The Fibonacci Sequence

Its History, Significance, and Manifestations in Nature

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A Senior Thesis submitted in partial fulfillment
of the requirements for graduation
in the Honors Program
Liberty University
Spring 2013

Acceptance of Senior Honors Thesis

This Senior Honors Thesis is accepted in partial fulfillment of the requirements for graduation from the Honors Program of Liberty University.

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Abstract

The discoveries of Leonard of Pisa, better known as Fibonacci, are revolutionary contributions to the mathematical world. His best-known work is the Fibonacci sequence, in which each new number is the sum of the two numbers preceding it. When various operations and manipulations are performed on the numbers of this sequence, beautiful and incredible patterns begin to emerge. The numbers from this sequence are manifested throughout nature in the forms and designs of many plants and animals and have also been reproduced in various manners in art, architecture, and music.

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Introduction

The mathematician Leonardo of Pisa, better known as Fibonacci, had a significant impact on mathematics. His contributions to mathematics have intrigued and inspired people through the centuries to delve more deeply into the mathematical world. He is best known for the sequence of numbers bearing his name. This thesis both examines more deeply the life and contributions of Fibonacci and depicts how his famous sequence is realized in nature, art, music, and architecture.

As the thirteenth century began, Europe started to awaken from the Dark Ages and move into the Renaissance. As the stifling effects of the Dark Ages began to be replaced by a growing interest in the scientific world, artists, scholars, architects, scientists, and mathematicians all began making revolutionary discoveries and advances in knowledge. One such person was Leonardo of Pisa, who contributed to the transformation of the mathematical world at that time.

The Life of Fibonacci

Early Life

Although his work is quite well known, surprisingly little is known of the life of Fibonacci. While neither date nor location of his birth is known for certain, it is likely that he was born sometime around the year 1170 near the city of Pisa, Italy. The name Fibonacci means “of the house of Bonacci,” however, he would have been known as Leonardo Pisano, or Leonardo of Pisa, in reference to the city of his birth. His father, Guilielmo Bonacci, was a Pisan merchant. Pisa at that time was a thriving hub of international trade, and the merchants were some of the most important members of its society. The Dark Ages had ended, ushering in a renewed interest in commerce as the countries of Europe again began to develop and prosper (Keith, 2011, pp. 27-29). Growing up surrounded by the hustle and bustle of commerce, Leonardo would have been constantly exposed to the importance of numbers, as merchants set prices and measured their goods, and customs officials set taxes on imports (p. 35).

When Leonardo was a teenager, his father was appointed to work representing Pisa and Italian merchants in a customs house in Bugia, on the coast of northern Africa, located in present day Algeria. There was a strong Arabic presence in Bugia, and Arab mathematicians were able to pass on much of the knowledge that was unknown to Europeans, who had been trapped in the Dark Ages, and had made few scientific advancements for hundreds of years (Henderson, 2007, pp. 1-2). Leonardo soon joined his father in Bugia, where he was educated and taught the skill of calculating in preparation to become a merchant. This was a necessary skill because every republic had a different unit of money, making conversion between currencies a required ability to be able to carry on business negotiations. It was during his education there that Leonardo first learned of the Hindi numerals 1-9 and the Arab numeral 0 (Posamentier & Lehmann, 2007, pp. 18-19). Most of Europe at that time used Roman numerals, but Leonardo recognized this new system of numbering, which was also used by the Arabic merchants, because it was much more efficient and easier to work with. Computations with Roman numerals were exceedingly cumbersome. Addition and subtraction were time consuming, and multiplication and division were very complicated. Most Europeans would do their calculations with an abacus and then record the final answer in Roman numerals. With this new Hindu-Arabic system of numbering, computations were much more efficient, and the steps could be recorded along with the answer. Leonardo's education under a Muslim teacher included not just the classical Greek mathematics, but also the works of Indian and Arabic scholars, and he was introduced to algebra through a book written by a Persian mathematician (Bradley, 2006, pp. 118-119). Leonardo's education in Bugia gave him a solid background in mathematics and sparked his interest in the subject that would become his lifelong passion.

Mathematical Works

Over the course of his life, Fibonacci wrote several books, including *Liber Abaci*, which publicized the advantages of the Hindu numerals and discussed various mathematical problems, a book on geometry, which included trigonometry and proofs, a book on flowers, and a book on number theory, which brought him much recognition as an extremely talented mathematician. By far the most well-known of his works is *Liber Abaci*, which means "book of calculating" or "book of computation." This book was quite possibly one of the most influential mathematical works of the Middle Ages. Handwritten in Latin, *Liber Abaci* was first publicized in 1202 and later revised in 1228 (Posamentier & Lehmann, 2007, pp. 19-20). The first seven chapters of the book introduced the Hindu-Arabic numerals and showed how to use them to perform various mathematical calculations, illustrating their efficiency and ease of use. Fibonacci first explained how to read and write the numbers and then showed how to add, subtract, multiply, and divide using whole numbers and fractions. The next four chapters explained how these techniques and the Hindu-Arabic numbering system would greatly simplify business transactions (Bradley, 2006, p. 120). These practical problem-solving techniques told merchants how to calculate profits and how to convert from one currency to another, as merchants dealing in port cities could have any number of currencies in their possession at any given time, and Italy itself had a number of different currencies. Weights used to measure goods were also not standardized, so it was necessary to know how to convert between those as well (Henderson, 2007, pp. 3-4). The last four chapters dealt with techniques from varying branches of mathematics, including algebra, geometry, and number theory, and presented an array of problems and puzzles in which the techniques could be used (Bradley, 2006, p. 120). In this book, Fibonacci introduced two new ideas that are now foundational to algebraic study, although they were not widely used until the 16th century. In certain problems, he uses single letters to represent variables, and in one problem, he utilizes negative numbers (Bradley, 2006, p. 122). Fibonacci's comprehensive and detailed explanations and multitude of examples regarding the new numbering system were widely received and contributed toward persuading Europeans to discard the Roman numeral system in favor of the better, more efficient Hindu-Arabic system. Europe was entering a time of great intellectual, financial, and commercial change, and when *Liber Abaci* was published, it was sent out into a very receptive environment (Devlin, 2011, pp. 85-86).

Fibonacci Sequence

In spite of his influential contributions to the field of European mathematics, Fibonacci is not most remembered for any of these reasons, but rather for a single sequence of numbers that provided the solution to a problem included in *Liber Abaci*. Like most of the problems in the book, Fibonacci did not invent this problem himself, but his solution to it has forever immortalized him in the mathematical world (Devlin, 2011,

p. 143). The problem, dealing with the regeneration of rabbits, calculated the number of rabbits after a year if there is only one pair the first month. The problem states that it takes one month for a rabbit pair to mature, and the pair will then produce one pair of rabbits each month following. Fibonacci's solution stated that in the first month there would be only one pair; the second month there would be one adult pair and one baby pair; the third month there would be two adult pairs and one baby pair; and so forth (Posamentier & Lehmann, 2007, pp. 25-26). When the total number of rabbits for each month is listed, one after the other, it generates the sequence of numbers for which Fibonacci is most famous:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377…

This string of numbers is known as the Fibonacci sequence, and each successive term is found by adding the two preceding terms together. The Fibonacci sequence is the oldest known recursive sequence, which is a sequence where each successive term can only be found through performing operations on previous terms. Interestingly, Fibonacci does not comment on the recursive nature of this sequence. The relationship between the terms was not identified in publication until four hundred years later. At the time of the publication of *Liber Abaci*, no special notice was taken of these numbers. It was not until the mid-1800s that mathematicians began to be intrigued by what would later be known as the Fibonacci numbers (Posamentier & Lehmann, 2007, pp. 26-27).

A closer inspection of the numbers making up the Fibonacci sequence brings to light all sorts of fascinating patterns and mathematical properties. Fibonacci himself makes no mention of these patterns in his book, but the following patterns are a few that have been brought to light over years of examination of the numbers in the sequence.

Any two consecutive Fibonacci numbers are relatively prime, having no factors in common with each other (Garland, 1987, p. 67). For example:

$$5, 8, 13, 21, 34$$
$$5 = 1 \cdot 5$$
$$8 = 2 \cdot 2 \cdot 2$$
$$13 = 1 \cdot 13$$
$$21 = 3 \cdot 7$$
$$34 = 2 \cdot 17$$

Summing together any ten consecutive Fibonacci numbers will always result in a number which is divisible by eleven (Posamentier & Lehmann, 2007, p. 33).

$$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143$$

$$\frac{143}{11} = 13$$

$$89 + 144 + 233 + 377 + 610 + 987 + 1{,}597 + 2{,}584 + 4{,}181 + 6{,}675 = 17{,}567$$

$$\frac{17{,}567}{11} = 1{,}597$$

Following tradition, F_n will be used to represent the n-th Fibonacci number in the sequence.

n	F_n
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89
12	144
13	233
14	377
15	610

Every third Fibonacci number is divisible by two, or F_3. Every fourth Fibonacci number is divisible by three, or F_4. Every fifth Fibonacci number is divisible by five, or F_5. Every sixth Fibonacci number is divisible by eight, or F_6, and the pattern continues. In general, every nth Fibonacci number is divisible by the nth number in the Fibonacci sequence, or F_{mn} is divisible by F_n (Garland, 1987, p. 69).

Fibonacci numbers in composite-number positions are always composite numbers, with the exception of the fourth Fibonacci number. In other words if n is not a prime, the nth Fibonacci number will not be a prime (Posamentier & Lehmann, 2007, p. 35).

$$F_6 = 8$$

$$F_9 = 34$$

$$F_{16} = 987$$

The reciprocal of the eleventh Fibonacci number, 89, can be found by adding the Fibonacci sequence in such a fashion that each Fibonacci number contributes one digit to the repeating decimal of the reciprocal, $\frac{1}{89}$ (Garland, 1987, p. 69).

```
      0.0112358
              13
               21
                34
                 55
                  89
                  144
                   233
                    377
                     610
                      987
      -------------------
1/89 = 0.01123595505617787
```

Multiplying any Fibonacci number by two and subtracting the next number in the sequence will result in the answer being the number two places before the original (Garland, 1987, p. 70).

$$\ldots 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 \ldots$$

$$2 \cdot F_6 - F_7 = (2 \cdot 8) - 13 = 16 - 13 = 3 = F_4$$

$$2 \cdot F_{11} - F_{12} = (2 \cdot 89) - 144 = 178 - 144 = 34 = F_9$$

$$2 \cdot F_n - F_{n+1} = F_{n-2}$$

Summing consecutive odd-positioned Fibonacci numbers, starting with the first odd-positioned number, F_1, will result in a number that is the next Fibonacci number in the sequence after the last term in the sum (Posamentier & Lehmann, 2007, pp. 38-39).

$$F_1 + F_3 = 1 + 2 = 3 = F_4$$

$$F_1 + F_3 + F_5 = 1 + 2 + 5 = 8 = F_6$$

$$F_1 + F_3 + F_5 + F_7 = 1 + 2 + 5 + 13 = 21 = F_8$$

A similar pattern emerges when summing consecutive, even-positioned Fibonacci numbers beginning with F_2, only this time, the result is a number that is one less than the Fibonacci number following the last even number in the sum (Posamentier & Lehmann, 2007, pp. 37-38).

$$F_2 + F_4 = 1 + 3 = 4 = F_5 - 1$$

$$F_2 + F_4 + F_6 = 1 + 3 + 8 = 12 = F_7 - 1$$

$$F_2 + F_4 + F_6 + F_8 = 1 + 3 + 8 + 21 = 33 = F_9 - 1$$

The product of any Fibonacci number multiplied by the number two places after it will be one more or one less than the square of the Fibonacci number between the two. When the number to be squared is an even-positioned Fibonacci number, one is added, and when it is odd-positioned, one is subtracted (Posamentier & Lehmann, 2007, pp. 44-45).

$$\ldots 3, 5, 8, 13, 21, 34, 55, 89 \ldots$$

$$F_4 \cdot F_6 = 3 \cdot 8 = 24 \text{ and } {F_5}^2 = 5^2 = 25$$

$$F_9 \cdot F_{11} = 34 \cdot 89 = 3{,}026 \text{ and } {F_{10}}^2 = 55^2 = 3{,}025$$

$$F_{n-1} \cdot F_{n+1} = F_n \pm 1$$

When the square of a Fibonacci number is subtracted from the square of the number two places after it, the result is a Fibonacci number.

$${F_6}^2 - {F_4}^2 = 8^2 - 3^2 = 55 = F_{10}$$

$${F_7}^2 - {F_5}^2 = 13^2 - 5^2 = 144 = F_{12}$$

$${F_{15}}^2 - {F_{13}}^2 = 610^2 - 233^2 = 317{,}811 = F_{28}$$

$${F_n}^2 - {F_{n-2}}^2 = F_{2n-2}$$

When the subscripts in each example are inspected, a pattern begins to surface. The sum of the first two subscripts is equal to the third subscript in each equation (Posamentier & Lehmann, 2007, p. 42).

Similarly, when the squares of two consecutive Fibonacci numbers are added, the sum is also a Fibonacci number.

$${F_3}^2 + {F_4}^2 = 2^2 + 3^2 = 13 = F_7$$

$${F_6}^2 + {F_7}^2 = 8^2 + 13^2 = 233 = F_{13}$$

$${F_{13}}^2 + {F_{14}}^2 = 233^2 + 377^2 = 196{,}418 = F_{27}$$

$${F_n}^2 + {F_{n+1}}^2 = F_{2n+1}$$

In this case as well, the subscripts in each equation are related by addition. As was the case in the previous example, the sum of the first two subscripts is equal to the third subscript (Posamentier & Lehmann, 2007, p. 43).

If any two consecutive Fibonacci numbers are squared and then added together, the result is a Fibonacci number, which will form a sequence of alternate Fibonacci numbers (Garland, 1987, p. 72).

$$\begin{array}{lll} 1^2 = 1 & & \\ \quad + & = 2 & \\ 1^2 = 1 & & \\ \quad + & = 5 & \\ 2^2 = 4 & & \\ \quad + & = 13 & \\ 3^2 = 9 & & \end{array}$$

When any four consecutive numbers in the Fibonacci sequence are considered, the difference of the squares of the two numbers in the middle is equal to the product of the two outer numbers (Garland, 1987, p. 74).

$$\dots 5, 8, 13, 21 \dots$$

$$F_7{}^2 - F_6{}^2 = 13^2 - 8^2 = 105 = 5 \cdot 21 = F_5 \cdot F_8$$

$$F_{n+1}{}^2 - F_n{}^2 = F_{n-1} \cdot F_{n+2}$$

The squares of the successive terms in the sequence, starting with F_1, add up to a number that can be written as a product of two consecutive Fibonacci numbers. Specifically, the sum is equal to the product of the last number that is squared and the Fibonacci number immediately following.

$$1^2 + 1^2 = 1 \cdot 2$$

$$1^2 + 1^2 + 2^2 = 2 \cdot 3$$

$$1^2 + 1^2 + 2^2 + 3^2 = 3 \cdot 5$$

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 5 \cdot 8$$

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8 \cdot 13$$

This provides an easy way to find the sum of squares of Fibonacci numbers. The sum is simply the product of the last squared number in the sum and the number that would come after it in the Fibonacci sequence (Posamentier & Lehmann, 2007, pp. 40-41).

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

For any three consecutive Fibonacci numbers, subtracting the cube of the smallest from the sum of the cubes of the two greater will result in another Fibonacci number (Garland, 1987, pp. 67-77).

$$\dots 1, 2, 3, 5, 8, 13 \dots$$

$${F_3}^3 + {F_4}^3 - {F_2}^3 = 2^3 + 3^3 - 1^3 = 8 + 27 - 1 = 34 = F_9$$

$${F_6}^3 + {F_7}^3 - {F_5}^3 = 8^3 + 13^3 - 5^3 = 512 + 2197 - 125 = 2{,}584 = F_{18}$$

$${F_{n+1}}^3 + {F_{n+2}}^3 - {F_n}^3 = F_{3n+3}$$

Summing any number of consecutive Fibonacci numbers will result in a number that is one less than the Fibonacci number two places beyond the last one added.

$$\dots 1, 1, 2, 3, 5, 8, 13 \dots$$

$$F_1 + F_2 + F_3 + F_4 + F_5 = 1 + 1 + 2 + 3 + 5 = 12 = 13 - 1 = F_7 - 1$$

This gives a general formula for a simple way to find the sum of any number of Fibonacci numbers (Posamentier & Lehmann, 2007, pp. 36-37).

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Discovering the value of a Fibonacci number, given its location in the sequence, can be very time consuming and tedious, particularly if it has a later placement in the sequence. Finding the fifth Fibonacci number is not difficult. Finding the fiftieth is much more cumbersome, as the process involves finding and summing the previous forty-nine terms. In 1843, the French mathematician Jacques-Philippe-Marie Binet discovered a formula which could find any Fibonacci number without having to find any of the previous numbers in the sequence. This formula finds the n-th Fibonacci number using a number called the golden ratio, $\frac{1+\sqrt{5}}{2}$, and its inverse (Posamentier & Lehmann, 2007, pp. 293, 296).

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

(Posamentier & Lehmann, 2007, p. 300).

Because the Fibonacci sequence is a linear, homogeneous, recurrence relation of the second degree, the above formula can be derived as follows:

Recurrence relation: $f_n = f_{n-1} + f_{n-2}$

Initial conditions: $f_0 = 0,\ f_1 = 1$

Assume that $f_n = r^n$ is a solution

Then $r^n = r^{n-1} + r^{n-2} \quad \Longrightarrow \quad r^2 = r + 1 \quad \Longrightarrow \quad r^2 - r - 1 = 0$

Using the quadratic formula to solve this equation results in $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

$$f_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \Longrightarrow \quad f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$f_0 = \alpha_1 + \alpha_2 = 0$

$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$\alpha_1 = -\alpha_2$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\frac{1-\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} = \frac{1-5}{2+2\sqrt{5}} = -\frac{2}{1+\sqrt{5}} = -\frac{1}{\phi}$$

$$\therefore f_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right]$$

(Rosen, 2012, pp. 514-517).

The more the Fibonacci sequence is studied, the more fascinating and intriguing patterns begin to surface. As various mathematical operations are performed on the numbers, all sorts of relationships between the numbers come to light. This is one of the many reasons this string of numbers has captivated the mathematical world for centuries.

The Golden Ratio

The Fibonacci numbers also have a geometric manifestation in the form of the golden ratio. The golden ratio can be found by partitioning a line segment in such a way that the longer portion (L) is to the shorter portion (S) as the entire line segment is to the longer portion. This relationship is generally expressed by the formula $\frac{L}{S} = \frac{L+S}{L}$. To find the numerical value for the golden ratio, let $x = \frac{L}{S}$. Then $x = 1 + \frac{1}{x}$. Finally, solving for x using the quadratic equation gives the numerical value for the golden ratio, which is often denoted by the Greek letter phi (Posamentier & Lehmann, 2012, pp. 13-14).

$$\phi = \frac{L}{S} = x = \frac{1+\sqrt{5}}{2} = 1.6180339887\ ...$$

Interestingly, after the reciprocal of phi is simplified, it turns out to be only one less than phi.

$$\frac{1}{\phi} = \frac{S}{L} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}+1}{2} - 1 = \phi - 1 = 0.6180339887\ldots$$

This reveals a very unique relationship between phi and its reciprocal. $\phi - \frac{1}{\phi} = 1$, but it is also true that $\phi \cdot \frac{1}{\phi} = 1$. Phi and the reciprocal of phi are the only two numbers whose difference and product are both equal to one (Posamentier & Lehmann, 2012, p. 15).

As it turns out, phi can also be calculated using Fibonacci numbers. Dividing a Fibonacci number by the preceding Fibonacci number will result in a number that approaches phi. The larger the numbers used, the closer the result will be to the actual value of phi.

$$\frac{F_8}{F_7} = \frac{21}{13} = 1.6153846153\ldots$$

$$\frac{F_{14}}{F_{13}} = \frac{377}{233} = 1.6180257511\ldots$$

$$\frac{F_{20}}{F_{19}} = \frac{6{,}765}{4{,}181} = 1.6180339632\ldots$$

This can be shown to be true in general by taking the limit as n approaches infinity of any Fibonacci number divided by the preceding one.

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{5}}\left[\phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1}\right]}{\frac{1}{\sqrt{5}}\left[\phi^{n} - \left(-\frac{1}{\phi}\right)^{n}\right]}$$

$$= \lim_{n\to\infty} \frac{\phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1}}{\phi^{n} - \left(-\frac{1}{\phi}\right)^{n}}$$

$$= \lim_{n\to\infty} \frac{\phi\left[\phi^n - \frac{1}{\phi}\left(-\frac{1}{\phi}\right)^{n+1}\right]}{\phi^n - \left(-\frac{1}{\phi}\right)^n}$$

$$= \phi \lim_{n\to\infty} \frac{\phi^n + \left(-\frac{1}{\phi}\right)^{n+2}}{\phi^n - \left(-\frac{1}{\phi}\right)^n} \cdot \frac{\frac{1}{\phi^n}}{\frac{1}{\phi^n}}$$

$$= \phi \lim_{n\to\infty} \frac{1 + \frac{(-1)^{n+2}}{\phi^{2n+2}}}{1 + \frac{(-1)^n}{\phi^{2n}}}$$

$$= \phi \cdot \frac{1+0}{1+0} = \phi \cdot 1 = \phi$$

$$\therefore \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$$

Conversely, if a Fibonacci number is divided by the following Fibonacci number, the result will be close to the reciprocal of phi. Again, the larger the two numbers used, the closer the result will be to the reciprocal of phi (Posamentier & Lehmann, 2007, pp. 107-109).

$$\frac{F_7}{F_8} = \frac{13}{21} = 0.6190476190 \ldots$$

$$\frac{F_{13}}{F_{14}} = \frac{233}{377} = 0.6180371353 \ldots$$

$$\frac{F_{19}}{F_{20}} = \frac{4{,}181}{6{,}765} = 0.6180339985 \ldots$$

Fibonacci numbers become even more closely linked to the golden ratio when powers of phi are considered. First, ϕ^2 is written in terms of ϕ, which after simplification yields $\phi^2 = \phi + 1$. Each successive power of phi can then be written in terms of factors of previous powers of phi. The result of each power is a multiple of ϕ plus a constant. It turns out that the coefficient of phi and the constant are consecutive Fibonacci numbers in sequential order (Posamentier & Lehmann, 2007, pp. 113-114).

$$\phi^3 = \phi \cdot \phi^2 = \phi(\phi + 1) = \phi^2 + \phi = (\phi + 1) + \phi = 2\phi + 1$$

$$\phi^4 = \phi^2 \cdot \phi^2 = 3\phi + 2$$

$$\phi^5 = \phi^3 \cdot \phi^2 = 5\phi + 3$$

$$\phi^6 = \phi^3 \cdot \phi^3 = 8\phi + 5$$

$$\phi^7 = \phi^4 \cdot \phi^3 = 13\phi + 8$$

Golden rectangle. Throughout the course of history, there is a rectangle whose proportions are found most pleasing to the eye. It is neither too fat nor too skinny, neither too long nor too short. People will subconsciously choose this rectangle over another one with different proportions. This rectangle, considered the most perfectly shaped rectangle, is known as the golden rectangle (Garland, 1987, pp. 19-20). This rectangle is one in which the ratio of length to width is the golden ratio and follows the formula $\frac{w}{l} = \frac{l}{w+l}$. In the late 1800s, Gustav Fechner, a German psychologist, invested a good deal of time into researching the subject. He measured thousands of common rectangles, from playing cards and books to windows and writing pads, and he ultimately found that in most of them, the ratio of length to width was close to phi. Fechner also conducted a study in which he asked a large number of people to choose the rectangle out of a group of rectangles was the most pleasing to the eye. His findings showed that the largest percentage of people preferred the rectangle with a ratio of 21:34. These numbers are consecutive Fibonacci numbers, and their ratio approaches the reciprocal of phi. The rectangle most preferred by people was a golden rectangle (Posamentier & Lehmann, 2007, pp. 115-117).

Golden angle. The golden angle is the angle which divides a complete circle of 360° into central angle portions corresponding to the golden ratio. This golden angle, represented by the symbol ψ, is found when 360° is multiplied by the reciprocal of phi, and that result is then subtracted from 360° (Posamentier & Lehmann, 2007, pp. 148-149).

$$\psi = 360° - (360°)\left(\frac{1}{\phi}\right) = 137.5077640501\ldots° \approx 137.5°$$

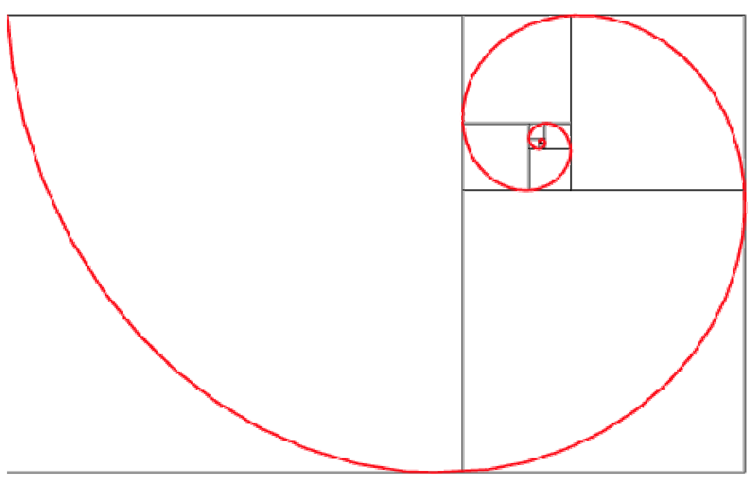
This golden angle is approached when 360°, multiplied by the ratio of two consecutive Fibonacci numbers, is subtracted from 360°. As with the golden ratio, this approximation of the golden angle becomes more accurate as the Fibonacci numbers used grow larger (Adam, 2003, p. 218).

$$360° - (360°)\left(\frac{3}{5}\right) = 144°$$

$$360° - (360°)\left(\frac{13}{21}\right) = 137.1428571428\ldots°$$

$$360° - (360°)\left(\frac{34}{55}\right) = 137.\overline{45}°$$

Golden spiral. The golden spiral, also known as a logarithmic spiral, is a spiral whose form remains the same, even as it continues to grow in size. When the length increases, the radius also increases proportionally, so the actual shape of the spiral is unchanged. This spiral is also referred to as an equiangular spiral because its curve intersects each radius vector from the center of the spiral at the same constant angle (Huntley, 1970, p. 168). This golden spiral's construction can be approximated using both a golden rectangle and Fibonacci squares. To construct it with a golden rectangle, a golden rectangle is divided up by cutting off successive squares. For example, a rectangle of length 89 and width 55, which is composed of two Fibonacci numbers and is very close to a golden rectangle, can be sectioned into a square with a side length of 55 and a rectangle with side lengths of 55 and 34. The new rectangle is divided into a square with a side length of 34 and a rectangle with side lengths of 34 and 21. This new rectangle is divided into a square with a side length of 21 and a rectangle with side lengths of 21 and 13. This pattern continues, and when quarter-circle arcs are drawn between opposing corners of each square, they form a spiral, as shown in the figure below (Posamentier & Lehmann, 2012, pp. 104-105). Alternatively, the spiral can also be approximated using squares with side lengths of the sequential Fibonacci numbers. It begins with a square of length 1. Another square of length 1 is attached to that. A square of length 2 is attached to the sides of the previous two squares where it fits, as 1 + 1 = 2. Then a square of length 3 is attached to the squares of lengths 1 and 2, where 1 + 2 = 3. A square of length 5 is attached to the squares of lengths 2 and 3, a square of length 8 is attached to the squares of lengths 3 and 5, and so forth. Quarter-circle arcs are then drawn to sequentially connect the opposing corners of the squares, as can be seen in the figure below (Garland, 1987, pp. 14-15).

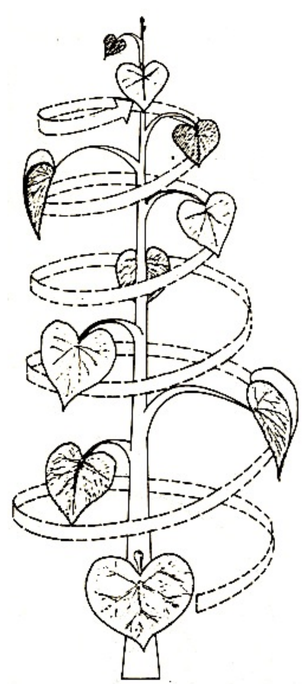


(Wolfram MathWorld)

Fibonacci Sequence in Nature

Several hundred years after Fibonacci's lifetime, the time of the Renaissance emerged, when people began to pay more analytical attention to the natural world surrounding them. As they studied the structures and forms of various plants, animals, and humans, they noticed that Fibonacci number patterns were showing up in measurements throughout creation.

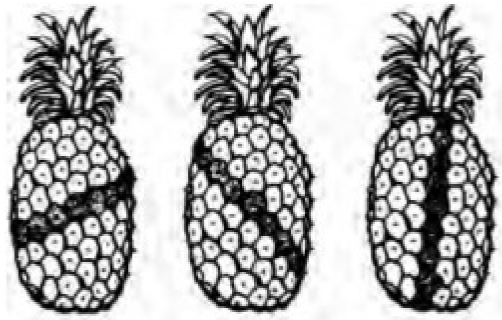
Plants. One place where Fibonacci numbers consistently appear is in the leaf arrangement on plants, a field of study known as phyllotaxis. As leaves go up a plant stem, they follow a spiral arrangement. Starting at one leaf, let x be the number of turns of the spiral before a leaf is reached that is directly above the first leaf. Let y be the number of leaves encountered along the spiral between the first leaf and the last leaf in this arrangement, not counting the first. This ratio of x/y is known as the divergence of the plant (Devlin, 2011, p. 146).



(Askipedia).

In this phyllotactic ratio, the numerator and denominator are very often Fibonacci numbers. For example, leaves are generated after about 3/8 of a revolution for poplar, willow, and pear trees, 1/3 for beech and hazel, 2/5 for oak, cherry, and apple, 1/2 for elm and lime, and 5/13 for almond. Other phyllotactic ratios include 3/5, 5/13, and 8/13 (Adam, 2003, p. 217). Also, the total number of leaves or petals is often a Fibonacci number. To name just a few, primroses, larkspur, and buttercups have five petals, delphiniums have eight, marigolds have thirteen, asters and chicory have twenty-one, and various types of daisies can have thirteen, twenty-one, thirty-four, fifty-five, or eighty-nine petals (Devlin, 2011, p. 145). The Fibonacci numbers are present in the leaf or petal arrangement of most plants. It has been speculated that the reason these numbers are often present in such arrangements could be to maximize the amount of light received or the space allotted for each leaf or petal on the plant. A stem growing upwards will generate leaves, which branch out at regular angular intervals, spiraling up the stalk. If the leaves on a stem all grew with angular intervals that were multiples of 360°, then they would be growing, one directly above the other. The top few leaves would then block the lower leaves and prevent them from receiving as much sunlight and moisture (Adam, 2003, p. 217).

Fibonacci ratios also appear in the spiral patterns of many plants. When a growing plant shoot is examined, the very tip of the shoot is called the apex, and clustered all around the apex are tiny lumps known as primordia. As the plant grows, these primordia move away from the apex and eventually become petals or leaves. These primordia, which are arranged in a spiral, determine the growth pattern of the plant. Ones which appear earlier will migrate farther away from the apex. When the angles between successive primordia are measured with the center of the apex as the vertex, the angles between any two successive primordial, known as divergence angles, are basically equal. As it turns out, the measurement of those angles is approximately 137.5°, which is the golden angle. This arrangement results in two interconnected spirals, one of which winds clockwise, and the other counterclockwise (Stewart, 1998, pp. 124-126). The spiraling scales on pinecones provide a clear example. A careful examination will show that there are actually two sets of spirals. One set goes from left to right, and the other set goes from right to left. One of these sets of spirals rises steeply up the side of the pinecone, and the other rises much more gradually. The number of steep and the number of gradual spirals up the side of the pinecone are almost always Fibonacci numbers, and often they are consecutive in the Fibonacci sequence. For example, some pinecones have three gradual and five steep spirals, while others have eight gradual and thirteen steep spirals. A similar sort of spiraling occurs in the outer petals of artichokes and various other flower buds. Like pinecones, there is one set of spirals going steeply in one direction and another set of spirals going gradually in the other direction, and the number of spirals in each set is a Fibonacci number (Garland, 1987, pp. 9-10). The same can also be found true of pineapples. A pineapple is covered in hexagonally shaped scales, known as bracts. These bracts form spirals in three different directions, each passing through opposing sides of the hexagon. Five spirals rise gradually in one direction, eight spirals rise at a medium rate in a second direction, and thirteen spirals rise steeply in the third direction, giving three consecutive Fibonacci numbers for the three different sets (Posamentier & Lehmann, 2007, pp. 63-64).



(Livio, 2002, p. 111)

Additionally, the leaf stubs on the trunks of palm trees also form spirals, and the number of leaf stub spirals is almost always a Fibonacci number. Depending on the kind of palm tree, there can be one, two, three, five, eight, thirteen, or twenty-one spirals present (Posamentier & Lehmann, 2007, p. 73).

A similar Fibonacci spiraling tendency also surfaces when examining the centers of flowers, the spines of various types of cacti, and the leaves on certain succulents. In this case, one set of spirals can be found going in a clockwise direction, and a second set is found going in a counterclockwise direction. The number of spirals going clockwise and the number of spirals going counterclockwise are consecutive Fibonacci numbers.

This is most clearly shown in the sunflower. The seeds at the center of the flower head spiral clockwise and counterclockwise. While the numbers of spiral sets depends on the age and development of the sunflower, they are always Fibonacci numbers. The two numbers can vary from 13 and 21, to 34 and 55, to 89 and 144 (Posamentier & Lehmann, 2007, pp. 67-69). It was discovered that if the divergence angle was less than 137.5°, it would result in gaps in the seed head of the sunflower, and only one direction of spirals would be apparent. Similarly, if the divergence angle was greater than 137.5°, gaps would again appear in the seed head, and this time, spirals would only be visible winding in the other direction. It turns out that only at the golden angle can the seeds on the seed head be packed without gaps and both directions of spirals appear (Stewart, 1998, pp. 126-127). When the divergence angle equals the golden angle, it results in "the most efficient packing,…that makes the most solid and robust seed head." (Stewart, 1998, p. 127).

When a budding rose is viewed from above, we see that the petals are unfolding in a spiraling pattern. If the angles between any two successive petals are measured, it is found that the angles are about 137.5°, the golden angle (Hemenway, 2005, p. 135). For centuries, scientists and mathematicians have tried to discover the reasons behind the common appearance of Fibonacci numbers in plant growth and development. In 1992, Yves Couder and Adrien Douady, two French mathematicians, traced the cause of the Fibonacci numbers appearances to inherent constraints on plant development. Their work showed that "the apparent mathematical patterns in plants do indeed arise from universal laws of the physical world. They are not merely genetic accidents reinforced by evolution" (Stewart, 1998, p. 123).

The golden spiral, found in pinecones, flower seed heads, and pineapples, can be found in countless other places in nature as well. The curl of a growing fern follows the pattern of a logarithmic spiral, starting out tightly furled, but loosening as it grows. This same spiral can be traced in ocean waves curling forward upon themselves before crashing on the shore. The spiral form within a galaxy conforms to a golden spiral as well, as does the spiraling shape of a storm (Garland, 1987, pp. 30-31).

Animals. Fibonacci numbers also manifest themselves in the animal kingdom. This sequence of numbers, which first made its appearance in a problem about the regeneration of rabbits, also shows up in the regeneration of other living creatures. The numbers can be discovered by an inspection of the family tree of the male bee. There are three types of bees living in a bee hive: the queen, who produces eggs; the male bees, who do no work; and the female bees, who do all the work (Posamentier & Lehmann, 2007, p. 59). The female bees develop from fertilized eggs, meaning they have both mothers and fathers. The male bees, on the other hand, develop from unfertilized eggs, meaning they have only mothers but no fathers. They do, however, have grandfathers, as each female bee has a father (Garland, 1987, p. 13). So one male bee has one mother, two grandparents, three great-grandparents, five great-great-grandparents, and eight great-great-great-grandparents. The number of bees in each preceding generation is a Fibonacci number (Hemenway, 2005, p. 135).

One of the most intriguing appearances of the Fibonacci sequence in the animal kingdom is in the spiral which indicates animal growth. One of the best examples of the golden spiral can be found in the shell of the chambered nautilus. As the nautilus grows larger, the chamber in which it lives must necessarily become larger as well, while still maintaining the same shape. As the shell increases in size, the radius of each successive chamber increases in size as well, yet the angles of intersection between each radius and the outer wall of the shell stay the same. This results in chambers that are shaped similarly, but sequentially increase in size, thus creating an equiangular spiral that exhibits Fibonacci proportions. This spiral can be found in many other places throughout the animal kingdom such as parrot beaks, elephant tusks, the tail of a seahorse, and the horns of bighorn sheep. Other manifestations of the spiral include spider webs, cat claws, the growth patterns of many seashells, and an insect’s path as it approaches a light source. All these spirals possess the basic characteristics of the golden spiral, as they all increase in size while still maintaining the same shape, and most exhibit the Fibonacci proportions in their spirals (Garland, 1987, pp. 16, 31).

Fibonacci numbers and the golden ratio also play a part in nature exhibited by pentagons. If the measure of a side of a pentagon is a Fibonacci number and the measure of an interior diagonal is the next Fibonacci number in the sequence, then the pentagon is a regular pentagon. The higher the numbers, the closer the ratio between them is to the golden ratio. Additionally, where the interior diagonals of the pentagon intersect, they segment each other into two consecutive Fibonacci numbers. For example, in a pentagon with side lengths of 89 and diagonals of 144, two intersecting diagonals will divide each other into segments of 55 and 89. These pentagons appear in many places in nature, often manifested as a star-like shape. The center design of a sand dollar bears the shape of a pentagon, as do the shapes of many starfish, the seed placement in the cross section of an apple, and the forms of many flowers (Garland, 1987, pp. 17-18).

These examples illustrate only a few instances where Fibonacci numbers are found in nature. The manifestations of the Fibonacci numbers and the golden ratio are seemingly endless. When one begins looking for these occurrences, they suddenly can be found everywhere. Snowflakes are constructed according to the golden ratio. Pine needles often grow in groups of 2, 3, or 5. The number of segments in most plant pods is a Fibonacci number (Garland, 1987, p. 18). These manifestations occur far too often to be pure chance or coincidence. Instead they indicate the mathematical nature of a world formed with order and precision.

Fibonacci Sequence in Architecture, Art, and Music

Fibonacci numbers and the golden ratio have been used in works of art and architecture for centuries. The golden ratio, closely intertwined with the Fibonacci numbers, has been shown to provide proportions most pleasing to the eye. Architects often used the golden rectangle as well, but to avoid the difficulties involved with working with the irrational number, phi, golden rectangles were usually constructed using Fibonacci numbers, which provide a very close approximation of phi (Posamentier & Lehmann, 2007, p. 232).

Architecture. One of the earliest examples can be found in the Great Pyramid at Giza. Let b be the base of a triangle which goes from the midpoint of a side of the pyramid to the center of the square base. Let a be the diagonal up the side of the pyramid from the same midpoint of the side to the very top of the pyramid. For the Great Pyramid, the approximate lengths of a and b are 612.01 feet and approximately 377.9 feet, respectively. $\frac{a}{b} = \frac{612.01}{377.9} = 1.62$, which is very close to the golden ratio (Livio, 2002, pp. 56-57). Whether this indicates that the ancient Egyptians knew about the golden ratio, or simply that they chose those dimensions because they were visually appealing is a point of great debate. Another well-known example of the golden ratio in architecture is the Parthenon of ancient Greece. Located on the Acropolis in Athens, the Parthenon was built as a temple to house the statue of the Greek goddess Athena. The dimensions of the front of the building fit into a golden rectangle, and the structure of the building lends itself to being partitioned off into all sorts of golden rectangles. Much of the ornamentation involves the golden ratio in its measurements. Exactly how much of this was intentional on the part of the ancient architects remains uncertain (Posamentier &

Lehmann, 2007, pp. 232-234). The designs of many buildings built during the Renaissance involve Fibonacci numbers or the golden ratio. For example, the Cathedral in Florence involves the Fibonacci numbers 55, 89, and 144, as well as 17, which is half of 34, and 72, which is half of 144. The strongest example can be found in the windows, which have proportions of 89 and 55. $\frac{89}{55} = 1.6181818 \ldots$, which is very close to the golden ratio (Posamentier & Lehmann, 2007, pp. 239-240).

Art. The golden ratio also figures quite prominently in works of art, both in sculptures and in paintings. In the case of the statue *Apollo Belvedere*, the measurements from his feet to his navel and from his navel to the top of his head form the golden ratio, as do the measurements from his navel to his shoulders and from his shoulders to the top of his head. The entire figure of the statue *Aphrodite of Melos* is divided into the golden ratio by her navel. Again, how much of this was intentional by the sculptors is uncertain (Posamentier & Lehmann, 2007, pp. 245-246). The golden ratio can be found in art everywhere, from the Middle Ages paintings of Madonna to ancient Chinese bowls to Syrian floor mosaics to Indian statues of Buddha (Garland, 1987, pp. 27-29). Leonardo da Vinci, a man of science as well as a brilliant painter, utilized the golden ratio in the majority of his work. In his well-known sketch of the Vitruvian man, the ratio of the side of the square which corresponds to the man’s arm span and height to the radius of the circle which contains his outstretched arms and legs is the golden ratio (Posamentier & Lehmann, 2007, p. 257). In the *Mona Lisa*, a golden rectangle can be used to enclose the space from the top of her head to the top of her bodice. Dividing this rectangle into a square results in a square that precisely encloses her head, with her left eye at the center (Atalay, 2006, pp. 176-177).

These are but a few of countless examples of how the golden ratio, Fibonacci numbers, and golden rectangles are involved in the construction and architecture of buildings, as well as in the structure of sculptures and paintings, both ancient and modern.

Music. Fibonacci numbers are clearly illustrated when looking at the keyboard of a piano. The first six numbers in the Fibonacci sequence can be found by looking at just one octave of keys. Each octave is composed of 13 keys, 8 of which are white and 5 of which are black, and the black keys are partitioned into groups of 2 and 3 (Garland, 1987, p. 33). The violins made by Antonio Stradivarius are the most sought after of all violins, and today they can cost several million dollars. The proportions and components of these instruments have been carefully studied by those who wish to replicate them, and it turns out the violin is divided into proportions of 2, 3, 5, 8, and 13 (Posamentier & Lehmann, 2007, p. 291). The true relationship between music and Fibonacci numbers can only be found when the actual musical compositions are examined. In many of Chopin's preludes, the climax of the music is located very near the place where the golden ratio would divide the length of the piece. This is especially true of his *Prelude No. 1 in C major*, which has 34 measures. The climax of this piece occurs in measure 21, and the ratio of the two comes close to the golden ratio, as $\frac{34}{21} = 1.619 \ldots.$ Something similar happens in his *Prelude No. 9 in E major*. This piece contains 48 beats, and the climax occurs on beat 29. The ratio of these two numbers also comes close to the golden ratio, as $\frac{48}{29} = 1.655 \ldots.$ This occurs in a number of preludes, although there are also many in which it does not happen (Posamentier & Lehmann, 2007, pp. 272-273). The first movement of Beethoven's Fifth Symphony is divided into the golden ratio by the opening five measures, which repeat 372 measures later, and again after 228 measures. There are 377 measures before the middle repetition and 233 measures after the middle repetition, providing a ratio of $\frac{377}{233} = 1.618$ ... (Garland, 1987, p. 38). Out of Mozart's seventeen piano sonatas employing what is known as the sonata-allegro form, six are exactly divided into the golden ratio, and eight are very close. Since a total of 82% of his sonatas are divisible by the golden section, it would seem that the use of the golden ratio was very important to Mozart in his compositions (Posamentier & Lehmann, 2007, pp. 277-278). These composers, along with Haydn, Wagner, and Bartok, represent only a few of the musicians, both modern and classical, who have compositions that are divided into the golden ratio (Posamentier & Lehmann, 2007, pp. 279-285). The use of the golden ratio seems more intentional in music than in art.

Conclusion

Although little is known of the life of Leonardo of Pisa, his work has left an indelible impression upon the world. The discovery for which he is best known – the sequence of numbers bearing his name – brought him no recognition during his life, and indeed did not really become a topic of interest until hundreds of years later. This unique and fascinating string of numbers possesses all sorts of intriguing properties, which can be discovered by applying various mathematical procedures to the numbers in the sequence. Fibonacci numbers are present throughout the world in which we live, and the patterns which can be formed from them both astonish and perplex the mind. The Fibonacci numbers are beautiful to study in and of themselves, but there is a higher beauty to them as well. These numbers highlight the incredible order and mathematical complexity of the world we live in, which all points to the Creator. Such intricate patterns could not have evolved by mere chance, but are the work of a God of order, who created all things.

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