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# Behavior of the Base Points of Parametrizations under Projections\*

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## Abstract

Given a rational parametrization  $\mathcal{P}(\bar{t})$ ,  $\bar{t} = (t_1, \dots, t_r)$ , of an  $r$ -dimensional unirational variety, we analyze the behavior of the variety of the base points of  $\mathcal{P}(\bar{t})$  in connection to its generic fibre, when successively eliminating the parameters  $t_i$ . For this purpose, we introduce a sequence of generalized resultants whose primitive and content parts contain the different components of the projected variety of the base points and the fibre. In addition, when the dimension of the base points is strictly smaller than 1 (as in the well known cases of curves and surfaces), we show that the last element in the sequence of resultants is the univariate polynomial in the corresponding Gröbner basis of the ideal associated to the fibre; assuming that the ideal is in  $t_1$ -general position and radical.

## 1 Introduction

We start this introduction by describing and motivating the concepts of fibre and base point as well as its relations. We consider an algebraically close field  $\mathbb{K}$  of characteristic zero and a unirational variety  $\mathcal{V} \subset \mathbb{K}^n$  of dimension  $r = \dim(\mathcal{V}) < n$ . With unirational we mean that there exists a tuple of rational functions (i.e. a rational parametrization)

$$\mathcal{P}(\bar{t}) = \left( \frac{p_1(\bar{t})}{q(\bar{t})}, \dots, \frac{p_n(\bar{t})}{q(\bar{t})} \right) \in \mathbb{K}(\bar{t})^n, \text{ where } \bar{t} = (t_1, \dots, t_r)$$

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(say w.l.o.g. that  $\gcd(p_1, \dots, p_n, q) = 1$  and that none  $p_i/q$  is constant) depending on  $r$  independent parameters,  $t_1, \dots, t_r$ , such that the rank of the jacobian of  $\mathcal{P}(\bar{t})$  is  $r$ , and such that

- for almost all (i.e. for a non-empty open Zariski subset of  $\mathcal{V}$ ) points  $P \in \mathcal{V}$  there exists, at least one,  $\bar{t}^0 \in \mathbb{K}^r$  such that  $P = \mathcal{P}(\bar{t}^0)$ , and
- for all  $\bar{t}^0$ , where  $\mathcal{P}(\bar{t})$  is defined,  $\mathcal{P}(\bar{t}^0) \in \mathcal{V}$ .

Associated with  $\mathcal{P}(\bar{t})$  we want to define a map. Clearly it can be done as follows:

$$\begin{aligned} \Phi_{\mathcal{P}} : \mathbb{K}^r \setminus \Lambda &\rightarrow \mathcal{V} \\ \bar{t}^0 &\mapsto \mathcal{P}(\bar{t}^0) \end{aligned}$$

where  $\Lambda = \{\bar{t} \in \mathbb{K}^r \mid q(\bar{t}) = 0\}$ . Note that, because of the two requirements above,  $\Phi_{\mathcal{P}}$  is well defined and  $\Phi_{\mathcal{P}}(\mathbb{K}^r \setminus \Lambda)$  is dense (in the Zariski topology) in  $\mathcal{V}$ . At this point, the first natural questions appear: what is the cardinality of  $\Phi_{\mathcal{P}}^{-1}(P)$  for a generic point  $P \in \Phi_{\mathcal{P}}(\mathbb{K}^r \setminus \Lambda)$ ? If this cardinality is bigger than 1, can we replace  $\mathcal{P}(\bar{t})$  by another parametrization where this cardinality is 1?

The cardinality of  $\Phi_{\mathcal{P}}^{-1}(P)$  is the **degree** of  $\Phi_{\mathcal{P}}$  (see e.g. [9] for a formal definition) and the set  $\Phi_{\mathcal{P}}^{-1}(P)$  is called the **(generic) fibre** of  $\Phi_{\mathcal{P}}$ ; we denote the fibre by  $\mathfrak{F}_{\mathcal{P}}(\bar{h})$ , where the generic point  $P$  has been taken as  $\mathcal{P}(\bar{h})$ , being  $\bar{h}$  a new tuple of parameters. When the degree of  $\Phi_{\mathcal{P}}$  is 1, we say that the parametrization is **rational** and, hence, the second question asks whether the concepts of rationality and unirationality are equivalent; question that is related to Lüroth's theorem and Castelnuovo's theorem and that we do not deal with here.

Coming back to the definition of  $\Phi_{\mathcal{P}}$ , one may try to get information on the parameter values in  $\Lambda$ . Intuitively, they might be related to the points of  $\mathcal{V}$  at infinity. For a deeper analysis of this, we pass to the projective space. That is, we consider the projective closure  $\mathcal{V}^H$  of  $\mathcal{V}$ . Moreover we consider the projective parametrization  $\mathcal{P}^H(\bar{t}^H)$  associated with  $\mathcal{P}(\bar{t})$ ; that is,  $\bar{t}^H = (t_0 : t_1 : \dots : t_r)$  and

$$\mathcal{P}^H(\bar{t}^H) = (p_1^H(\bar{t}^H) : \dots : p_n^H(\bar{t}^H) : q^H(\bar{t}^H)),$$

where  $p_i^H, q^H$  are the homogenization of  $p_i, q$ , respectively, multiplied by a suitable power of  $t_0$  such that all the homogeneous polynomials  $p_i^H, q^H$  have the same degree and  $\gcd(p_1^H, \dots, p_n^H, q^H) = 1$ . In this situation, we try to define a projective map, using  $\mathcal{P}^H(\bar{t}^H)$ , from  $\mathbb{P}^r(\mathbb{K})$  on  $\mathcal{V}^H$ . This can be done as follows:

$$\begin{aligned} \Phi_{\mathcal{P}^H} : \mathbb{P}^r(\mathbb{K}) \setminus \mathcal{B}(\mathcal{P}^H) &\rightarrow \mathcal{V}^H \\ \bar{t}^{H,0} &\mapsto \mathcal{P}^H(\bar{t}^{H,0}) \end{aligned}$$

where  $\mathcal{B}(\mathcal{P}^H) = \{\bar{t}^H \in \mathbb{P}^r(\mathbb{K}) \mid p_1^H(\bar{t}^H) = \dots = p_n^H(\bar{t}^H) = q^H(\bar{t}^H) = 0\}$ . The points in  $\mathcal{B}(\mathcal{P}^H)$  are called the **(projective) base points** of  $\mathcal{P}^H(\bar{t}^H)$ . Since, our starting object

was  $\mathcal{P}(\bar{t})$ , we are interested in  $\mathcal{B}(\mathcal{P}^H) \cap \{\bar{t}^H \in \mathbb{P}^r(\mathbb{K}) \mid t_0 \neq 0\}$ , that we can identify with the set  $\{\bar{t} \in \mathbb{K}^r \mid p_1(\bar{t}) = \dots = p_n(\bar{t}) = q(\bar{t}) = 0\}$ ; in the next paragraph we will extend this definition considering this variety defined over a superfield of  $\mathbb{K}$ . We call these points the **(affine) base points** of  $\mathcal{P}(\bar{t})$ , and we denote the set of affine base points by  $\mathfrak{B}(\mathcal{P})$ . These are the points we deal with here; see e.g. [15] for further comments on projective base points.

But, how are the base points related to the generic fiber? In order to define the fibre we have considered a new tuple of parameters  $\bar{h}$ . Let  $\mathbb{F}$  be the algebraic closure of  $\mathbb{K}(\bar{h})$ . Then, the generic fibre consists in those  $\bar{t}^0 \in \mathbb{F}^r$  such that  $\mathcal{P}(\bar{t}^0) = \mathcal{P}(\bar{h})$ . Therefore if  $V_1$  is variety defined by the polynomials  $\{p_i(\bar{t})q(\bar{h}) = q(\bar{t})p_i(\bar{h})\}_{i=1,\dots,r}$  and  $V_2$  the the variety defined by  $\{q(\bar{t})\}$ , both over  $\mathbb{F}$ , then

$$\mathfrak{F}_{\mathcal{P}}(\bar{h}) = V_1 \setminus V_2.$$

On the other hand, the base points is the variety defined over  $\mathbb{K}$  by  $\{p_1, \dots, p_r, q\}$ . Now, let us see the  $\mathfrak{B}(\mathcal{P})$  defined over  $\mathbb{F}$  instead that over  $\mathbb{K}$ ; we call it again  $\mathfrak{B}(\mathcal{P})$ . Then

$$\mathfrak{F}_{\mathcal{P}}(\bar{h}) = V_1 \setminus \mathfrak{B}(\mathcal{P}).$$

Beside the above motivation on the definability of the rational map associated to the parametrization, why are the base points so important? The computation of the fibre, the degree of the map, the implicit equations, the singularities of  $\mathcal{V}$ , etc, all these questions can be translated to elimination theory problems and, consequently, approached by means of Gröbner bases or characteristic sets. Nevertheless, many authors have been and are trying to approach these associated problems by means of resultants (classical resultants,  $u$ -resultants, multivariate resultants, etc) appearing, for instance, in the development of the  $\mu$ -base, moving curves and moving surfaces theory, etc. However, most of these methods, hit difficulties under the presence of base points (see [1], [2], [3], [4], [5], [6], [7], [14], [15]). Nevertheless, for the surface case ( $r = 2$ ), our approaches, based on generalized resultants, to compute the implicit equation as well as the degree of the map (see [10], [11], [12]) do work even under the presence of base points. All these algorithmic methods play a crucial role in many applications, as for instance in computer aided geometric design, and therefore the theoretical understanding of the base points helps in the improvement of these potential practical applications.

Let us take a closer look at the base points of  $\mathcal{P}(\bar{t})$ . If  $r = 1$ , since  $\gcd(p_1, \dots, p_n, q) = 1$ , one has that  $\mathfrak{B}(\mathcal{P}) = \emptyset$ . So the curve case is trivial. If  $r = 2$ ,  $\mathfrak{B}(\mathcal{P})$  is either empty or consists in the intersection points of  $(n + 1)$  plane curves without common components, namely those defined by  $p_1(t_1, t_2), \dots, p_n(t_1, t_2), q(t_1, t_2)$ . Therefore, if  $r = 2$  then either  $\mathfrak{B}(\mathcal{P}) = \emptyset$  or  $\dim(\mathfrak{B}(\mathcal{P})) = 0$ . The situation is more complicated when  $r > 2$  since  $\mathfrak{B}(\mathcal{P})$  is the intersection of  $(n + 1)$   $(r - 1)$ -dimensional

varieties without common components. Thus, either  $\mathfrak{B}(\mathcal{P}) = \emptyset$  or  $\dim(\mathfrak{B}(\mathcal{P})) < r - 2$ . In Example 1,  $n = 5$ ,  $r = 3$  and  $\dim(\mathfrak{B}(\mathcal{P})) = 1$ , and in Example 4,  $n = 5$ ,  $r = 3$  and  $\dim(\mathfrak{B}(\mathcal{P})) = 0$ .

Motivated by this last fact, in this paper, we analyze the extension of the ideas in [11], [12] to the case where  $r > 2$ . We introduce a sequence of generalized resultants associated to  $\mathcal{P}(\bar{t})$  that ends in a univariate polynomial in  $t_1$  (see Section 4), and such that allows us to study how the successive projections of the points in the generic fibre of  $\mathcal{P}(\bar{t})$ , as well as of the points in  $\mathfrak{B}(\mathcal{P})$ , behave (see Section 5). The fibre is zero-dimensional and hence its projections. However, the base points variety may have high dimensional components such that their projections fill the whole projection space, and hence the information with the fiber is lost. For instance, in Example 1 where  $n = 5$ ,  $r = 3$  and  $\dim(\mathfrak{B}(\mathcal{P})) = 1$ , the second successive projection (i.e.  $(t_1, t_2, t_3) \mapsto (t_1, t_2) \mapsto t_1$ ) yields to the whole line  $\mathbb{K}$ . To avoid this phenomenon, at each elimination step, one has to detect the hypersurface components of the projection of the base points and excluded them from the process. More precisely, say that we have eliminated  $t_i, \dots, t_r$  and we proceed to eliminate  $t_{i+1}$ . Then, the corresponding generalized resultant factors as its content times its primitive part. Associated to each factor we introduce a variety in the  $i$ -dimensional space; let us call them  $\mathfrak{C}_i$  and  $\mathfrak{M}_i$ , respectively. Alternatively, the fibre and  $\mathfrak{B}(\mathcal{P})$  are also projected onto the same space. Then the behavior of the projections of the fibre and the base points is essentially as follows:

- the fibre projects into the primitive part variety  $\mathfrak{M}_i$  (see Theorem 4),
- the hypersurface components (if any) of the base points project into the content part variety  $\mathfrak{C}_i$  (see Theorem 3),
- while the low dimensional components of the base points go into  $\mathfrak{M}_i$  (see Theorem 4).

For analyzing the next elimination step, we control and indeed exclude the components embedded in  $\mathfrak{C}_i \cap \mathfrak{M}_i$ . All these problems are studied in Section 5. Finally, in Section 6, we prove that in the cases where either  $\mathfrak{B}(\mathcal{P}) = \emptyset$  or  $\dim(\mathfrak{B}(\mathcal{P})) = 0$ , the last element in the sequence of resultants is the univariate polynomial in the corresponding Gröbner basis of the ideal associated to the fibre; assuming that the ideal is in  $t_1$ -general position and radical. This can be seen as a generalization of the results for curves and surfaces in [11] to the case  $r > 3$ .

We cannot finish this introduction without saying that, although from our ideas one can derive an algorithm for the computation of the degree (when  $\dim(\mathfrak{B}(\mathcal{P})) < 1$ ), this algorithm is not efficient in its current form. And we should emphasize that our main goal in this paper, being more theoretical than algorithmic, is to provide the first

steps towards the establishment of the theoretical framework to better understanding the behavior of the base points.

Parts of our proofs presented in this paper are very technical. So, we leave most of the details of the reasonings to Section 7.

## 2 Notation and Preliminary Remarks

Throughout this paper, we will use the following notation and terminology.  $\mathbb{K}$  is an algebraically closed field of characteristic zero.  $\mathcal{V} \subset \mathbb{K}^n$  is a unirational algebraic variety, of dimension  $r = \dim(\mathcal{V})$ , rationally parametrized by

$$\mathcal{P}(\bar{t}) = \left( \frac{p_1(\bar{t})}{q(\bar{t})}, \dots, \frac{p_n(\bar{t})}{q(\bar{t})} \right) \in \mathbb{K}(\bar{t})^n,$$

where  $\bar{t} = (t_1, \dots, t_r)$ , and such that  $\gcd(p_1, \dots, p_n, q) = 1$ .

**Remark 1.** We **assume** (see below) that none of the rational functions  $p_i/q$  is constant. Furthermore, although the reasonings in this paper can be adapted for  $r \in \{1, 2\}$ , for simplicity in the explanation, we **assume** that  $r > 2$ ; note that essentially case  $r = 1$  is treated in [16] and  $r = 2$  in [11].

Associated with  $\mathcal{P}(\bar{t})$ , we have the rational map

$$\begin{aligned} \Phi_{\mathcal{P}} : \mathbb{K}^r &\longrightarrow \mathcal{V} \\ \bar{t} &\longmapsto \mathcal{P}(\bar{t}). \end{aligned}$$

Observe that  $\Phi_{\mathcal{P}}(\mathbb{K}^r)$  is dense in  $\mathcal{V}$  and that the jacobian of  $\Phi_{\mathcal{P}}$  has rank  $r$ ; being both remarks a consequence of the fact that  $\mathcal{P}(\bar{t})$  is a rational parametrization. We denote by  $\deg(\Phi_{\mathcal{P}})$  the degree of  $\Phi_{\mathcal{P}}$  (see Section 1).

Moreover, we consider the following polynomials, where the new variables  $\bar{h} = (h_1, \dots, h_r)$  and  $\bar{Z} = (Z_1, \dots, Z_{n-2})$  are introduced; note that  $n - 2 > 1$ :

- ★  $G_i(\bar{t}, \bar{h}) = p_i(\bar{t})q(\bar{h}) - p_i(\bar{h})q(\bar{t}) \in \mathbb{K}[\bar{h}][\bar{t}]$ , for  $i \in \{1, \dots, n\}$ ,
- ★  $G(\bar{t}, \bar{h}, \bar{Z}) = G_2(\bar{t}, \bar{h}) + Z_1 G_3(\bar{t}, \bar{h}) + \dots + Z_{n-2} G_n(\bar{t}, \bar{h}) \in \mathbb{K}[\bar{h}, \bar{Z}][\bar{t}]$ .

**Proposition 1.** *If  $p_i/q$  is not constant (see Remark 1), then  $G_i$  is not constant.*

*Proof.*  $G_i$  is identically zero iff  $p_i/q$  is constant. Thus  $G_i$  is not zero. Now, if  $G_i(\bar{t}, \bar{h}) = \lambda \in \mathbb{K}$ , then  $0 = G_i(\bar{t}, \bar{t}) = \lambda$  which is impossible because of our previous remark.  $\square$

For a field  $\mathbb{L}$  we denote by  $\overline{\mathbb{L}}$  its algebraic closure. Let  $\mathbb{F} = \overline{\mathbb{K}(\overline{h})}$ . Moreover, if  $\mathcal{G}$  is a finite set of polynomials over  $\mathbb{L}$ , we represent by  $\mathbb{V}_{\mathbb{L}}(\mathcal{G})$ , the algebraic variety defined by  $\mathcal{G}$  over  $\overline{\mathbb{L}}$ . We introduce the algebraic sets:

- ★ for each  $i \in \{1, \dots, n\}$ ,  $\mathcal{W}_i^{\overline{h}} = \mathbb{V}_{\mathbb{F}}(G_i) \subset \mathbb{F}^r$ .
- ★  $\mathcal{W}_{n+1} = \mathbb{V}_{\mathbb{F}}(q) \subset \mathbb{F}^r$ ; note that  $\mathcal{W}_{n+1}$  is empty if and only if  $\mathcal{P}(\overline{t})$  is a polynomial parametrization.
- ★ We denote by  $\mathfrak{B}(\mathcal{P})$  the algebraic set of base points of the parametrization  $\mathcal{P}(\overline{t})$ , i.e.  $\mathfrak{B}(\mathcal{P})$  is the variety defined by  $\{p_1, \dots, p_n, q\}$ . We will see  $\mathfrak{B}(\mathcal{P})$ , as we have done with  $\mathcal{W}_{n+1}$ , embedded in  $\mathbb{F}^r$ . So  $\mathfrak{B}(\mathcal{P}) = \mathbb{V}_{\mathbb{F}}(\{p_1, \dots, p_n, q\})$ . Note that then

$$\mathfrak{B}(\mathcal{P}) = \mathcal{W}_1^{\overline{h}} \cap \dots \cap \mathcal{W}_n^{\overline{h}} \cap \mathcal{W}_{n+1}$$

For every  $\overline{\alpha} \in \mathbb{K}^r$  such that  $\mathcal{P}(\overline{\alpha})$  is defined, we denote by  $\mathfrak{F}_{\mathcal{P}}(\overline{\alpha})$  the fibre of  $\overline{\alpha}$  via  $\Phi_{\mathcal{P}}$ ; i.e

$$\mathfrak{F}_{\mathcal{P}}(\overline{\alpha}) = \{\overline{t} \in \mathbb{K}^r \mid \mathcal{P}(\overline{t}) = \mathcal{P}(\overline{\alpha})\}.$$

Note that  $\deg(\Phi_{\mathcal{P}})$  is the cardinality of a generic fibre.

In addition, we consider a non-empty open Zariski set of  $\mathbb{K}^r$ , that we denote by  $\Omega(\mathcal{P})$ , such that for  $\overline{\alpha} \in \Omega(\mathcal{P})$  it holds that  $\text{card}(\mathfrak{F}_{\mathcal{P}}(\overline{\alpha})) = \deg(\Phi_{\mathcal{P}})$  (see Theorem 7.16 in [9]). Abusing of the notation, we will denote by  $\mathfrak{F}_{\mathcal{P}}(\overline{h})$  the generic fibre

$$\mathfrak{F}_{\mathcal{P}}(\overline{h}) = \{\overline{t} \in \mathbb{F}^r \mid \mathcal{P}(\overline{t}) = \mathcal{P}(\overline{h})\}.$$

Note that  $\mathfrak{F}_{\mathcal{P}}(\overline{h}) = (\mathcal{W}_1^{\overline{h}} \cap \dots \cap \mathcal{W}_n^{\overline{h}}) \setminus \mathfrak{B}(\mathcal{P})$ ; see e.g. Theorem 2 in [16].

Finally, if  $A$  is a subset of an affine space, we will denote by  $A^*$  its Zariski closure. Moreover, for a polynomial  $g(x)$  with coefficients over a unique factorization domain, we denote by  $\text{LCoeff}(g, x)$  its leading coefficient w.r.t.  $x$ .

### 3 General Assumptions and Preliminary Results

Throughout this paper, we assume (see also Remark 1) the following **general assumptions**:

A-1 None of the rational function  $p_i/q$  is constant and  $\gcd(p_1, \dots, p_n, q) = 1$ .

A-2 Let  $\mathcal{M}$  be the subset of those polynomials in  $\{p_1, \dots, p_n, q\}$  that are not constant. We assume that the hypersurfaces in  $\mathbb{K}^r$ , defined by each of the polynomials in  $\mathcal{M}$ , do not pass through the point at infinity  $(0 : \dots : 0 : 1 : 0)$ , where the homogeneous variables are  $(t_1, \dots, t_r, w)$ ; note that this is equivalent to require that for every  $g \in \mathcal{M}$  it holds that  $\deg_{t_r}(g)$  is the total degree of  $g$  and  $\text{LCoeff}(g, t_r) \in \mathbb{K}$ .

A-3  $G_1$  does not divide  $G_3$ .

These assumptions imply the proposition.

**Proposition 2.** *If assumptions A-1 and A-2 hold, the following statements hold*

(i) *Let  $G_i^H(\bar{t}, w, \bar{h})$  denote the homogenization of  $G_i(\bar{t}, \bar{h})$  as a polynomial in  $\mathbb{K}[\bar{h}][\bar{t}]$ . Then  $G_i^H(0, \dots, 0, 1, 0, \bar{h}) \neq 0$  for  $i = 1, \dots, n$ .*

(ii) *For  $i = 1, \dots, n$ ,  $\deg_{t_r}(G_i) > 0$  and  $\text{LCoeff}(G_i, t_r) \in \mathbb{K}[\bar{h}]$ .*

*Proof.* (i) If either  $p_i$  or  $q$  is constant, it follows from A-2. Otherwise, homogenizing and taking into account the total degrees of  $p_i$  and  $q$ , the result follows from A-1, A-2. (ii) We express  $G_i^H$  as  $g_m(\bar{t}, \bar{h}) + \dots + g_0(\bar{t}, \bar{h})w^m$ , where  $g_i$  is homogeneous in  $\bar{t}$  of degree  $i$ . By (i)  $g_m$  does depend on  $t_r$  and  $\alpha = \text{LCoeff}(G_i, t_r)$  only depends on  $\bar{h}$ .  $\square$

Next, we see that the above assumptions do not imply any loss of generality.

A-1. Say w.l.o.g. that  $\mathcal{P}(\bar{t}) = (p_1/q, \dots, p_s/q, \lambda_{s+1}, \dots, \lambda_n)$  with  $\lambda_i \in \mathbb{K}$  and  $p_i/q$  non-constant, we consider the projection

$$\pi : \mathcal{V} \rightarrow \pi(\mathcal{V})^*; (x_1, \dots, x_n) \mapsto (x_1, \dots, x_s)$$

and the parametrization  $\mathcal{H}(\bar{t}) = \pi(\mathcal{P}(\bar{t}))$  of  $\pi(\mathcal{V})^*$ . Now, since  $\pi$  is birational then  $\deg(\Phi_{\mathcal{P}}) = \deg(\Phi_{\pi(\mathcal{P})})$ . Therefore, we can work with  $\mathcal{H}(\bar{t})$  where A-1 holds.

A-2. For every  $g \in \mathcal{M}$ , let  $\text{tdeg}(g)$  denote the total degree of  $g$ , and let  $g_{\text{tdeg}(g)}(\bar{t})$  denote the homogeneous form of maximum degree of  $g(\bar{t})$ ; i.e. of degree  $\text{tdeg}(g)$ . Let  $\bar{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{K}^n$ , with  $\alpha_r \neq 0$ , be such that  $g_{\text{tdeg}(g)}(\bar{\alpha}) \neq 0$  for all  $g \in \mathcal{M}$ ; note that, by definition,  $g_{\text{tdeg}(g)}$  is not identically zero, and hence  $\bar{\alpha}$  always exists. We then consider the linear parameter change  $\bar{t} = L(\bar{t}^*)$  defined by  $\bar{t} = (t_1^* + \alpha_1 t_r^*, \dots, t_{r-1}^* + \alpha_{r-1} t_r^*, \alpha_r t_r^*)$ . Now, for all  $g \in \mathcal{M}$ , it holds that  $g(L(\bar{t}^*))$  is not constant, and  $\text{tdeg}(g) = \text{tdeg}(g(L))$ . Moreover, the homogeneous form of  $g(L)$  of degree  $\text{tdeg}(g(L))$  is of the form

$$g_{\text{tdeg}(g)}(\bar{\alpha})(t_r^*)^{\text{tdeg}(g)} + h(t_1^*, \dots, t_{r-1}^*).$$

Since  $g_{\text{tdeg}(g)}(\bar{\alpha}) \neq 0$ , A-2 holds for  $\mathcal{Q}(\bar{t}^*) = \mathcal{P}(L(\bar{t}^*))$ . Moreover,  $\deg(\Phi_{\mathcal{P}}) = \deg(\Phi_{\mathcal{Q}})$ .

A-3. This assumption is not used till Section 4. Moreover, in Remark 2, we see that as a consequence of the other assumptions there always exists  $G_i$  such that  $G_1$  does not divide  $G_i$ . Hence a simple change of coordinates yields to the required condition.

Therefore, one has the following theorem.

**Theorem 1.** *The above assumptions can be assumed without loss of generality.*

In the following example, we illustrate the above ideas

**Example 1.** *We consider the 3-dimensional rational variety  $\mathcal{V} \subset \mathbb{C}^5$  (so,  $n = 5$  and  $r = 3$ ) given by the parametrization*

$$\mathcal{P}(\bar{t}) = \left( \frac{t_1}{t_2}, \frac{t_1^2 t_3}{t_2}, \frac{t_1^2}{t_2 t_3}, t_2 t_3, \frac{t_1^3}{t_2} \right)$$

[General Assumptions] *Although the assumptions A-1 and A-3 are satisfied, A-2 does not hold. Therefore, we perform the linear transformation*

$$\bar{t} = L(\bar{t}) = (t_1 + t_3, t_2 - t_3, t_3)$$

to replace  $\mathcal{P}(\bar{t})$  by the new parametrization

$$\mathcal{P}(\bar{t}) = \left( \frac{t_1 + t_3}{t_2 - t_3}, \frac{(t_1 + t_3)^2 t_3}{t_2 - t_3}, \frac{(t_1 + t_3)^2}{(t_2 - t_3) t_3}, (t_2 - t_3) t_3, \frac{(t_1 + t_3)^3}{t_2 - t_3} \right)$$

*This new parametrization fulfills all the general assumptions. The polynomials  $G_i$  are*

$$\begin{aligned} G_1(\bar{t}, \bar{h}) &= (t_1 + t_3) t_3 (h_2 - h_3) h_3 - (h_1 + h_3) h_3 (t_2 - t_3) t_3 \\ G_2(\bar{t}, \bar{h}) &= (t_1 + t_3)^2 t_3^2 (h_2 - h_3) h_3 - (h_1 + h_3)^2 h_3^2 (t_2 - t_3) t_3 \\ G_3(\bar{t}, \bar{h}) &= (t_1 + t_3)^2 (h_2 - h_3) h_3 - (h_1 + h_3)^2 (t_2 - t_3) t_3 \\ G_4(\bar{t}, \bar{h}) &= (t_2 - t_3)^2 t_3^2 (h_2 - h_3) h_3 - (h_2 - h_3)^2 h_3^2 (t_2 - t_3) t_3 \\ G_5(\bar{t}, \bar{h}) &= (t_1 + t_3)^3 t_3 (h_2 - h_3) h_3 - (h_1 + h_3)^3 h_3 (t_2 - t_3) t_3 \\ G(\bar{t}, \bar{h}, \bar{Z}) &= G_2 + Z_1 G_3 + Z_2 G_4 + Z_3 G_5 \end{aligned}$$

[Base Points] *We analyze the base points. For this purpose, we consider the ideal  $I$ , in  $\mathbb{C}[\bar{t}]$ , generated by  $\{p_1, \dots, p_5, q\}$ , and we take the Gröbner basis  $\mathcal{G}$  of  $I$  w.r.t. the lex order with  $t_3 > t_2 > t_1$ :*

$$\mathcal{G} = \{t_1^2 (t_2 + t_1), t_1 (t_1 + t_3), -t_1^2 + t_2 t_3, -(t_1 - t_3) (t_1 + t_3)\}.$$

$I$  decomposes as

$$I = \langle t_1, t_3 \rangle \cap \langle t_1 + t_3, t_2 + t_1 \rangle \cap \langle t_1, t_2, t_3 \rangle.$$

Thus, the base points decomposes as union of two lines, namely

$$\mathfrak{B}(\mathcal{P}) = \{(0, \lambda, 0) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\} \cup \{(-\lambda, \lambda, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\}.$$

In particular, we deduce that  $\dim(\mathfrak{B}(\mathcal{P})) = 1$ .

[Fibre] *We deal now with  $\mathfrak{F}_{\mathcal{P}}(\bar{h})$ . For this, we consider the ideal  $J$ , in  $\mathbb{C}(\bar{h})[\rho, \bar{t}]$ , generated by  $\{G_1, \dots, G_5, \rho q - 1\}$ , and we take the Gröbner basis  $\mathcal{F}$  of  $J$  w.r.t. the lex order with  $\rho > t_3 > t_2 > t_1$ :*

$$\mathcal{F} = \{-h_1^2 + t_1^2, -t_1 h_2 + h_1 t_2, h_1 t_3 - t_1 h_3, -1 + (h_2 h_3 - h_3^2) \rho\}$$



From  $\mathcal{F}$ , we get that

$$\mathfrak{F}_{\mathcal{P}}(\bar{h}) = \{\bar{h}, -\bar{h}\}.$$

Therefore,  $\deg(\Phi_{\mathcal{P}}) = 2$ . □

In the following lemma, assuming the general assumptions, we summarize the basic properties of the varieties  $\mathcal{W}_i^{\bar{h}}$ .

**Lemma 1.** *It holds that*

1.  $(\mathcal{W}_1^{\bar{h}} \cap \dots \cap \mathcal{W}_n^{\bar{h}}) \setminus \mathfrak{B}(\mathcal{P}) \subset (\mathbb{F} \setminus \mathbb{K})^r$  and is zero-dimensional.
2. Let  $\mathcal{F}$  be a finite subset of  $\mathbb{K}[\bar{t}]$ . If the ideal generated by  $\mathcal{F}$  is zero dimensional, then  $\mathbb{V}_{\mathbb{F}}(\mathcal{F}) \subset \mathbb{K}^r$ .
3. If  $\dim(\mathfrak{B}(\mathcal{P})) = 0$ , then  $\mathfrak{B}(\mathcal{P}) \subset \mathbb{K}^r$ .
4. For  $\bar{\alpha} \in \Omega(\mathcal{P})$ , let  $\mathcal{W}_i^{\bar{\alpha}} = \mathbb{V}_{\mathbb{K}}(G_i(\bar{t}, \bar{\alpha}))$ . Then,  $\dim(\mathcal{W}_1^{\bar{\alpha}} \cap \dots \cap \mathcal{W}_n^{\bar{\alpha}}) < r - 1$ .
5.  $\dim(\mathcal{W}_1^{\bar{h}} \cap \dots \cap \mathcal{W}_n^{\bar{h}}) < r - 1$ .

**Remark 2.** *Because of Lemma 1 (5),  $\gcd(G_1, \dots, G_n) = 1$ . Therefore there exists  $G_i$ , with  $i > 1$  such that  $G_1$  does not divide  $G_i$ . In particular, as imposed in assumption A-3, we can assume w.l.o.g. that  $i = 3$ .*

## 4 Generic Resultant Sequence

In this section we introduce the notion of generic resultant sequence, and we study its first properties. For this purpose, we need some additional notation. For  $i = 1, \dots, r-2$  (recall that  $r > 2$ ; see Remark 1), let  $\bar{W}_i = (Z_{1i}, \dots, Z_{(n-2)i})$  be a tuple of new variables. Let  $\bar{W} = (\bar{W}_1, \dots, \bar{W}_{r-2})$ . For  $j \in \{1, \dots, r-1\}$  we use the notation  $\bar{t}^j = (t_1, \dots, t_j)$ . We also denote by  $\text{pp}_{\text{var}}(M)$  and  $\text{cont}_{\text{var}}(M)$  the primitive part and the content of the polynomial  $M$  w.r.t. to the set of variables  $\text{var}$ . For the following construction we observe that, by Prop. 2 (ii),  $G_1$  and  $G$  do depend on  $t_r$ . Let

- $R_{r-1} = \text{res}_{t_r}(G_1, G)$ ,  $S_{r-1} = \text{pp}_{\bar{Z}}(R_{r-1})$ .  
 $R_{r-1} \in \mathbb{K}[\bar{t}^{r-1}, \bar{h}, \bar{Z}]$ , and  $S_{r-1} \in \mathbb{K}[\bar{t}^{r-1}, \bar{h}, \bar{Z}] \setminus \mathbb{K}[\bar{h}, \bar{Z}]$ ; see Theorem 2 (3)
- $$\begin{cases} R_{r-2} = \begin{cases} S_{r-1} & \text{if } S_{r-1} \text{ does not depend on } t_{r-1}, \text{ otherwise} \\ \text{res}_{t_{r-1}}(S_{r-1}(\bar{t}^{r-1}, \bar{h}, \bar{W}_{r-2}), S_{r-1}(\bar{t}^{r-1}, \bar{h}, \bar{Z})) \end{cases} \\ S_{r-2} = \begin{cases} S_{r-1} & \text{if } S_{r-1} \text{ does not depend on } t_{r-1}, \text{ otherwise} \\ \text{pp}_{\bar{Z}}(R_{r-2}) \end{cases} \end{cases}$$
 $R_{r-2}, S_{r-2} \in \mathbb{K}[\bar{t}^{r-2}, \bar{h}, \bar{W}_{r-2}, \bar{Z}].$

- $$\begin{cases} R_{r-3} = \begin{cases} S_{r-2} & \text{if } S_{r-2} \text{ does not depend on } t_{r-2}, \text{ otherwise} \\ \text{res}_{t_{r-2}}(S_{r-2}(\bar{t}^{r-2}, \bar{h}, \bar{W}_{r-2}, \bar{W}_{r-3}), S_{r-2}(\bar{t}^{r-1}, \bar{h}, \bar{W}_{r-2}, \bar{Z})) \end{cases} \\ S_{r-3} = \begin{cases} S_{r-2} & \text{if } S_{r-2} \text{ does not depend on } t_{r-2}, \text{ otherwise} \\ \text{pp}_{\bar{Z}}(R_{r-3}) \end{cases} \\ R_{r-3}, S_{r-3} \in \mathbb{K}[\bar{t}^{r-3}, \bar{h}, \bar{W}_{r-2}, \bar{W}_{r-3}, \bar{Z}]. \\ \vdots \end{cases}$$
- $$\begin{cases} R_2 = \begin{cases} S_3 & \text{if } S_3 \text{ does not depend on } t_3, \text{ otherwise} \\ \text{res}_{t_3}(S_3(\bar{t}^3, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_2), S_3(\bar{t}^3, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_3, \bar{Z})) \end{cases} \\ S_2 = \begin{cases} S_3 & \text{if } S_3 \text{ does not depend on } t_3, \text{ otherwise} \\ \text{pp}_{\bar{Z}}(R_2) \end{cases} \\ R_2, S_2 \in \mathbb{K}[\bar{t}^2, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_2, \bar{Z}]. \end{cases}$$
- $$\begin{cases} R_1 = \begin{cases} S_2 & \text{if } S_2 \text{ does not depend on } t_2, \text{ otherwise} \\ \text{res}_{t_2}(S_2(\bar{t}^2, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_1), S_2(\bar{t}^2, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_2, \bar{Z})) \end{cases} \\ S_1 = \text{cont}_{\bar{W}_{r-2}, \dots, \bar{W}_1, \bar{Z}}(R_1) \\ R_1 \in \mathbb{K}[t_1, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_1, \bar{Z}], \text{ and } S_1 \in \mathbb{K}[t_1, \bar{h}]. \end{cases}$$
- $S_0 = \text{pp}_{\bar{h}}(S_1) \in \mathbb{K}[t_1, \bar{h}].$

**Definition 1.** We say that  $\{S_0, (S_1, R_1), \dots, (S_{r-1}, R_{r-1})\}$  is the generic resultant sequence (shorten, in the following, by GRS) of  $\mathcal{P}(\bar{t})$ . We denote it by  $\text{GRS}(\mathcal{P})$ .

Before establishing the basic properties on  $\text{GRS}(\mathcal{P})$ , we state some technical lemmas on resultants and generalized resultants.

**Lemma 2.** Let  $\mathbb{L}$  be a unique factorization domain. It holds that

1. Let  $M_1, \dots, M_\ell \in \mathbb{L}[x]$ ,  $\ell \geq 3$ ,  $M_1$  non-constant, and  $\text{gcd}(M_1, \dots, M_\ell) = 1$ .  $\text{res}_x(M_1, M_2 + Z_1 M_3 + \dots + Z_{\ell-2} M_\ell)$  does not depend on  $\{Z_1, \dots, Z_{\ell-2}\}$  iff  $M_1$  divides all  $M_i$  with  $i \geq 3$ .
2. Let  $\Delta$  be a tuple of variables,  $M \in \mathbb{L}[\Delta][x] \setminus \mathbb{L}[x]$  without factors in  $\mathbb{L}[x]$ , and  $N(\Delta^*, \Delta) = \text{res}_x(M(\Delta^*, x), M(\Delta, x))$ , where  $\Delta^*$  is a tuple of new variables.  $N$  depends on  $\Delta$ , and on  $\Delta^*$ , and has no factor depending only on  $\Delta^*$  nor only on  $\Delta$ .

**Theorem 2. (Basic Properties of GRS)** Let  $\text{GRS}(\mathcal{P})$  be the GRS of  $\mathcal{P}(\bar{t})$ . It holds that

1.  $R_i$ , for  $1 \leq i \leq r-1$ , depends on  $\bar{Z}$ ; in particular it is not zero.

2.  $S_i$ , for  $0 \leq i \leq r-1$ , is not zero, and for  $2 \leq i \leq r-1$  they depend on  $\bar{Z}$ , and have no factor in  $\mathbb{K}[\bar{t}, \bar{h}]$ .
3.  $S_{r-1}$  depends on  $\bar{t}^{r-1}$ .

The results in the next sections use that the generic resultant sequence satisfies certain conditions on the dependencies on the variable  $\bar{t}^i$  as well as the requirement of having constant (i.e. in  $\mathbb{K}(\bar{h}, \bar{W})$ ) leading coefficients; this motivates the notion of normality.

**Definition 2.** We say that  $\text{GRS}(\mathcal{P})$  is normal if, for  $i \in \{2, \dots, r-1\}$ ,  $\deg_{t_i}(S_i) > 0$  and  $\text{LCoeff}(S_i, t_i)$  does not depend on  $\bar{t}^{i-1}$ .

If  $\text{GRS}(\mathcal{P})$  is not normal, we perform a linear transformation  $\bar{t} = L(\bar{t}')$  such that the GRS of the transformed parametrization  $\mathcal{P}(\bar{t}') = \mathcal{P}(L(\bar{t}'))$  is normal. Note that, under a linear transformation, the degree of the induced rational maps is preserved and both, base points and fibres, are under control. We have not proved that such a linear transformation exists, although empirically we have seen that for random linear transformations one yields normality.

## 5 Base Points, Fibres and $\text{GRS}(\mathcal{P})$

In this section we study the connection of the base points and the fibre with the varieties defined from

$$\text{GRS}(\mathcal{P}) = \{S_0, (S_1, R_1), \dots, (S_{r-1}, R_{r-1})\}.$$

To be more precise we will see how the different projections of  $\mathfrak{B}(\mathcal{P})$  of  $\mathfrak{F}_{\mathcal{P}}(\bar{h})$  and the varieties defined the polynomials in  $\text{GRS}(\mathcal{P})$  are related. For this purpose, for  $r \geq \ell > i \geq 1$ , we denote by  $\pi_i$  the projection map

$$\pi_i : \mathbb{F}^\ell \rightarrow \mathbb{F}^i; \pi_i(\bar{t}^\ell) = \bar{t}^i,$$

and by  $\text{coeffs}_{\text{var}}(f)$  the set of coefficients of a polynomial  $f$  w.r.t. to the set of variables  $\text{var}$ . Moreover, we consider the fields

$$\mathbb{F}_j = \begin{cases} \overline{\mathbb{K}(\bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_j)} & \text{if } 2 \leq j < r-1 \text{ and } r > 3 \\ \mathbb{F} & \text{if } j = r-1 \end{cases}$$

In addition, throughout this section we **assume** that  $\text{GRS}(\mathcal{P})$  is normal. In this situation, for  $j \in \{2, \dots, r-1\}$ , we consider the following varieties:

- Related to  $\text{GRS}(\mathcal{P})$ .

$$\mathfrak{R}_j = \mathbb{V}_{\mathbb{F}_j}(\text{coeffs}_{\bar{Z}}(R_j)), \mathfrak{M}_j = \mathbb{V}_{\mathbb{F}_j}(\text{coeffs}_{\bar{Z}}(S_j)), \mathfrak{C}_j = \mathbb{V}_{\mathbb{F}_j}(\text{cont}_{\bar{Z}}(R_j)).$$

Note that  $\mathfrak{R}_j = \mathfrak{C}_j \cup \mathfrak{M}_j$ .

- Related to  $\mathfrak{F}_{\mathcal{P}}(\bar{h})$ . We decompose  $\pi_j(\mathfrak{F}_{\mathcal{P}})$  as  $\pi_j(\mathfrak{F}_{\mathcal{P}}(\bar{h})) = (\mathfrak{F}_{\mathcal{P}})^P_j \cup (\mathfrak{F}_{\mathcal{P}})^I_j$  where

$$\begin{cases} (\mathfrak{F}_{\mathcal{P}})^P_j = \pi_j(\mathfrak{F}_{\mathcal{P}}(\bar{h})) \setminus \pi_j(\mathfrak{B}(\mathcal{P}))^* & \text{(pure part of the projection)} \\ (\mathfrak{F}_{\mathcal{P}})^I_j = \pi_j(\mathfrak{F}_{\mathcal{P}}(\bar{h})) \cap \pi_j(\mathfrak{B}(\mathcal{P}))^* & \text{(impure part of the projection)} \end{cases}$$

- Related to  $\mathfrak{B}(\mathcal{P})$ . Since  $\gcd(p_1, \dots, p_n, q) = 1$ ,  $\dim(\mathfrak{B}(\mathcal{P})) \leq r - 2$ . Thus,  $\dim(\overline{\pi_{r-1}(\mathfrak{B}(\mathcal{P}))}) \leq r - 2$ . Then, we decompose  $\pi_{r-1}(\mathfrak{B}(\mathcal{P}))$  as

$$\mathbb{F}^{r-1} \supset \pi_{r-1}(\mathfrak{B}(\mathcal{P}))^* = \mathfrak{H}_{r-1} \cup \mathfrak{L}_{r-1}$$

where

- $\mathfrak{H}_{r-1}$  is either the hypersurface contained in  $\pi_{r-1}(\mathfrak{B}(\mathcal{P}))$  if  $\dim(\pi_{r-1}(\mathfrak{B}(\mathcal{P}))) = r - 2$  or, otherwise, the empty set, and
- $\mathfrak{L}_{r-1}$  is the union of all the components of  $\pi_{r-1}(\mathfrak{B}(\mathcal{P}))^*$  of dimension strictly smaller than  $r - 2$ .

Additionally, we decompose  $\mathfrak{M}_{r-1}$  and  $\mathfrak{L}_{r-1}$  as

$$\mathfrak{M}_{r-1} = \mathfrak{M}_{r-1}^P \cup \mathfrak{M}_{r-1}^I, \quad \mathfrak{L}_{r-1} = \mathfrak{L}_{r-1}^P \cup \mathfrak{L}_{r-1}^I$$

as follows:

- $\mathfrak{M}_{r-1}^P$  is the union of the components of  $\mathfrak{M}_{r-1}$  not included in  $\mathfrak{H}_{r-1}$ , and  $\mathfrak{M}_{r-1}^I$  is the union of the components of  $\mathfrak{M}_{r-1}$  included in  $\mathfrak{H}_{r-1}$ .
- Similarly,  $\mathfrak{L}_{r-1}^P$  is the union of the components of  $\mathfrak{L}_{r-1}$  not included in  $\mathfrak{H}_{r-1}$ , and  $\mathfrak{L}_{r-1}^I$  is the union of the components of  $\mathfrak{L}_{r-1}$  included in  $\mathfrak{H}_{r-1}$ .

If  $\dim(\pi_{r-2}(\mathfrak{B}(\mathcal{P}))) = r - 2$ , then  $\pi_{r-2}(\mathfrak{B}(\mathcal{P}))^* = \mathbb{F}^{r-2}$ . However,  $\dim(\pi_{r-2}(\mathfrak{L}_{r-1}^P)) \leq r - 3$ . Thus, we decompose it as

$$\mathbb{F}^{r-2} \supset \pi_{r-2}(\mathfrak{L}_{r-1}^P)^* = \mathfrak{H}_{r-2} \cup \mathfrak{L}_{r-2}$$

where

- $\mathfrak{H}_{r-2}$  is either the hypersurface contained in  $\pi_{r-2}(\mathfrak{L}_{r-1}^P)^*$  if  $\dim(\pi_{r-2}(\mathfrak{L}_{r-1}^P)) = r - 3$  or, otherwise, the empty set, and
- $\mathfrak{L}_{r-2}$  is the union of all the components of  $\pi_{r-2}(\mathfrak{L}_{r-1}^P)^*$  of dimension strictly smaller than  $r - 3$ .

Additionally, we decompose  $\mathfrak{M}_{r-2}$  and  $\mathfrak{L}_{r-2}$  as

$$\mathfrak{M}_{r-2} = \mathfrak{M}_{r-2}^P \cup \mathfrak{M}_{r-2}^I, \quad \mathfrak{L}_{r-2} = \mathfrak{L}_{r-2}^P \cup \mathfrak{L}_{r-2}^I$$

as follows:

- $\mathfrak{M}_{r-2}^P$  is the union of the components of  $\mathfrak{M}_{r-2}$  not included in  $\pi_{r-2}(\mathfrak{M}_{r-1}^I)^* \cup \mathfrak{H}_{r-2}$ , and  $\mathfrak{M}_{r-2}^I$  is the union of the components of  $\mathfrak{M}_{r-2}$  included in  $\pi_{r-2}(\mathfrak{M}_{r-1}^I)^* \cup \mathfrak{H}_{r-2}$ .
- Similarly,  $\mathfrak{L}_{r-2}^P$  is the union of the components of  $\mathfrak{L}_{r-2}$  not included in  $\pi_{r-2}(\mathfrak{M}_{r-1}^I)^* \cup \mathfrak{H}_{r-2}$ , and  $\mathfrak{L}_{r-2}^I$  is the union of the components of  $\mathfrak{L}_{r-2}$  included in  $\pi_{r-2}(\mathfrak{M}_{r-1}^I)^* \cup \mathfrak{H}_{r-2}$ .

Repeating the argument, we decompose  $\pi_j(\mathfrak{L}_{j+1}^P)$  as  $\mathbb{F}^j \supset \pi_j(\mathfrak{L}_{j+1}^P)^* = \mathfrak{H}_j \cup \mathfrak{L}_j$ , and we introduce  $\mathfrak{M}_j^P, \mathfrak{M}_j^I, \mathfrak{L}_j^P, \mathfrak{L}_j^I$  analogously.

**Remark 3.** *By definition,  $\mathfrak{R}_j$  and  $\mathfrak{M}_j, \mathfrak{M}_j^P, \mathfrak{M}_j^I$  are  $\mathbb{F}_j$ -definable. Furthermore, from the theorem of the closure (see pg. 122 in [8]) and taking into account that  $\mathfrak{B}(\mathcal{P})$  is  $\mathbb{K}$ -definable, one deduces that  $\mathfrak{H}_j$  and  $\mathfrak{L}_j$  are also  $\mathbb{K}$ -definable. Thus,  $\mathfrak{L}_j^P, \mathfrak{L}_j^I$  are also  $\mathbb{K}$ -definable. See also Theorem 5 for the  $\mathbb{K}$ -definability of  $\mathfrak{C}_j$ .*

**Remark 4.** *Observe that if  $\dim(\mathfrak{B}(\mathcal{P})) = 0$  then for  $j \in \{2, \dots, r-1\}$  it holds that  $\mathfrak{H}_j = \mathfrak{M}_j^I = \mathfrak{L}_j^I = \emptyset$ ,  $\mathfrak{M}_j^P = \mathfrak{M}_j, \mathfrak{L}_j^P = \mathfrak{L}_j$  and  $\dim(\mathfrak{L}_j) = 0$ .*

**Example 2.** *In this example, we illustrate the above varieties. For this purpose, we continue working with Example 1 and, hence, we use the notation introduced there. First we observe, that  $\text{GRS}(\mathcal{P})$  is normal. Since  $r = 3$  the associated varieties are*

- *For the resultant sequence:  $\mathfrak{R}_2 = \mathfrak{C}_2 \cup \mathfrak{M}_2$ .*
- *For the fibre:  $\pi_2(\mathfrak{F}_{\mathcal{P}}(\bar{h})) = (\mathfrak{F}_{\mathcal{P}})_2^P \cup (\mathfrak{F}_{\mathcal{P}})_2^I$ .*
- *For the base points:  $\pi_2(\mathfrak{B}(\mathcal{P}))^* = \mathfrak{H}_2 \cup \mathfrak{L}_2$  with the related decompositions*
  - $\mathfrak{M}_2 = \mathfrak{M}_2^P \cup \mathfrak{M}_2^I$
  - $\mathfrak{L}_2 = \mathfrak{L}_2^P \cup \mathfrak{L}_2^I$ .

[Varieties associated to  $\text{GRS}(\mathcal{P})$ ] *The content of  $R_2$  w.r.t.  $\bar{Z}$  is  $t_1^2(t_2 + t_1)$ . So,*

$$\mathfrak{C}_2 = \{(0, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\} \cup \{(-\lambda, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\}$$

*On the other hand, the coefficients of  $S_2$  w.r.t.  $\bar{Z}$  are*

$$\begin{aligned} \text{coeffs}_{\bar{Z}}(S_2) &= \{-(h_1 + h_3)^2 (h_2 + h_1)^2 (-t_1 h_2 + h_1 t_2), \\ &\quad -(h_1 + h_3)^2 \Delta_1 \Delta_2, \\ &\quad -(h_2 - h_3)^2 \Delta_1 \Delta_2, \\ &\quad (h_1 + h_3)^3 (h_2 + h_1 + t_1 + t_2) (h_2 + h_1 - t_1 - t_2) \Delta_1\} \end{aligned}$$

*where*

$$\begin{aligned} \Delta_1 &= t_1 h_3 + h_3 t_2 - t_1 h_2 + h_1 t_2 \\ \Delta_2 &= h_2^2 h_3 + 2h_1 h_2 h_3 + t_1^2 h_2 + t_2 h_2 t_1 - t_1^2 h_3 - 2t_1 h_3 t_2 - h_1 t_2 t_1 - t_2^2 h_3 + h_1^2 h_3 - h_1 t_2^2 \end{aligned}$$

Furthermore, the Gröbner basis of coeffs $_{\mathbb{Z}}(S_2)$  w.r.t. lex order with  $t_2 > t_1$ , as ideal in  $\mathbb{C}(\bar{h})[t_1, t_2]$  is

$$\mathcal{M} = \{-h_2^2 t_2 + t_2^3, -h_1 t_2 + t_1 h_2\}.$$

Therefore,

$$\boxed{\mathfrak{M}_2 = \{(0, 0), (h_1, h_2), (-h_1, -h_2)\}}$$

[Fibre] In Example 1, we have seen that  $\mathfrak{F}_{\mathcal{P}}(\bar{h}) = \{\bar{h}, -\bar{h}\}$ . Thus

$$\pi_2(\mathfrak{F}_{\mathcal{P}}(\bar{h})) = \{(h_1, h_2), (-h_1, -h_2)\}$$

and, taking into account  $\pi_2(\mathfrak{B}(\mathcal{P}))^*$  (see below), we get that

$$\boxed{(\mathfrak{F}_{\mathcal{P}})_2^P = \{(h_1, h_2), (-h_1, -h_2)\}, \text{ and } (\mathfrak{F}_{\mathcal{P}})_2^I = \emptyset}$$

[Base Points] In Example 1 we have seen that

$$\mathfrak{B}(\mathcal{P}) = \{(0, \lambda, 0) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\} \cup \{(-\lambda, \lambda, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\}.$$

Therefore, projecting the lines, one gets that

$$\pi_2(\mathfrak{B}(\mathcal{P}))^* = \{(0, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\} \cup \{(-\lambda, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\}.$$

Thus,  $\pi_2(\mathfrak{B}(\mathcal{P}))^* = \mathfrak{H}_2 \cup \mathfrak{L}_2$ , where

$$\boxed{\mathfrak{H}_2 = \{(0, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\} \cup \{(-\lambda, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\}, \text{ and } \mathfrak{L}_2 = \emptyset.}$$

Furthermore, see above,  $\mathfrak{M}_2$  decomposes as  $\mathfrak{M}_2 = \mathfrak{M}_2^P \cup \mathfrak{M}_2^I$ , where

$$\boxed{\mathfrak{M}_2^P = \{(h_1, h_2), (-h_1, -h_2)\}, \text{ and } \mathfrak{M}_2^I = \{(0, 0)\}.$$

Obviously  $\mathfrak{L}_2^P = \mathfrak{L}_2^I = \emptyset$ . □

Next, we analyze the relations among the varieties we have introduced in connection to the projection of the fibre and of the base point variety. As mentioned in the introduction, we will see that essentially the behavior is as follows

- the projection of the fibre goes into the pure part  $\mathfrak{M}_i^P$  of the primitive variety  $\mathfrak{M}_i$  (see Theorem 4),
- the high dimensional components of the base points project into the content variety  $\mathfrak{C}_i$  (see Theorem 3),
- while the low dimensional components of the base points go into  $\mathfrak{M}_i^P$  (see Theorem 4).

We know that

$$\mathcal{W}_1^{\bar{h}} \cap \cdots \cap \mathcal{W}_n^{\bar{h}} = \mathfrak{F}_{\mathcal{P}}(\bar{h}) \cup \mathfrak{B}(\mathcal{P}).$$

We start with the next lemma where a similar decomposition holds for  $\mathfrak{M}_j^P$ . This first lemma, indeed, establishes that all points in the pure part of the primitives varieties are either projections of base points or of fibre points. This will be used in the next theorems to first state that the content varieties are essentially defined by the projection of the sufficiently high dimensional components of the base points (see Theorem 3), and to afterwards provide a clearer decomposition of the pure part of the primitive part variety (see Theorem 4).

**Lemma 3.** *For  $j \in \{2, \dots, r-1\}$ ,  $\mathfrak{M}_j^P = (\mathfrak{M}_j^P \cap \pi_j(\mathfrak{F}_{\mathcal{P}}(\bar{h}))) \cup (\mathfrak{M}_j^P \cap \mathfrak{L}_j^P)$ .*

As consequence of this lemma, we get the following theorem. We recall that the varieties introduced above are defined for  $j \in \{2, \dots, r-1\}$ , where we have assumed that  $r \geq 3$ .

**Theorem 3.** (Decomposition of  $\mathfrak{C}_j$ ) *It holds that*

1. *If  $\dim(\mathfrak{B}(\mathcal{P})) < r-2$  then  $\mathfrak{C}_{r-1} = \emptyset$ .*
2. *If  $r > 3$  and  $\dim(\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I)) < j-1$  for  $j \in \{2, \dots, r-2\}$ , then  $\mathfrak{C}_j = \emptyset$ .*
3. *If  $\dim(\pi_{r-1}(\mathfrak{B}(\mathcal{P}))^*) = r-2$ , then  $\mathfrak{C}_{r-1} = \mathfrak{H}_{r-1}$ .*
4. *If  $r > 3$  and  $\dim(\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I)) = j-1$  for  $j \in \{2, \dots, r-2\}$ , then  $\mathfrak{C}_j$  is the hypersurface included in  $\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I)^*$ .*

**Remark 5.** *In Example 1, we have seen that  $\dim(\mathfrak{B}(\mathcal{P})) = 1 = r-2$ , and in Example 2 we have seen that  $\mathfrak{C}_2 = \mathfrak{H}_2$ ; compare to Theorem 3 (3).*

**Corollary 1.** *If  $\dim(\mathfrak{B}(\mathcal{P})) = 0$ , for  $j \in \{2, \dots, r-1\}$ ,  $\mathfrak{C}_j = \emptyset$ .*

*Proof.* By Theorem 3 (1), since  $r > 2$  (see Remark 1),  $\mathfrak{C}_{r-1} = \emptyset$ . For  $j \in \{2, \dots, r-2\}$ , the result follows from Theorem 3 (2), taking into account that  $\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I) = \pi_j(\mathfrak{M}_{j+1} \cap \mathfrak{L}_{j+1}) \subset \pi_j(\mathfrak{L}_{j+1})$  (see Remark 4), from where  $\dim(\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I)) \leq \dim(\pi_j(\mathfrak{L}_{j+1})) = 0 < j-1$ .  $\square$

**Theorem 4.** (Decomposition of  $\mathfrak{R}_j, \mathfrak{M}_j$ ) *It holds that*

1.  $\mathfrak{R}_{r-1} = \pi_{r-1}(\mathfrak{B}(\mathcal{P}))^* \cup (\mathfrak{F}_{\mathcal{P}})_{r-1}^P$ .
2. *If  $r > 3$  and  $j \in \{2, \dots, r-2\}$ ,  $\mathfrak{R}_j = \pi_j(\mathfrak{M}_{j+1})^*$ .*

3. For  $j \in \{2, \dots, r-1\}$ ,  $\mathfrak{M}_j^P = (\mathfrak{F}_{\mathcal{P}})_j^P \cup \mathfrak{L}_j^P \cup \mathfrak{Q}_j$ , where  $\mathfrak{Q}_j \subset (\mathfrak{F}_{\mathcal{P}})_j^I$ .

**Example 3.** We illustrate the theorem by means of Example 1 and 2. Let  $\mathcal{L}_1 = \{(0, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\}$  and  $\mathcal{L}_2 = \{(-\lambda, \lambda) \mid \lambda \in \overline{\mathbb{C}(\bar{h})}\}$ . Then (compare to Theorem 4 (1)),

$$\begin{aligned} \mathfrak{R}_2 &= \mathfrak{C}_2 \cup \mathfrak{M}_2 = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{(0, 0), (h_1, h_2), (-h_1, -h_2)\} = \\ &= \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{(h_1, h_2), (-h_1, -h_2)\} = \pi_2(\mathfrak{B}(\mathcal{P}))^* \cup (\mathfrak{F}_{\mathcal{P}})_2^P \end{aligned}$$

Moreover (compare to Theorem 4 (3)),  $\mathfrak{M}_2 = (\mathfrak{F}_{\mathcal{P}})_2^P \cup \mathfrak{L}_2 \cup \mathfrak{Q}_2$ , with  $\mathfrak{Q}_2 = \emptyset$ .  $\square$

**Theorem 5.** ( $\mathbb{K}$ -definability of  $\mathfrak{C}_j$ )

1.  $\mathfrak{C}_{r-1}$  is  $\mathbb{K}$ -definable.
2. If  $r > 3$  and  $\dim(\pi_j(\mathfrak{M}_{j+1}^I)^*) < j-1$  for  $j \in \{2, \dots, r-2\}$ , then  $\mathfrak{C}_j$  is  $\mathbb{K}$ -definable.

*Proof.* By Theorem 3, we only need to prove the theorem if  $j < r-1$  and  $\dim(\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I)) = j-1$ . By Theorem 3,  $\mathfrak{C}_j$  is the hypersurface included in  $\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I)^*$ . Thus, reasoning as in the proof of Theorem 4,  $\mathfrak{C}_j$  is the hypersurface included in  $\Delta = \mathfrak{H}_j \cup \pi_j(\mathfrak{M}_{j+1}^I)^*$ . Thus,  $\mathfrak{C}_j \subset \mathfrak{H}_j$ . Now the theorem follows from Remark 3.  $\square$

## 6 Connection of $\text{GRS}(\mathcal{P})$ to Gröbner Bases

In the previous section we have analyzed some varieties of the  $\text{GRS}(\mathcal{P})$  in connection to the base points and a generic fibre. In this section, we study the connection to Gröbner bases. We **assume** that

$$\text{GRS}(\mathcal{P}) = \{S_0, (S_1, R_1), \dots, (S_{r-1}, R_{r-1})\}$$

is normal. In addition, in the sequel, we **assume** that either  $\mathfrak{B}(\mathcal{P}) = \emptyset$  or  $\dim(\mathfrak{B}(\mathcal{P})) = 0$ . We start with the following lemma.

**Lemma 4.** (Up and down property) *It holds that*

1. If  $P \in \mathfrak{F}_{\mathcal{P}}(\bar{h})$ , then  $\pi_1(P)$  is a root of  $S_0(t_1)$ .
2. If  $\alpha$  is a root of  $S_0(t_1)$ , then there exists  $P \in \mathfrak{F}_{\mathcal{P}}(\bar{h})$  such that  $\pi_1(P) = \alpha$ .

From Lemma 4, one directly gets the following theorem

**Theorem 6.**  $\pi_1(\mathfrak{F}_{\mathcal{P}}(\bar{h})) = \mathbb{V}_{\mathbb{F}}(S_0)$ .



From this result we get the following corollaries.

**Corollary 2.** *Let  $I$  be the ideal, in  $\mathbb{K}(\bar{h})[\rho, \bar{t}]$ , generated by  $\{G_1, \dots, G_n, \rho q - 1\}$ , and  $\mathcal{G}$  be a reduced Gröbner basis of  $I$  w.r.t. the lex order with  $\rho > t_r > \dots > t_1$ . Let  $\{g_1(t_1)\} = \mathcal{G} \cap \mathbb{K}(\bar{h})[t_1]$ , then the square-free part of  $g_1$  and of  $S_0$  are equal, up to multiplication by a non-zero element in  $\mathbb{K}$ .*

*Proof.* First we observe that  $I$  is zero dimensional, since its variety over  $\mathbb{F}$  is  $\mathfrak{F}_{\mathcal{P}}(\bar{h})$ . Now, the result follows from the closure theorem, the elimination property of Gröbner bases, Hilbert's Nullstellensatz, and Theorem 6.  $\square$

In the next corollary we use the notion of  $t_i$ -regular position of an ideal of  $\bar{t}$ -multivariate polynomials over a field (see [18] page 194), that says that if  $I$  is zero-dimensional then it is in  $t_i$ -regular position if any two zeros of  $I$ , over the algebraic closure of the ground field, have different  $t_i$ -coordinate. As commented in [18], we observe that nearly every linear change of coordinates will set the ideal in regular position.

**Corollary 3.** *Let  $I$  be as in Corollary 2. If  $I$  is  $t_1$ -regular and radical then*

$$\deg(\Phi_{\mathcal{P}}) = \deg_{t_1} \left( \frac{S_0}{\gcd \left( S_0, \frac{\partial S_0}{\partial t_1} \right)} \right).$$

*Proof.* It follows from Corollary 2 and the Shape Lemma (see Theorem 8.4.6., page 195, in [18]; observe that the notion used in this paper of reduced Gröbner basis is the notion of normed reduced Gröbner basis used in [18]).  $\square$

**Example 4.** *Through this example, we illustrate the results in this section. We consider the 3-dimensional rational variety  $\mathcal{V} \subset \mathbb{C}^5$  (so,  $n = 5$  and  $r = 3$ ) given by the parametrization*

$$\mathcal{P}(\bar{t}) = \left( \frac{t_3}{t_2 - t_3}, \frac{(t_1 + t_3)^4}{t_2 - t_3}, \frac{(t_1 + t_3)^2}{t_2 - t_3}, \frac{t_3^4}{t_2 - t_3}, \frac{t_3^3}{t_2 - t_3} \right),$$

*that satisfies our assumptions. The polynomials  $G_i$  are*

$$\begin{aligned} G_1(\bar{t}, \bar{h}) &= t_3(h_2 - h_3) - h_3(t_2 - t_3) \\ G_2(\bar{t}, \bar{h}) &= (t_1 + t_3)^4(h_2 - h_3) - (h_1 + h_3)^4(t_2 - t_3) \\ G_3(\bar{t}, \bar{h}) &= (t_1 + t_3)^2(h_2 - h_3) - (h_1 + h_3)^2(t_2 - t_3) \\ G_4(\bar{t}, \bar{h}) &= t_3^4(h_2 - h_3) - h_3^4(t_2 - t_3) \\ G_5(\bar{t}, \bar{h}) &= t_3^3(h_2 - h_3) - h_3^3(t_2 - t_3) \\ G(\bar{t}, \bar{h}, \bar{Z}) &= G_2 + Z_1 G_3 + Z_2 G_4 + Z_3 G_5 \end{aligned}$$

[Base Points] We analyze the base points. For this purpose, we consider the ideal  $I$ , in  $\mathbb{C}[\bar{t}]$ , generated by  $\{p_1, \dots, p_5, q\}$ , and we take the Gröbner basis  $\mathcal{G}$  of  $I$  w.r.t. the lex order with  $t_3 > t_2 > t_1$ :

$$\mathcal{G} = \{t_1^2, t_2, t_3\}$$

Thus,

$$\mathfrak{B}(\mathcal{P}) = \{(0, 0, 0)\}$$

and, hence,  $\dim(\mathfrak{B}(\mathcal{P})) = 0$ . Therefore,  $\mathfrak{S}_2 = \mathfrak{M}_2^I = \mathfrak{L}_2^I = \emptyset$  and  $\mathfrak{L}_2^P = \mathfrak{L}_2 = \{(0, 0)\}$ .

[Fibre] We deal now with  $\mathfrak{F}_{\mathcal{P}}(\bar{h})$ . For this, we consider the ideal  $J$ , in  $\mathbb{C}(\bar{h})[\rho, \bar{t}]$ , generated by  $\{G_1, \dots, G_5, \rho q - 1\}$ , and we take the Gröbner basis  $\mathcal{F}$  of  $J$  w.r.t. the lex order with  $\rho > t_3 > \dots > t_1$ :

$$\mathcal{F} = \{-h_1^2 + 2t_1h_3 + t_1^2 - 2h_1h_3, -h_2 + t_2, t_3 - h_3, -1 + (h_2 - h_3)\rho\}$$

One can check that  $J$  as ideal in  $\mathbb{C}(\bar{h})[\rho, \bar{t}]$  is radical and  $t_1$ -regular. From  $\mathcal{F}$ , we get that

$$\mathfrak{F}_{\mathcal{P}}(\bar{h}) = \{(h_1, h_2, h_3), (-h_1 - 2h_3, h_2, h_3)\}.$$

Thus

$$\pi_2(\mathfrak{F}_{\mathcal{P}}(\bar{h})) = \{(h_1, h_2), (-h_1 - 2h_3, h_2)\}$$

and,

$$(\mathfrak{F}_{\mathcal{P}})_2^P = \pi_2(\mathfrak{F}_{\mathcal{P}}(\bar{h})), \quad (\mathfrak{F}_{\mathcal{P}})_2^I = \emptyset$$

[Varieties associated to  $\text{GRS}(\mathcal{P})$ ] Since  $r = 3$ , we only analyze  $\mathfrak{M}_2 = \mathfrak{M}_2^P$ ; recall that  $\mathfrak{R}_2 = \mathfrak{M}_2$  and that  $\mathfrak{C}_2 = \mathfrak{M}_2^I = \emptyset$ . First we observe that  $\text{GRS}(\mathcal{P})$  is normal. On the other hand, the Gröbner basis of coeffs  $\bar{z}(S_2)$  w.r.t. lex order with  $t_2 > t_1$ , as ideal in  $\mathbb{C}(\bar{h})[t_1, t_2]$  is

$$\mathcal{M} = \{-t_2(h_2 - t_2), t_1^2h_2 - h_1^2t_2 - 2h_1h_3t_2 + 2h_3t_2t_1\}.$$

Therefore,

$$\mathfrak{M}_2 = \{(0, 0), (h_1, h_2), (-h_1 + 2h_3, h_2)\}$$

that decomposes as (see Theorem 4)

$$\mathfrak{M}_2 = \mathfrak{M}_2^P = (\mathfrak{F}_{\mathcal{P}})_2^P \cup \mathfrak{L}_2^P \cup \mathfrak{Q}_0 \quad \text{with } \mathfrak{Q}_0 = \emptyset,$$

or as

$$\mathfrak{R}_2 = \mathfrak{M}_2 = \pi_2(\mathfrak{B}(\mathcal{P}))^* \cup (\mathfrak{F}_{\mathcal{P}})_2^P.$$

[Connection to Gröbner bases] The polynomials  $S_1$  and  $S_0$  are

$$\begin{aligned} S_1(t_1, t_2) &= t_1^2(-t_1 + h_1)(h_1 + 2h_3 + t_1) \\ S_0(t_1) &= (-t_1 + h_1)(h_1 + 2h_3 + t_1). \end{aligned}$$

On the other hand, the univariate polynomial (in  $t_1$ ) of the ideal of the fibre, namely  $J$ , is (compare to Corollary 2)

$$-h_1^2 + 2t_1h_3 + t_1^2 - 2h_1h_3 = -S_0(t_1),$$

and  $\deg(\Phi_{\mathcal{P}}) = \deg_{t_1}(S_0) = 2$  (see Corollary 3).  $\square$

## 7 Appendix

In this section we give the details of some technical proofs in the paper. More precisely, of the proofs of Lemmas 1, 2, 3 and 4, and Theorems 2, 3 and 4.

The proofs of Lemma 3 and Theorems 3 and 4 are, at some points, very technical. So here, we give the skeleton of the proofs, as well as the intuitive ideas. For more details see [13] (Section 7).

### [Proof of Lemma 1]

(1) Let  $I$  be the ideal of  $\mathbb{K}(\bar{h})[\rho, \bar{t}]$ , generated by  $\mathcal{A}(\bar{h}, \bar{t}, \rho) = \{G_1, \dots, G_n, q\rho - 1\}$ , where  $\rho$  is a new variable. Since  $\mathfrak{F}_{\mathcal{P}}(\bar{h}) = (\mathcal{W}_1^{\bar{h}} \cap \dots \cap \mathcal{W}_n^{\bar{h}}) \setminus \mathfrak{B}(\mathcal{P})$ ,  $I$  is zero-dimensional. Let  $\mathcal{G}(\bar{h}, \bar{t}, \rho)$  be a reduced Gröbner basis of  $I$  w.r.t. the lex order with  $\rho > t_r > \dots > t_1$ ; reduced in the sense of Def. 5 at [8], p. 90. There exists an open subset  $\Sigma$  of  $\mathbb{K}^r$  such that for  $\bar{h}^0 \in \Sigma$ ,  $\mathcal{G}(\bar{h}^0, \bar{t}, \rho)$  is the Gröbner basis of  $\mathcal{A}(\bar{h}^0, \bar{t}, \rho)$ , see e.g., ex. 7, p. 283, in [8]. Moreover, since  $I$  is zero-dimensional,  $\mathcal{G}(\bar{h}, \bar{t}, \rho) \cap \mathbb{K}(\bar{h})[t_1] = \{g(\bar{h}, t_1)\}$  and every solution of  $g$  over  $\mathbb{F}$  can be continued to a solution of the full system (see e.g. [18] pg. 194). Now, let  $\bar{\alpha} = (a_1, \dots, a_n) \in \mathfrak{F}_{\mathcal{P}}(\bar{h})$  be such that at least one  $a_i$  is constant, say  $a_1 \in \mathbb{K}$ . Let  $\mathcal{W}^*$  be the algebraic set generated, over  $\mathbb{K}$ , by  $\mathcal{P}(a_1, t_2, \dots, t_r)$ . Note that  $\dim(\mathcal{W}^*) < r$ , and that  $t_1 - a_1$  divides  $g(\bar{h}, t_1)$ . We consider the open set  $\Omega(\mathcal{P}) \cap \Sigma$ . Let  $\bar{h}^0 \in \Omega(\mathcal{P}) \cap \Sigma$ . Then,  $\mathcal{P}(\bar{h}^0)$  is well defined, and  $\mathcal{G}(\bar{h}^0, \bar{t}, \rho)$  is a Gröbner basis of  $\mathcal{A}(\bar{h}^0, \bar{t}, \rho)$ . Since  $g(\bar{h}^0, a_1) = 0$ ,  $a_1$  is extended to a solution  $(A, \rho^0)$  of the full system, where the first component of  $A$  is  $a_1$ . Therefore,  $\mathcal{P}(\bar{h}^0) = \mathcal{P}(A)$  and hence  $\mathcal{P}(\Omega(\mathcal{P}) \cap \Sigma) \subset \mathcal{W}^*$  which is a contradiction.

(2) It is a direct consequence of the triangular structure of the reduced Gröbner basis, w.r.t. the lex order, of a zero-dimensional ideal  $\mathbb{K}$ -definable.

(3) It follows from statement (2).

(4) Let  $\mathcal{W}_1^{\bar{\alpha}} \cap \dots \cap \mathcal{W}_n^{\bar{\alpha}}$  contain a hypersurface in  $\mathbb{K}^r$ , and  $M(\bar{t})$  its defining polynomial. Then, there exist  $N_i \in \mathbb{K}[\bar{t}]$  such that

$$G_i(\bar{t}, \bar{\alpha}) = p_i(\bar{t})q(\bar{\alpha}) - p_i(\bar{\alpha})q(\bar{t}) = M(\bar{t})N_i(\bar{t}), \text{ for } i = 1, \dots, n$$

Observe that, since  $\gcd(p_1, \dots, p_n, q) = 1$ , then  $\gcd(q(\bar{t}), M(\bar{t})) = 1$ . Now, we consider the set  $\Lambda_{\bar{\alpha}} := \{\bar{\beta} \in \mathbb{K}^n / M(\bar{\beta}) = 0, q(\bar{\beta}) \neq 0\}$ .  $\Lambda_{\bar{\alpha}} \neq \emptyset$  is an open subset of  $\mathbb{V}_{\mathbb{K}}(M)$ . Moreover  $\Lambda_{\bar{\alpha}} \subset \mathfrak{F}_{\mathcal{P}}(\bar{\alpha})$ , which is impossible since  $\text{card}(\Lambda_{\bar{\alpha}}) = \infty$  and  $\mathfrak{F}_{\mathcal{P}}(\bar{\alpha})$  is zero dimensional because  $\bar{\alpha} \in \Omega(\mathcal{P})$ .

(5) follows from (4).

### [Proof of Lemma 2]

(1) Let  $M = M_2 + Z_1 M_3 + \dots + Z_{\ell-2} M_\ell$ , and let  $\mathcal{R}$  be the set of all the roots of  $M_1$  in the algebraic closure of the quotient field of  $\mathbb{L}$ . The result follows from  $0 \neq \text{res}_x(M_1, M) = \text{LCoeff}(M_1)^{\deg_x(M)} \prod_{\alpha \in \mathcal{R}} M(\alpha)$ .

(2) Since  $M$  does not have factors in  $\mathbb{L}[x]$  then  $\gcd(M(\Delta^*, x), M(\Delta, x)) = 1$ , and hence  $N \neq 0$ . Let  $\mathcal{T}$  be the set of non-constant monomials in  $\Delta$  appearing in  $M$ . We express

$M$  as  $M(\Delta, x) = a(x) + \sum_{T \in \mathcal{T}} a_T(x)T$ . If  $\mathcal{R}$  is the set of all roots of  $M(\Delta^*, x)$ , in the algebraic closure of the quotient field of  $\mathbb{L}[\Delta^*]$ , as univariate polynomial in  $x$ , then

$$N = \text{LCoeff}(M(\Delta^*, x), x)^{\deg_x(M)} \prod_{\alpha \in \mathcal{R}} M(\Delta, \alpha)$$

Since  $N \neq 0$ , if  $N$  does not depend on  $\Delta$ , then  $a_T(\alpha) = 0$  for all  $T \in \mathcal{T}$  and for all  $\alpha \in \mathcal{R}$ . So,  $M(\Delta, x) - a(x) = B(\Delta, \Delta^*, x)M(\Delta^*, x)$  for some polynomial  $B$ . If  $a = 0$ ,  $M(\Delta^*, x)$  divides  $M(\Delta, x)$ . Thus  $M(\Delta^*, x) \in \mathbb{L}[x]$ , and hence  $M(\Delta, x) \in \mathbb{L}[x]$  which is a contradiction. If  $a \neq 0$  then  $M(\Delta^*, x)(1 - B(\Delta^*, \Delta^*, x)) = a(x)$ . So,  $M$  divides  $a$  which is again a contradiction. So  $N$  depends on  $\Delta$ , and reasoning similarly we get that also depends on  $\Delta^*$ . For the second part, let  $C(\Delta^*)$  be a factor of  $N$  depending only on  $\Delta^*$ ; similarly if it only depends on  $\Delta$ . Let  $P$  be a solution of  $C$  over the algebraic closure  $\mathbf{M}$  of the quotient field of  $\mathbf{L}$ . Then,  $N(P, \Delta) = 0$  and since  $\text{LCoeff}(M, x)(\Delta) \neq 0$  there exists  $a \in \mathbf{M}$  such that  $M(P, a) = 0 = M(\Delta, a) = 0$ . This implies that  $(x - a)$  divides  $M$ , which is a contradiction.

### [Proof of Theorem 2]

We prove (1) and (2) simultaneously. We start with the case  $i = r - 1$ . By Lemma 1 (5),  $\gcd(G_1, \dots, G_r) = 1$ , and by Prop.1  $G_1$  is not constant. Therefore, by assumption A-3, and Lemma 2 (1) applied to  $R_{r-1}$ , we get that  $R_{r-1}$  depends on  $\bar{Z}$  and so  $S_{r-1}$  does. Moreover, by definition,  $S_{r-1}$  does not have factors in  $\mathbb{K}[\bar{t}, \bar{h}]$ . Now, for  $i = r - 2$ , applying Lemma 2 (2) to  $S_{r-1}$ , and taking  $\mathbb{L} = \mathbb{K}[\bar{t}^{r-2}, \bar{h}]$  and  $\Delta = \bar{Z}$ , one gets the result. Similarly, for  $R_i$  with  $i < r - 2$  and for  $S_i$  with  $2 < i < r - 2$ . Finally, since  $R_1$  is not zero it follows that  $S_1, S_0$  are not zero either.

To prove (3), let us assume that  $S_{r-1}$  does not depend on  $\bar{t}$ . By (2),  $S_{r-1}$  depends on  $\bar{Z}$ . Let  $\bar{\alpha} \in \mathbb{F}^{r-2}$  be such that  $S_{r-1}$  vanishes at  $\bar{Z} = \bar{\alpha}$ . By Prop. 2, there exists  $a$  such that  $G_1(\bar{t}^{r-1}, a, \bar{h}, \bar{\alpha}) = G(\bar{t}^{r-1}, a, \bar{h}, \bar{\alpha}) = 0$ . Moreover, since  $G_1(\bar{t}^{r-1}, t_r, \bar{h}, \bar{\alpha}) \in \mathbb{F}[\bar{t}^{r-1}][t_r]$ , one has that  $a$  does not depends on  $\bar{Z}$ , and hence  $G_i(\bar{t}^{r-1}, a, \bar{h}, \bar{\alpha}) = 0$  for  $i = 1, \dots, n$ . Let us see that  $q(\bar{t}^{r-1}, a) \neq 0$ . Indeed, if it vanishes, then  $p_i(\bar{t}^{r-1}, a) = 0$  for  $i = 1, \dots, n$ . Therefore  $(t_r - a)$  divides to the  $\gcd(p_1, \dots, p_n, q)$  which is a contradiction. This implies that  $\mathcal{P}(\bar{t}) = \mathcal{P}(\bar{t}^{r-1}, a)$ . But this is impossible because  $a$  belongs to the algebraic closure of  $\mathbb{F}[\bar{t}^{r-1}]$  and hence it does not depend on  $t_r$  while  $\mathcal{P}(\bar{t})$  does since  $\dim(\mathcal{V}) = r$ .

### [Proof of Lemma 3]

The right-left inclusion is clear. The other inclusion is proved by induction. In the first step of the induction (i.e.  $r = j - 1$ ) we observe that, because of the definition of  $\mathfrak{M}_j^P$ ,  $\Omega = \mathfrak{M}_{r-1}^P \setminus \mathfrak{H}_{r-1}$  is dense in  $\mathfrak{M}_{r-1}^P$ , and hence it is enough to reason with  $\Omega$  which implies that only the pure components, namely  $\mathfrak{M}_{r-1}^P, \mathfrak{L}_{r-1}^P$  are relevant in the decomposition. The rest of the this part of the proof follows from the normality (that implies that points in  $\Omega$  extend to points in  $\mathfrak{B}(\mathcal{P})$ ) and from the decomposition  $\pi_{r-1}(\mathfrak{B}(\mathcal{P}))$  in terms of  $\mathfrak{H}_{r-1}$  and  $\mathfrak{L}_{r-1}$ .

For the second part of the induction, the role of  $\Omega$  is played by  $\mathfrak{M}_{j-1}^P \setminus (\pi_{j-1}(\mathfrak{M}_j^I)^* \cup \mathfrak{H}_{j-1})$ , that is dense in  $\mathfrak{M}_{j-1}^P$  and that, as above, implies that only the pure components are relevant. The rest of the proof follows again from the normality and that by the projection of the decomposition provided by the induction hypothesis.

**[Proof of Theorem 3]**

The proof of (1) and (2) are analogous. Let us focus on (2). Let assume that  $\text{cont}_{\bar{Z}}(R_j)$  have factors depending on  $\bar{t}^j$  and let  $\mathcal{A} \subset \mathbb{F}_j^j$  be the irreducible variety defined by any of these irreducible factors. Then, we reason with the dense  $\Omega = \mathcal{A} \setminus \pi_j(\mathfrak{F}_{\mathcal{P}}(\bar{h}))$ . By the normality, the points in  $\Omega$  extend to points in  $\mathfrak{M}_{j+1} \setminus \pi_{j+1}(\mathfrak{F}_{\mathcal{P}}(\bar{h}))$ ; and, using Lemma 3, to  $(\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I$ . Then,  $j-1 = \dim(\Omega) \leq \dim(\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I))$  which is a contradiction.

In order to prove (3), we observe that  $R_{r-1}$  vanishes on  $\pi_{r-1}(\mathfrak{B}(\mathcal{P}))$ ; in particular on  $\mathfrak{H}_{r-1}$ . Furthermore, since  $\mathfrak{H}_{r-1} \subset \mathbb{F}^{r-1}$ ,  $\mathfrak{H}_{r-1} \subset \mathfrak{R}_{r-1}$ . Furthermore, since  $\dim(\mathfrak{H}_{r-1}) = r-2$ ,  $\mathfrak{H}_{r-1} \subset \mathfrak{C}_{r-1}$ . Now, let us assume that  $\mathfrak{C}_{r-1}$  has another component (say irreducible)  $\mathfrak{H}_{r-1}^* \neq \mathfrak{H}_{r-1}$ . We consider the  $(r-2)$ -dimensional set  $\Omega = \mathfrak{H}_{r-1}^* \setminus \pi_{r-1}(\mathfrak{B}(\mathcal{P}))^* = \mathfrak{H}_{r-1}^* \setminus (\mathfrak{H}_{r-1} \cup \mathfrak{L}_{r-1})$ . Then, using the normality,  $\Omega$  extends to  $\Omega^e \subset \mathfrak{F}_{\mathcal{P}}(\bar{h}) \cup \mathfrak{B}(\mathcal{P})$ . Using that  $\dim(\Omega^e) \geq r-2$  and that  $\dim(\mathfrak{B}(\mathcal{P})) = r-2$ , one can take  $P^* \in \Omega^e \cap \mathfrak{B}(\mathcal{P})$ . Thus,  $\pi_{r-1}(P^*) \in \Omega \cap \pi_{r-1}(\mathfrak{B}(\mathcal{P})) = \emptyset$ , which is a contradiction.

To prove (4), let  $\Omega$  be the hypersurface included in  $\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I)^*$ . The inclusion  $\Omega \subset \mathfrak{C}_j$  follows as in (3) when seeing that  $\mathfrak{H}_{r-1} \subset \mathfrak{C}_{r-1}$ . The other inclusion is also as in (3), but taking  $\Omega = \mathfrak{T} \setminus (\Omega \cup \pi_j(\mathfrak{F}_{\mathcal{P}}))$  where  $\mathfrak{T}$  is supposed to be an irreducible factor of  $\mathfrak{C}_j$  different to  $\Omega$ .

In order to prove (3) let  $H \in \mathbb{K}[\bar{t}^{r-1}]$  be the defining polynomial of  $\mathfrak{H}_{r-1}$ .  $G_1$  and  $G$  vanish on  $\mathfrak{B}(\mathcal{P})$ , so  $R_{r-1}$  vanishes on  $\pi_{r-1}(\mathfrak{B}(\mathcal{P}))$ ; in particular on  $\mathfrak{H}_{r-1}$ . Furthermore, since  $\mathfrak{H}_{r-1} \subset \mathbb{F}^{r-1}$ , all coefficients of  $R_{r-1}$  w.r.t.  $\bar{Z}$  vanish on  $\mathfrak{H}_{r-1}$ . Thus,  $\mathfrak{H}_{r-1} \subset \mathfrak{R}_{r-1}$ . Furthermore, since  $\dim(\mathfrak{H}_{r-1}) = r-2$ ,  $\mathfrak{H}_{r-1} \subset \mathfrak{C}_{r-1}$ . Now, let us assume that  $\text{cont}_{\bar{Z}}(R_{r-1})$  has another factor  $H^*$ , coprime with  $H$ , and depending on  $\bar{t}^{r-1}$ . Let  $\mathfrak{H}_{r-1}^* = \mathbb{V}_{\mathbb{F}}(H^*)$ . We take  $\Omega = \mathfrak{H}_{r-1}^* \setminus \pi_{r-1}(\mathfrak{B}(\mathcal{P}))^* = \mathfrak{H}_{r-1}^* \setminus (\mathfrak{H}_{r-1} \cup \mathfrak{L}_{r-1})$ . Since  $\mathfrak{H}_{r-1}^* \neq \mathfrak{H}_{r-1}$ ,  $\dim(\Omega) = r-2$ . Every  $P \in \Omega$ , by the normality, extends to  $P^* \in \mathcal{W}_1^{\bar{h}} \cap \dots \cap \mathcal{W}_n^{\bar{h}} = \mathfrak{F}_{\mathcal{P}}(\bar{h}) \cup \mathfrak{B}(\mathcal{P})$ ; note that  $P$  does not depend on  $\bar{Z}$  and  $G_1 \in \mathbb{F}[\bar{t}]$ . Let  $\Omega^e \subset \mathfrak{F}_{\mathcal{P}}(\bar{h}) \cup \mathfrak{B}(\mathcal{P})$  be the set of extended points from  $\Omega$ .  $\dim(\Omega^e) \geq r-2$ . Therefore, since  $r > 2$  and  $\dim(\mathfrak{B}(\mathcal{P})) = r-2$ ,  $\Omega^e \cap \mathfrak{B}(\mathcal{P}) \neq \emptyset$ . Let  $P^* \in \Omega^e \cap \mathfrak{B}(\mathcal{P})$ . Thus,  $\pi_{r-1}(P^*) \in \Omega \cap \pi_{r-1}(\mathfrak{B}(\mathcal{P})) = \emptyset$ , which is a contradiction. Therefore,  $\mathfrak{H}_{r-1} = \mathfrak{C}_{r-1}$ .

To prove (4), let  $\Omega$  be the hypersurface included in  $\pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cup \mathfrak{M}_{j+1}^I)^*$  and let  $H$  be its defining polynomial.  $S_{j+1}(\bar{t}^{j+1}, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_j), S_{j+1}(\bar{t}^{j+1}, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_{j+1}, \bar{Z})$  vanish on  $\mathfrak{M}_{j+1}$ ; in particular,  $R_j$  vanishes on  $\Omega \subset \pi_j(\mathfrak{M}_{j+1})^*$ . Furthermore, since  $\Omega \subset \mathbb{F}_j^j$ , all coefficients of  $R_j$  w.r.t.  $\bar{Z}$  vanish on  $\Omega$ . Thus,  $\Omega \subset \mathfrak{R}_j$ . Furthermore, since  $\dim(\Omega) = j-1$ ,  $\Omega \subset \mathfrak{C}_j$ . Now, let us assume that  $\text{cont}_{\bar{Z}}(R_j)$  has another factor  $H^*$ , coprime with

$H$ , and depending on  $\bar{t}^j$ ; say that  $\mathfrak{T} = \mathbb{V}_{\mathbb{F}_j}(H^*)$ . We consider the non-empty open set  $\Omega = \mathfrak{T} \setminus (\Omega \cup \pi_j(\mathfrak{F}_{\mathcal{P}}))$ ; note that  $\dim(\pi_j(\mathfrak{F}_{\mathcal{P}})) = 0$  and that  $\gcd(H, H^*) = 1$ . Every  $P \in \Omega$  extends to a common solution  $P^*$  of  $S_{j+1}(\bar{t}^{j+1}, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_j)$  and  $S_{j+1}(\bar{t}^{j+1}, \bar{h}, \bar{W}_{r-2}, \dots, \bar{W}_{j+1}, \bar{Z})$ . Since  $P$  does not depend on  $\bar{Z}$ ,  $P^* \in \mathfrak{M}_{j+1}$ . By construction  $P^* \notin \pi_{j+1}(\mathfrak{F}_{\mathcal{P}})$ . Thus, by Lemma 3,  $P^* \in (\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cap \mathfrak{M}_{j+1}^I$ . Therefore, if we denote by  $\Omega^e$  the set of extended solutions of  $\Omega$ ,  $\Omega^e \subset (\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cap \mathfrak{M}_{j+1}^I$ . Furthermore,  $\Omega = \pi_j(\Omega^e) \subset \pi_j((\mathfrak{M}_{j+1}^P \cap \mathfrak{L}_{j+1}^P) \cap \mathfrak{M}_{j+1}^I)$ . But  $\dim(\Omega) = j - 1$ , and hence  $\Omega \subset \Omega$  which is a contradiction.

**[Proof of Theorem 4]**

(1) and (2) follow from the normality and from the fact that the involved points do not depend on  $\bar{Z}$ .

(3) We prove it by induction. The idea is to find two suitable expressions of  $\mathfrak{R}_{r-1}$  such that subtracting from them  $\mathfrak{H}_{r-1}$  we clean up the impure components and we achieve the desired decomposition. By the definition of  $\mathfrak{R}_{r-1}$  and by Theorem 3, we have  $\mathfrak{R}_{r-1} = \mathfrak{H}_{r-1} \cup \mathfrak{M}_{r-1}^P \cup \mathfrak{M}_{r-1}^I$ . On the other hand, by (1),  $\mathfrak{R}_{r-1} = \mathfrak{H}_{r-1} \cup \mathfrak{L}_{r-1}^P \cup (\mathfrak{F}_{\mathcal{P}})_{r-1}^P$ . Now, if we subtract in both sides of  $\mathfrak{H}_{r-1} \cup \mathfrak{M}_{r-1}^P \cup \mathfrak{M}_{r-1}^I = \mathfrak{H}_{r-1} \cup \mathfrak{L}_{r-1}^P \cup (\mathfrak{F}_{\mathcal{P}})_{r-1}^P$  the variety  $\mathfrak{H}_{r-1}$  we clean up the impure components, and taking closures we yields that  $\mathfrak{M}_{r-1}^P = \mathfrak{L}_{r-1}^P \cup (\mathfrak{F}_{\mathcal{P}})_{r-1}^P \cup \Omega_{r-1}$ , where  $\Omega_{r-1} = \emptyset$ .

Let the result be true for  $j + 1 \leq r - 1$ . The idea is as above: we find two different expressions of  $\mathfrak{R}_j$ , and we determine a variety big enough to clean up the impure components to derive the final decomposition. More precisely, using the induction hypothesis and (2), we get  $\mathfrak{C}_j \cup \mathfrak{M}_j = (\mathfrak{F}_{\mathcal{P}})_j^P \cup \mathfrak{H}_j \cup \mathfrak{L}_j \cup \Omega_0 \cup \pi_j(\mathfrak{M}_{j+1}^I)^*$ . On the other hand, using again the induction hypothesis, we get the  $\mathfrak{C}_j$  is either empty or the hypersurface included in  $\mathfrak{H}_j \cup \mathfrak{L}_j \cup \pi_j(\Omega_{j+1} \cap \mathfrak{L}_{j+1}^P) \cup \pi_j(\mathfrak{M}_{j+1}^I)^*$ ; that is, the hypersurface included in  $\Delta = \mathfrak{H}_j \cup \pi_j(\mathfrak{M}_{j+1}^I)^*$ . In this situation, we subtract  $\Delta$  to get  $\mathfrak{M}_j \setminus \Delta = (\mathfrak{F}_{\mathcal{P}})_j^P \cup (\mathfrak{L}_j^P \setminus \Delta) \cup (\Omega_0 \setminus \Delta)$ . Taking Zariski closures we get that  $\mathfrak{M}_j^P = (\mathfrak{F}_{\mathcal{P}})_j^P \cup \mathfrak{L}_j^P \cup \Omega_j$  with  $\Omega_j = \Omega_0 \setminus \Delta \subset \Omega_0 \subset (\mathfrak{F}_{\mathcal{P}})_j^I$ .

**[Proof of Lemma 4]**

(1) Let  $P \in \mathfrak{F}_{\mathcal{P}}(\bar{h})$  and  $P_i = \pi_i(P)$ . The proof goes as follows: (a) we prove by induction that, for  $i \in \{2, \dots, r - 1\}$ ,  $P_i \in \mathfrak{M}_i$ ; (b) we prove that  $S_0(P_1) = 0$ .

(a) Since  $G_i$  vanishes at  $P$ , and  $P$  does not depend on  $\bar{Z}$ ,  $P_{r-1} \in \mathfrak{R}_{r-1} = \mathfrak{C}_{r-1} \cup \mathfrak{M}_{r-1}$ . So, by Corollary 1,  $P_{r-1} \in \mathfrak{M}_{r-1}$ . Now, let  $P_i \in \mathfrak{M}_i$ . Then  $P_i$  does not depend on  $\bar{Z}$  and, hence,  $P_{i-1} \in \mathfrak{R}_{i-1} = \mathfrak{C}_{i-1} \cap \mathfrak{M}_{i-1}$ . Now, by Corollary 1,  $P_{i-1} \in \mathfrak{M}_{i-1}$ .

(b) Since  $P_2 \in \mathfrak{M}_2$ ,  $P_2$  does not depend on  $\bar{Z}$ , and hence  $R_1(P_1) = 0$ . Moreover, since  $P_1$  does not depend on  $\bar{Z}$ ,  $\bar{W}_\ell$ , and since  $R_1$  is univariate in  $t_1$ , one has that  $S_1(P_1) = 0$ . Finally, since  $S_1$  is univariate in  $t_1$  and  $P_1$  does depend on  $\bar{h}$  (see Lemma 1 (1)), one concludes that  $S_0(P_1) = 0$ .

(2) Since  $\bar{h} \in \mathfrak{F}_{\mathcal{P}}(\bar{h})$ , by (1),  $S_0(h_1) = 0$ . Hence  $S_0$  is not constant. Moreover, since  $S_0$  is primitive w.r.t.  $\bar{h}$ , all roots of  $S_0$  are in  $\mathbb{F} \setminus \mathbb{K}$ . On the other hand, by definition,  $\mathfrak{L}_2^P$  is either empty or zero-dimensional. In this situation, if  $S_0(\alpha) = 0$ ,

then  $S_1(\alpha) = 0$ , and hence  $R_1(\alpha) = 0$ . Moreover, by the normality assumption,  $\alpha$  extends to a common solution  $(\alpha, \beta)$  not depending on  $\overline{Z}$ . So,  $(\alpha, \beta) \in \mathfrak{M}_2^P = \mathfrak{M}_2$  (see Remark 4). By Theorem 4,  $(\alpha, \beta) \in \mathfrak{Q}_2 \cup \mathfrak{L}_2^P \cup (\mathfrak{F}_P)_2^P$ . Moreover, by Remark 3,  $\mathfrak{L}_2^P$  is  $\mathbb{K}$ -definable. So, since  $\mathfrak{L}_2^P$  is zero-dimensional and  $\alpha \notin \mathbb{K}$ , one has that  $(\alpha, \beta) \notin \mathfrak{L}_2^P$ . Thus,  $(\alpha, \beta) \in \mathfrak{Q}_2 \cup (\mathfrak{F}_P)_2^P \subset \pi_2(\mathfrak{F}_P)$ . Therefore, there exists  $P \in \mathfrak{F}_P$  such that  $\pi_1(P) = \alpha$ .

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