A Short Journey Through the Riemann Integral

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April 23, 2014

Abstract

An introductory-level theory of integration was studied, focusing primarily on the well-known Riemann integral and ending with the Lebesgue integral. An examination of the Riemann integral's basic properties and necessary conditions shows that this integral is not very strong. This conclusion leads to Lebesgue's necessary and sufficient condition for Riemann integrability, perhaps where the Riemann integral ends and the Lebesgue integral begins: a function defined on a closed integral $[a, b]$ is Riemann integrable if and only if the function is discontinuous on a set of measure zero. The proof of this uses that the infinite union of sets of measure zero has a measure of zero. To distinguish between the Riemann and Lebesque integral, the classical example of the dirac-delta function displays the strength of the Lebesgue integral over the Riemann integral. In conclusion, every function that is Riemann integrable is Lebesgue integrable, but the converse is not true.

1 Introduction: Integration

When talking about integration, the question being asked is what the area is between a real-valued function and the line $y = 0$, at least for functions defined on the real numbers. This paper seeks to understand the theory behind integration, how to build a useful tool in studying calculus. Typically the student learns the calculational method (specifically the anti-derivative) in pre-calculus or calculus I, but the tools the Riemann developed in his analysis explain why those calculational methods work. A major result of his analysis is both parts of the fundemental theorem of calculus, and his integral is useful in many other areas of theoretical study. However, as the conclusion of this paper approaches it becomes evident that there are many instances when Riemann's idea cannot be used. There are several other types of integrals, some a more advanced version of the Riemann, but others formed from a new idea completely. In regards to the latter, Henri Lebesgue came up with an idea of integration that is much more advanced and much stronger than the Riemann integral, covering a much wider range of functions. However, this paper does not dive too far into what the integral really is. A proper introduction would require much background information in other areas of mathematics. Therefore, in this paper I am focusing on the Riemann integral and answering the question of when this integral will actually work. I will start out explaining what the integral is, provide some properties, and the reveal an example where this integral fails. In conclusion, I will provide a theorem from Lebesgue which is a criteria for Riemann integration and then get into some topics for further discussion.

2 The Riemann Integral

The basic idea behind the Riemann integral is to look at a bounded function on a closed interval $[a, b]$ into $\mathbb{R}$, and to create rectangles with a base from a partition of $[a, b]$ and the height from the max and min values of the function within each base. The summation of the areas of the rectangles can approximate the area under the curve given by the function.

Definition: Upper/Lower Sums. Let f be a bounded, real-valued function defined on a closed interval $[a, b]$. We say the set $P = \{a = x_0, x_1, ..., x_{n-1}, x_n = b\}$ is a **partition** of $[a, b]$ and for any $i \in \{1, ..., n\}$ let

$$M_i = sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$m_i = inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

We call an **Upper Sum** of f with respect to P:

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

and the **Lower Sum** of f with respect to P:

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i$$

where $\Delta x_i = (x_i - x_{i-1})$ for each i.

Remark: The "sup" and "inf" used above are called the greatest lower bound and least upper bound, respectively, and to avoid a careful, detailed and unnecessary explanation for my purposes here, they can be thought of as the maximum and minimum values, respectively.

Definition: Upper/Lower Integrals. For a bounded function f on a closed interval $[a, b]$, the **Upper Integral** is denoted as

$$U(f) = inf\{U(f, P) : \text{P is a partition of } [a, b]\}$$

and the **Lower Integral** is denoted as

$$L(f) = sup\{L(f, P) : \text{P is a partition of } [a, b]\}$$

The function f is **Riemann Integrable** if and only if $U(f) = L(f)$, and when this is true its value is denoted by

$$\int_a^b f = U(f) = L(f)$$

2.1 Some Immediate Observations of the Riemann Integral

Based on the defintions given, there should be some immediate questions to be asked:

1. What if a function f is defined on an open interval? A half-open interval?
2. What if a function is not bounded?
3. If the function is negative at some points, how does that affect the interpretation of the integral?

To answer the first question, without being defined on a closed interval, the Fundemental Theorem of Calculus fails because we need uniform continuity of the function. Concerning the second observation, if the function is not bounded, then there would be no way to calculate the area of at least one rectangle in the partial sums. Hence the area of an unbounded function exceeds any real number.

Both of these issues lead to a definition of an "improper integral:"
Let f be defined on $(a, b]$ and integrable on $[c, d]$ for all $c \in (a, b]$. The **improper integral** of f on $(a, b]$ is

$$\int_a^b f = \lim_{c \to a+} \int_c^b f.$$

When the improper integral exceeds any finite number, it is said to **diverge**. When it doesn't, it **converges**. An interesting observation is that f does not need to be defined at a. We could also look at functions defined on the whole real number line by a similar idea.

As for the third question, negative values in the range of f does not compromise the definitions given at the start. Riemann integration can still be attained by such a function, but it may skew the way we look at the area of the function under its curve, as its negative area will surely "take away" from the positive area.

2.2 Properties/Conditions for Riemann Integration

Looking at area approximations with small partitions is not very effective. The whole idea of Riemann integration is to look at the sums as the partitions increase in size. In fact, many definitions of the Riemann integral involve the limits of the sum, which beckons the efficiency of working with sequences as learned in Real Analysis. For instance, consider the following proposition:

Proposition 2: Let f be a bounded function on $[a, b]$. Suppose that there exists a sequence $\{P_n\}$ of partitions of $[a, b]$ such that

$$lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$$

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{R}$ such that for any $n > N$ it follows

$$|U(f, P_n) - L(f, P_n)| = U(f, P_n) - L(f, P_n) < \varepsilon$$

which means that f is Riemann integrable. Furthermore, since

$$L(f, P_n) \leq L(f) = U(f) \leq U(f, P_n) \quad \forall n,$$

$$0 \leq L(f) - L(f, P_n) \leq U(f, P_n) - L(f, P_n) \quad \to 0$$

So by squeeze theorem, $L(f) = lim_{n\to\infty}L(f, P_n) = lim_{n\to\infty}U(f, P_n)$.

So we can view the integral as a limit of the area of the rectangles. As we shall see with the following definitions and results, the whole idea of the Riemann integral is to make the partions really big so we can get an accurate approximation of the area.

Now we give definitions of refinements and some comparitive observations between the partition and its refinement.

Defintion: Refinements. Let f be a bounded function on a closed interval $[a, b] \in \mathbb{R}$ with a partion P of $[a, b]$. A **refinement** Q of P is a partition of $[a, b]$ with $P \subseteq Q$. Note that equality of the two partitions is entirely useless.

An immediate consquence of this defintion is given in the following proposition:

Proposition 1: $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$
Proof: Let $P = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be a partion of $[a, b]$ and let Q_1 be another partition of $[a, b]$ constructed by adding a point $t \in [a, b], t \notin P$ to P. Then there exists some $i \in \{1, ..., n\}$ such that $x_{i-1} < t < x_i$. So let

$$y_1 = inf\{f(x) : x \in [x_{i-1}, t]\},$$

$$y_2 = inf\{f(x) : x \in [t, x_i]\}.$$

Remember $m_i = inf\{f(x) : x \in [x_{i-1}, x_i]\}$, which means that $y_1, y_2 \geq m_i$ (We are looking at *every* point in the interval, not just the endpoints). Now

$$\begin{aligned}
L(f, Q_1) - L(f, P) &= \sum_{j=1}^{i-1} m_j \Delta x_j + y_1(t - x_{i-1}) + y_2(x_i - t) + \sum_{j=1}^{i+1} m_j \Delta x_j - \sum_{i=j}^{n} m_j \Delta x_j \\
&= y_1(t - x_{i-1}) + y_2(x_i - t) - m_i(x_i - x_{i-1}) \\
&= y_1(t - x_{i-1}) + y_2(x_i - t) - m_i(t - x_{i-1}) - m_i(x_i - t) \\
&= (y_1 - m_i)(t - x_{i-1}) + (y_2 - m_i)(x_i - t) \\
&\geq 0 \quad \text{Since each term is nonnegative.}
\end{aligned}$$

Hence, $L(f, P) \leq L(f, Q_1)$.
A similar argument will show that $U(f, Q_1) \leq U(f, P)$, and it is easily seen by defintion that

$L(f, Q_1) \leq U(f, Q_1)$. If a partition Q_n has n more points that P, then we can just repeat this argument n times. ◆

Now what does this proposition actually tell us? For the lower sums, this means that the more points we have for a partition, the sum approximation of the integral becomes more accurate and the lower rectangles are becoming "bigger," or rather closer to the actual curve. The same is true for the upper sums, but for these the rectangles are shrinking in area. But even these statements, although important to notice, do not say much about a function that is Riemann integrable. This next theorem provides a sufficient condition for such a function.

Theorem 1: Let f be a bounded function of the closed interval $[a, b] \in \mathbb{R}$. Then f is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof: Suppose first that f is Riemann integrable and let $\varepsilon > 0$. By definition, $U(f) = L(f)$. Using supremum and infimum definitions, there exists some partition P of $[a, b]$ such that

$$L(f, P) > L(f) - \frac{\varepsilon}{2},$$

and a partition Q of $[a, b]$ such that

$$U(f, Q) < U(f) + \frac{\varepsilon}{2}.$$

Then for the refinement $R = P \cup Q$ we have

$$L(f, R) \geq L(f, P) > L(f) - \frac{\varepsilon}{2}$$

and

$$U(f, R) \leq U(f, Q) < U(f) + \frac{\varepsilon}{2}.$$

Putting these together we obtain:

$$U(f, R) - L(f, R) \leq U(f, Q) - l(f, P) < (U(f) + \frac{\varepsilon}{2}) + (-L(f) + \frac{\varepsilon}{2}) = \varepsilon.$$

For the converse, with $\varepsilon > 0$, suppose that there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$, or $U(f, P) < L(f, P) + \varepsilon$. Then,

$$U(f) \leq U(f, P) < L(f, P) + \varepsilon \leq L(f) + \varepsilon$$

so that, put more visibly,

$$U(f) \leq L(f) + \varepsilon.$$

But since ε is arbitrary, it follows that $U(f) \leq L(f)$, and since $L(f) \leq U(f)$, $U(f) = L(f)$, which means that f is Riemann integrable. ◆

In conclusion, if we can make the upper sum and lower sum of f with repect to some partition arbitrarily small, it is reasonable to say that the upper integral and lower integral are the same value, and hence is the Riemann integral. But what kind of functions does this work for? As we shall soon see, there are certain limitations for a function that this will not work for. Before we get to those limitations, it will be important to discuss the idea of continuity of a function from $\mathbb{R}$ to $\mathbb{R}$. While there are several equivalent conditions for continuity, I will focus on one definition that will be most helpful for this discussion.

Definition: Continuous Function. A function $f : D \to \mathbb{R}$ is continuous at a point $c \in \mathbb{R}$ provided that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any $x \in D$ satisfying $|x - c| < \delta$ it follows that

$|f(x) - f(c)| < \varepsilon$. If this holds for every $c \in D$ then f is a continuous function.

What this is saying is that if we take any open interval, call it V, around a point in the range of f we can find an open interval, call it U, in the domain of f such that f maps U to a set contained in V. The function that is continuous is not very erratic; there are no sudden "jumps" in the range values, nor are there certain points of the domain that the function is not defined for. An even nicer kind of function is given by the following definition.

Definition: Uniformly Continuous Function. A function $f : D \to \mathbb{R}$ is uniformly continuous provided that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any $x, y \in D$ satisfying $|x - y| < \delta$ it follows that $|f(x) - f(y)| < \varepsilon$.

When a function is continuous on a closed interval in $\mathbb{R}$, it follows that f is uniformly continuous and that the function attains a maximum and minimum value. Hence continuity on a closed interval is an overkill for Riemann integrability, since we can just increase the partition large enough to make the Δx_i's sufficiently small, which would keep the differences of the range values for the Upper and Lower sums sufficiently small as well. A formal proof is given below.

Theorem 2 Let f be a continuous function on $[a, b]$. Then f is Riemann integrable on $[a, b]$.
Proof: Suppose f is a continuous function of the closed interval $[a, b] \in \mathbb{R}$. Then f is uniformly continuous. Hence, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$ it follows that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Hence, as mentioned in the above comment, let $P = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be a partition of $[a, b]$ such that $\Delta x_i < \delta$ for every i. As mentioned in the observations, f also attains its maximum and minimum on $[a, b]$, and therefore for each subinterval created by the partition. More precisely, for any interval $[x_{i-1}, x_i]$ there exists points $a_i, z_i \in [x_{i-1}, x_i]$ such that $m_i = f(a_i)$, $M_i = f(z_i)$. But since $|a_i - z_i| < \delta$, it follows that $|f(z_i) - f(a_i)| = f(z_i) - f(a_i) < \frac{\varepsilon}{b-a}$. Now we have

$$U(f, P) - L(f, P) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i = \sum_{i=1}^{n}(f(z_i) - f(a_i))\Delta x_i < \sum_{i=1}^{n} \frac{\varepsilon}{b-a}\Delta x_i = \frac{\varepsilon}{b-a}(b - a) = \varepsilon$$

so that f is Riemann integrable by Theorem 1. ◆

Now that we have a basic idea of the Riemann integral and some defining characterstics of it, we can take a look at some of its limitations that will lead to a nice criteria for Riemann integrability.

3 Limitations of the Riemann Integral

1. Fails when a function is discontinuous on a set of nonzero measure (Lebesgue's criteria).

   Example (*Dirichlet function (figure 1)*):

   $$f(x) = \begin{cases} 0 & : x \in \mathbb{Q} \\ 1 & : x \notin \mathbb{Q} \end{cases}$$

   We have, for any partition P of the domain,

   $$L(f, P) = 0 \text{ and } U(f, P) = 1,$$

   because the minimum value in each subinterval is 0, while the maximum value for each subinterval is 1, since there will be an infinite amount of both rational and irrational numbers in each subinterval. Therefore, Riemann integrability fails for this function. Notice also that the Dirichlet function is discontinuous at every point in its domain. To see this, take an open set around any (of the two) point in the range, let's just say a neighborhood centered around $f(m) = 1$, where m is a rational number, with a radius of the neighborhood strictly less than 1. Now take any open set in the domain centered

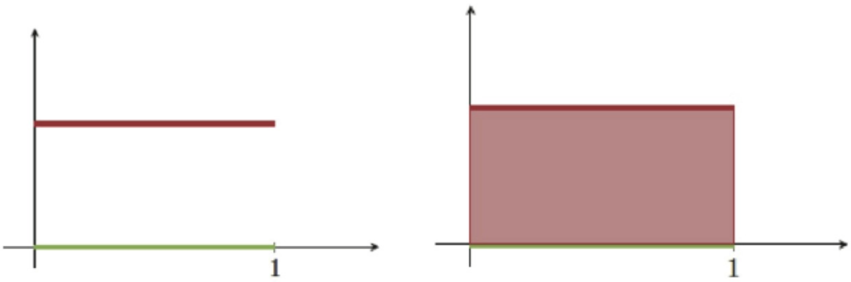

Figure 1: Dirichlet function defined on $[0, 1]$

around m. It does not matter how small you make this open set around m, it will always contain an irrational number, in fact it will contain an infinite amount of irrational numbers! Then when you map the set, there will be many values that go to 0, which means that the mapping of the set will miss the neighborhood around $f(m)$, as this neighborhood does not contain the point 0 because of the radius we gave it. Therefore, this function is not continuous at m, or at any other point in the domain. Hence the Dirichlet function is nowhere continuous. There are two resoultions to this classical example: we can redifine the function to make it only discontinuous at the rationals, or we could use a different integral. We will get back to these later on.

2. Convergence of simple functions may not be integrable (Fourier Series/Fourier Transforms)

3. Foundationally the Riemann integral is not as strong as Lebesgue, as the Lebesgue uses an abstract concept of measure to be able to extend the integration for a much richer mathematical theory. Lebesgue is also much stronger because of the use of convergent sequences in the theory, which provides much more rigor and therefore strength in its analysis (at least from how I, in my naivity, perceive it). Speaking of Measure Theory, I will talk about a result from measure theory that will be useful in establishing the criteria for Riemann integrability.

4 Sets of Measure Zero

In order to understand the culmination of the Riemann integral, a look at measurable sets is necessary. To start the discussion, consider the interval $[0, 1]$. It is intuitively clear that this set takes up some space on the real number line, and we say it has a measure of 1. How about $[0, 1] \cup [4, 5]$? One result of measure theory is that the mesaure of the union of disjoint mesaurable sets is just the summation of the individual measures, so in this case the measure $\mu([0, 1] \cup [4, 5]) = \mu([0, 1]) + \mu([4, 5]) = 1 + 1 = 2$. However, for the Riemann intergal, our main interest lies in sets of mesaure zero, to which we shall now turn our attention.

Consider the set $S_1 = \{1\}$. We can picture this point on the number line, but upon the realization that there are an uncountably infinite amount of points in any neighborhood of 1, it seems as though just one point does not occupy any space at all on the line. In fact, for an arbitrarily small positive number, say ε, we can find some interval $I_1 = (1 - \frac{\varepsilon}{3}, 1 + \frac{\varepsilon}{3})$ which has $S_1 \subseteq I_1$ and $\mu(I_1) = (1 + \frac{\varepsilon}{3}) - (1 - \frac{\varepsilon}{3}) = \frac{2}{3}\varepsilon < \varepsilon$. We can now formalize this observation in a definition.

Definition: Lebesgue Measure Zero. The set $S \subseteq \mathbb{R}$ has Lebesgue Measure Zero, i.e. $\mu(S) = 0$, if and only if for any $\varepsilon > 0$ there exists a family of intervals $\{I_k\}$ such that:

(i) $S \subseteq \bigcup_{j=1}^{\infty} I_j$

(ii) $\sum\limits_{j=1}^{\infty} \mu(I_j) < \varepsilon$.

Remark: It is not hard to show that any denumerable set contained in $\mathbb{R}$ has Lebesgue measure zero. To do this, we find intervals as we did above for every number in the set and use a geometric series to prove (ii) of the definition. A more interesting and useful theorem is the following:

Theorem 3: If a denumerable collection of subsets of $\mathbb{R}$, $\{S_1, S_2, ..., S_n, S_{n+1}, ...\}$, has the property that each set in the family has Lebesgue measure zero, the the set $S = \bigcup_{j=1}^{\infty} S_j$ has Lebesgue measure zero.
Proof: For each $S_i \in \{S_j\}_{j=1}^{\infty}$, there exists a family of intervals $\{I_k^i\}$ associated with S_i such that

$$S_i \subseteq \bigcup_{k=1}^{\infty} I_k^i \qquad \text{and} \qquad \sum_{k=1}^{\infty} \mu(I_k^i) < \frac{\varepsilon}{2^{i+2}},$$

by definition. Let the set $H^i = \bigcup_{k=1}^{\infty} I_k^i$ for every i. Now consider the family of sets $\{H^1, H^2, ..., H^n, ...\}$. If there is an element $x \in \bigcup_{j=1}^{\infty} S_j$, then $x \in S_t$ for some $t \in \mathbb{N}$. This implies $x \in \bigcup_{j=1}^{\infty} I_j^t = H^t$. But $H^t \subseteq \bigcup_{j=1}^{\infty} H^j$ which implies $x \in \bigcup_{j=1}^{\infty} H^j$. Hence $S = \bigcup_{j=1}^{\infty} S_j \subseteq \bigcup_{j=1}^{\infty} H^j$.
Also, for any $m \in \mathbb{N}$,

$$\mu(H^m) < \frac{\varepsilon}{2^{m+2}}.$$

Hence,

$$\sum_{j=1}^{\infty} \mu(H^j) \leq \frac{\varepsilon}{2^{1+2}} + \frac{\varepsilon}{2^{2+2}} + ... + \frac{\varepsilon}{2^{n+2}} + ... = \frac{\varepsilon}{4} \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{\varepsilon}{4} < \varepsilon. \qquad \blacklozenge$$

Now the question of discontinuity on a set of measure zero can be understood, but there is still some more prerequisites that must be met before getting to the Lebesgue's theorem.

5 Lebesgue's Criteria For Riemann Integrability

Before jumping right into the theorem, an extremely useful tool called the oscillation of a function should be discussed.
Definition: Oscillation of f on S. Let $f : D \to \mathbb{R}$ be a bounded function and let $C \subseteq D$. The oscillation of f on S is defined as

$$W(f; C) = sup\{|f(x) - f(y)| : x, y \in C\}.$$

Remember in the discussion of continuous functions, we were concerned about a neighborhood around a point in the domain. In order to see how the oscillation of a function relates to the function's continuity, we define the oscillation at a specific point.

Definition: Oscillation of f at c. With $c \in C$, the oscillation of f at c is defined as

$$w(f; c) = inf\{W(f; V(c; \delta)) : \delta > 0\}$$

where the neighborhood $V(c; \delta) = \{x \in D : |x - c| < \delta\}$.

Remark: What is the second definition actually saying? We are looking at neighborhoods around a point in the domain. $W(f; C)$ calculates the greatest difference in range, so $w(f; c)$ calculates the greatest difference in the range when their respective values in the domain are made arbitrarily close together. So as the δ gets smaller and smaller, the differences in the range *should* be getting smaller and smaller (it is certain the differences are not increasing), at least for continuous functions as we shall prove shortly. Hence an immediate conclusion can be that $w(f; c) \leq W(f; V(c; \delta))$. Also, this function is nonnegative, as $f(c) - f(c) = 0$ and 0 will be greater than any negative difference. These insights will be useful when proving the following lemma.

Lemma: If $f : D \to \mathbb{R}$, then f is continuous at c if and only if $w(f; c) = 0$.
Proof: Suppose f is continuous at $c \in D$. Let $\varepsilon > 0$. Then there exists some $\delta > 0$ such that for any $x \in D$

with $|x-c| < \delta$ we have $|f(x) - f(c)| < \frac{\varepsilon}{2}$. This tells us that for every $x, y \in V(c;\delta)$, $|f(x) - f(c)| < \frac{\varepsilon}{2}$ and $|f(y) - f(c)| < \frac{\varepsilon}{2}$. Hence,

$$|f(x) - f(y)| = |f(x) - f(c) + f(c) - f(y)| \leq |f(x) - f(c)| + |f(y) - f(c)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and therefore, $W(f;V(c;\delta)) = sup\{|f(x) - f(y)| : x, y \in V(c;\delta)\} < \epsilon$. In conclusion,

$$0 \leq w(f;c) \leq W(f;V(c;\delta)) < \varepsilon.$$

Since ε is arbitrary, $w(f;c) = 0$.
Consider the converse, and suppose that $w(f;c) = 0$ and let $\varepsilon > 0$. By definition, this means that $inf\{W(f;V(c;\delta)) : \delta > 0\} = 0$. But if this is true then there must be some $\delta > 0$ such that $W(f;V(c;\delta)) < \varepsilon$, which implies that for any $x \in V(c;\delta)$, $|f(x) - f(c)| < \varepsilon$. Hence f is continuous at c. ◆

Now we can gather all of our results and put it into a very important theorem for Riemann integration. This proof will use the oscillation of a function to look at points of discontinuity, and then use the theorem discussed from measure theory to prove the points of discontinuity are in a set of measure zero.

Theorem 4: Let f be a bounded, real-valued function on a closed interval $[a,b]$. Then f is Riemann integrable if and only if f is only discontinuous on a set in $[a,b]$ that has Lebesgue measure zero.

Proof: ($\Rightarrow$) Suppose first that f is Riemann integrable. Define a family of sets $\{H_1, H_2, ..., H_n, ...\}$ each in $[a,b]$ such that $H_j = \{x \in [a,b] : w(f;x) < \frac{1}{2^j}\}$. If we can show each of these H_j's has Lebesgue measure zero, then it will follow by Theorem 3 that $\bigcup_{i=1}^{\infty} H_i$ has Lebesgue measure zero.
Since f is Riemann integrable, there exists a partition $P_j = \{a = x_0^j, x_1^j, ..., x_n^j = b\}$ of $[a,b]$ such that

$$U(f,P_j) - L(f,P_j) < \frac{\varepsilon}{4^j}.$$

Suppose there exists an $x \in H_j \cap (x_{i-1}^j, x_i^j)$ for some $i \in 0, 1, ..., n$. Since (x_{i-1}^j, x_i^j) is open, there exists a $\delta > 0$ such that $V(x;\delta) \subseteq (x_{i-1}^j, x_i^j)$. Now for this x,

$$\frac{1}{2^j} < w(f;x) \leq W(f;V(x;\delta)) = sup\{|f(x) - f(y)| : x, y \in V(c;\delta)\} \leq M_i^j - m_i^j.$$

Hence $\frac{1}{2^j} \leq M_i^j - m_i^j$ and if we denote the set of i's such that the intersection of H_j and (x_{i-1}^j, x_i^j) is nonempty by T, then it follows that

$$\frac{1}{2^j} \sum_{i \in T} (x_i^j - x_{i-1}^j) \leq U(f,P_j) - L(f,P_j) = \sum_{i=1}^{n} (M_i^j - m_i^j)(x_i^j - x_{i-1}^j) < \frac{\varepsilon}{4^j}.$$

Hence

$$\sum_{i \in T} (x_i^j - x_{i-1}^j) < \frac{\varepsilon}{2^j}$$

Let $F = \bigcup_{i \in T} H_j \cap (x_{i-1}^j, x_i^j)$, and notice that $H_j \subseteq F \cup P_j$. As we have already seen, $\mu(P_j) = 0$, so that $\mu(H_j) \leq \mu(F \cup P_j) < \frac{\varepsilon}{2^j}$.
Now let $D = \{x \in [a,b] : w(f;x) > 0\}$, which is in fact the same as $\bigcup_{i=1}^{\infty} H_i$, and we find that

$$\mu(D) = \mu(\bigcup_{i=1}^{\infty} H_i) = \sum_{i=1}^{\infty} \mu(H_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$$

Therefore D, the set of discontinuous points in $[a,b]$, has Lebesgue measure zero.

($\Leftarrow$) Now consider the converse; suppose f is discontinuous on a set of measure zero. Let $f : [a,b] \to \mathbb{R}$ be a bounded function, i.e. $|f(x)| \leq M$ for some $M \in \mathbb{R}$, and suppose that f is discontinuous only on a set of

Lebesgue measure zero. Let D denote the set of discontinuous points and let $\varepsilon > 0$. By definition it follows that there is a collection of real intervals $\{I_k\}_{k=1}^{\infty}$ such that $D \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum\limits_{k=1}^{\infty} \mu(I_k) < \frac{\varepsilon}{2M}$. For each $t \in [a, b]$ there are two possibilities:

1. $t \notin D$. Then f is continuous at t, which means there is some $\delta(t) > 0$ such that $x \in V(t; \delta(t)) \Rightarrow |f(x) - f(t)| < \frac{\varepsilon}{2}$. If this is true, then

$$0 \leq M_t - m_t < \varepsilon$$

2. $t \in D$. Then choose a $\delta(t) > 0$ such that $V(t; \delta(t)) \subseteq I_k$ for some k. Also, since $f(x)$ is bounded by M, it follows that

$$0 \leq M_t - m_t \leq 2M$$

Now let $P = \{a = x_0, x_1, ..., x_{n-1}, x_n = b\}$ be a partition of $[a, b]$ such that when we choose $t_i \in [x_{i-1}, x_i]$ we have $(x_i - x_{i-1}) < \delta(t_i)$.

To distinguish between discontinuous and continuous points, we define two sets:

$$S_c = \{i : t_i \notin D\} \text{ and } S_d = \{i : t_i \in D\}.$$

Notice that $[x_{i-1}, x_i] \subseteq V(t_i, \delta(t_i))$ for every i. Then it follows that $M_i - m_i \leq M_{t_i} - m_{t_i}$ for all i. Moreover, when $i \in S_d$, $[x_{i-1}, x_i] \subseteq V(t_i, \delta(t_i)) \subseteq I_{k(i)}$ for some $k(i)$ and $M_i - m_i \leq 2M$. When $i \in S_c$, $M_i - m_i \leq \varepsilon$. In conclusion:

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{i \in S_c} (M_i - m_i)(x_i - x_{i-1}) + \sum_{i \in S_d} (M_i - m_i)(x_i - x_{i-1}) \\
&< \sum_{i \in S_c} \varepsilon(x_i - x_{i-1}) + \sum_{i \in S_d} 2M \cdot \mu(I_{k(i)}) \\
&\leq \varepsilon(b - a) + 2M(\frac{\varepsilon}{2M}) = \varepsilon(b - a + 1)
\end{aligned}$$

Since ε is arbitrary, it follows that f is Riemann integrable.

5.1 Dirichlet Function Revisited

With Lebesgue's useful tool, consider a modification of the Dirichlet function (Figure 2)[1]:

$$f(x) = \begin{cases} \frac{1}{n} & : x \in \mathbb{Q} \text{ is in lowest terms} \\ 0 & : x \notin \mathbb{Q} \end{cases}$$

I will prove that this modified verson, defined on $[0, 1]$, is discontinuous on a set of measure zero:

Proof: Let t be an irrational number in $[0, 1]$. There exists a sequence of rationals $\{a_n\}$ that converge to t. Now let $\varepsilon > 0$. Then there exists a $b \in \mathbb{N}$ such that $\frac{1}{b} < \varepsilon$ by the Archimedian Property. Notice that there are only finite number of fractions in the interval $[0, 1]$ whose denominator in lowest terms is less than b, and therefore we can create a δ-neighborhood around t that will be absent of such fractions. Since $\lim a_n = t$, there exists a number N such that for any $n > N$ we have $|a_n - t| < \delta$. Then it follows that for any $n > N$ we have

$$|f(a_n) - f(x)| = |f(a_n) - 0| = |f(a_n)| < \frac{1}{b} < \varepsilon.$$

Hence f is continuous at any irrational. At this point we can conclude that f is Riemann integrable, because even if the functions are discontinuous on the rationals, the rationals have Lebesgue measure zero since they are denumerable, and hence f would be discontinuous only on a set of measure zero. As it turns out, however, the function is discontinuous on the rationals.

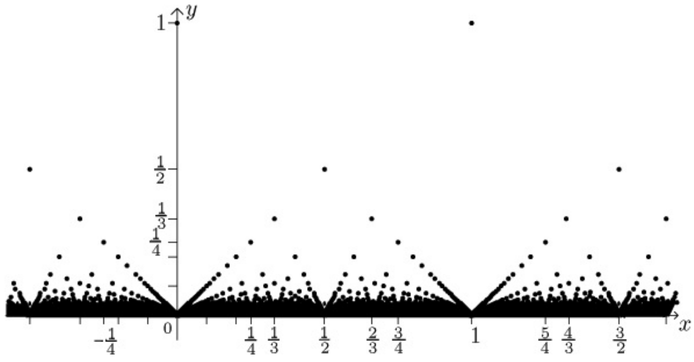

Figure 2: Modified Dirichlet discontinuous only on rationals.

There is another way, however, to integrate the (unmodified) Dirichlet function, but we must move past the Riemann integral. Without further adieu, we turn our attention to the Lebesgue integral for a (petty) introduction to this beast of a machine.

6 Lebesgue Integral

The Riemann integral partitions the domain, whereas the Lebesgue integral essentially partitions the range of a function. To picture this, one can think of rectangles side-by-side for the Riemann integral, and stacked on top of each other for the Lebesgue integral (Figure 3).[2]

To get started, there are a few important concepts that must be understood. The Lebesgue integral uses characteristic functions, step functions, and sequences along with tools from measure theory to understand the area under the curve. This theory is very useful in probability theory, as it deals with measure and looks at data that may or may not be continuous. We start out with some definitions and will end this short look at the Lebesgue with another look at the Dirichlet function.

Characteristic function:

$$\chi_A(x) = \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases}$$

Lebesgue integral of the characteristic function:

$$\int \chi_A = \mu(A)$$

Simple function: A simple function f is a function whose range is denumerable.

Notice if f is a simple function, we can rewrite f has a linear combination of characteristic functions. To illustrate, suppose

$$f(x) = \begin{cases} 1 & : x \in [0,1) \\ 2 & : x \in [1,2] \end{cases}$$

Then we can write $f = 1 \cdot \chi_{[0,1)} + 2 \cdot \chi_{[1,2]}$, and the Lebesgue integral would just be

$$\int f = \int (1 \cdot \chi_{[0,1)} + 2 \cdot \chi_{[1,2]}) = 1 \cdot \mu([0,1)) + 2 \cdot \mu([1,2]) = 1 \cdot 1 + 2 \cdot 1 = 3$$

[1]Reference: math.feld.cvut.cz/mt/txtb/4/txe3ba4s.htm
[2]Reference: wikipedia.org/wiki/LebesgueIntegration

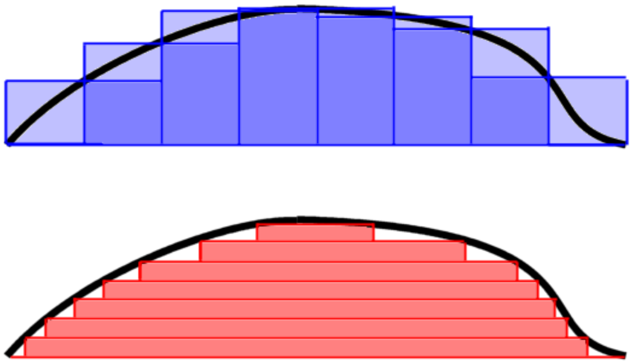

Figure 3: Riemann (top) vs. Lebesgue (bottom).

With this integral, the dirac-delta function is no problem to integrate. As we have shown, since the set of rationals is denumerable, it's measure is zero. The measure of the irrationals on the interval $[0, 1]$ is just 1, so when we integrate:

$$\int f = \int (0 \cdot \chi_{\mathbb{Q}} + 1 \cdot \chi_{\mathbb{R}-\mathbb{Q}}) = 0 \cdot \mu(\chi_{\mathbb{Q}}) + 1 \cdot \mu(\chi_{\mathbb{R}-\mathbb{Q}}) = 0 \cdot 0 + 1 \cdot 1 = 1$$

So again, just to reiterate, Lebesgue looks at the range values of a function and its inverse image, i.e. the values in the domain that map to the range values, and multiplies the measure of the inverse image and the value in the range to get the area of the rectangle. As functions become more complicated, we turn to sequences of functions and look at limits of step functions to approximate the areas.

To conclude this section, there is a very profound theorem that I will not prove: **Every function that is Riemann integrable is Lebesgue integrable**. Obviously, the converse is not true, as this can be seen by the Dirichlet function. Therefore we can say that the Lebesgue is much stronger as it covers a much wider class of functions. Does this mean every function is Lebesgue integrable? The answer is no (One example if $f(x) = \frac{1}{x}$). However, the only requirement really needed for a function to be Riemann integrable is that there exists a sequence of step functions that converge to the given function.

7 Conclusion

The Riemann integral is a widely known and used integral in undergraduate mathematics. The concept is pretty straightforward and is generally nice to work with (especially when the function is uniformly continuous. This integral is certainly strong, as it is used for the fundemental theorem of calculus and can be used (according to Dr. Skoumbourdis) in studying theoretical physics, among many other areas. However, as can be seen by the Dirichlet function, there are some functions whose area seems easy to guess but is actually impossible to integrate with Riemann's partial sums. Lebesgue's criteria for Riemann integrability is essentially where the Riemann integral ends and more advanced techniques must be sought out. One of those techniques is the Lebesgue integral, which partitions the range instead of the domain and uses measure theory to approximate the area. If I continued studying this topic, I would like to gain a much better understanding of the Lebesgue integral and the idea of a convergent sequence of step functions. Also, measure theory has been very fascinating, and although I only looks at sets of measure zero for this project, I hope to be able to explore more of this topic in graduate school.

8 References

1. Bartle, R. G. & Sherbert, D. R. (2011). *Introduction to Real Analysis.* Hoboken, New Jersey: Wiley.
2. Ray, S. R. (2005). *Analysis: With an Introduction to Proof.* Upper Saddle River, New Jersey: Pearson Education.
3. Dr. Evangelos Skoumbourdis