On some Family of Multivalent Functions.

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§ 1. Family E_p of regular functions.

Let the function

$$(1.1) \qquad F(z)=\sum_{n=p}^{\infty} a_n z^n. \qquad (a_p=1)$$

be regular in the unit-circle, where p is any positive integer. [1]
We denote by S_p or K_p respectively the family of functions F(z) by which the unit-circle is mapped into a star-like region with respect to the origin or a convex region, then the following theorems are well known. [2]

Theorem 1.

The necessary and sufficient condition that F(z) should belong to the family S_p is

$$R\left[z\frac{F'(z)}{F(z)}\right]>0 \qquad (|z|<1).$$

Theorem 2.

The necessary and sufficient condition that F(z) should belong to the family K_p is

$$1+R\left[z\frac{F''(z)}{F'(z)}\right] - \frac{p-1}{p} R\left[z\frac{F'(z)}{F(z)}\right]>0 \qquad (|z|<1).$$

Now we denote by E_p the family of functions F(z) having the following two properties,

1° The mapped region of $|z|<1$ by F(z) is p-valent,

2° At any point of the mapped curve of $|z|=r$, where r is any positive number less than 1, by F(z), the curvature is positive and finite determinate.

And we call F(z) a quasi-convex function. We can understand, by this definition, that a quasi-convex function F(z) is convex on Riemann surface of F(z). Then we have

Theorem 3.

The necessary and sufficient condition that F(z) should belong to the family E_p is

$$1+R\left[z\frac{F''(z)}{F'(z)}\right]>0 \qquad (|z|<1).$$

Proof: If

$$1+R\left[z\frac{F''(z)}{F'(z)}\right] = R\left[z\frac{(zF'(z))'}{zF'(z)}\right]>0,$$

then

$$zF'(z) = pz^p + (p+1)\, a_{p+1}z^{p+1} + \cdots\cdots$$

and we have

$$\left[\frac{F'(z)}{z^{p-1}}\right]_{z=0} = P \neq 0.$$

Thefore $\frac{F'(z)}{z^{p-1}}$ does not vanish in $|z|<1$ and $F'(z)\neq o$ in $o<|z|<1$. [3]

Now we denote by ρ the curvature at any point on the mapped curve C of $|z|=r$ $(0<r<1)$, then

$$\rho=\frac{1}{|zF'(z)|}R\left[1+z\frac{F''(z)}{F'(z)}\right]>0,$$

which is the property 2°.

As $F'(z)\neq o$ on $|z|=r$, C is a regular curve and the angle from the real-axis to the tangent line at any pnint on C is given by $\arg izF'(z)$. So that we have, as z describes on $|z|=r$,

$$\int d\arg izF'(z)=\int d\arg Z^p+\int d\arg\frac{F'(z)}{Z^{p-1}}=\int d\arg Z^p=2p\pi,$$

where $\frac{F'(z)}{Z^{p-1}}$ does not vanish, as cited above.

Therefore the mapped curve C is closed and p-valent, and, being r arbitrary, the mapped region of $|z|<1$ is p-valent which proves the property 1°.

Conversely, if we have properties 1°, 2°, then the curvature

$$\rho=\frac{1}{|zF'(z)|}R\left[1+z\frac{F''(z)}{F'(z)}\right]$$

at any point z on $|z|=r$ $(o<r<1)$ is positive, and, being r arbitrary, we have

$$1+R\left[z\frac{F''(z)}{F'(z)}\right]>o \qquad (|z|<1).$$

Thus our theorem is completly proved.

§ 2. Relations among S_p, K_p, and E_p,

If F(z) belongs to any one of Sp, Kp and Ep, then F(z) is p-valent in $|z|<1$, and consequently F(z) does not vanish in $o<|z|<1$. Hence there exists the regular function h(z) such that

$$h(z)=z\sqrt[p]{\frac{F(z)}{z^p}}\,,\qquad F(z)=[h(z)]^p,\quad h(o)=o,\quad h'(o)=1\quad(|z|<1),$$

and we have, for h(z), the relations

$$(2.1)\quad\begin{cases}R\left[z\frac{F'(z)}{F(z)}\right]=pR\left[z\frac{h'(z)}{h(z)}\right]\\ 1+R\left[z\frac{F''(z)}{F'(z)}\right]=1+R\left[z\frac{h''(z)}{h'(z)}\right]+(p-1)R\left[z\frac{h'(z)}{h(z)}\right]\\ 1+R\left[z\frac{F''(z)}{F'(z)}\right]-\frac{p-1}{p}R\left[z\frac{F'(z)}{F(z)}\right]=1+R\left[z\frac{h''(z)}{h'(z)}\right].\end{cases}$$

We get immediately, from these relatins, the following

Theorem 4.

Suppose that $F(z)\in S_P$, then $h(z)\in S_1$ and suppose that $F(z)\in K_P$, then $h(z)\in K_1$.

If we assume that $F(z)\in K_p$ and therefore $h(z)\in K_1$, then

$$R\left[z\frac{h'(z)}{h(z)}\right]>\frac{1}{2}$$

by the theorem due to M. Strohäcker [4], and, from (2,1),

$$1+R\left[z\frac{F''(z)}{F'(z)}\right]>\frac{1}{2}(p-1)\geqq o, \qquad R\left[z\frac{F'(z)}{F(z)}\right]>\frac{p}{2}.$$

This proves the following

Theorem 5.

Suppose that $F(z)\in K_p$, then $F(z)\in E_p$ and $F(z)\in S_p$.

Next we assume that $E(z)\in E_p$ and denote by φ the angle from the real-axis to the tangent line at $F(z)$, where $Z=re^{i\theta}$, on the mapped curve of $|z|=r$, then

$$\frac{d\varphi}{d\theta}=\frac{d}{d\theta}\arg izF'(z)=1+R\left[z\frac{F''(z)}{F'(z)}\right]>o.$$

So that the tangent line rotates such that φ increases as Z describes in positive sense on $|z|=r$. And, as the curvature ρ is positive, the radius vector $F(z)$ rotates in positive sense. Hence we have

$$R\left[z\frac{F'(z)}{F(z)}\right]=\frac{d}{d\theta}\arg F(z)>o.$$

which reduces to the following theorem.

Theorem 6.

Suppose that $F(z)\in E_P$, then $F(z)\in S_P$.

We can represent the theorem 5 and 6 symbolically by

$$K_p\subset E_p\subset S_p,$$

and, especially for the case $p=1$,

$$K_1\equiv E_1\subset S_1$$

by (2.1).

Lastly we have the theorem as follows,from the relation

$$R\left[z\frac{(zF'(z))'}{zF'(z)}\right]=1+R\left[z\frac{F''(z)}{F'(z)}\right]$$

Theorem 7.

If $F(z)\in E_p$, then $\frac{1}{p}zF'(z)\in S_P$ and if $F(z)\in S_P$, then

$p\int\frac{F(z)}{z}dz\in E_P$.

§ 3. The circle of quasi-convexity for thə functions $\in S_p$.

Suppose that $F(z)\in S_p$, then

$$R\left[z\frac{F'(z)}{F(z)}\right]>o,$$

and

$$\left[z\frac{F'(z)}{F(z)}\right]_{z=o}=p.$$

Therefore,

$$p\frac{1-r}{1+r}\leqq R\left[z\frac{F'(z)}{F(z)}\right]\leqq p\frac{1+r}{1-r} \qquad (|z|\leqq r),$$

ahd we have, by the well known theorem of G. Julia,

$$\left| z\frac{F''(z)}{F'(z)} - z\frac{F'(z)}{F(z)} + 1 \right| \leqq 2\frac{|z|}{1-|z|^2}\frac{R\left[z\frac{F'(z)}{F(z)}\right]}{\left|z\frac{F'(z)}{F(z)}\right|}.$$

Combining these inequalities, we get

$$1 + R\left[z\frac{F''(z)}{F'(z)}\right] \geqq \frac{pr^2-2(p+1)r+p}{1-r^2} \qquad (|z|\leqq r),$$

and hence, for $|z|< \sigma_p = \frac{1}{p}(p+1-\sqrt{2p+1})$,

$$1 + R\left[z\frac{F''(z)}{F'(z)}\right] > o,$$

and that, for the function

$$F(z) = \frac{z^p}{(1-z)^{2p}} \in S_p,$$

we have

$$1 + R\left[z\frac{F''(z)}{F'(z)}\right] = o \qquad (Z=-\sigma_p)$$

Thus we conclude the following

Theorem 8.

Suppose that $F(z) \in S_p$, then $F(z) \in E_p$ for $|z|<\sigma_p = \frac{1}{p}(p+1-\sqrt{2p+1})$. And σ_p is the greatest number for quasi-convexity.

§ 4. Distortion theorem and Coefficient problem for E_p.

For the function $F(z) \in E_p$, we have, by the theorem 7,

$$\frac{1}{p}zF'(z) \in S_p,$$

and consequently,

$$\frac{|z|^p}{(1+|z|)^{2p}} \leqq \frac{1}{p}\left|zF'(z)\right| \leqq \frac{|z|^p}{(1-|z|)^{2p}}.$$

From this relation, we can prove the following theorem by the analogous method for the case $p=1$.

Theorem 9.

Let $F(z) = \sum_{n=P}^{\infty} a_n z^n$ $(a_p=1) \in E_p$, then

$$\frac{p|z|^{P-1}}{(1+|z|)^{2P}} \leqq |F'(z)| \leqq \frac{p|z|^{P-1}}{(1-|z|)^{2P}}$$

$$p\int_0^{|z|}\frac{Z^{p-1}}{(1+Z)^{2p}}dZ \leqq |F(z)| \leqq p\int_0^{|z|}\frac{Z^{p-1}}{(1-Z)^{2p}}dZ.$$

And the equality signs are true for

$$F(z) = p\int_0^z\frac{Z^{p-1}}{(1-z)^{2p}}\,dZ \in E_p.$$

We have already known, for $F(z) \in S_p$, that

$$|a_{p+K}| \leqq \frac{2p(2p+1)\cdots(2p+k-1)}{k!}, \quad K = 1, 2, \cdots\cdots$$

and the equality holds for $F(z) = \frac{z^p}{(1-z)^{2p}}$. [5]

We can represent this theorem, by Majorant symbol, as follows,

$$F(z) \ll \frac{z^p}{(1-z)^{2p}}.$$

Now, if $F(z) \in E^q$, then $\frac{1}{p} ZF'(z) \in S_q$ and therefore

$$\frac{1}{p} z F'(z) \ll \frac{z^p}{(1-z)^{2p}}.$$

From this relation, we have

$$F(z) \ll p\int_0^Z \frac{z^{p-1}}{(1-z)^p}\, dz$$

i, e,

$$F(z) \ll p\int_0^Z \left[z^{p-1} + \sum_{K=1}^{\infty} \frac{2p(2p+1)\cdots(2p+k-1)}{k!} z^{p+K-1} \right] dz$$

$$= z^p + \sum_{K=1}^{\infty} \frac{2p(2p+1)\cdots(2p+k-1)}{k!\,(p+k)} z^{p+K}.$$

Consequently the following result is obtained.

Theorem 10.

Let $F(z) = \sum_{n=p}^{\infty} a_n z^n \ (a_p=1) \in E_p$, *then*

$$|a_{p+k}| \leqq \frac{2p(2p+1)\cdots(2p+k-1)}{k!\,(p+k)}, \quad (k=1,2,\cdots)$$

and equality holds true for the function

$$F(z) = p\int_0^z \frac{Z^{p-1}}{(1-Z)^{2p}}\, dZ.$$

§ 5. Some lemmas.

In this section, some inequalities, all of which hold for $|z| \leqq r$, will be given for the preparation of next section.

Let $F(z)$ be bounded ($|F(z)| < M$), then we have

Lemma 1.

$$mr^p \frac{1-Mr}{M-r} \leqq |F(z)| \leqq Mr^p \frac{1+Mr}{M+r}.$$

Now putting $\varphi(z) = \frac{F(z)}{Z^p}$, then $|\varphi(z)| \leqq M$, $\varphi(o) = 1$ and therefore

$$|\varphi'(z)| \leqq \frac{M^2 - |\varphi(z)|^2}{M(1-r^2)}.$$

i,e,

$$(5,1) \qquad \left| \frac{F'(z)}{Z^p} - p\frac{F(z)}{Z^{p+1}} \right| \leqq \frac{M^2 r^{2p} - |F(z)|^2}{Mr^{2p}(1-r^2)}.$$

Combining (5,1) with lemma 1, we have

Lemma 2.

$$|F'(z)| \geqq M\gamma^{p-1}\frac{p(\gamma)}{(M-\gamma)^2},$$

where $p(\gamma) = pM - \{(p+1)M^2 + (p-1)\}\gamma + pM\gamma^2$.

and

Lemma 3.

$$|\lambda(z)| \geqq \frac{M^2\gamma^{2p}}{(M-\gamma)^2}(1-2M\gamma+\gamma^2),$$

where $\lambda(z) = zF'(z) - (p-1)F(z)$.

From Lemma 1, 2, (5,1), we get

Lemma 4.

$$\nu(z) \geqq \frac{M^3\gamma^{2(p-1)}}{(M-\gamma)^2}(1-\gamma^2)(M-2\gamma+M\gamma^2)$$

,*where* $\nu(z) = M^2\gamma^{2(p-1)}|F(z)|^2$.

and

Lemma 5.

$$\left|\frac{\lambda(z)}{zF'(z)}\right| \geqq R\left[\frac{\lambda(z)}{zF'(z)}\right] \geqq M\frac{1-2M\gamma+\gamma^2}{P(\gamma)}.$$

Next we have, applying (5,1) and lemma 1 to

$$\frac{|\lambda(z)|}{M\gamma^{p-1}+|F(z)|} = \frac{|F(z)|}{M\gamma^{p-1}+|F(z)|}\left|Z\frac{F'(z)}{F(z)} - (p+1)\right|,$$

Lemma 6.

$$\frac{|\lambda(z)|}{M\gamma^{p-1}+|F(z)|} \geqq \frac{\gamma(1-2M\gamma+\gamma^2)}{(1-\gamma^2)(M-\gamma)}.$$

Lastly lemma 2 and 4 conclude

Lemma 7.

$$\frac{\nu(z)}{M(1-\gamma^2)\gamma^{p-2}|F'(z)|} \leqq \frac{M\gamma(M-2\gamma+M\gamma^2)}{(1-\gamma^2)P(\gamma)}$$

The equality sign in lemma 1 ~ 7 holds at $z=\gamma$ for the function

$$F(z) = Mz^p\frac{1-Mz}{M-z}.$$

§ 6. The circle of quasi-convexity for the bounded functions.

Let F(z) be bounded ($|F(z)|<M$), then the auther has proved [6] that the circle of convexity of F(z) for $K_1 \equiv E_1$ is given by $|z|<\rho_1$, whre ρ_1 is the positive root less than 1 of the equation

$$M-(4M^2-1)Z+3MZ^2-Z^3=o.$$

Now we shall generalise this theorem for the functions of F_p.

If we define the function $\phi(z)$ regular in $|z|<1$ by

$$\phi(z)=M^2\frac{z^{p-1}F(w) - w^{p-1}F(z)}{M^2\,\overline{z}^{p-1}W^{p-1}-\overline{F(z)}F(w)},\qquad W=\frac{-S+z}{1-zS},\ (|z|<1),$$

then $|\phi(s)|<M$, $\phi(0)=0$ and, by the theorem of bounded functions,

$$M-\frac{|\phi'(0)|^2}{M}\geqq\frac{1}{2}|\phi''(0)|.$$

Thie inequality is reduced to

$$(6.1)\qquad R\left[z\frac{F''(z)}{F'(z)}\right]\geqq(p-1)(p-2)R\left[\frac{F(z)}{zF'(z)}\right]+\frac{2|z|}{1-|z|^2}R\left[z\frac{\lambda(z)}{F'(z)}\right]+2M\frac{|\lambda(z)|^2|z|^{p-2}}{\nu(z)|F'(z)|}$$

$$+2R\left[\frac{(p-1)M^2|z|^{2(p-1)}-\overline{zF(z)F'(z)}}{\nu(z)}\cdot\frac{\lambda(z)}{zF'(z)}\right]-2\frac{\nu(z)}{M(1-\gamma^2)^2|z|^{p-2}|F'(z)|},$$

where $F'(z)\neq o$ and $\lambda(z)=zF'(z)-(p-1)F(z)$, $\nu(z)=M^2|z|^{2(p-1)}-|F(z)|^2$.
The fourth term of the right-hand side in (6.1) is reduced to

$$2R\left[\left(p-1-\frac{\overline{F(z)\lambda(z)}}{\nu(z)}\right)\frac{\lambda(z)}{zF'(z)}\right],$$

which is not less than

$$2(p-1)R\left[\frac{\lambda(z)}{zF'(z)}\right]-2\left|\frac{\overline{F(z)}\lambda(z)}{\nu(z)}\right|\left|\frac{\lambda(z)}{zF'(z)}\right|,$$

and

$$(p-1)\frac{F(z)}{zF'(z)}=1-\frac{\lambda(z)}{zF'(z)}.$$

Therefore we have

$$(6.2)\qquad R\left[z\frac{F''(z)}{F'(z)}\right]\geqq p-2+\left[\frac{2|z|^2}{1-|z|^2}+p\right]R\left[\frac{\lambda(z)}{zF'(z)}\right]+2\left|\frac{\lambda(z)}{zF'(z)}\right|\frac{|\lambda(z)|}{M|z|^{p-1}+|F(z)|}$$

$$-2\frac{\nu(z)}{M(1-|z|^2)^2|z|^{p-2}|F'(z)|}.$$

Applylngl emma 1 ~ 7 in §5, the inequality

$$(6.3)\qquad 1+R\left[Z\frac{F''(z)}{F'(z)}\right]\geqq\frac{K(\gamma)}{P(\gamma)(M-\gamma)}$$

holdsfor $|z|\leqq\gamma$, where

$$(6.4)\qquad K(\gamma)=M^2p^2-M\{(p+1)^2M^2+2p^2-2p-1\}\gamma+\{(2p^2+2p-1)M^2+(p-1)^2\}\gamma^2-Mp^2\gamma^3.$$

The equality sign in (6.3) holds at $z=\gamma$ for the function

$$F(z)=Mz^p\frac{1-Mz}{M-z}.$$

As it is easiy proved that the equation $K(\gamma)=0$ has only one positive root ρ_p less than 1 and $K(\gamma)$ and $P(\gamma)$ and positive for $o\leqq\gamma<\rho_p$, we have the result as follows.

Theorem 11.

Let $F(z)=\sum_{n-p}^{\infty}a_nz^n\ (a_p=1)$ *qe regular and bounpep* ($|F(z)|<M$) *in* $|z|<1$, *then* $F(z)$ *is quasi-convex in* $|z|<\rho_p$, *where* ρ_p *is a positive rooot less than* 1 *of the equation* $K(\gamma)=0$. *And* ρ_p *is tbe possible number for quasi-convexity.*

If we assume that p is any real mumber, then we can prove

$$\frac{d\rho_p}{dp} > 0,$$

so that ρ_p increases with p and hence

$$\rho_1 < \rho_2 < \cdots\cdots < \rho_p < \rho_{p+1} < \cdots\cdots.$$

Consequently we have the following corollary of theorem 11.

Corollary.

Let F(z) be any function in theorem 11, then F(z) is quasi-convex in $|z| < \rho_1$ *for all positive integer p, where* ρ_1 *is a positive root less than 1 of the equation*

$$M-(4M^2-1)Z+3MZ^2-Z^3=0.$$

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