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Covering of surfaces parametrized without projective base points

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ABSTRACT

We prove that every affine rational surface, parametrized by means of an affine rational parametrization without projective base points, can be covered by at most three parametrizations. Moreover, we give explicit formulas for computing the coverings. We provide two different approaches: either covering the surface with a surface parametrization plus a curve parametrization plus a point, or with the original parametrization plus two surface reparametrizations of it.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms

General Terms

Algorithm

Keywords

Rational algebraic surface, parametrization coverings, base points

1. INTRODUCTION

When dealing with algebraic surfaces in applications, as for instance in Computer Aided Geometric Design, parametric representations play a very important role. Some examples are tracing, surface fitting, intersections of surfaces (see [9] and [7]). One important feature of a parametrization of a surface is *normality*, that is, the fact that its image is the whole algebraic surface. This is relevant for algorithms that produce information on the image, for example the calculation of the set of singular points as in [11]: a non-surjective parametrization would mean leaving out potential singularities of the surface if they happen to be outside the image of the parametrization. The same phenomenon can appear also when computing the intersection points of two surfaces where one of them is given parametrically.

Example 1. Consider the Steiner surface S (see Figure 1) given parametrically by

$$\left(\frac{s^2}{q(s,t)}, \frac{s^2+t^2}{q(s,t)}, \frac{s^2+st+s+t}{q(s,t)}\right), \quad q = s^2+t^2+s-t+1.$$

Its intersection with the plane $y = 1$ can easily be determined to be an ellipse. However, if we implicitize and substitute $y = 1$ in the equation of the surface we obtain

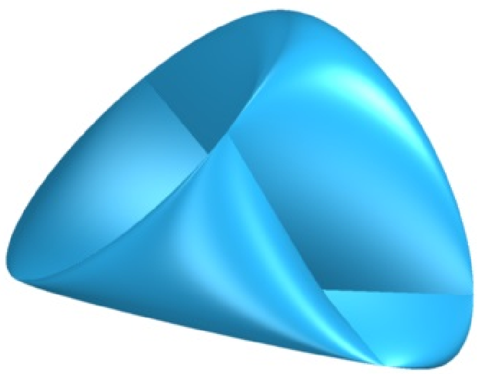

Figure 1: The Steiner surface S

$$(2x^2 - 2xz + z^2 - x)(2x^2 + 2xz + z^2 - 3x - 2z + 1) = 0$$

which is the union of two ellipses E_1, E_2 respectively (see Figure 2), the second factor being the equation of the intersection found originally. Thus the initial parametrization does not cover, at least, the ellipse E_1. See Example 6 in Section 4 for more details.

The problem of determination of the normality of surface parametrizations is a difficult one. Indeed, we do not have an example of a surface that cannot be parametrized surjectively. In other words, we do not know whether every rational surface can always be parametrized by means of a normal (surjective) rational parametrization.

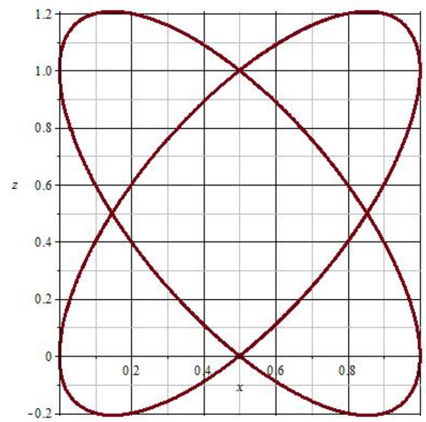

Figure 2: Intersection of S with the plane $y = 1$

This problem was estudied in [8] for algebraic varieties of arbitrary dimension over an algebraically closed field of characteristic zero. The method presented there is based on the Ritt-Wu's decomposition algorithm, and they provide normal parametrizations for conics and some quadrics. Normal parametrizations for the remaining quadrics are given in [2].

In [2] the authors also provide a method to construct normal parametrizations over the reals for parametrizations where no real point on the variety corresponds only to complex parameter values. They leave the study of the normality of parametrizations in the other case as an open problem. In the case that they solve, their method gives 2^n injective parametrizations that together cover the real part of the surface, including the points coming from points at infinity. In contrast, we will work over the complex field and, when no projective base points exist, we cover the surface with at most three parametrizations.

In [14] a complete analysis for the case of plane curves over fields of characteristic zero is presented. An extension to space curves can be found in [1]; alternative results for space curves are also in [12]. See also Sections 6.3. and 7.3 of [16]. On the other hand, in [10] the notion of pseudo-normality is introduced. This concept provides necessary conditions for a surface parametrization to be normal. Furthermore, in that article, algorithms for deciding pseudo-normality are given, and necessary and sufficient conditions for a pseudo-normal parametrization to be normal are provided. In particular, it is stated that pseudo-normal polynomial surface parametrizations are normal. The following two examples illustrate non-normality.

Example 2. One parametrization of the revolution cone $x^2 + y^2 = z^2$ is

$$\mathcal{P}(s,t) = \left(\frac{2st}{1+t^2}, \frac{s(1-t^2)}{1+t^2}, s \right),$$

where for each value of s we have a circle of radius s minus one point. The image of $\mathcal{P}$ is the cone minus the line $x = y + z = 0$, except for the origin which is indeed in the image (for $s = 0$ and any t), see Figure 3. This example shows that the set of missing points is not always Zariski closed. Indeed it is, in general, a constructible set on the surface.

Example 3. The parametrization given in Example 1 covers the surface S except the following set: the points in the

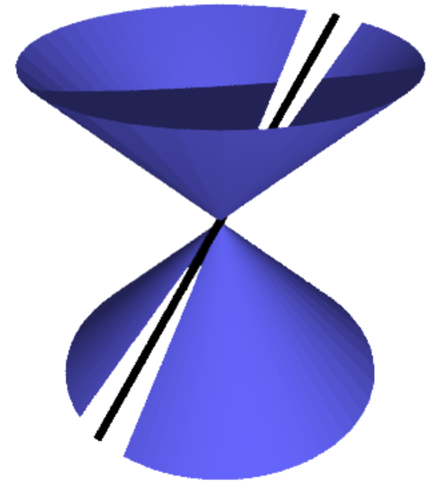

Figure 3: Missing line in the cone parametrization

ellipse $(y = 1) \wedge (2x^2 - 2xz + z^2 - x = 0)$ minus the points

$$\left(\frac{1}{2},\ 1,\ 0\right), \quad \left(\frac{1}{2} \mp \frac{\sqrt{2}}{4},\ 1,\ \frac{1}{2}\right).$$

Remark 1. In the particular case when $\mathcal{P}$ is proper (i.e. injective) with inverse $\mathcal{Q}$, the points not in the image are contained in the curves defined by the denominators of $\mathcal{P}$ and the denominators of $\mathcal{P}(\mathcal{Q})$; see Example 6. More precisely, one can proceed as follows:

1. Compute a representant of the inverse of $\mathcal{P}$; say
$$\mathcal{Q}(x,y,z) = \left(\frac{A_1(x,y,z)}{B_1(x,y,z)}, \frac{A_2(x,y,z)}{B_2(x,y,z)} \right).$$
2. Compute the denominators $D_i(x,y,z)$ of $\mathcal{P}(\mathcal{Q}(x,y,z))$.
3. The intersection of the algebraic surface and $V(\mathrm{lcm}(D_1, D_2, D_3, B_1, B_2))$ is a lower-dimensional algebraic set containing the set of non reachable points.

Remark 2. In any case, given a point (not necessarily on the surface), using Gröebner basis techniques it is simple (but possibly computationally demanding) to check if it belongs to the image of the parametrization. The same technique works if we want to test the points of a parametric curve on the surface:

1. A Gröbner basis is calculated as in the point case, but including the parameter c of the curve in the list of variables of the ring.
2. The basis behaves well under specialization, except at most at finitely many values of c (see in [6] the Extension Theorem, also Exercise 6.3.7 in p. 283). Therefore from the basis we decide the reachability of generic points of the curve by the surface parametrization.
3. One calculates individual Gröbner bases for the points not decidable by specialization of the previous basis. If the curve is not parametrized surjectively one can also test the missed points.

A variation of this technique solves the problem of testing all the points in a curve given implicitly. The problem of finding such a curve (or another set containing the missing points) is solved below.

Our approach to the problem of surjectivity of parametrizations is to cover the algebraic surface with a finite number of affine parametrizations. Indeed, that every rational surface can be covered by finitely many parametrizations is a consequence of [3]. Thus the problem becomes:

Problem. Given an affine parametrization of a surface, find a finite set of parametrizations such that the surface is contained in the union of their images.

Our coefficient field is algebraically closed of characteristic zero; for other fields (for example $\mathbb{R}$, of obvious interest) the curve case already suffers from complications that make the analysis very difficult, see [2], [14].

We will show how to produce a proper closed subset that contains the points missed by the parametrization, and how to cover it by additional parametrizations, under the assumption that the original parametrization does not have any base points in projective space. We recall that a projective base point is a projective parameter value $(s:t:v)$ where all numerators and denominators of the parametrization vanish; see Definition 2 in Section 2. The existence of base points in the surface parametrization is usually a difficulty in some problems as implicitization, moving surfaces analysis, etc. Only in some cases, like ruled surfaces, there has been progress in base point removal, see [5] and its reference [13]; see also [4] for the surface implicitization problem. Therefore in most situations it is assumed that the given parametrization has none base points; this is also our assumption. As an intermediate step, one can reparametrize in such a way that all affine base points are sent to infinity, see [15].

The main result of this paper is the following.

THEOREM 1. *An affine surface parametrization without projective base points covers the Zariski closure of its image except at most one rational curve, which can be described parametrically. Since a curve parametrization covers the whole algebraic curve except for at most one point, the surface can be covered with three parametrizations of dimensions 2, 1 and 0, respectively.*

It is of interest to cover the surface in such a way that every point is in the image of a two-dimensional parametrization (for example for local analysis). The following version provides this.

THEOREM 2 (ALTERNATE). *An affine surface parametrization without projective base points covers the Zariski closure of its image except at most one rational curve, which can be covered by at most two more surface parametrizations (reparametrizations of the given one). Therefore the surface can be covered with three bidimensional parametrizations.*

The parametrizations mentioned in the theorems above are explicitly constructed in Section 3.

The structure of the paper is as follows. In Section 2 we introduce basic notations, definitions and results. The main results are proved in Section 3.

2. PRELIMINARIES

Let us fix some notation through a few definitions.

Definition 1. Let k be an algebraically closed field of characteristic zero. and $S \subset k^3$ an affine algebraic surface. A *parametrization of S* is a triple of rational functions that determines a rational dominant map

$$\begin{array}{cccc} \mathcal{P}: & k^2 & \dashrightarrow & S \\ & (s,t) & \mapsto & \left(\dfrac{p_1(s,t)}{q(s,t)}, \dfrac{p_2(s,t)}{q(s,t)}, \dfrac{p_3(s,t)}{q(s,t)}\right). \end{array}$$

We assume w.l.o.g. that $\gcd(p_1,p_2,p_3,q)=1$. We denote as $\overline{S}$ the projective closure of S in $\mathbb{P}^3(k)$. The function $\mathcal{P}$ has a projective counterpart, $\overline{\mathcal{P}}$:

$$\begin{array}{cccc} \overline{\mathcal{P}}: & \mathbb{P}^2(k) & \dashrightarrow & \mathbb{P}^3(k) \\ & \mathbf{s}=(s:t:u) & \mapsto & (\overline{p_1}(\mathbf{s}):\overline{p_2}(\mathbf{s}):\overline{p_3}(\mathbf{s}):\overline{q}(\mathbf{s})) \end{array}$$

where the four components are the polynomial homogeneizations of the numerators and denominator of $\mathcal{P}$ such that their gcd is 1 and they have the same degree. Note that $\overline{\mathcal{P}}$ may be undefined at some points of $\mathbb{P}^2(k)$, since its four components may have a common zero.

Definition 2. The common zeros of the components of $\overline{\mathcal{P}}$ are called *projective base points*. Such a point $(s:t:u)$ is also called an *affine base point* if $u \neq 0$.

Since the gcd of the four homogeneous polynomials is 1, by Bézout's theorem it follows that there can be at most finitely many projective base points.

Definition 3. An (affine) surface parametrization is called *normal* if it is surjective on S, that is, for every $p \in S$ there exist $s_0, t_0 \in k$ such that $\mathcal{P}(s_0,t_0)=p$.

Definition 4. Let $\mathcal{P}$ be a parametrization that is not normal. A closed proper subset $C \subset S$ is called a *critical set of* $\mathcal{P}$ if $C \supset S \setminus \mathcal{P}(k^2)$.

Example 4. In Example 2 the line $x=y+z=0$ is a critical set of the parametrization, as is any other (reducible) curve in the cone containing that line.

Example 5. In Example 1, E_1 is a critical set.

In Theorem 1 the rational curve mentioned is a critical set of the parametrization. In the next section we give explicit descriptions of this curve.

3. MAIN RESULTS

We will use the notations introduced in the previous section.

THEOREM 3. *Let $\mathcal{P}$ be a non-normal parametrization of a surface S without projective base points. Then the rational curve*

$$\{\overline{\mathcal{P}}(s:t:0) \mid (s:t) \in \mathbb{P}(k)\} \cap S.$$

is a critical set.

PROOF. Since there are no projective base points, by [17, Theorem 5.2.2, p. 57], $\overline{\mathcal{P}}$ is surjective on $\overline{S}$. Therefore, every affine point of S is the image by $\overline{\mathcal{P}}$ of some point in $\mathbb{P}^2(k)$, but not necessarily the image by $\mathcal{P}$ of a point in k^2. The only points of S that may not be images of points in k^2 are therefore the images of the line at infinity:

$$\overline{\mathcal{P}}\left(\mathbb{P}^2(k) \setminus k^2\right) \cap S = \{\overline{\mathcal{P}}(s : t : 0) \mid (s : t) \in \mathbb{P}(k)\} \cap S$$

This is then a critical set. Since $\mathcal{P}$ is not normal, this set is not empty.

Recall that $\overline{\mathcal{P}}(s : t : u) = (\overline{p_1} : \overline{p_2} : \overline{p_3} : \overline{q})$ where the four components are homogeneous polynomials in s, t, u of the same degree n. Let $p \in S$ be the image of some $(s_0 : t_0 : 0)$. Then if

$$\overline{q} = Q_n(s,t) + Q_{n-1}(s,t) \cdot u + \cdots + Q_0 \cdot u^n$$

where each Q_i is homogeneous of degree i, necessarily $\overline{q}(s_0 : t_0 : 0) \neq 0$, so $Q_n(s_0, t_0) \neq 0$. Then $\deg q = n$ and we define $l_i = \deg q - \deg p_i \geq 0$ for $i = 1, 2, 3$. Using also the homogeneous forms of the $\overline{p_i}$ with respect to u we can write (the first subindex of each P is $1, 2, 3$ and the second subindex is its degree)

$$\overline{\mathcal{P}} = \Big(P_{1,n-l_1}(s,t) \cdot u^{l_1} + \cdots + P_{1,0} \cdot u^n : \ \ldots \ : \ \ldots \ :$$

$$: \underbrace{Q_n(s,t)}_{\neq 0} + \cdots + Q_0 \cdot u^n\Big).$$

Then at the line at infinity we have

$$\overline{\mathcal{P}}(s : t : 0) = (P_{1,n-l_1}(s,t) \cdot \delta_{l_1,0} : P_{2,n-l_2}(s,t) \cdot \delta_{l_2,0} :$$

$$: P_{3,n-l_3}(s,t) \cdot \delta_{l_3,0} : Q_n(s,t))$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. The set $C = \{\overline{\mathcal{P}}(s : t : 0) : s, t \in k\}$ is a rational projective curve provided that it does not degenerate to a point. But if it was constant the principal homogeneous forms would have to be proportional, and any pair (s,t) that was a root of one of them would automatically create a base point, contrary to the hypothesis.

Therefore C contains all the points that may not be images of affine parameter values, so $C \cap S$ is a critical set. Since $Q_n \neq 0$, it is also an affine rational curve. $\square$

Now that we have an explicit description of a critical set, we can cover the surface parametrically.

COROLLARY 1. *Using the notations in the previous theorem, let*

$$C_1(s) = \left(\frac{P_{1,n-l_1}(s,1)}{Q_n(s,1)} \cdot \delta_{l_1,0},\ \frac{P_{2,n-l_2}(s,1)}{Q_n(s,1)} \cdot \delta_{l_2,0},\right.$$

$$\left.\frac{P_{3,n-l_3}(s,1)}{Q_n(s,1)} \cdot \delta_{l_3,0}\right),$$

and

$$p = \left(\frac{P_{1,n-l_1}(1,0)}{Q_n(1,0)} \cdot \delta_{l_1,0},\ \frac{P_{2,n-l_2}(1,0)}{Q_n(1,0)} \cdot \delta_{l_2,0},\right.$$

$$\left.\frac{P_{3,n-l_3}(1,0)}{Q_n(1,0)} \cdot \delta_{l_3,0}\right).$$

Then $S = \mathcal{P}(k^2) \cup C_1(k) \cup \{p\}$.

PROOF. The projective curve C at the end of the proof of the theorem covers $S \setminus \mathcal{P}(k^2)$, and $C_1 \cup p$ are the affine points of C. $\square$

Remark 3. Every algebraic curve parametrization admits a normal reparametrization, possibly at the cost of extending the coefficient field; see [16, Theorem 6.26] or [14, Theorem 3].

On the other hand, it is possible to cover the whole surface with two-dimensional paramerizations. This may be useful when analyzing the behavior of the surface around the point.

THEOREM 4. *In the hypotheses of the previous theorem, let*

$$\mathcal{P}'(s,t) = \mathcal{P}\left(\frac{s}{t}, \frac{1}{t}\right), \quad \mathcal{P}'' = \mathcal{P}\left(\frac{1}{s}, t\right).$$

Then $S = \mathcal{P}(k^2) \cup \mathcal{P}'(k^2) \cup \mathcal{P}''(k^2)$.

PROOF. Recall that

$$\mathcal{P}(s,t) = \left(\frac{P_{1,n-l_1}(s,t) + P_{1,n-l_1-1}(s,t) + \cdots}{Q_n(s,t) + Q_{n-1}(s,t) + \cdots},\ \ldots,\ \ldots\right).$$

The birational map $(s,t) \leftarrow (\frac{s}{t}, \frac{1}{t})$ converts it to

$$\mathcal{P}'(s,t) = \left(\frac{\dfrac{P_{1,n-l_1}(s,1)}{t^{n-l_1}} + \dfrac{P_{1,n-l_1-1}(s,1)}{t^{n-l_1-1}} + \cdots}{\dfrac{Q_n(s,1)}{t^n} + \dfrac{Q_{n-1}(s,1)}{t^{n-1}} + \cdots},\ \ldots,\ \ldots\right)$$

$$= \left(\frac{t^{l_1}\left(P_{1,n-l_1}(s,1) + t \cdot P_{1,n-l_1-1}(s,1) + \cdots\right)}{Q_n(s,1) + t \cdot Q_{n-1}(s,1) + \cdots},\ \ldots,\ \ldots\right)$$

but for $t = 0$ we obtain precisely the curve C_1 of Corollary 1.

Now, let $P_{i,n-l_i}(s,t) = a_{i,n_i} s^{n_i} + \cdots$ for $i = 1, 2, 3$ and $Q_n(s,t) = d_n s^n + \cdots$. Then the point p in Corollary 1 is precisely

$$\left(\frac{a_{1,n_1}}{d_n} \cdot \delta_{l_1,0},\ \frac{a_{2,n_2}}{d_n} \cdot \delta_{l_2,0},\ \frac{a_{3,n_3}}{d_n} \cdot \delta_{l_3,0}\right)$$

but the birational map $(s,t) \leftarrow (\frac{1}{s}, t)$ converts $\mathcal{P}$ to

$$\mathcal{P}''(s,t) = \left(\frac{s^{s-n_1}\left(a_{1,n_1} + s \cdot (\cdots)\right)}{d_n + s \cdot (\cdots)},\ \ldots,\ \ldots\right)$$

which covers p when $s = 0$. $\square$

Remark 4. At the end of Example 6 it is shown that $\mathcal{P}$ and $\mathcal{P}'$ may not suffice to cover all the surface.

We now point out the following convenient normality conditions which are obtained from the proof of Theorem 3.

THEOREM 5. *If $\mathcal{P}$ has no projective base points and it holds that* $\max(\deg p_i)_{i=1,2,3} > \deg q$ *then $\mathcal{P}$ is normal.*

COROLLARY 2. *If $\mathcal{P}$ has no projective base points and is polynomial then it is normal.*

4. EXAMPLES

In this section, we illustrate the previous ideas with two examples.

Example 6. We take once more the parametrization in Example 1:

$$\left(\frac{s^2}{q(s,t)},\ \frac{s^2+t^2}{q(s,t)},\ \frac{s^2+st+s+t}{q(s,t)}\right),\quad q=s^2+t^2+s-t+1.$$

It has no projective base points, since in projective space

$$V(s^2, s^2+t^2, s^2+st+su+tu, s^2+t^2+su-tu+u^2)=\emptyset.$$

A Gröebner basis computation proves that it is proper. In order to produce its inverse one has several choices of elements of the basis to solve for; the resulting denominators vary, the simplest ones being

$$(z-1)(y-1)(3y-2z-2)^2z^2,\qquad (y-1)(3y-2z-2)z.$$

Thus by Remark 1 one critical set is the intersection of S with the product of those denominators (geometrically, the union of several plane sections of S) and the corresponding denominators after substituting the inverse in the parametrization. One of the resulting curves is the section with $y=1$, which contains the two ellipses shown in Example 1.

Using the results described above we will find a smaller critical set. Following Corollary 1, considering

$$P_{1,0}=s^2,\quad P_{2,0}=s^2+t^2,\quad P_{3,0}=s^2+st,\quad Q_2=s^2+t^2$$

one critical set is the union of the curve $C_1(s)$ and the point p given by

$$C_1(s)=\left(\frac{s^2}{s^2+1},\ 1,\ \frac{s^2+s}{s^2+1}\right),\quad p=(1,1,1).$$

Implicitizing, $C_1\cup\{p\}$ is precisely E_1 in Example 1.

Following Remark 2 another Gröebner basis computation provides the points in C_1 not attainable by $\mathcal{P}$. Using the implicit equation of E_1 together with $y=1$, elimination yields the equation $(2t-1)(2t^2-1)=0$, thus $t=1/2$, $t=\pm1/\sqrt{2}$ (with corresponding values for s obtainable from the Gröebner basis) giving rise to the points

$$\left(\frac{1}{2},\ 1,\ 0\right),\quad \left(\frac{1}{2}\mp\frac{\sqrt{2}}{4},\ 1,\ \frac{1}{2}\right)$$

which are the only ones in the image of C_1 that are attainable by $\mathcal{P}$. It remains to check the point p, which turns out not to be in the image of $\mathcal{P}$. Note that $p\neq C_1(s)$ for every value of s, and so we need C_1 and p to cover the missing points of $\mathcal{P}$.

Finally we construct the bidimensional parametrizations given in Theorem 4, namely,

$$\mathcal{P}(s,t),\quad \mathcal{P}'(s,t)=\mathcal{P}\left(\frac{s}{t},\frac{1}{t}\right),\quad \mathcal{P}''(s,t)=\mathcal{P}\left(\frac{1}{s},t\right).$$

Yet another Gröbner basis computation proves that the point p is not in the image of $\mathcal{P}'$, proving that the third parametrization in Theorem 4 is necessary. Indeed, $(1,1,1)=\mathcal{P}''(0,t)$ for every value of t; note that

$$\mathcal{P}''=\left(\frac{1}{s^2t^2-s^2t+s^2+s+1},\frac{s^2t^2+1}{s^2t^2-s^2t+s^2+s+1},\frac{s^2t+st+s+1}{s^2t^2-s^2t+s^2+s+1}\right).$$

Example 7. Consider again the cone parametrized in Example 2. We compute its projective base points:

$$V(2stu,\ s(u^2-t^2),\ s(u^2+t^2),\ u^3+t^2u)=$$
$$\{(0:1:0),(1:0:0),(0:i:1),(0:-i:1)\}.$$

If we try to apply Corollary 1, the denominators become zero. Interestingly, it turns out that the line given in Example 2 as a critical set is the seam curve corresponding to $(0:1:0)$. This can be checked by considering the pencil of lines through this projective point and calculating the limits as one approaches it in different directions.

5. ACKNOWLEDGMENTS

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