Quantum Markov Process on a Lattice

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We develop a systematic description of Weyl and Fano operators on a lattice phase space. Introducing the so-called ghost variable even on an odd lattice, odd and even lattices can be treated in a symmetric way. The Wigner function is defined using these operators on the quantum phase space, which can be interpreted as a spin phase space. If we extend the space with a dichotomic variable, a positive distribution function can be defined on the new space. It is shown that there exits a quantum Markov process on the extended space which describes the time evolution of the distribution function.

1. Introduction

The quantum Markov process for an integer spin system was constructed on the lattice phase space by Cohendet et al. [1]. The time evolution of the pseudo-distribution function, i.e., the Wigner function on the space can be derived from the process. However, the extension of their method to a half integer spin system is not straightforward. There are difficulties in the construction of the Wigner function on an even lattice. A way to avoid this difficulties is to introduce the so-called "ghost variable" [2]. The Weyl and Fano operators can be constructed on an even lattice if we consider fictitious lattice points between the original ones (see also [3]). In the present paper, we develop the systematic treatment of the Wigner function on both odd and even lattices and show that a quantum Markov process for a half integer spin system can also be constructed in a similar manner.

2. Weyl and Fano operators with ghost variables on a lattice

We consider a lattice composed of N lattice points. A lattice with odd or even lattice points is called odd or even lattice, respectively. We denote the set of integers and half integers by $\bar{\mathbf{Z}}$. For odd N, we define $J^o, \bar{J}^o$ as the sets composed of integers, integers and half integers between $-(N/2)$ and $(N/2)+1$, respectively, and $\bar{J}^o_+$ be the non-negative part of $\bar{J}^o$. For even N, we define $J^e, \bar{J}^e, \bar{J}^e_+$ similarly between $-(N/2)$ and $(N-1)/2$. The symbols $J, \bar{J}, \bar{J}_+$ are the abbreviations of $J^o, \bar{J}^o, \bar{J}^o_+$ for odd N and $J^e, \bar{J}^e, \bar{J}^e_+$ for even N. We consider a Kronecker's delta function $\bar{\delta}^{(N)}_{n,m}$ on $\bar{\mathbf{Z}}$, defined by $\bar{\delta}^{(N)}_{m,n} \equiv \delta_{2m,2n}$ in mod $2N$, where $m, n \in \bar{\mathbf{Z}}$. We denote one of primitive roots of unity of order N by ω, e.g., $\omega = \exp\left(\frac{2\pi i}{N}\right)$.

The (n, m) components of basic phase, shift and skew operators Q, P, T are defined as

$$Q \equiv (\omega^n \bar{\delta}^{(N)}_{n,m}), P \equiv (\bar{\delta}^{(N)}_{n+1,m}), T \equiv (\bar{\delta}^{(N)}_{n+m,0}), \quad (1)$$

for odd N, and

$$Q \equiv (\omega^n \bar{\delta}^{(N)}_{n,m}), P \equiv ((1 - 2\bar{\delta}^{(N)}_{n,\frac{N-1}{2}})\bar{\delta}^{(N)}_{n+1,m}),$$

$$T \equiv (\bar{\delta}^{(N)}_{n+m,0}), \quad (2)$$

for even N, where $m, n \in J$. The Froquet or Bloch angle is π in the latter case [4].

3. Wigner function on a lattice

3.1. Weyl operators

We define the Weyl operators by

$$W_{m,n} \equiv \omega^{-2mn} Q^{2n} P^{-2m} = \omega^{2mn} P^{-2m} Q^{2n}, \quad (3)$$

for both odd and even N, where $m, n \in \bar{J}$. We can see the Weyl operators satisfy the relations for reflection and shift:

$$W^\dagger_{m,n} = W_{-m,-n} = T W_{m,n} T, \quad (4)$$

$$W_{m,n}W_{m',n'} = \omega^{-2(mn'-nm')}W_{m+m',n+n'}. \quad (5)$$

The N^2 operators $W_{m,n}$ with (m,n) in the basic region $\bar{J}_+ \times \bar{J}_+$ are complete and orthogonal in the trace norm. Generally, $W_{m,n}$ with $(m,n) \in \bar{\mathbf{Z}} \times \bar{\mathbf{Z}}$ and $W_{\mathrm{mod}(m,\frac{N}{2}),\mathrm{mod}(n,\frac{N}{2})}$ in the basic region are related as

$$W_{m,n} = \theta(m,n)W_{\mathrm{mod}(m,\frac{N}{2}),\mathrm{mod}(n,\frac{N}{2})}, \quad (6)$$

where $\theta(m,n) = (-1)^{\tau(m,n)}$, and

$$\tau(m,n) = 2n\left[\frac{2m}{N}\right]+2m\left[\frac{2n}{N}\right]+\left[\frac{2m}{N}\right]\cdot\left[\frac{2n}{N}\right], \quad (7)$$

for odd N,

$$\tau(m,n) = 2n\left[\frac{2m}{N}\right]+2m\left[\frac{2n}{N}\right]+\left[\frac{2m}{N}\right]+\left[\frac{2n}{N}\right], \quad (8)$$

for even N.

The Weyl operators have the next symplectic property, i.e., they are rotated by $\pi/2$ anticlockwise under the Fourier transformation,

$$\mathcal{F}W_{m,n}\mathcal{F}^\dagger = W_{-n,m}, \ \mathcal{F} = \frac{1}{\sqrt{N}}(\omega^{mn}). \quad (9)$$

3.2. Fano operators

We define the Fano operators by

$$\Delta_{m,n} = W_{m,n}T = \omega^{-2mn}Q^{2n}TP^{2m}, \quad (10)$$

for both odd and even N, where $m,n \in \bar{J}$. They are hermite but over complete. Those in the basic region are orthogonal and complete. The Fano operators have the same symplectic property as the Weyl operators,

$$\mathcal{F}\Delta_{m,n}\mathcal{F}^\dagger = \Delta_{-n,m}. \quad (11)$$

We can see that the Fano operators have proper marginal properties:

$$\sum_{n\in\bar{J}}\Delta_{m,n} = 0, \ \text{for half integer } m \ \text{(ghost)} \quad (12)$$

$$\sum_{n\in\bar{J}}\Delta_{m,n} = 2N(\delta^{(N)}_{i,m}\delta^{(N)}_{j,m}), \ \text{for integer } m \quad (13)$$

$$\sum_{m\in\bar{J}}\Delta_{m,n} = 0, \ \text{for half integer } n \ \text{(ghost)} \quad (14)$$

$$\sum_{m\in\bar{J}}\Delta_{m,n} = 2(\omega^{m(i-j)}), \ \text{for integer } n \quad (15)$$

for odd N, and

$$\sum_{n\in\bar{J}}\Delta_{m,n} = 0, \ \text{for integer } m \ \text{(ghost)} \quad (16)$$

$$\sum_{n\in\bar{J}}\Delta_{m,n} = 2N(\delta^{(N)}_{i,m}\delta^{(N)}_{j,m}), \text{for half integer } m \quad (17)$$

$$\sum_{m\in\bar{J}}\Delta_{m,n} = 0, \ \text{for integer } n \ \text{(ghost)} \quad (18)$$

$$\sum_{m\in\bar{J}}\Delta_{m,n} = 2(\omega^{n(i-j)}), \ \text{for half integer } n \quad (19)$$

for even N.

3.3. Time evolution of the Wigner function

We define the Wigner function by

$$\mathcal{W}(m,n) = \frac{1}{N}\mathrm{tr}(\rho\Delta_{m,n}), \quad (20)$$

which is real valued and bounded by $1/N$,

$$-\frac{1}{N} \le \mathcal{W}(m,n) \le \frac{1}{N}. \quad (21)$$

The average value of an observable $\mathcal{O}$ is given by

$$\langle\mathcal{O}\rangle = N\sum_{m,n\in\bar{J}_+}\mathcal{O}(m,n)\mathcal{W}(m,n), \quad (22)$$

where $\mathcal{O}(m,n) = \frac{1}{N}\mathrm{tr}(O\Delta_{m,n})$. We rewrite the time evolution equation of the density matrix ρ,

$$i\frac{\partial\rho_t}{\partial t} = [H,\rho_t], \quad (23)$$

in terms of the Wigner function. We expand the Hamiltonian H by using the Weyl operators in the basic region, and the density matrix by the Fano operators in the same region. Equating the coefficient of $\Delta_{m,n}$, the time evolution equation of the Wigner function is given by

$$\frac{d}{dt}\mathcal{W}_{mn} = -2\sum_{m'n'\in\bar{J}_+}\tilde{\mathcal{H}}^+(m'',n'')$$

$$\sin\left\{\frac{2\pi i}{N}\{2(m''n'-n''m')-\alpha(m'',n'')\}\right\}$$

$$\theta(m''+m',n''+n')\mathcal{W}_{m'n'}, \qquad (24)$$

where $m''=\mathrm{mod}(m-m',\frac{N}{2})$, $n''=\mathrm{mod}(n-n',\frac{N}{2})$ and $\tilde{\mathcal{H}}^{+}(m,n)$ and $\alpha(m,n)$ are determined by the polar decomposition $\tilde{\mathcal{H}}(m,n)=\tilde{\mathcal{H}}^{+}(m,n)\omega^{\alpha(m,n)}$.

4. Construction of a Markov process

We extend the basic region with the dichotomic variable $\sigma\in\{\pm1\}\equiv B$, i.e., $\bar{J}_+\times\bar{J}_+\to\bar{J}_+\times\bar{J}_+\times B$, and consider a new real valued function $G(m,n,\sigma)$ on the extended space,

$$G(m,n,\sigma)=\frac{1}{4N}\left\{\frac{2}{N}+\sigma\mathcal{W}(m,n)\right\}. \qquad (25)$$

It can be seen that $G(m,n,\sigma)$ is positive and satisfies the inequality,

$$\frac{1}{4N^2}\leq G(m,n,\sigma)\leq\frac{3}{4N^2}, \qquad (26)$$

and normalization condition,

$$\sum_{(m,n)\in\bar{J}_+\times\bar{J}_+,\ \sigma\in\{\pm1\}}G(m,n,\sigma)=1. \qquad (27)$$

The positivity of $G(m,n,\sigma)$ follows from the boundedness of the Wigner function Eq.(21). The average value of an observable $\mathcal{O}$ is given by

$$\langle\mathcal{O}\rangle=2N^2\sum_{m,n\in\bar{J}_+,\sigma\in B}\sigma\mathcal{O}(m,n)G(m,n,\sigma). \qquad (28)$$

It is natural to call $G(m,n,\sigma)$ as a distribution function on the extended space $\bar{J}_+\times\bar{J}_+\times B$. The time evolution of $G(m,n,\sigma)$ is given by

$$\frac{d}{dt}G(m,n,\sigma)=\sum_{m'n'\in\bar{J}_+,\sigma'\in\{\pm1\}}-\mathrm{sgn}(\sigma\sigma')$$
$$\tilde{\mathcal{H}}^{+}(m'',n'')\sin\left\{\frac{2\pi i}{N}\{2(m''n'-n''m')-\alpha(m'',n'')\}\right\}$$
$$\times\theta(m''+m',n''+n')G(m',n',\sigma'). \qquad (29)$$

Introducing a generating operator as

$$\mathcal{A}_t(m,n,\sigma;m',n',\sigma')=\tilde{\mathcal{H}}^{+}(m'',n'')\left[\frac{1}{G(m',n',\sigma')}\right.$$
$$-\mathrm{sgn}(\sigma\sigma')\times\sin\left\{\frac{2\pi i}{N}2(m''n'-n''m')\right.$$
$$\left.\left.-\frac{2\pi i}{N}\alpha(m'',n'')\right\}\theta(m''+m',n''+n')\right], \qquad (30)$$

for $(m,n)\neq(m',n')$,

$$\mathcal{A}_t(m,n,\sigma;m,n,\sigma')=0, \qquad (31)$$

for $(m,n)=(m',n')$, $\sigma\neq\sigma'$,

$$\mathcal{A}_t(m,n,\sigma;m,n,\sigma)=-\frac{2}{G(m,n,\sigma)}$$
$$\sum_{(m',n')\in\bar{J}_+\times\bar{J}_+\setminus(0,0)}\tilde{\mathcal{H}}^{+}(m',n') \qquad (32)$$

for $(m,n,\sigma)=(m',n',\sigma')$, the equation for $G(m,n,\sigma)$ can be written briefly as

$$\frac{d}{dt}G(m,n,\sigma)=\sum_{m',n'\in\bar{J}_+,\ \sigma\in\{\pm1\}}$$
$$\mathcal{A}_t(m,n,\sigma;m',n',\sigma')G(m',n',\sigma'). \qquad (33)$$

It can be checked that the generator $\mathcal{A}_t(m,n,\sigma;m',n',\sigma')$ satisfies the Markov condition,

$$\sum_{(m,n,\sigma)\in\bar{J}_+\times\bar{J}_+\times B}\mathcal{A}(m,n,\sigma;m',n',\sigma')=0, \qquad (34)$$

$$\mathcal{A}(m,n,\sigma;m',n',\sigma')\geq0, \qquad (35)$$

if $(m,n,\sigma)\neq(m',n',\sigma')$, which assures the existence of a background quantum Markov process on the extended lattice quantum phase space.

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