福井大学 工学部 研究報告 第50巻 第2号2002年9月
Mem. Fac. Eng. Fukui Univ., Vol. 50, No. 2 (September 2002)

An Algorithm for the Set of All Generators of an Arbitrary Firing Count Vector in Petri Nets

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(Received August 22, 2002)

In this paper, an effective method to obtain all nonnegative integer minimal support vectors (U_4, V_4) at level 4 for an arbitrary homogeneous / inhomogeneous solution of a matrix equation $Ax = b$ $(A \in Z^{m\times n},\ b \in Z^{m\times 1})$ starting from nonnegative rational number minimal support vectors (U_3, V_3) at level 3 is proposed. Although V_4 has been derived from all minimal vectors (U_5, V_5) of level 5 which are obtained starting from vectors at level 1, 2, or 3, so far, this proposed method for (U_4, V_4) gives us a big shortcut comparing with them. However, it is pointed out that obtaining (U_5, V_5) of level 5 from (U_4, V_4) of level 4 is difficult.

Key words: Petri Nets, Matrix Equations, Generators, Minimal Support Vectors, Minimal Vectors

1. Introduction

Petri nets are a promising model applicable to many concurrent systems[1]. The fundamental properties of a system modeled with a Petri net are obtained by solving a matrix equation $Ax = b$ $(A \in Z^{m\times n}, b \in Z^{m\times 1})$. An $(n\times 1)$ nonnegative integer solution $x \in Z_+^{n\times 1}$ of $Ax = b$ means a firing count vector of a Petri net. An arbitrary nonnegative integer solution $x \in Z_+^{n\times 1}$ of $Ax = b$ is expressed at level 4 (level 3, resp.) by each minimal support vector for a nonnegative integer (rational number, resp.) homogeneous / inhomogeneous solution, and is also expressed at level 5 by each minimal vector for a nonnegative integer homogeneous / inhomogeneous solution. The level is classified by attitude of the generating vectors and the expansion coefficients for $x \in Z_+^{n\times 1}$ [2]. The analyses of Petri nets become more efficient by executability check of each generating vectors (i.e., minimal support vectors or minimal vectors). Therefore, the exact and useful methods to obtain all generating vectors are needed. However, there is no method by now to obtain minimal support vectors at level 4 for a nonnegative integer inhomogeneous solution by the bottom-up with respect to levels, but the top-down methods from level 5 are known [5], [6], [8], [9].

In this paper, first, relations between "each minimal support vector for a nonnegative rational number homogeneous / inhomogeneous solution (level 3)" and "each minimal support vector of a nonnegative integer inhomogeneous solution (level 4)" are shown. Next, a new method which generates all minimal support vectors for a nonnegative integer inhomogeneous solution at level 4 starting from nonnegative rational number vectors at level 3 is proposed. Moreover, it is also shown that the method for obtaining minimal vectors of level 5 from these minimal support vectors of level 4 is complex and is not useful. Then, algorithms for minimal vectors of level 5 starting from vectors of level 1 to 3 are effective and useful as long as we concern the level 5 generators.

This paper is organized as follows. Some preliminaries, i.e., notations and some definitions about mathematics and Petri nets are given in Sect.2. Section 3 shows the previous results about generators at each level for $Ax = b$ or $\tilde{A}\tilde{x} = 0^{m\times 1}$. Section 4 shows the relations between (U_3, V_3) and (U_4, V_4). Section 5 proposes a new algorithm for (U_4, V_4) from (U_3, V_3). Section 6 shows usefulness of algorithms to obtain minimal vectors of level 5 from vectors of level 1 to 3. Finally, conclusions are given in Sect. 7.

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2. Preliminaries [1], [2]

2.1 Mathematical Notations

$Q^{m\times n}$, $Z^{m\times n}$, $Q_+^{m\times n}$, and $Z_+^{m\times n}$ is the set of $m\times n$ matrices with rational number elements, integer number elements, nonnegative rational number elements, and nonnegative integer elements, respectively. $0^{m\times n}$ is $m\times n$ matrix with zero elements and $E^{m\times m}$ is $m\times m$ unit matrix. $I(k):=\{1,\cdots,k\}$ is the index set. $x(i)$ is, for example, the i-th element of $x\in Q^{n\times 1}$, $i\in I(n)$.

The following notations for vector inequalities are used in this paper, where $x=(x(1),\cdots,x(n))^T$ and $y=(y(1),\cdots,y(n))^T$ are used for representation.

$x>y \Leftrightarrow x(i)>y(i)\ \ \forall i\in I(n)$,

$x\geqq y \Leftrightarrow x(i)\geqq y(i)\ \ \forall i\in I(n)$,

$x\geq y \Leftrightarrow x\geqq y$ and $\exists i\in I(n)$ s.t. $x(i)>y(i)$.

The following definitions with respect to solutions of $Ax=b$ are used at each level in this paper.

① A homogeneous solution of $Ax=b$; an $n\times 1$ matrix x is called a homogeneous solution, where A is an $m\times n$ matrix and $b=0^{m\times 1}$ is the $m\times 1$ zero matrix.

② An inhomogeneous solution of $Ax=b$; an $n\times 1$ matrix x is called an inhomogeneous solution, where A is an $m\times n$ matrix and $b\neq 0^{m\times 1}$ is the nonzero $m\times 1$ matrix.

③ A particular solution of $Ax=b$; at each level, an inhomogeneous solution x of $Ax=b$ is called a particular solution if and only if x is never expressed by the sum which contains at least a homogeneous solution.

④ In this paper, level i $(i=1,2,\cdots,5)$ means the complexity of calculation for the generators when attitude of each expansion coefficient with respect to generators at level i is restricted, where if $j>i$ is true, complexity of level j is greater than that of level i.

⑤ Support of $x\in Z_+^{n\times 1}$; the set of elements corresponding to nonzero entities in a solution $x\geq 0^{n\times 1}$ is called the support of a solution $x\geq 0^{n\times 1}$ and is denoted support(x). A support is said to be minimal if no proper nonempty support of the support is also a support.

⑥Minimal vector; a vector $x=(x(i))\in Z_+^{n\times 1}$ is said to be minimal if there is no other vector $x_1=(x_1(i))\in Z_+^{n\times 1}$ such that $x_1(i)\leqq x(i)$ for all elements $i\in I(n)$.

⑦ A minimal support (i.e., an elementary) vector; given a minimal support of a vector, there exists a unique minimal vector corresponding to the minimal support. We call such a vector a minimal support (i.e., an elementary) vector.

Every minimal support (i.e., every elementary) vector is a minimal vector, but the converse is not always true.

2.2 Preliminaries about P/T Petri Nets

Here, minimum terminologies about P/T Petri nets are given and for more details [1] should be referred. A P/T Petri net N is a 4-tuple, $N=(P,T,F,W)$, where P is a finite set of places, T is a finite set of transitions, $F\subseteq(P\times T)\cup(T\times P)$ is a set of arcs, and W is a set of weights of arcs. Note also that $|P|=m$, $|T|=n$, $P\cap T=\phi$, and $P\cup T\neq\phi$. A P/T Petri net system is denoted $\Sigma(N,M_0,M_d)$, where M_0, $M_d\in Z_+^{m\times 1}$ are initial and destination markings, respectively.

3. Previous Results about Generators at Each Level [2]

3.1 Generators (U_L,V_L) for $Ax=b$, $L=1,2,3,4,5$

General expression for an $(n\times 1)$ solution x of $Ax=b$ at level L is formally given as follows, where $A\in Z^{m\times n}$, $b\in Z^{m\times 1}$, and $L=1,\cdots,5$;

$$x=x_{HL}+x_{PL},$$
$$x_{HL}=\textstyle\sum_{i=1}^{l_L}\alpha_i u_i^{(L)},$$
$$U_L:=\{u_i^{(L)}\mid \text{the level-}L\text{ bases } u_i^{(L)} \text{ of } Au_i^{(L)}=0^{m\times 1},\ i\in I(l_L)\},\ l_L=|U_L|.$$
$$x_{PL}=\textstyle\sum_{j=1}^{k_L}\beta_j v_j^{(L)},\ \sum_{j=1}^{k_L}\beta_j=1,$$
$$V_L:=\{v_j^{(L)}\mid \text{ the level-}L\text{ particular solutions } v_j^{(L)} \text{ of } Av_j^{(L)}=b\neq 0^{m\times 1},\ j\in I(k_L)\},\ k_L:=|V_L|.$$

Let us call (U_L,V_L) the level-L generator for $Ax=b$ in this paper.

Level 1:

x, x_{H1}, x_{P1}, $u_i^{(1)}$, $v_j^{(1)}\in Q^{n\times 1}$; α_i, $\beta_j\in Q^{1\times 1}$; $\beta_j=1$, $l_1=|U_1|=n-r$, $k_1=|V_1|=1$, $r=\mathrm{rank}(A)$.

The total number of generators is $s_1 = k_1 + l_1 = n+1-r$, where l_1 is the number of Q -bases. $u_i^{(1)} \in U_1 :=$ { $u_i^{(1)} \in Q^{n\times 1}$ | rational number vectors of $Ax = 0^{m\times 1}$ }, $v_1^{(1)} = \{v_1^{(1)} \in Q^{n\times 1}$ | a rational number particular solution vector of $Ax = b \neq 0^{m\times 1}$ }.

Level 2:

$x, x_{H2}, x_{P2}, u_i^{(2)}, v_j^{(2)} \in Z^{n\times 1}$; $\alpha_i, \beta_j \in Z^{1\times 1}$; $\beta_j = 1$, $l_2 = |U_2| = n-r$, $k_2 = |V_2| = 1$, $r = \text{rank}(A)$.

The total number of generators is $s_2 = k_2 + l_2 = n+1-r$ and $s_2 = s_1$. Here, l_2 is the number of Z -bases. $u_i^{(2)} \in U_2 :=$ { $u_i^{(2)} \in Z^{n\times 1}$ | integer vectors of $Ax = 0^{m\times 1}$ }, $v_1^{(2)} =$ { $v_1^{(2)} \in Z^{n\times 1}$ | an integer particular solution vector of $Ax = b \neq 0^{m\times 1}$ }.

Level 3:

$x, x_{H3}, x_{P3}, u_i^{(3)}, v_j^{(3)} \in Q_+^{n\times 1}$; $\alpha_i, \beta_j \in Q_+^{1\times 1}$;

$l_3 = |U_3| \geqq n-r$, $k_3 = |V_3| \geqq 1$, $r = \text{rank}(A)$.

The total number of generators is $s_3 = k_3 + l_3 \geqq s_2 = s_1$, where l_3 is the number of all nonnegative rational number bases, and k_3 is the number of all nonnegative rational number particular solutions. $u_i^{(3)} \in U_3 := \{u_i^{(3)} \in Q_+^{n\times 1}$ | nonnegative rational number minimal support vectors of $Ax = 0^{m\times 1}$ }, $v_j^{(3)} \in V_3$:= { $v_j^{(3)} \in Q_+^{n\times 1}$ | nonnegative rational number minimal support vectors of $Ax = b \neq 0^{m\times 1}$ }.

Now, if we use some informal notations in order to illustrate the mutual relationships among levels, we have the followings, where U_4 and V_4 are defined in the next item. $U_3 = \{U_3 \cap V_4, U_3 \setminus U_4\}$, where $u_i^{(3)} \in U_3 \cap U_4) = \{u_i^{(3)} \in Z_+^{n\times 1}\}$ and $u_i^{(3)} \in (U_3 \setminus U_4) = \{u_i^{(3)} \in Q_+^{n\times 1}\}$. $V_3 = \{V_3 \cap V_4, V_3 \setminus V_4\}$, where $v_j^{(3)} \in (V_3 \cap V_4) = \{v_j^{(3)} \in Z_+^{n\times 1}\}$ and $v_j^{(3)} \in (V_3 \setminus V_4) = \{v_j^{(3)} \in Q_+^{n\times 1}\}$.

Note that we use a lot of similar notations for easy reading in this paper.

Level 4:

$x, x_{H4}, x_{P4}, u_i^{(4)}, v_j^{(4)} \in Z_+^{n\times 1}$; α_i, $\beta_j \in Q_+^{1\times 1}$;

$l_4 = |U_4| = l_3 \geqq n-r$, $k_4 = |V_4| \geqq k_3$, $r =$ rank(A).

The total number of generators is $s_4 = l_4 + k_4 \geqq s_3$, where l_4 is the number of all nonnegative integer and minimal-support bases (i.e., all elementary or minimal-support T-invariants) and k_4 is the number of all nonnegative integer and minimal-support particular solutions. $u_i^{(4)} \in U_4 = \{U_3 \cap U_4, U_4 \setminus U_3\} :=$ { $u_i^{(4)} \in Z_+^{n\times 1}$ | nonnegative integer minimal support vectors (minimal support T-invariants) of $Ax = 0^{m\times 1}$ }, $u_i^{(4)} = u_i^{(3)} \in (U_3 \cap U_4)$, $v_j^{(4)} \in V_4 = \{V_3 \cap V_4, V_4 \setminus V_3\} :=$ { $v_j^{(4)} \in Z_+^{n\times 1}$ | nonnegative integer minimal support vectors of $Ax = b \neq 0^{m\times 1}$ }. $v_j^{(4)} = v_j^{(3)} \in (V_3 \cap V_4)$.

Level 5:

$x, x_{H5}, x_{P5}, u_i^{(5)}, v_j^{(5)} \in Z_+^{n\times 1}$; α_i, $\beta_j \in Z_+^{1\times 1}$; $l_5 \geqq l_4 = l_3 \geqq n-r$, $k_5 \geqq k_4 \geqq k_3$, $r = \text{rank}(A)$.

The total number of generators is $s_5 = k_5 + l_5 \geqq s_4$, where l_5 is the number of all nonnegative integer and minimal bases (i.e., all minimal T-invariants) and k_5 is the number of all nonnegative integer and minimal particular solutions. $u_i^{(5)} \in U_5 = \{U_3 \cap U_4, U_4 \setminus U_3, U_5 \setminus U_4\} := \{u_i^{(5)} \in Z_+^{n\times 1}$ | nonnegative integer minimal vectors (minimal T-invariants) of $Ax = 0^{m\times 1}$ }, $u_i^{(5)} = u_i^{(4)} = u_i^{(3)} \in (U_3 \cap U_4)$, $u_i^{(5)} = u_i^{(4)} \in (U_4 \setminus U_3)$, $v_j^{(5)} \in V_5 = \{V_3 \cap V_4, V_4 \setminus V_3, V_5 \setminus V_4\} :=$ { $v_j^{(5)} \in Z_+^{n\times 1}$ | nonnegative integer minimal vectors of $Ax = b \neq 0^{m\times 1}$ }. $v_j^{(5)} = v_j^{(4)} = v_j^{(3)} \in (V_3 \cap V_4)$, $v_j^{(5)} = v_j^{(4)} \in (V_4 \setminus V_3)$.

3.2 Generators $\widetilde{U}_L$ for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$

In order to obtain simultaneously V_L (i.e., "generators of a inhomogeneous solution") as well as U_L (i.e., "generators of a homogeneous solution") for $Ax = b$, we discuss an augmented incidence matrix $\widetilde{A} = [A, -b] \in Z^{m\times(n+1)}$ in $\widetilde{A}\widetilde{x} = 0^{m\times 1}$ instead of $A \in Z^{m\times n}$ in $Ax = b$. Now, let us define $\widetilde{U}_L$ (i.e., "the set of generators of an $(n+1)\times 1$ homogeneous solution $\widetilde{x}$ for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$") at level $L = 1, 2, 3, 5$.

Here, note that it is the level $L = 1,2,3$ that $\widetilde{l}_L$ (i.e., "the number of generators for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$"), is equal to s_L (i.e., "the number of generators for $Ax = b$"). Therefore, we denote this fact $\widetilde{U}_L \leftrightarrow (U_L, V_L)$ in this paper, where $L = 1,2,3$. But at level 5, in general, $\widetilde{l}_5$ (i.e.," the number of nonnegative integer minimal vectors for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$"), is more than or equal to s_5 (i.e., "the number of generators for $Ax = b$").

Because all of vectors $u_i \in U_L$ for $Au_i = 0^{m\times 1}$ and all of vectors $v_j \in V_L$ for $Av_j = b \neq 0^{m\times 1}$ correspond to a part of vectors $\widetilde{x}_i \in \widetilde{U}_L$ for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$, then all of $u_i \in Z_+^{n\times 1}$ and all of $v_j \in Z_+^{n\times 1}$ for $Ax = b$ are gotten from vectors $\widetilde{x}_i \in \widetilde{U}_L$ for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$ as follows.

Now we can lastly find the set of generators (U_L, V_L) at level L for $x \in Z_+^{n\times 1}$ in $Ax = b$ from $\widetilde{x}_i \in \widetilde{U}_L$.

(1) If $\widetilde{x}_i(n+1) = 0^{1\times 1}$ on $\widetilde{x}_i \in \widetilde{U}_L$, then find $u_i :=$

$(\tilde{x}_i(1),\cdots,\tilde{x}_i(n))^T \in U_L$, at $L=1, 2, 3, 5$.

(2) If $\tilde{x}_j(n+1)=1^{1\times1}$ on $\tilde{x}_j \in \tilde{U}_L$, then find $v_j := (\tilde{x}_j(1),\cdots,\tilde{x}_j(n))^T \in V_L$, at $L=1, 2, 3, 5$.

(3) If $\tilde{x}_k(n+1)>1^{1\times1}$ on $\tilde{x}_k \in \tilde{U}_L$, then $x_j := (\tilde{x}_k(1), \cdots,\tilde{x}_k(n))^T$ is not a solution of $Ax=b$ at $L=5$.

Here, at level 3 each minimal support vector is adjusted to the $(n+1)$ th element equals unity. Pay attention to those vectors of $\tilde{A}\tilde{x}=0^{m\times1}$ at level 4 are nonnegative integer minimal support vectors which are obtained from nonnegative rational number minimal support vectors of $\tilde{A}\tilde{x}=0^{m\times1}$ at level 3 with nonnegative integer weights.

4. Properties of Generator at Level 3,4,5 [3]

Let q be the number of nonzero elements of $x \in Z_+^{n\times1}$ for $Ax=b$, and let A' be the sub-matrix which is composed of columns of A, of which columns are corresponding to nonzero elements of x. $\rho(*)$ implies rank$(*)$.

4.1 Properties of a Nonnegative Integer Homogeneous Solution

In this subsection, an arbitrary nonnegative integer minimal vector $u_i \in U_5 := \{u_i \in Z_+^{n\times1} \mid$ nonnegative integer minimal vectors $x=u_i$ of $Ax=0^{m\times1}$, $A\in Z^{m\times n}\}$ is a target, where $U^0 \supseteq U_5=\{U_4, U_5\setminus U_4\} \supseteq U_4=\{U_3 \cap U_4, U_4\setminus U_3\}$. Here, $U^0 := \{x\in Z_+^{n\times1} \mid Ax=0^{m\times1}, A\in Z^{m\times n}\}$ is the set of all nonnegative integer homogeneous solutions, i.e., all T-invariants.

For U^0, the next properties have been given in [3].

[Theorem 1] [3]

(1) $u_i \in U_4 (\subseteq U_5 \subseteq U^0)$
$$\rightleftarrows n \geqq q(u_i) = \rho(A'(u_i))+1,$$
(2) $u_i \in U^0 \setminus U_4$
$$\rightleftarrows n \geqq q(u_i) \geqq \rho(A'(u_i))+2,$$
(3) $u_i \in U_5 \setminus U_4 (\subseteq U^0)$
$$\longrightarrow n \geqq q(u_i) \geqq \rho(A'(u_i))+2. \quad \blacksquare$$

4.2 Properties of a Nonnegative Integer Inhomogeneous Solution

In this subsection, an arbitrary nonnegative integer minimal vector $v_j \in V_5 := \{v_j \in Z_+^{n\times1} \mid$ nonnegative integer minimal vectors $v_j=x$ of $Ax=b\neq 0^{m\times1}$, $A\in Z^{m\times n}$, $b\in Z^{m\times1}\}$ is a target, where $X^0 \supseteq V_5=\{V_4, V_5\setminus V_4\} \supseteq V_4=\{V_3\cap V_4, V_4\setminus V_3\}$. Here, $X^0 := \{x\in Z_+^{n\times1} \mid Ax=b\neq 0^{m\times1}, A\in Z^{m\times n}, b\in Z^{m\times1}\}$ is the set of all nonnegative integer inhomogeneous solutions.

For X^0, the next properties are known [4].

[Theorem 2] [4]

(1) $v_j \in V_3 \cap V_4 (\subseteq V_4 \subseteq V_5 \subseteq X^0)$
$$\rightleftarrows n \geqq q(v_j) = \rho(A'(v_j)),$$
(2) $v_j \in X^0 \setminus (V_3 \cap V_4)$
$$\rightleftarrows n \geqq q(v_j) \geqq \rho(A'(v_j))+1. \quad \blacksquare$$

[Theorem 3] [4]

(1) $v_j^{(4)} \in V_4 \setminus V_3 \rightleftarrows$ Condition (A) is satisfied, where $q(v_j)=\rho(A'(v_j))+1$,

(2) $v_j^{(5)} \in V_5 \setminus V_4 \rightleftarrows$
Condition (a) is satisfied, where $q(v_j)=\rho(A'(v_j))+1$
or
Condition (B) or Condition (b) is satisfied, where $q(v_j) \geqq \rho(A'(v_j))+2$.

Condition (A);

v_j is expressed by the sum of one T-invariant $u_i^{(3)} \in U_3\setminus U_4$ with a nonnegative integer weight and one particular solution $v_j^{(3)} \in V_3 \setminus V_4$.

Moreover, v_j satisfies either the next condition (i) or (ii).

(i) supp. $(u_i^{(3)}) >$ supp. $(v_j^{(3)})$.

(ii) When supp. $(u_i^{(3)}) \not\geq$ supp. $(v_j^{(3)})$ is true and $v_j^{(3)}$ includes just g positive components which are corresponding to g zero components of $u_i^{(3)}$, those g column vectors of $A'(v_j^{(3)})$ are linearly independent to each column of $A'(u_i^{(3)})$.

Condition (a);

v_j is expressed by the sum of two particular solutions s.t. $v_j^{(4)} \in V_3 \cap V_4$ with nonnegative rational number weights.

Condition (B);

v_j satisfies either the following condition (i) or (ii).

(i) At least one column vector out of g column vectors of the above condition (ii) of Condition (A) is linearly dependent to at least one column vector of $A'(u_i^{(3)})$.

(ii) v_j is expressed by the sum of more than or equal to two T-invariants $u_i^{(3)} \in U_3 \setminus U_4$ with nonnegative integer weights and one particular solution $v_j^{(3)} \in V_3 \setminus V_4$.

Condition (b);

v_j is expressed by the sum of more than or equal to three particular solutions s.t. $v_j^{(4)} \in V_3 \cap V_4$ with non-negative rational number weights.

■

5. A Method to Obtain Generators at Level 4

5.1 Methods to Obtain Generators of Level 3

The algorithm, proposed in this paper, for obtaining all nonnegative integer minimal support vectors (U_4, V_4) at level 4 has a preposition that all nonnegative rational number minimal support vectors (U_3, V_3) for both a homogeneous / inhomogeneous solution at level 3 are known.

Methods for (U_3, V_3) are listed up as follows.

(1) Find extreme points of the convex polyhedron formed by the LP expression for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$ and $\sum_{i=1}^{n+1}\widetilde{x}(i) = 1$ [3].

(2) Algorithm $(\widetilde{U}_1 \to \widetilde{U}_3)$ [5].

(3) Extended Fourier-Motzkin method [6].

5.2 A Method to Obtain Generators (U_4, V_4)

We consider a method to obtain $u_i^{(4)} \in U_4 = \{U_3 \cap U_4,\ U_4 \setminus U_3\}$ and $v_j^{(4)} \in V_4 = \{V_3 \cap V_4,\ V_4 \setminus V_3\}$ of $Ax = b$. Each $u_i^{(4)} \in U_3 \cap U_4$ is equal to $u_i^{(3)} \in U_3 \cap U_4$. $u_i^{(4)} \in U_4 \setminus U_3$ is directly obtained from $u_i^{(3)} \in U_3 \setminus U_4$ by multiplying a nonnegative integer. $v_j^{(4)} \in V_3 \cap V_4$ is equal to $v_j^{(3)} \in V_3 \cap V_4$. $v_j^{(4)} \in V_4 \setminus V_3$ is shown by the sum of one T-invariant s.t. $u_i^{(3)} \in U_3 \setminus U_4$ with a nonnegative integer weight and one particular solution s.t. $v_j^{(3)} \in V_3 \setminus V_4$. Moreover, $v_j^{(4)} \in V_4 \setminus V_3$ satisfies the rank condition $q(v_j^{(4)}) = \rho(A'(v_j^{(4)})) + 1$ from Theorem 3 (1) in §4.2. Therefore an algorithm to obtain $v_j^{(4)} \in V_4 \setminus V_3$ is shown as follows.

Part 1

Find the set of all generators at Level 3 for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$: $\widetilde{U}_3 = \{\widetilde{x} \in Q^{(n+1)\times 1} \mid$ all of minimal support vectors for $\widetilde{A}\widetilde{x} = 0^{m\times 1}$ where $\widetilde{A} = [A, -b] \in Z^{m\times(n+1)}\}$. Choose all $\widetilde{x} \in \widetilde{U}_3$ with at least one element s.t. $\widetilde{x}(r) \in Q_+^{1\times 1} \setminus Z_+^{1\times 1}$, $r \in I(n+1)$ (it corresponds to $u_i^{(3)} \in U_3 \setminus U_4$ or $v_j^{(3)} \in V_3 \setminus V_4$) from $\widetilde{U}_3$ and let the set of these $\widetilde{x}$ be $\widetilde{U}_3'$. Go to part 2.

Part 2

Let G^1 be the $(n+1)\times \widetilde{l}_3'$ matrix consisting of each vector $\widetilde{x} \in Q_+^{(n+1)\times 1}$ and $\widetilde{x} \in \widetilde{U}_3'$ and let the column index set be $I^1 := \{1, 2, \cdots, \widetilde{l}_3'\}$.

step 0; $k := 1$, $w := \widetilde{l}_3'$.

step 1; After the algorithm has been processed the $(k-1)$-th loop (step1-5), we give the description of the next loop. Given the $n \times w$ matrix G^k and the column index set $I^k = \{1, 2, \cdots, w\}$ of the k-th loop.

Find $\{a_{kj} \mid j \in I^k \text{ in } G^k\}$. Let $I^{k[0]}$ and $I^{k[1]}$ be the index set of columns of which the $(n+1)$ th elements are zero and unity, respectively. If the k-th row of G^k has no element which belongs to $Q_+^{1\times 1} \setminus Z_+^{1\times 1}$, then $k = k+1$ and go to step 2. If the k-th row of G^k has at least one negative element, then go to step 3.

step 2; If $k \leqq n$, then go to step 1 else step 6.

step 3; Determine the set of all pairs of those elements of columns of G^k such that

$R = \{(s,t) \mid a_{ks}, a_{kt} \in Q_+^{1\times 1} \setminus Z_+^{1\times 1},\ s < t,\ s \in I^{k[0]}, t \in I^{k[1]}\}$

Then go to step 4.

step 4; Determine the nonnegative integer coefficient α s.t. $\alpha \cdot a_{ks} + a_{kt}$. Let $F = \{j \mid a_{kj} \in Z_+^{1\times 1} \text{ in } G^k\}$. Here q denotes the number of elements of F, i.e., $q = |F|$. Then the first q columns of the new matrix G^{k+1} are all g^j for $j \in F$, where g^j denotes the j-th column of G^k. Let g^s and g^t be the s-th and t-th columns of G^k, respectively. Adjoin the new column $g^s + g^t$ to the new matrix G^{k+1}. Let $\hat{I}^k$ be the above adjoined column index set. Then $q = q + |\hat{I}^k|$ and go to step 5.

step 5; $k = k+1$. If $k \leqq n$, then $w = q$ and go to step 1 else go to step 6.

step 6; If all elements of G^k are nonnegative and if a column vector v_j, which is obtained by deleting the $(n+1)$ th element from g^s, satisfies the rank condition $q(v_j) = \rho(A'(v_j)) + 1$, then the column vector is a minimal support vector (i.e., a level 4 generator for a particular solution) of a nonnegative integer inhomogeneous solution of $Ax = b$.

step 7; Stop.

5.3 Example

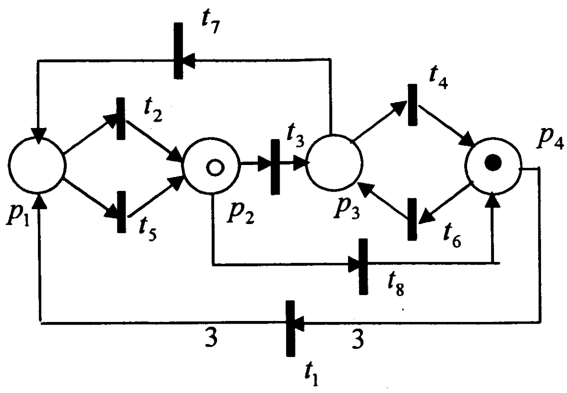

Fig. 1 An example of a Petri net

We have thirteen nonnegative rational number minimal support vectors $\widetilde{x} \in \widetilde{U}_3$ and $\widetilde{x} \in Q_+^{9\times 1}$ of $\widetilde{A}\widetilde{x} = 0^{4\times 1}$ in this Petri net as follows.

$$\widetilde{U}_3 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 0 & 0 & 1/3 & 1/3 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{matrix}\right\}$$

$$= \{\widetilde{x}_1, \cdots, \widetilde{x}_{13}\}.$$

Here, $\widetilde{x}_1$ to $\widetilde{x}_5$ correspond to $u_i^{(4)} = u_i^{(3)} \in U_3 \cap U_4$, $i = 1, \cdots, 5$, and $\widetilde{x}_6$ to $\widetilde{x}_9$ correspond to $u_i^{(3)} \in U_3 \setminus U_4$, $i = 6, \cdots, 9$. Therefore, $u_i^{(4)} \in U_4 \setminus U_3$ are obtained from $u_i^{(3)} \in U_3 \setminus U_4$, $i = 6, \cdots, 9$, with nonnegative integer weights as follows.

$$u_6^{(4)} = (1\,3\,3\,3\,0\,0\,0\,0)^T,\quad u_7^{(4)} = (1\,3\,0\,0\,0\,0\,0\,3)^T,$$
$$u_8^{(4)} = (1\,0\,3\,3\,3\,0\,0\,0)^T,\quad u_9^{(4)} = (1\,0\,0\,0\,3\,0\,0\,3)^T.$$

x_1 to x_9 obtained by deleting the zero ninth element of $\widetilde{x}_1$ to $\widetilde{x}_9$ satisfy the rank condition $q(x_i) = \rho(A'(x_i)) + 1$.

Moreover, $\widetilde{x}_{10}$ and $\widetilde{x}_{11}$ correspond to $v_j^{(4)} = v_j^{(3)} \in V_3 \cap V_4$, $j = 1,2$ and $\widetilde{x}_{12}$ and $\widetilde{x}_{13}$ correspond to $v_j^{(3)} \in V_3 \setminus V_4$, $j = 3,4$. x_{10} to x_{13} obtained by deleting the unity ninth element of $\widetilde{x}_{10}$ to $\widetilde{x}_{13}$ satisfy the rank condition $q(x_i) = \rho(A'(x_i))$. An example for the method to obtain $v_j^{(4)} \in V_4 \setminus V_3$ is shown as follows.

Let G^1 be the matrix consisted of all nonnegative rational number minimal support vectors $\widetilde{x}' \in \widetilde{U}_3'$ which are corresponding to $u_i^{(3)} \in U_3 \setminus U_4$ or $v_j^{(3)} \in V_3 \setminus V_4$.

$$G^1 = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 1/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

We pay attention to the first row of the matrix G^1. The first row have six elements which belong to $Q_+^{1\times 1} \setminus Z_+^{1\times 1}$, then we have $R = \{(1,5), (2,5), (3,5), (4,5), (1,6), (2,6), (3,6), (4,6)\}$. Next, we determine nonnegative integer coefficient α s.t. $\alpha \cdot a_{ks} + a_{kt} \in Z_+^{1\times 1}$ for each (s,t). Then we have $\alpha = 2$ for all (s,t) in this example. There is no column which satisfies $F = \{j \mid a_{1j} \in Z_+^{1\times 1} \text{ in } G^1\}$. Then the new matrix G^2 which adjoins the new column $g^s + g^t$ is shown as follows.

$$G^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 1 & 1 & 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Because all elements of the matrix G^2 are nonnegative integer, then the rank conditions of Section 4 are applied to each column vector obtained by deleting the unity ninth element. The first, second, seventh, and eighth column vectors $v_j \in Z_+^{8\times 1}$ obtained by deleting the unity ninth element satisfy the rank condition $q(v_j) = \rho(A'(v_j)) + 1$. Then $v_j^{(4)} \in V_4 \setminus V_3 (j = 3,4,5,6)$ are as follows.

$$v_3^{(4)} = (1\,3\,2\,2\,0\,0\,0\,0)^T,\quad v_4^{(4)} = (1\,3\,0\,0\,0\,0\,0\,2)^T,$$
$$v_5^{(4)} = (1\,0\,2\,2\,3\,0\,0\,0)^T,\quad v_6^{(4)} = (1\,0\,0\,0\,3\,0\,0\,2)^T.$$

Here, the third, forth, fifth, and sixth column vectors $v_j \in Z_+^{8\times 1}$ obtained by deleting the unity ninth element satisfy the another rank condition $q(v_j) = \rho(A'(v_j)) + 2$. Therefore these four vectors are $v_j^{(5)} \in V_5 \setminus V_4$. $(j = 7,8,9,10)$

■

6. Usefulness of Each Method to Obtain Minimal Vectors of Level 5 from Vectors of Level 1~3

A method to obtain nonnegative integer minimal vectors (U_5, V_5) of level 5 from rational number vectors (U_1, V_1) at level 1, integer vectors (U_2, V_2) at level 2, or nonnegative rational number vectors (U_3,V_3) at level 3 repeats the next fundamental operation [5], [7], [8], [9].

From Level 1, add two vectors without any weight to change elements which satisfy $Q^{l\times l} \setminus Z_+^{l\times l}$ into $Z_+^{l\times l}$.

From Level 2, add two vectors without any weight to change elements which satisfy $Z^{l\times l} \setminus Z_+^{l\times l}$ into $Z_+^{l\times l}$.

From Level 3, add two vectors without any weight to change elements which satisfy $Q_+^{l\times l} \setminus Z_+^{l\times l}$ into $Z_+^{l\times l}$.

On the other hand, in a method to obtain minimal vectors U_5 (V_5, resp.) of level 5 from minimal support vectors U_4 (V_4, resp.) at level 4, more than or equal to two vectors of U_4 (V_4, resp.) should be added with nonnegative rational number weights. Each element of each minimal vector and each minimal support vector is nonnegative integer. In other words, there is no difference between each element of (U_4,V_4) and that of (U_5,V_5) with respect to attitude for elements, in contrast with the methods for (U_5,V_5) from (U_1,V_1), (U_2,V_2), or (U_3,V_3). Then it is difficult to decide minimal support vectors of level 4 which are to be combined. Moreover, all possible combinations should be considered to obtain all minimal vectors (U_5,V_5) of level 5 from (U_4,V_4). Here, $u_i^{(5)} \in U_5 \setminus U_4$ is expressed by a linear combination with nonnegative rational number coefficients with respect to $u_i^{(4)} \in U_4 \setminus U_3$ and $v_j^{(5)} \in V_5 \setminus V_4$ is also expressed by a convex combination with respect to $v_j^{(4)} \in V_4 \setminus V_3$. In the above, all nonnegative rational number coefficients should be decided. However, it is also difficult to do the above operation.

According to the above reasons, even if (U_4,V_4) have been given, each method to obtain (U_5,V_5) from (U_1,V_1), (U_2,V_2), or (U_3,V_3) is more effective than a method to obtain (U_5,V_5) from (U_4,V_4).

7. Conclusions

We have shown a useful method which generates all minimal support vectors (i.e., V_4, the set of all minimal generators at level 4) of a nonnegative integer inhomogeneous solution starting from nonnegative rational number vectors (U_3,V_3) at level 3. This method is based on the two facts that each minimal support vector s.t. $v_j^{(4)} \in V_4 \setminus V_3$ is expressed by the sum of only two vectors; " one T-invariant $u_i^{(3)} \in U_3 \setminus U_4$ with nonnegative integer weight and one particular solution $v_j^{(3)} \in V_3 \setminus V_4$," and that $v_j^{(4)} \in V_4 \setminus V_3$ satisfies the rank condition $q(v_j^{(4)}) = \mathrm{rank}(A'(v_j^{(4)}))+1$. Then we have gotten a useful algorithm for (U_4, V_4) from (U_3, V_3) because U_4 is easily and directly obtained from U_3 s.t. $|U_4|=|U_3|$.

Note that V_4 has been just obtained by now from V_5 (this is the set of all nonnegative integer particular solutions, i.e., all nonnegative integer minimal generators for an arbitrary inhomogeneous solution at level 5) because the above basic facts about $v_j^{(4)} \in V_4 \setminus V_3$ have not been known. Therefore if we restrict ourselves to finding (U_4, V_4), the proposed algorithm for (U_4, V_4) from (U_3, V_3) is more effective and useful than those which obtain (U_4, V_4) from (U_5, V_5).

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