

# Topological Behavior of Families of Algebraic Curves Continuously Depending on a Parameter Under Certain Conditions.

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## Abstract

In previous works (see (1),(2)), the question of computing the different shapes arising in a family of algebraic curves algebraically depending on a real parameter was addressed. In this work we show how the ideas in these papers can be used to extend the results to a more general class of families of algebraic curves, namely families not algebraically but just continuously depending on a parameter. These families correspond to polynomials in the variables  $x, y$  whose coefficients are continuous functions of a parameter  $t \in U \subset \mathbb{R}$ , where  $U$  is in general the union of finitely many open intervals. Under certain conditions, here we provide an algorithm for computing a univariate real function  $R^*(t)$ , with the property that the topology of the family stays invariant along every real interval  $I \subset U$  not containing any real root of  $R^*(t)$ . In that situation, a partition of the parameter space ( $\mathbb{R}$ , in this case) where each element gives rise to a same shape arising in the family, can be computed. Then, these shapes can be described by using well-known methods ((5),(6),(8)). An important situation when the method is applicable is the case when the coefficients are algebraically independent, or can be expressed in terms of algebraically independent functions.

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## 1 Introduction

In previous works ((1), (2)) we addressed the following problem: given a family of algebraic curves algebraically depending on a parameter  $t \in \mathbb{R}$ , therefore represented by a real polynomial  $f(x, y, t)$ , compute the different shapes arising in the family, and the  $t$ -values corresponding to each shape. In (2) an algorithm for solving this question was provided (the same algorithm was adapted in (3) to the case when the family was rational, in order to work directly in parametric form, therefore in a more efficient way). By using the algorithm in (2), one can compute, for example, the different shapes arising in the family

$$f(x, y, t) = t(x + y^2 - 1) + (t - 1)^2(x^3 + y^2 - x^2) = 0$$

More precisely, in this case one has three topology types, shown in Figure 1, corresponding to  $t \in (-\infty, 0)$  (see Fig. 1, left),  $t = 0$  (see Fig. 1, center) and  $t \in (0, \infty)$  (see Fig. 1, right), respectively.

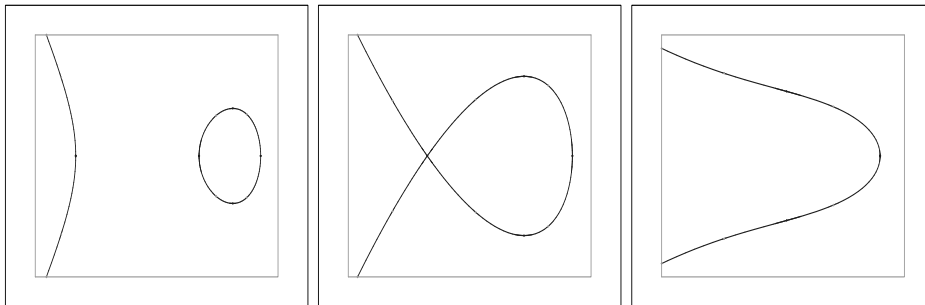


Fig. 1. Shapes in the family  $t(x + y^2 - 1) + (t - 1)^2(x^3 + y^2 - x^2) = 0$

The algorithm in (2) is of interest in contexts where geometrical objects depending on parameters may appear, for example in the topological analysis of the solutions of first-order differential equations, in robotics, in Computer Aided Geometric Design, in Control Theory, in Algebraic Biology, etc. Also of interest in these contexts, in this paper we consider the same problem but for a wider class of families of algebraic curves, also depending on a real parameter but where the dependence of the parameter is continuous, but not necessarily algebraic. So, we consider the problem of computing the shapes arising in families like, for instance,

$$f(x, y, t) = y^4 + \log(t^2 + 1)xy^2 - |t|y + t^2 + 1 = 0,$$

with  $t \in \mathbb{R}$ . The topology types arising in this family are computed in Example 2, Section 4 of this paper. In order to solve this question, based on (2) and

on classical results on the specialization of resultants, we first develop some results on families of algebraic curves algebraically depending on finitely many parameters. Then, exploiting these results and using properties of continuous functions, we provide, for a given family of the considered type, a method for computing a real function  $R(t^*)$  with the following property: whenever  $R(t^*)$  is not identically 0, the topology type of the family is invariant along every real interval  $I \subset \mathbb{R}$  where  $f$  is defined, not containing any real root of  $R(t^*)$ ; thus, the topology of the family along  $I$  can be described by taking a representative  $t_0 \in I$ , and computing the topology of the curve described by  $f(x, y, t_0)$  (see (5),(6),(8)).

An important case where  $R(t^*)$  can be guaranteed to be different from 0 happens when all the coefficients  $f(x, y, t)$  are algebraically independent, or can be expressed in terms of algebraically independent functions (see Corollary 11). Hence, under that condition the method is always successful. When  $R(t^*)$  is identically 0, the method fails. At the moment we are unaware of a solution for this special case, that we pose here as an open problem. Nevertheless, at the end of the paper (see Section 5) we make some observations on this special case, and we connect it with a topological problem.

The structure of the paper is the following. In Section 2 we briefly recall some ideas and results of (2) that are needed for developing the results of the paper. In Section 3, some results on families of algebraic curves depending on finitely many parameters are provided. In Section 4, these results are used to provide the main result of the paper. Some general considerations on the special case  $R(t^*) = 0$  are made in the last section of the paper.

## 2 Previous Results

Let  $F = F(x, y, t)$  be a real polynomial in the variables  $x, y, t$ . Given  $t_0 \in \mathbb{R}$  such that  $F(x, y, t_0) = F_{t_0}(x, y)$  is not identically 0,  $F_{t_0}(x, y)$  defines a plane algebraic curve  $\mathcal{C}_{t_0}$ . Hence, we can consider  $F$  as defining a family of algebraic curves  $\mathcal{F}$  algebraically depending on the parameter  $t$ . Clearly, as  $t_0$  moves in  $\mathbb{R}$  the shape of  $\mathcal{C}_{t_0}$  may change. So, it has sense to address the problem of detecting the different topology types arising in  $\mathcal{F}$ . Moreover, by Hardt's Semialgebraic Triviality Theorem (see Theorem 5.46 in (4)), the number of different topology types in the family must be necessarily finite. In order to solve this problem, the following definition is introduced:

**Definition 1** *Let  $\mathcal{A} \subset \mathbb{R}$ . We say that  $\mathcal{A}$  is a critical set of the family defined by  $F$ , if for all  $t_i, t_{i+1} \in \mathbb{R}$  satisfying that  $[t_i, t_{i+1}] \cap \mathcal{A} = \emptyset$ , the topology types of  $F_{t_i}$  and  $F_{t_{i+1}}$  are equal.*

Hence, a critical set is essentially a real set containing all the values of the parameter where the topology of the family may change. In (2) an almost identical definition is provided; the difference between the definition in (2) and the above one is that in (2) a critical set is defined as a finite real set satisfying the condition in Definition 1, and the definition here does not require the set to be finite. Notice that if a *finite* critical set  $\mathcal{A} = \{a_1, \dots, a_r\}$  is computed, then the different topology types in the family can be derived. Indeed, in that case the parameter space ( $\mathbb{R}$ , in this case) can be decomposed as

$$(-\infty, a_1) \cup \{a_1\} \cup (a_1, a_2) \cup \dots \cup \{a_r\} \cup (a_r, +\infty)$$

Then, taking a representative for each element of the above partition, and applying standard methods ((5), (6), (8)) for describing the topology of an algebraic curve, the topology types in the family can be computed. So, in order to determine the topology of the family, the crucial question is the computation of a finite critical set. This is exactly the problem addressed in (2). Hence, we recall now the main ideas in (2).

In (2) the following hypotheses on  $F$  (that we denote as (I) and (II)) are required: (I)  $F$  is square-free (as a polynomial in the variables  $x, y$ ); (II)  $\text{lcoeff}_y(F)$  does not depend on the variable  $x$ . Notice that (II) can always be achieved by applying if necessary a change of coordinates of the type  $x = aX + bY, y = cX + dY$ , which does not change the topology of the family; furthermore, (II) holds trivially if  $\deg_y(F) = 0$ . In (2), it is also required, as a first hypothesis, the non-existence of  $t_0 \in \mathbb{R}$  so that  $F_{t_0}(x, y)$  is identically 0. This hypothesis is formulated in order to ensure that every specialization of the parameter  $t$  defines a curve of the family; however, the results in (2) can be perfectly developed excluding this hypothesis. Thus, in the present work we will not take it into account.

Furthermore, we need to introduce two polynomials; here,  $D_w(G) := \text{Res}_w(G, \frac{\partial G}{\partial w})$ , and  $\sqrt{G}$  denotes the square-free part of  $G$ . Also, abusing of language, in the sequel we will refer to  $D_w(G)$  as the “discriminant” of  $G$  w.r.t. the variable  $w$  (notice that usually the discriminant denotes the result of dividing out the resultant  $D_w(G)$  by the leading coefficient). Then we define (this notation is slightly different to that of (2), and was suggested by an anonymous post-reviewer of the paper (2)),

$$M(x, t) := \begin{cases} \sqrt{D_y(F(x, y, t))} & \text{if } \deg_y(F) \neq 0 \\ F(x, t) & \text{if } \deg_y(F) = 0 \end{cases},$$

$$R(t) := \begin{cases} D_x(M(x, t)) & \text{if } \deg_x(M) \neq 0 \\ M(t) & \text{if } \deg_x(M) = 0 \end{cases}$$

With this definition, the following lemma holds.

**Lemma 2** *The polynomial  $R(t)$  is identically 0 iff  $F(x, y, t)$  is identically 0.*

**Proof.** By the definition of  $M$ , it holds that  $M$  is either square-free, or constant. Indeed, this is clear if  $\deg_y(F) \neq 0$ ; if  $\deg_y(F) = 0$ ,  $M$  is defined as  $F$ , and since by hypothesis  $F$  is square-free, then so is  $M$ . Hence, let us see  $(\Rightarrow)$ . If  $R = 0$  then  $M = M(t)$ , because otherwise we would have that  $D_x(M) = 0$  and therefore  $M$  would not be square-free. But then by definition of  $R$  we have  $R = M$ , and therefore  $M = 0$ . Now by the definition of  $M$ , and taking again into account that  $F$  is square-free by hypothesis, we conclude that  $F = 0$ .  $(\Leftarrow)$  is straightforward from the definitions of  $M, R$ .  $\square$

Thus, the following theorem, which provides a method for computing a finite critical set, holds (see Theorem 4 and Theorem 13 in (2)).

**Theorem 3** *Let  $F$  satisfy the hypotheses (I) and (II). Then the set of real roots of  $R$ , is a critical set of  $F$ . If  $R$  has no real roots, then the elements of the family show just one topology type.*

### 3 Families of Algebraic Curves Algebraically depending on parameters

In Section 2 we have revised known results concerning families of algebraic curves algebraically depending on a parameter. Based on these ideas, in this section we provide some results for families of algebraic curves algebraically depending on finitely many parameters. So, let  $F \in \mathbb{R}[x, y, t_1, \dots, t_n]$  be a real polynomial in the variables  $x, y, t_1, \dots, t_n$  satisfying the conditions (I), (II) introduced in Section 2, where  $t_1, \dots, t_n$  are parameters taking values in  $\mathbb{R}$ ; in other words,  $F$  is square-free as a polynomial in the variables  $x, y$  and  $\text{lcoeff}_y(F)$  does not depend on  $x$ . Hence, for each specification of  $(t_1, \dots, t_n) = (a_1, \dots, a_n) \in \mathbb{R}^n$  in  $F$  (except for those ones causing  $F$  to be identically 0), the polynomial  $F_{a_1, \dots, a_n}(x, y)$  obtained this way defines an algebraic curve. In particular, one may consider  $F$  as defining a family  $\mathcal{F}$  of algebraic curves algebraically depending on the parameters  $t_1, \dots, t_n$ . Also from Hardt's Semi-algebraic Triviality Theorem, it follows that the number of these types must be finite. So, our purpose is to apply the results in the above section in order to study the topology types in  $\mathcal{F}$ .

The definitions of the polynomials  $M, R$  provided in Section 2 are easily gen-

eralized for the case of several parameters. Hence, we define

$$M(x, t_1, \dots, t_n) := \begin{cases} \sqrt{D_y(F(x, y, t_1, \dots, t_n))} & \text{if } \deg_y(F) \neq 0 \\ F(x, t_1, \dots, t_n) & \text{if } \deg_y(F) = 0 \end{cases},$$

$$R(t_1, \dots, t_n) := \begin{cases} D_x(M(x, t_1, \dots, t_n)) & \text{if } \deg_x(M) \neq 0 \\ M(t_1, \dots, t_n) & \text{if } \deg_x(M) = 0 \end{cases}$$

Notice that  $\deg_y(F) \neq 0$  (resp.  $\deg_x(M) \neq 0$ ) iff  $\text{lcoeff}_y(F)$  (resp.  $\text{lcoeff}_x(M)$ ) is not identically 0. Furthermore, reasoning as in Lemma 2, we have the following result.

**Lemma 4** *The polynomial  $R(t_1, \dots, t_n)$  is identically 0 iff  $F(x, y, t_1, \dots, t_n) = 0$ .*

Now, in the sequel, if  $G$  is a polynomial depending on the parameters  $t_1, \dots, t_n$  (maybe depending also on other variables), we denote the evaluation of  $G$  at  $t_1 = a_1, \dots, t_n = a_n$  as  $G_{a_1, \dots, a_n}$ . So,  $F_{a_1, \dots, a_n}, M_{a_1, \dots, a_n}, R_{a_1, \dots, a_n}$  denote the evaluations of  $F, M, R$  at  $t_1 = a_1, \dots, t_n = a_n$ , respectively. Notice that  $F_{a_1, \dots, a_n} = F_{a_1, \dots, a_n}(x, y)$ , and  $M_{a_1, \dots, a_n} = M_{a_1, \dots, a_n}(x)$ . Thus, a first problem is to decide whether  $M$  and  $R$  behave well under the specialization of the parameters  $t_1, \dots, t_n$ . In order to make this idea more precise, we introduce the following additional notation; for  $(a_1, \dots, a_n) \in \mathbb{R}^n$ , we define

$$\tilde{M}(x, a_1, \dots, a_n) := \begin{cases} \sqrt{D_y(F_{a_1, \dots, a_n}(x, y))} & \text{if } \deg_y(F_{a_1, \dots, a_n}) \neq 0 \\ F_{a_1, \dots, a_n}(x) & \text{if } \deg_y(F_{a_1, \dots, a_n}) = 0 \end{cases},$$

$$\tilde{R}(a_1, \dots, a_n) := \begin{cases} D_x(M_{a_1, \dots, a_n}(x)) & \text{if } \deg_x(M_{a_1, \dots, a_n}) \neq 0 \\ M_{a_1, \dots, a_n} & \text{if } \deg_x(M_{a_1, \dots, a_n}) = 0 \end{cases}$$

Notice that the above polynomials are *really* the specializations of  $M, R$  at  $t_1 = a_1, \dots, t_n = a_n$ , since there we are taking into account how the degrees of  $F, M$  w.r.t  $y, x$ , respectively, specialize at  $t_1 = a_1, \dots, t_n = a_n$  (however, when we substitute  $t_1 = a_1, \dots, t_n = a_n$  in  $M, R$  to get  $M_{a_1, \dots, a_n}, R_{a_1, \dots, a_n}$  we are not considering this). Then the question is to check whether the equalities  $\tilde{M}(x, a_1, \dots, a_n) = M_{a_1, \dots, a_n}(x)$ ,  $\tilde{R}(a_1, \dots, a_n) = R_{a_1, \dots, a_n}$  hold or not, i.e. whether  $M$  and  $R$  behave well under specialization, or not. A simple example will show that these polynomials do not necessarily specialize well; for example, taking

$$F(x, y, a) = ay^2 + y + x + 1$$

one may see that  $M(x, a) = 4xa^2 + 4a^2 - a$ ,  $R(a) = 4a^2$ , and therefore that  $M_0(x) = M(x, 0) = 0$ ,  $R_0 = R(0) = 0$ . However,  $F(x, y, 0) = y + x + 1$ , and therefore  $\tilde{M}_0(x) = \tilde{M}(x, 0) = 1$ ,  $\tilde{R}_0 = \tilde{R}(0) = 1$  (from the definition of  $\tilde{R}$ ).

First, we need the following lemma.

**Lemma 5** *Let  $(a_1, \dots, a_n) \in \mathbb{R}^n$  so that  $R_{a_1, \dots, a_n} \neq 0$ . Then, the following statements are true:*

- (i)  $\deg_y(F_{a_1, \dots, a_n}) = \deg_y(F)$
- (ii)  $F_{a_1, \dots, a_n}(x, y)$  is either non-dependent on  $x, y$ , or square-free as a polynomial in  $x, y$ .

**Proof.** Let us see (i). If  $\deg_y(F) = 0$  then the statement is obvious, so assume that  $\deg_y(F) > 0$ . Then, let  $A := \text{lcoeff}_y(F)$ . By using the Sylvester form of the resultant, one deduces that  $A$  divides  $M$ . If  $\deg_x(M) = 0$  then by the definition of  $R$  it also holds that  $A$  divides  $R$ ; otherwise, denoting  $B := \text{lcoeff}_x(M)$ , also by the Sylvester form of the resultant one deduces that  $B$  divides  $R$ , and since  $A$  divides  $B$ ,  $A$  divides  $R$ . Then,  $A(a_1, \dots, a_n) \neq 0$  because otherwise  $R_{a_1, \dots, a_n} = 0$ , which cannot happen by hypothesis. Hence, (i) holds. Now let us see (ii), and for this purpose we distinguish the cases  $\deg_y(F) = 0$  and  $\deg_y(F) \neq 0$ . We start with  $\deg_y(F) = 0$ ; in this case, by the definition of  $M$  we have  $M = F$ . If  $\deg_x(F) = 0$  then  $F$  is non-dependent on  $x, y$ , and (ii) holds. Otherwise,  $\deg_x(F) \neq 0$  and by the definition of  $R$  we have  $R = D_x(F)$ . As we have seen,  $B = \text{lcoeff}_x(F)$  is a factor of  $R$ . Hence,  $\text{lcoeff}_x(F)$  cannot vanish at  $(a_1, \dots, a_n)$  because  $R_{a_1, \dots, a_n} \neq 0$  by hypothesis. Then, by Lemma 4.3.1, pg. 96 in (10) it follows that the resultant  $\text{Res}_x(F, F_x)$  specializes well, and this resultant is exactly  $D_x(M)$ . Hence, if  $F_{a_1, \dots, a_n}(x, y)$  is not square-free, we get that  $D_x(M)$  vanishes at  $(a_1, \dots, a_n)$ , which is in contradiction with the fact that  $R_{a_1, \dots, a_n} \neq 0$ . Now, consider the case  $\deg_y(F) \neq 0$ . Here,  $M = \sqrt{D_y(F)}$ . Now since statement (i) holds, we have that  $\text{lcoeff}_y(F)$  does not vanish at  $(a_1, \dots, a_n)$  and therefore also by Lemma 4.3.1, pg. 96 in (10), the resultant  $\text{Res}_y(F, F_y)$  specializes well; however this resultant is  $D_y(F)$ . Since, reasoning as before, the resultant  $\text{Res}_x(M, M_x)$  also specializes well in the considered case, we get that if  $F_{a_1, \dots, a_n}(x, y)$  is not square-free, then  $R_{a_1, \dots, a_n} = 0$ , which cannot happen.  $\square$

Now we can prove the following result on the specialization of  $M, R$ . In order to prove it, one would use Lemma 5, and similar considerations to those in the proof of Lemma 5; hence, it is left to the reader.

**Lemma 6** *Let  $(a_1, \dots, a_n) \in \mathbb{R}^n$  so that  $R_{a_1, \dots, a_n} \neq 0$ . Then, the following statements hold:*

- (i)  $\tilde{M}(x, a_1, \dots, a_n) = M_{a_1, \dots, a_n}(x)$ .
- (ii)  $\tilde{R}(a_1, \dots, a_n) = R_{a_1, \dots, a_n}$ .

A consequence of Lemma 5 and Lemma 6 is the following result, which is essential for our purposes. This proposition ensures that under certain condi-



tions, when we specialize all the parameters  $t_i$ 's except but one a critical set of the uniparametric family can be determined by computing the real roots of the specialization of  $R$ . For simplicity, in the statement of the result we assume that the only parameter that we do not specialize is  $t_1$ ; clearly, the statements holds if we consider any other  $t_i$ , with  $i = 2, \dots, n$ .

**Proposition 7** *Let  $(a_2, \dots, a_n) \in \mathbb{R}^{n-1}$ , where  $R(t_1, a_2, \dots, a_n)$  is not identically 0, and let  $G(x, y, t_1) = F(x, y, t_1, a_2, \dots, a_n)$ . Then, the set of real roots of  $R(t_1, a_2, \dots, a_n)$  is a critical set of the family defined by  $G$ .*

**Proof.** By statement (ii) of Lemma 5 it holds that  $G$  is square-free as a polynomial in  $x, y$ . Moreover, because of statement (i) of Lemma 5 we have that  $\deg_y(G) = \deg_y(F)$ . In particular,  $\text{lcoeff}_y(G)$  is the evaluation of  $\text{lcoeff}_y(F)$  at  $t_2 = a_2, \dots, t_n = a_n$ , and this evaluation is not identically 0. Since by hypothesis  $\text{lcoeff}_y(F)$  does not depend on  $x$ , then  $\text{lcoeff}_y(G)$  does not depend on  $x$ , either. Therefore,  $G$  satisfies the hypotheses (I) and (II); so, by Theorem 3, the set of real roots of  $\tilde{R}(t_1, a_2, \dots, a_n)$  is a critical set of the family defined by  $G$ . However, for all  $t_1 \in \mathbb{R}$  so that  $R(t_1, a_2, \dots, a_n) \neq 0$ , by Lemma 6 we have that  $\tilde{R}(t_1, a_2, \dots, a_n) = R(t_1, a_2, \dots, a_n)$ . Thus, the only  $t_1$ -values where  $\tilde{R}(t_1, a_2, \dots, a_n)$  can vanish are the real roots of  $R(t_1, a_2, \dots, a_n)$ , and hence the proposition follows.  $\square$

Then we are finally ready to prove the main result of this subsection. For this purpose, we denote the variety defined in  $\mathbb{R}^n$  by  $R(t_1, \dots, t_n) = 0$  as  $\mathcal{R}$ . Since by Lemma 4 the polynomial  $R$  is not identically 0, then  $\mathcal{R}$  is a proper variety, and divides  $\mathbb{R}^n$  into open connected regions  $\mathcal{P}_1, \dots, \mathcal{P}_\ell$  (see Figure 1, left). Then, the following theorem, whose proof is illustrated in Figure 2, right, holds.

**Theorem 8** *The topology type of  $\mathcal{F}$  is invariant along each  $\mathcal{P}_i$ .*

**Proof.** Let  $i \in \{1, \dots, \ell\}$ , and let us prove that the topology type of  $\mathcal{F}$  is constant over  $\mathcal{P}_i$ . Since  $\mathcal{P}_i$  is connected, given two points  $P, Q \in \mathcal{P}_i$  we can connect them by means of finitely many oriented segments  $L_1, \dots, L_r$  satisfying the following conditions: (i) the start point of  $L_1$  is  $P$ , and the end point of  $L_r$  is  $Q$ ; (ii) the end point of  $L_j$ , is the start point of  $L_{j+1}$ ; (iii) along each segment  $L_j$ , all the parameters but one are constant (i.e. each segment is parallel to some coordinate axis of  $\mathbb{R}^n$ ). Now let  $L_u$  be one of these segments, and w.l.o.g. assume that  $t_1$  is the only parameter which is not constant over  $L_u$ , i.e. the points of  $L_u$  satisfy  $t_2 = a_2, \dots, t_n = a_n$  with  $(a_2, \dots, a_n) \in \mathbb{R}^{n-1}$ . Then, over  $L_u$  we have that  $G = F|_{L_u}$  defines a uniparametric family of algebraic curves of parameter  $t_1$ . Now by Proposition 7, the set of real roots of  $R(t_1, a_2, \dots, a_n)$  form a critical set of  $G$ . Finally, since  $L_u$  lies in  $\mathcal{P}_i$ , then  $R$  does not vanish along  $L_u$ , and neither does  $R(t_1, a_2, \dots, a_n)$ . Therefore, by Theorem 3 the topology type of  $G$ , and therefore of  $F$ , stays invariant along



$L_u$ . Since this holds for  $u \in \{1, \dots, r\}$ , we conclude that the topology type of  $F$  at  $P$  and  $Q$  is the same. Finally,  $P$  and  $Q$  are generic, and therefore the statement holds.  $\square$

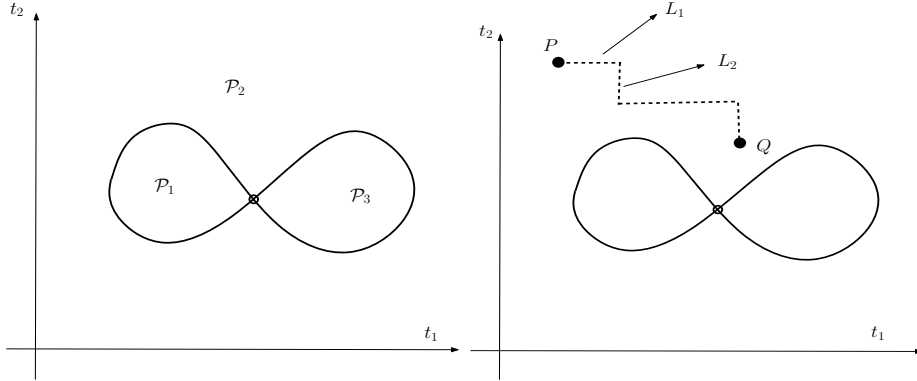


Fig. 2. Theorem 8: notation (left) and proof (right).

#### 4 Families of Algebraic Curves Continuously depending on a parameter.

Let  $U[t]$  be the ring of continuous functions in the variable  $t$ , over a subset  $U \subset \mathbb{R}$ ; for simplicity we will assume that  $U$  is a real open interval, though the results are easily generalized for the case when  $U$  is the union of finitely many real intervals. Now let us consider the ring  $U[t][x, y]$ ; the elements of  $U[t][x, y]$  are polynomials in the variables  $x, y$  with coefficients in  $U[t]$ , i.e.

$$f(x, y, t) = \sum_{i=0}^m \sum_{j=0}^n a_{ij}(t) x^i y^j$$

where the  $a_{ij}(t)$ 's are continuous. Given  $f \in U[t][x, y]$ , for every  $t_0 \in \mathbb{R}$  so that  $f(x, y, t_0) = f_{t_0}(x, y)$  does not identically vanish one has that  $f_{t_0}(x, y)$  defines an algebraic curve. So, one may consider that  $f \in U[t][x, y]$  defines a family  $\mathcal{F}_U$  of algebraic curves depending on the parameter  $t$ , with  $t \in U \subset \mathbb{R}$ . We say that such a family is a *family of algebraic curves continuously depending on the parameter  $t$* . Once again, by Semialgebraic Hardt's Triviality Theorem such a family contains finitely many topology types. So, here we address the algorithmic analysis of these topology types. Moreover, in the sequel we will also assume that  $\text{lcoeff}_y(f)$  does not depend on  $x$  (otherwise, as observed in Section 2, one can apply an affine change of coordinates involving just  $x, y$ , which therefore does not change the topology of the family).

Now let us consider the following definition, which is a generalization of Definition 1 to this new context.

**Definition 9** Let  $U \subset \mathbb{R}$  in the above conditions. Then, we say that  $\mathcal{A} \subset \mathbb{R}$  is a critical set of the family  $\mathcal{F}_U$ , if  $\mathcal{A}$  contains all the  $t$ -values where the topology type of  $\mathcal{F}_U$  over  $U$  changes.

Therefore, our aim is to compute a critical set of  $\mathcal{F}_U$ ; for this purpose, we will make use of the ideas in Section 3. We need to introduce some notation. Let  $f \in U[t][x, y]$ , having  $N$  different terms, that we will denote as  $b_1(t), \dots, b_N(t)$ . Moreover, whenever  $b_i(t)$  is non-constant, let us denote by  $F$  the result of performing the substitution  $b_i(t) := u_i$  (where  $u_i$  is a new variable) in  $f$ , for  $i = 1, \dots, N$ . Notice that  $F$  is a polynomial in the variables  $x, y, u_1, \dots, u_N$ ; one may also see  $F$  as defining a family of algebraic curves depending on the parameters  $u_1, \dots, u_N$ . Moreover, we assume that  $F$  is square-free as a polynomial in  $x, y$  (if it is not, we compute its square-free part and we carry on the analysis; notice that substituting back  $u_i := b_i(t)$ , the resulting polynomial defines the same family than  $f$ ). Hence, starting from  $F$  one can compute the polynomials  $M(x, u_1, \dots, u_N)$  and  $R(u_1, \dots, u_N)$  introduced in Section 3. We will write  $R_f(u_1, \dots, u_N)$  instead of simply  $R(u_1, \dots, u_N)$ , to emphasize that this polynomial has been obtained from  $f$ . Moreover, let  $R^*(t) := R_f(b_1(t), \dots, b_N(t))$ , i.e. the result of performing the “inverse substitution”  $u_i := b_i(t)$  in  $R_f$  for all  $i \in \{1, \dots, N\}$ . Then the following theorem, which is the main result of the paper, holds.

**Theorem 10** Let  $I \subset U$  be a real interval so that  $R^*(t)$  does not vanish over  $I$ . Then, the topology type of  $\mathcal{F}_U$  stays invariant along  $I$ .

**Proof.** Since  $R_f$  is a polynomial and the  $b_i(t)$ 's are continuous over  $I$ , it follows that  $R^*(t)$  is also continuous over  $I$ . Also, since by Lemma 4 it holds that  $R_f(u_1, \dots, u_N)$  is not identically 0, then it defines a proper algebraic variety over  $\mathbb{R}^N$  that we represent by  $\mathcal{V}_R$  (if  $R_f$  is constant,  $\mathcal{V}_R = \emptyset$ ). Now,  $I$  is connected and  $R^*(t)$  is continuous; therefore,  $R^*(I)$  is a connected subset of  $\mathbb{R}^N$ . Moreover, since  $R^*(t)$  does not vanish over  $I$ , this means that for  $t \in I$ , the curve  $\mathcal{L}$  of  $\mathbb{R}^N$  parametrized as  $x_1 = b_1(t), \dots, x_n = b_n(t)$  does not intersect  $\mathcal{V}_R$  when  $t \in I$ ; indeed, if  $P \in \mathcal{L} \cap \mathcal{V}_R$ , then  $P = (b_1(t^*), \dots, b_N(t^*))$  where  $R^*(t^*) = 0$ . And since  $R^*(I)$  is connected, we deduce that for  $t \in I$ ,  $\mathcal{L}$  is contained in one of the connected components of  $\mathbb{R}^N \setminus \mathcal{V}_R$ . Then the statement follows from Theorem 8.  $\square$

In other words, whenever  $R^*(t)$  is not identically 0 we have that the set  $\mathcal{A}$  of real roots of  $R^*(t)$  over  $U$  is a critical set of  $\mathcal{F}_U$ . Theorem 10 provides the following corollary. Here, the notion of *algebraic dependence* is used. Essentially, we say that several functions  $\alpha_1(t), \dots, \alpha_r(t)$  are algebraically dependent, if there exists a polynomial  $G(x_1, \dots, x_r)$  such that  $G(\alpha_1(t), \dots, \alpha_r(t)) = 0$ ; in other words, if there exists an algebraic relationship among them. For example,  $\alpha_1(t) = \sin(t)$  and  $\alpha_2(t) = \cos(t)$  are algebraically dependent, and  $G(x_1, x_2) = x_1^2 + x_2^2 - 1$  provides an algebraic relationship involving them.

We say that  $\alpha_1(t), \dots, \alpha_r(t)$  are algebraically independent, if they are not algebraically dependent. The problem of detecting whether a given set of explicit functions is algebraically independent or not has been considered, and is still open. The reader may find some results on this question in (7); in that paper, one may find several examples of families of real functions which are algebraically independent. So, the following corollary holds.

**Corollary 11** *If the  $a_{i,j}(t)$ 's are algebraically independent, then the method provided for computing a critical set, is successful.*

**Proof.** If  $R^*(t)$  is identically 0, then  $R(b_1(t), \dots, b_n(t)) = 0$  provides an algebraic relationship satisfied by  $b_1(t), \dots, b_n(t)$ . So, an algebraic relationship among the  $a_{i,j}(t)$ 's exists. However, if the  $a_{i,j}(t)$ 's are algebraically independent, this cannot happen.  $\square$

The following observations should also be taken into account:

- (i) Unlike it happens in the case when the dependence of  $t$  is algebraic, i.e. when  $f \in \mathbb{R}[x, y, t]$ , it may happen that  $R^*(t)$  has infinitely many real roots.
- (ii) If  $R^*(t)$  has  $r$  real roots in  $U$ , then there are at most  $2r + 1$  topology types of  $\mathcal{F}_U$  over  $I$ . Moreover, in order to determine them one computes the real roots of  $R^*(t)$ , and proceeds as suggested in Section 2.
- (iii) The set of **exp-log** functions (see (9) for further information) is the smallest set of functions  $\mathbb{R} \rightarrow \mathbb{R}$  containing *exp*, *log*, the identity function and the constant functions, closed under addition, multiplication and composition of functions. It is known that every exp-log function has finitely many real roots. Hence, if the coefficients of  $f$  are independent exp-log functions, then  $R^*(t)$  is a non-identically 0 exp-log function as well, and the critical set determined by applying Theorem 10 is finite. Moreover, in this case there are efficient algorithms to compute the real roots of  $R^*(t)$  (see (9)).

The following examples illustrate Theorem 10.

**Example 1** *Consider the family  $\mathcal{F}$  of algebraic curves defined by*

$$f(x, y, t) = y^4 + e^t xy^2 - \sqrt{t}y + x^2$$

*Observe that the coefficients of  $f$  are continuous for  $t \in U = (0, \infty)$ . Hence, let us compute the topology types of  $\mathcal{F}_U$ . For this purpose, we consider  $u_1 := e^t$  and  $u_2 := \sqrt{t}$ , and*

$$F(x, y, u_1, u_2) := y^4 + u_1 xy^2 - u_2 y + x^2$$

*Hence, we get  $M = -4u_1^3 u_2^2 x^3 - 27u_2^4 + 16u_1^4 x^6 - 128u_1^2 x^6 + 144u_1 x^3 u_2^2 + 256x^6$ , and  $R = -8916100448256 u_2^2 0(u_1 - 2)^6 (u_1 + 2)^6 (12 + u_1^2)^9$ . So,*

$$R^*(t) = -8916100448256 \cdot t^{10} (e^t - 2)^6 (e^t + 2)^6 (e^{2t} + 12)^9$$

which has just one real root in  $U$ , namely  $\log 2$  (here,  $\log$  denotes the natural logarithm). Hence, from Theorem 10 it holds that  $\mathcal{A} = \{\log 2\}$  is a critical set. Then, we get at most 4 topology types, corresponding to the following cases: (I)  $t = 0$ ; (II)  $t \in (0, \log 2)$ ; (III)  $t = \log 2$ ; (IV)  $t \in (\log 2, \infty)$ . These topology types are shown in Figure 3 (the pictures here have been obtained with Maple's package `algcurves`).

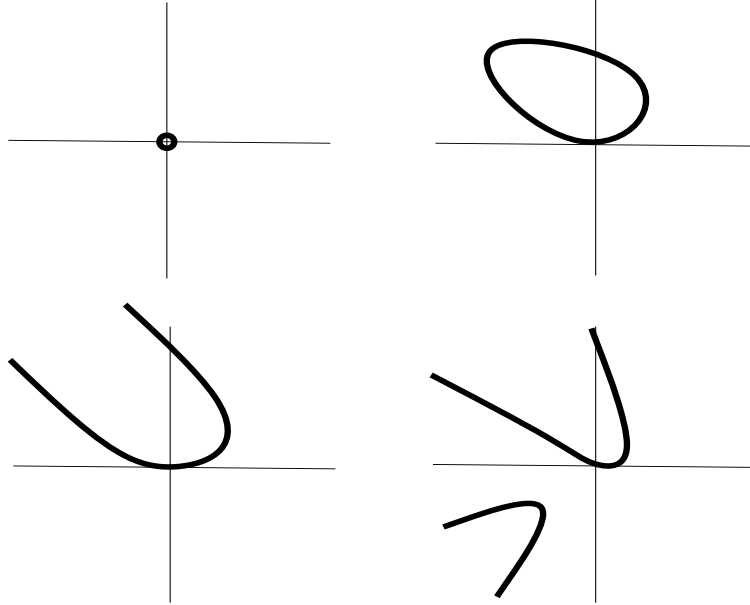


Fig. 3. Topology Types in Example 1: (I) up, left; (II) up, right; (III) down, left; (IV) down, right

**Example 2** Let  $\mathcal{F}$  be the family defined by

$$f(x, y, t) = y^4 + \log(t^2 + 1)xy^2 - |t|y + t^2 + 1$$

The coefficients of  $f$  are continuous for  $t \in \mathbb{R}$ . In this case, we get that

$$R^*(t) = -4096 \log(t^2 + 1)^{16} (t^2 + 1) |t|^4 (4096(t^2 + 1)^3 + 27|t|^4)^3$$

The only real root of  $R$  is 0; hence,  $\mathcal{A} = \{0\}$  is a critical set. Therefore, we get at most 3 topology types, corresponding to the cases  $t \in (-\infty, 0)$ ,  $t = 0$ ,  $t \in (0, \infty)$ , respectively. One may see that for  $t = 0$ , the corresponding curve in the family is empty over the reals; however, for  $t \neq 0$ , the topology type is shown in Figure 4.

## 5 Open Questions

Whenever  $R^*(t)$  is not identically 0, a critical set of the family can be computed. However,  $R^*(t)$  can be identically 0. For example, consider the family

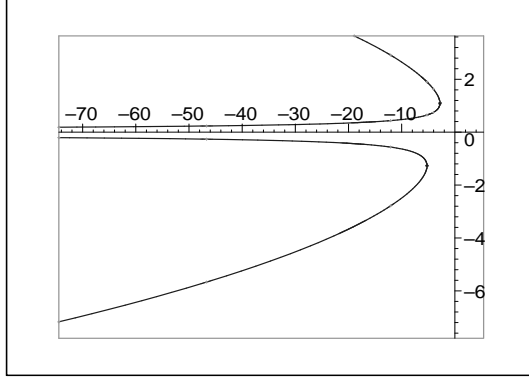


Fig. 4. Topology Type for  $t = 0$  in Example 2

defined by

$$f(x, y, t) = t^{1/4}x^2 + t^{1/2}xy + (1/4)t^{3/4}y^2 + e^t$$

In this case, setting  $u_1 := t^{1/4}$ ,  $u_2 := t^{1/2}$ ,  $u_3 := (1/4)t^{3/4}$ ,  $u_4(t) = e^t$  we have that  $R(u_1, u_2, u_3) = 16u_3^4u_4(-4u_3u_1 + u_2^2)^2$ . Hence,  $R^*(t) = 0$ . The reason for  $R^*(t)$  being identically 0 is that  $-4u_3u_1 + u_2^2 = 0$  expresses an algebraic relationship among  $u_1 := t^{1/4}$ ,  $u_2 := t^{1/2}$ ,  $u_3 := (1/4)t^{3/4}$ . In this particular case the problem is easily solved by simply choosing  $u_1, u_2, u_3$  in a different way; for example, taking  $u_1 := t^{1/4}$ ,  $u_2 := e^t$  we get  $f(x, y, u_1, u_2) = u_1x^2 + u_1^2xy + (1/4)u_1^3y^2 + u_2$ , and  $R^*(t) = te^t$ . However, at the moment we cannot guarantee that a “good” choosing for the  $u_i$ ’s exists in all cases, and we are also unaware of a characterization of the cases where such a good choosing exists.

Nevertheless, one may see that a general solution of the case  $R^*(t) = 0$  can be related to another problem, which up to our knowledge is currently unsolved. Let us explain this question in more detail. In Section 3, we addressed the computation of the topology types arising in an implicitly defined family of algebraic curves depending on parameters  $u_1, \dots, u_n$ , on the open ( $n$  dimensional) regions determined over the parameter space  $\mathbb{R}^n$  by the variety  $R(u_1, \dots, u_n) = 0$ . However, we did not address the computation of the topology of the family *over* the variety  $R(u_1, \dots, u_n) = 0$  itself. In fact, up to our knowledge, this problem has not been solved yet, and seems to require other tools than those used in our paper. In order to solve it, one should have a method for determining a partition of the variety  $R(u_1, \dots, u_n) = 0$  into cells (of different dimensions) so that the topology of the family was invariant over each cell. If such a description were available, one might determine the cells travelled by the curve in  $\mathbb{R}^n$  defined by  $u_1 = b_1(t), \dots, u_n = b_n(t)$ ; that way, the topology types in the family would be those corresponding to the cells travelled by the curve, and hence a complete solution to the problem addressed in the paper could be given.

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