A Simple Expression for the Trajectories under the Effects of Space Charge in a Quadrupole Lens

By *Masahiro Ueda* and *Noriyuki Yamazaki*
Department of Electronics, Faculty of Engineering, Ehime University, Matsuyama, Ehime, Japan

Abstract

A simple method for the numerical analysis of the second order non-linear coupled differential equations, which represent the trajectories in a quadrupole lens under the effects of space charge, is shown. The resultant equations of the trajectories are expressed by the usual matrix form. The numerical results are shown for a few typical cases.

Inhalt

Einfacher Ausdruck für die Elektronenbahnen in einer Quadrupollinse unter dem Einfluß von Raumladung. Es wird eine einfache Methode zur numerischen Lösung von gekoppelten nichtlinearen Diffenentialgleichungen zweiter Ordnung gegeben, die die Bahnen in einer Quadrupollinse unter dem Einfluß von Raumladung darstellen. Die Bahngleichungen ergeben sich in der üblichen Matritzenform. Numerische Ergebnisse werden für einige typische Fälle mitgeteilt.

Quadrupole lens systems have had very wide applications not only in the field of high-energy particle beam [1] but in the field of low-energy particle beam [2] in recent year. For a precise analysis of the particle trajectories in this lens e.g., the focusing characteristics of the lens, it is important to take account of the effects of space charge in the latter case where the beam perveance is low. There are little attention except the paper by *J. H. Bick* [3] to these effects on particle trajectories in the quadrupole lens. He analyzed those effects on trajectories theoretically by using the first order theory, but it is rather difficult to solve them numerically without proper approximations because the resultant equations of the trajectories in both planes inherent to the quadrupole lens are non-linear second order coupled differential equations. In this letter a simple method for the numerical analysis is proposed and the utility of this method is shown.

Fig. 1 shows the illustration of the trajectories in the two planes. The equations of the trajectories are given according to [3] as follows:

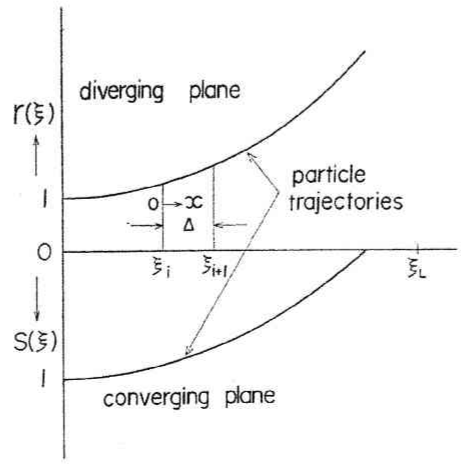

Fig. 1. Illustration of the particle trajectories in the quadrupole lens.

$$\left.\begin{aligned} &\frac{d^2 r}{d\xi^2} - r - \frac{\beta}{r+s} = 0 \\ &\frac{d^2 s}{d\xi^2} + s - \frac{\beta}{r+s} = 0 \\ &\beta = \frac{I}{\pi\varepsilon_0 v^2 a_0 G} \end{aligned}\right\} \tag{1}$$

where r and s are the normalized coordinates in the diverging and the converging planes, respectively, a_0 is the outermost coordinate of the beam at $\xi = 0$, ε_0 is the dielectric constant, v is the particle velocity, G is the coefficient of the magnetic scalar potential, I is the current of the beam and β represents the normalized space charge.

For most cases in the quadrupole lenses, the available region of ξ is limited within the region where $s(\xi)$ is positive for proper values of β, r(0), s(0), $\dot{r}(0)$ and $\dot{s}(0)$. The whole region of ξ, $\xi_L = 4$ here, is then divided into N equal sections, Δ, as shown in Fig. 1. As in the usual case, we expand $r(\xi)$ and $s(\xi)$ in a section as,

$$\left.\begin{aligned} &r(\xi_i + x) = r(\xi_i) + x\dot{r}(\xi_i) + \frac{x^2}{2}\ddot{r}(\xi_i) + \dots \\ &s(\xi_i + x) = s(\xi_i) + x\dot{s}(\xi_i) + \frac{x^2}{2}\ddot{s}(\xi_i) + \dots \\ &\xi_i \leqq \xi = \xi_i + x \leqq \xi_{i+1} = \xi_i + \Delta \\ &i = 1,2, \dots, N. \end{aligned}\right\} \tag{2}$$

where dot shows the differentiation with respect to x. For small x, the sum of $r(\xi_i + x)$ and $s(\xi_i + x)$ can be approximated as

$$r(\xi_i + x) + s(\xi_i + x) \simeq r(\xi_i) + s(\xi_i) + x[\dot{r}(\xi_i) + \dot{s}(\xi_i)], \tag{3}$$

since $\ddot{r}(\xi_i)$, $\ddot{s}(\xi_i)$ and other higher order derivatives are not so large in the case considered here. It is reasonable to put $x = \Delta/2$ in (3) as x varies from 0 to Δ. Equation (1) is then linearized to two independent second order differential equations, and is rewritten as

$$\left.\begin{aligned}
&\frac{d^2 r(\xi_i + x)}{dx^2} - r(\xi_i + x) - f(\xi_i) = 0 \\
&\frac{d^2 s(\xi_i + x)}{dx^2} + s(\xi_i + x) - f(\xi_i) = 0 \\
&f(\xi_i) = \frac{\beta}{r(\xi_i) + s(\xi_i) + \frac{\Delta}{2}[\dot{r}(\xi_i) + \dot{s}(\xi_i)]}
\end{aligned}\right\} \tag{4}$$

where $f(\xi_i)$ shows the effects of space charge on the trajectories at $\xi = \xi_i$. Equation (4) can easily be solved analytically. The coordinates and the slopes of the trajectories at $\xi = \xi_{i+1}$ are then given from the ones at $\xi = \xi_i$ in the usual matrix form,

$$\left.\begin{aligned}
&\begin{bmatrix} r(\xi_{i+1}) \\ \dot{r}(\xi_{i+1}) \end{bmatrix} = \begin{bmatrix} \cosh\Delta, \sinh\Delta, \cosh\Delta - 1 \\ \sinh\Delta, \cosh\Delta, \sinh\Delta \end{bmatrix} \begin{bmatrix} r(\xi_i) \\ \dot{r}(\xi_i) \\ f(\xi_i) \end{bmatrix} \\
&\begin{bmatrix} s(\xi_{i+1}) \\ \dot{s}(\xi_{i+1}) \end{bmatrix} = \begin{bmatrix} \cos\Delta, \sin\Delta, -\cos\Delta + 1 \\ -\sin\Delta, \cos\Delta, \sin\Delta \end{bmatrix} \begin{bmatrix} s(\xi_i) \\ \dot{s}(\xi_i) \\ f(\xi_i) \end{bmatrix}
\end{aligned}\right\} \tag{5}$$

Table 1 shows the numerical results for typical cases i.e., $r(0) = s(0) = 1$, $\dot{r}(0) = \dot{s}(0) = 0$ and $r(0) = s(0) = 1$, $\dot{r}(0) = \dot{s}(0) = 0.4$. The table shows that N has little contribution to the accuracy of this method when $N \geqq 20$.

Table 1

Numerical results of the trajectories at $\xi = 0.8$ for $\beta = 0.4$. (a) and (b) correspond to the cases $r(0) = s(0) = 1$, $\dot{r}(0) = \dot{s}(0) = 0$ and $r(0) = s(0) = 1$, $\dot{r}(0) = \dot{s}(0) = 0.4$, respectively.

Case	N	$r(\xi_i)$	$s(\xi_i)$	$\dot{r}(\xi_i)$	$\dot{s}(\xi_i)$
(a)	10	1.4044	0.7568	1.0629	−0.5765
	20	1.4042	0.7567	1.0621	−0.5773
	40	1.4042	0.7567	1.0619	−0.5775
(b)	10	1.7529	1.0376	1.5760	−0.3168
	20	1.7534	1.0380	1.5759	−0.3173
	40	1.7535	1.0381	1.5759	−0.3174

Fig. 2 shows the typical trajectories and its slopes. It is seen from these figures that the effects of space charge on the trajectories and its slopes can not be neglected even in a single lens and then these effects must be taken into account when one wish to calculate the focusing characteristics of the lens system consisting of two or more single lenses.

It is found from these results that the method proposed here is useful for the evaluation of rather troublesome effects of space charge on the focusing characteristics of the quadrupole lens system. We are now working the further calculation for the focusing characteristics of the three elements quadrupole lens.

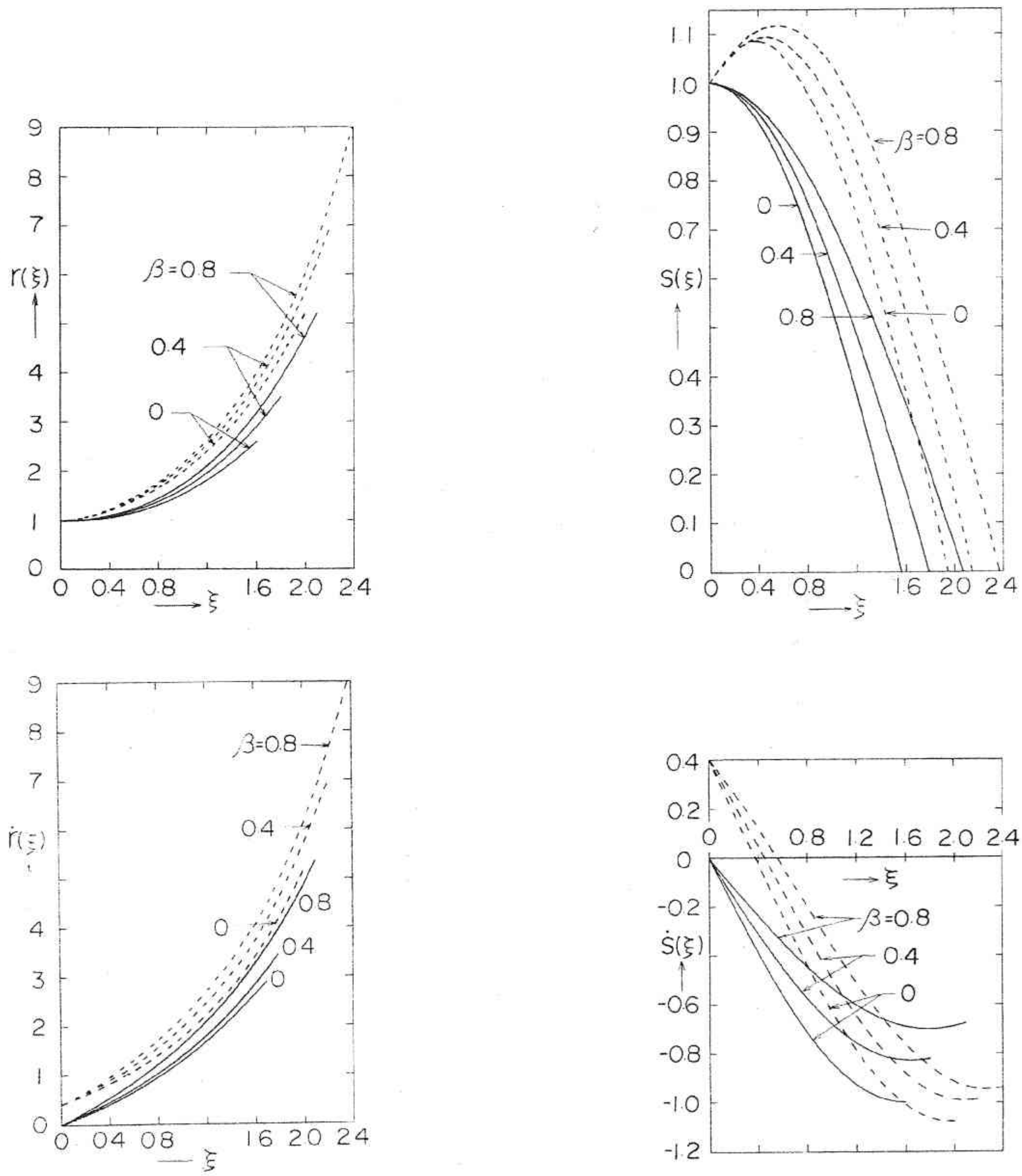

Fig. 2. Coordinates and its slopes of the trajectories for two typical cases. Solid and broken curves correspond to the cases $r(0) = s(0) = 1$, $\dot{r}(0) = \dot{s}(0) = 0$ and $r(0) = s(0) = 1$, $\dot{r}(0) = \dot{s}(0) = 0.4$, respectively.

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References

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[3] *J. H. Bick*, "The First-Order Theory of Quadrupole Lenses Including the Effects of Space Charge", IEEE Trans. *ED–12* (1965) 408.