in \mathfrak{K}_M$, and $\overline{PQ}$ be any Hauptsehne of the mapped region of the unit circle by $f(z)$, then*

$$\overline{PQ} \geqq 2M\,(M - \sqrt{M^2 - 1}\,)$$

and there exists the function $\in \mathfrak{K}_M$ for which the equality sign holds in the above expression.

This is a generalization of the theorem for univalent functions to the case of multivalent functions.

References.

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〔2〕 Y. Sasaki, *On the Hauptsehne of the Region to which the Unit-circle is Mapped by the Bounded Function.* Proc. Japan Acad. Vol. 27 (1951), No. 5.

〔3〕 M. Biernacki. *Sur les domains curverts par des fonctions multivalentes,* Bull. Sci. Math., (2) 70. (1946).

〔4〕 S. Ozaki and T. Yosida, *On some property of multivalent functions,* Sci. Rep. T. B. D. (A) No. 4 (1949).