Bézout Factors and L^1-Optimal Controllers for Delay Systems Using a Two-Parameter Compensator Scheme

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***Abstract*—The authors consider in this paper the simultaneous problem of optimal robust stabilization and optimal tracking for single-input/single-output (SISO) systems in an L^∞-setting using a two-parameter compensator scheme. Optimal robustness is linked to the work done by Georgiou and Smith in the L^2-setting. Optimal tracking involves the resolution of L^1-optimization problems. The authors consider in particular the robust control of delay systems. They determine explicit expressions of the Bézout factors for general delay systems which are in the Callier–Desoer class $\hat{\mathcal{B}}(0)$. Finally, they solve several general L^1-optimization problems and give an algorithm to solve the optimal robust control problem for a large class of delay systems.**

***Index Terms*— Bézout factors, delay system, optimal robust stabilization, L^1-optimization, tracking, two-parameter compensator scheme.**

I. Preliminaries

We have RHP:$\{x + jy, x > 0, y \in \mathbb{R}\}$.

L^p denotes the complex-valued measurable functions on the nonnegative real axis such that

$$\left(\int_0^\infty |f(t)|^p \, dt\right)^{1/p} < \infty.$$

L^∞ denotes the complex-valued measurable functions on the nonnegative real axis such that $ess \ \sup_{t\in\mathbb{R}_+} |f(t)| < \infty$. C_c denotes the subspace consisting of continuous functions of compact support. $\alpha(f)$ is defined for $f \in L^p$ as $\alpha(f) = \sup\{t \in \mathbb{R}_+, f(x) = 0 \text{ a.e. on } (0,t)\}$. A linear continuous-time system G is defined as a linear integral convolution operator G from L^p to L^p.

The system G is L^p-stable if

$$\|G\|_{\langle p\rangle} \triangleq \sup_{x\in L^p, x\neq 0} \frac{\|Gx\|_{L^p}}{\|x\|_{L^p}} < \infty.$$

$\mathcal{A}(\beta)$ denotes the space of distributions of the form

$$g(t) = g_a(t) + \sum_{i=0}^{\infty} g_i\delta(t - t_i)$$

where $t_i \in [0,\infty)$, $0 \le t_0 < t_1 < \cdots$, $\delta(t-t_i)$ is a delayed Dirac function, $g_i \in \mathbb{C}, e^{-\beta\cdot}g_a(\cdot) \in L^1[0,\infty)$, and $\Sigma_{i=0}^{\infty} |g_i|e^{-\beta t_i} < \infty$.

$\mathcal{A}(\beta)$ is equipped with the norm

$$\|g\|_{\mathcal{A}(\beta)} = \int_0^\infty e^{-\beta t}|g_a(t)| + \sum_{i=0}^{\infty} |g_i|e^{-\beta t_i}.$$

$\mathcal{A}(0)$ is simply denoted $\mathcal{A}$.

The Laplace transform $\mathcal{L}$ of g is denoted $\hat{g}$

$$\hat{g}(s) = \int_{-\infty}^{\infty} e^{-st}g(t)\,dt.$$

Often we shall be considering transforms of functions defined only on $[0,\infty)$, in which case we shall regard them as being defined to be zero on $(-\infty, 0)$:

$$\hat{\mathcal{A}}_-(\beta) = \{\hat{g}/\hat{g} \in \hat{\mathcal{A}}(\beta_1) \quad \text{for some} \quad \beta_1 < \beta\}.$$

$$\hat{\mathcal{A}}_\infty(\beta) = \{\hat{g} \in \hat{\mathcal{A}}_-(\beta)/\exists\, \rho \text{ s.t.} \cdot \inf_{\{s\in\{\Re(s)\ge\beta\}, |s|\ge\rho\}} |\hat{g}(s)| > 0\}.$$

The Callier–Desoer class $\hat{\mathcal{B}}(\beta)$ is defined as [4]

$$\hat{\mathcal{B}}(\beta) = \{\hat{g} = \hat{n}\hat{d}^{-1} \text{ where } \hat{n} \in \hat{\mathcal{A}}_-(\beta) \text{ and } \hat{d} \in \hat{\mathcal{A}}_\infty(\beta)\}.$$

Let T^1 denote the space of all linear time-invariant causal continuous-time L^1-stable systems equipped with the operator norm.

Distributions in $\mathcal{A}$ generate a subspace $T^1_{\mathcal{A}}$ of T^1. $T^1_{\mathcal{A}}$ is isometrically isomorphic to $\mathcal{A}$. Let $G \in T^1_{\mathcal{A}}$, then we have $\|G\|_{\langle 1\rangle} = \|G\|_{\langle\infty\rangle} = \|g\|_{\mathcal{A}} = \|\hat{g}\|_{\hat{\mathcal{A}}}$ (see [16]).

By a convenient abuse of notation we identify $T^1_{\mathcal{A}}$ and $\hat{\mathcal{A}}$ and use the same notation G for the operator G and the transfer function $\hat{g}$.

A causal system G in the quotient field of $\hat{\mathcal{A}}$ is said to have a coprime factorization (N, D) over $\hat{\mathcal{A}}$ if $G = (N/D), D \neq 0, N, D \in \hat{\mathcal{A}}$ and there exists $X, Y \in \hat{\mathcal{A}}$ such that $-NX + DY = 1$.

A coprime factorization (N, D) over $\hat{\mathcal{A}}$ is said to be normalized if

$$N(s)\overline{N(s)} + D(s)\overline{D(s)} = 1$$

for any $s = j\omega$ (as N and D are in $\hat{\mathcal{A}}$, $N(j\omega)$ and $D(j\omega)$ are continuous and bounded on $\mathbb{R}$). We can deduce from [16] that each G having a coprime factorization over $\hat{\mathcal{A}}$ has a

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normalized coprime factorization over $\hat{\mathcal{A}}$ which is unique to within multiplication by ± 1, as the proof of [16, Th. 4.2] is still valid in the complex case.

It proceeds as follows [16], [20]. Let (N, D) be any coprime factorization over $\hat{\mathcal{A}}$, define $F(j\omega) = 1/(|N(j\omega)|^2 + |D(j\omega)|^2)$, and write $\ln\ F(j\omega) = V(j\omega) + \overline{V(j\omega)}$, then U defined by $U(j\omega) = \exp V(j\omega)$ is such that $G = (NU/DU)$ is a normalized coprime factorization.

The graph topology on $\hat{\mathcal{A}}$ is defined by Vidyasagar [24] by introducing a basic neighborhood $B(N, D; \gamma)$ of a system G corresponding to the coprime factorization (N, D) of G over $\hat{\mathcal{A}}$ and to the number $\gamma > 0$ [which must be smaller than a certain positive number depending only on (N, D)] as

$$B(N, D; \gamma) = \left\{ G_1 \ \text{ which admit a coprime factorization,} \right. \\ \left. G_1 = \frac{N_1}{D_1}, \left\| \begin{matrix} N_1 - N \\ D_1 - D \end{matrix} \right\|_{\hat{\mathcal{A}}} < \gamma \right\}$$

where for $X, Y \in \hat{\mathcal{A}}$

$$\left\| \begin{matrix} X \\ Y \end{matrix} \right\|_{\hat{\mathcal{A}}} \equiv \sqrt{\|X\|_{\hat{\mathcal{A}}}^2 + \|Y\|_{\hat{\mathcal{A}}}^2}$$

for example.

Partington and Mäkilä [16] extended the result of Vidyasagar concerning convergence in the graph topology in the finite energy setting [24, Lemma 7.2.20] to the L_∞-setting. We state this result [16, Lemma 3.2].

The graph topology is also known as the gap topology [24].

Lemma 0.1 [16]: Let G (respectively, $\{G_i\}$) be a (respectively a sequence of) causal transfer function admitting a coprime factorization in $\hat{\mathcal{A}}$. Then, the following statements are equivalent.

1) $\{G_i\}$ converges to G in the bounded-input/bounded-output (BIBO) gap topology.
2) For every coprime factorization (N, D) of G over $\hat{\mathcal{A}}$, there exist coprime factorizations (N_i, D_i) over $\hat{\mathcal{A}}$ such that $N_i \longrightarrow N$, $D_i \longrightarrow D$ in $\hat{\mathcal{A}}$.
3) There exist a coprime factorization (N, D) of G over $\hat{\mathcal{A}}$ and a sequence $\{(N_i, D_i)\}$ of coprime factorizations of G_i over $\hat{\mathcal{A}}$ such that $N_i \longrightarrow N$, $D_i \longrightarrow D$ in $\hat{\mathcal{A}}$.

II. Introduction

In this paper, we consider the robust control of infinite-dimensional single-input/single-output (SISO) systems with a special emphasis on delay systems and particularly on the delay integrator as it is the system which motivated our study. In fact, through an industrial problem of car depollution we were faced with the problem of optimally robustly stabilizing a delay integrator as well as making it optimally track a constant signal output over time of unit amplitude. This is a simply stated problem, but there is no existing method for it in the literature.

Optimal robust stabilization for the standard feedback problem in the L^2-setting has been considered by Georgiou and Smith [11] when uncertainty is based on perturbations on the coprime factors of the system (or on perturbations in the gap metric). In the case of delay systems of the form $G(s) = e^{-sT}R(s)$ where $R(s)$ is a strictly proper rational function and $T > 0$, an explicit formula for the optimal robust controller is given as well as the value of the optimal robustness margin. However, this formula is not obtained using the Youla parameterization of the stabilizing controllers and the resolution of an optimization problem in H_∞ as set in the generic case but is a closed form expression. Thus it is impossible to modify this expression to be able to act on the tracking quality of this controller.

At this point, the idea is to consider a two-parameter scheme instead of the standard feedback configuration. In this scheme too, the controllers are parameterized in terms of the Bézout factors of the system. Kamen *et al.* [15] and more recently Brethe and Loiseau [3] and Glüsing-Luerßen [13] considered the existence of coprime factorizations of time-delay systems with commensurate time delays. They proved that the set of entire functions over $\mathbb{R}(s)[e^{-s}]$ is a Bézout domain. In [3] we can find an algorithm to compute the coprime factorizations for such delay systems. The stabilizing controllers (of the standard feedback scheme) obtained through the standard Youla parameterization produce control laws which contain commensurate and distributed delays.

Now, the practical problem leads us to consider an L^∞-setting firstly because we consider persistent signals and secondly because we intend to measure the L^∞-quality of the tracking. It was Vidyasagar [25] and Dahleh and Pearson [5]–[8] who first mentioned the interest of developing an equivalent theory to the well-known H_∞-theory arising in the L^2-setting: in practical situations, very often signals are naturally not of bounded energy but of bounded magnitude. Moreover, one might want to control the magnitude of an error signal rather than its integral square. As L^∞-optimal control gives rise to optimization problems in the algebra $L^1 + \Sigma_{i=0}^{\infty} c_i \delta(t - t_i)$, they solved several L^1 (ℓ^1)-optimization problems for continuous (discrete-time) finite-dimensional systems. Staffans [23] studied equivalent problems for discrete-time infinite-dimensional systems but as far as we know the case of continuous-time infinite-dimensional systems has not been studied. The problem of optimal robust stabilization of infinite-dimensional systems in an L^∞-setting has been studied in [20], [16], and [2]. In [2], we established a link between optimal controllers in this setting and those determined by Glover–McFarlane [17] and Georgiou–Smith [11] in the L^2-setting as well as convergence results of optimal controllers and robustness margins of finite-dimensional systems to those of infinite-dimensional systems. Those results will be helpful in the study of the present paper.

The paper is organized as follows. In Section III, we formulate the double problem of optimal robust stabilization and optimal tracking through a two-parameter compensator scheme for general infinite-dimensional systems. In Section IV we propose a family of (eventually normalized) coprime factorizations and Bézout factors for a class of delay systems. In Section V, we solve for different classes of systems the L^1-optimization problem that was posed in Section III. Finally, we give in Section VI a general algorithm to solve the proposed robust control problem and illustrate it on a simple example.

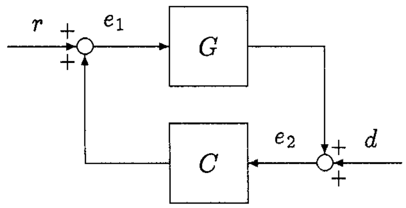

Fig. 1. Standard feedback configuration.

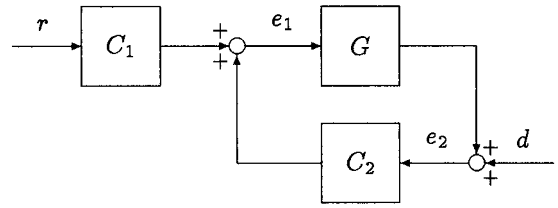

Fig. 2. Infeasible implementation of a two-parameter compensator.

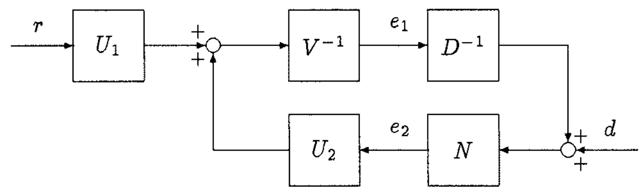

Fig. 3. Feasible implementation of a two-parameter compensator.

III. Robust Stabilization and Tracking for Infinite-Dimensional Systems

We consider in this section, the simultaneous problem of robust stabilization and tracking.

Given a linear continuous-time infinite-dimensional system G defined as

$$G\colon \quad L^\infty \to L^\infty$$
$$u \mapsto g * u$$

we know from [20] that G is BIBO-stabilizable by a feedback controller admitting a coprime factorization if and only if G admits a coprime factorization over $\hat{\mathcal{A}}$.

Now, let us suppose (N, D) is a normalized coprime factorization of G over $\hat{\mathcal{A}}$ and (X, Y) are corresponding Bézout factors

$$-NX + DY = 1$$

and

$$N(s)N(-s) + D(s)D(-s) = 1, \qquad \text{for any} \quad s = j\omega.$$

It is well-known (see [24, p. 141]) that, given e the plant input, y the plant output, and u the external input, the greatest general feedback linear compensator scheme is given by

$$e = C_1 u + C_2 y$$

where C_1 and C_2 are linear operators on L^∞.

Here, we denote the external inputs r (reference signal) and d (disturbance).

If we take $C_1 = C_2$ (denoted C), we obtain the standard feedback configuration of Fig. 1.

In this case, the set of all stabilizing compensators is parameterized as $C = (X + DN)(Y + NQ)^{-1}$ where $Q \in \hat{\mathcal{A}}$.

The general feedback law can be implemented as in Fig. 2.

As explained in [24], this implementation makes no sense unless C_1 is stable. Let $(V, (U_1, U_2))$ be a coprime factorization of (C_1, C_2) so that $C_1 = (U_1/V)$ and $C_2 = (U_2/V)$. A feasible implementation of the general feedback law is given in Fig. 3.

By [24] (where Theorem 15 is still valid in infinite dimensions), we know that the set of all two-parameter compensators that stabilize G is given by $(U(Y + NQ)^{-1}, (X + DN)(Y + NQ)^{-1})$ with $U, Q \in \hat{\mathcal{A}}$.

The input–output relation corresponding to Fig. 3 is described as follows:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} (VD - U_2N)^{-1}U_1D & (VD - U_2N)^{-1}U_2D \\ (VD - U_2N)^{-1}U_1N & (VD - U_2N)^{-1}VD \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix}$$

and it is easy to see that the stabilization is governed by C_2 only.

We suppose $[G, (C_1, C_2)]$ is stable and consider $G_1 = (N + \Delta N)/(D + \Delta D)$. In this case the input–output matrix

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

is given by

$$\begin{aligned} m_{11} &= ((VD - U_2N) + (V\Delta D + U_2\Delta N))^{-1} \\ &\quad \cdot U_1(D + \Delta D) \\ m_{12} &= ((VD - U_2N) + (V\Delta D + U_2\Delta N))^{-1} \\ &\quad \cdot U_2(D + \Delta D) \\ m_{21} &= ((VD - U_2N) + (V\Delta D + U_2\Delta N))^{-1} \\ &\quad \cdot U_1(N + \Delta N) \\ m_{22} &= ((VD - U_2N) + (V\Delta D + U_2\Delta N))^{-1} \\ &\quad \cdot V(D + \Delta D). \end{aligned}$$

The stability of $[G_1, (C_1, C_2)]$ depends on the invertibility of $((VD - U_2N) + (V\Delta D + U_2\Delta N))$ in $\hat{\mathcal{A}}$.

Clearly, [2, Propositions 6.1–6.3] giving results on the robust stabilization through a standard feedback scheme as well as Proposition 6.4 giving convergence results are still valid here. We recall these propositions in our context of the two-parameter compensator scheme.

Proposition 2.1:

1) If

$$(\|\Delta N\|^2_{\hat{\mathcal{A}}} + \|\Delta D\|^2_{\hat{\mathcal{A}}})^{1/2} < \frac{1}{\left\| \begin{matrix} X + DQ \\ Y + NQ \end{matrix} \right\|_\infty}$$

then $[G_1, (C_1, C_2)]$ is stable.

2) If $(Y+NQ)\Delta D-(X+DQ)\Delta N=1$ for some s_0 on $j\mathbb{R}$ then $[G_1,(C_1,C_2)]$ is unstable.

Proposition 2.2: Let $C=(U(Y+NQ)^{-1},(X+DN)(Y+NQ)^{-1})$, with $U,Q\in\hat{\mathcal{A}}$. Then the following are equivalent.

1) $[G_1,(C_1,C_2)]$ is stable for all transfer functions $G_1=(N+\Delta N)(D+\Delta D)^{-1}$ where $\Delta N,\ \Delta D\in\hat{\mathcal{A}}$ and $\|(\Delta N,\Delta D)\|_{\hat{\mathcal{A}}}<b$.

2)
$$\left\|\begin{matrix}X+DQ\\ Y+NQ\end{matrix}\right\|_\infty\le\frac{1}{b}.$$

In the case of systems with kernel in $L^1+\mathbb{C}\delta$, we have the following.

Proposition 2.3: Let $Q^{\rm opt}$ be defined by

$$\left\|\begin{pmatrix}X\\ Y\end{pmatrix}+\begin{pmatrix}D\\ N\end{pmatrix}Q^{\rm opt}\right\|_\infty =\inf_{Q\in H_\infty}\left\|\begin{pmatrix}X\\ Y\end{pmatrix}+\begin{pmatrix}D\\ N\end{pmatrix}Q\right\|_\infty.$$

Then $X+DQ^{\rm opt}\in\mathcal{L}(L^1(0,\infty)+\mathbb{C}\delta)$ and $Y+NQ^{\rm opt}\in\mathcal{L}(L^1(0,\infty)+\mathbb{C}\delta)$. Moreover

$$b^{\rm opt}(G)=\frac{1}{\left\|\begin{pmatrix}X\\ Y\end{pmatrix}+\begin{pmatrix}D\\ N\end{pmatrix}Q^{\rm opt}\right\|_\infty}.$$

Let $(G_n)_n$ be a sequence of transfers such that $G_n\underset{n\to\infty}{\longrightarrow}G$ in the BIBO gap topology and $C_{2_n}^{\rm opt}$ (respectively, $C_2^{\rm opt}$) be the optimal robust controller of G_n (respectively, G) relative to coprime factor perturbations.

Proposition 2.4: If the greatest singular value of D^*X+N^*Y is of multiplicity one then:

1) $b^{\rm opt}(G_n)\underset{n\to\infty}{\longrightarrow}b^{\rm opt}(G)$;

2) $C_{2_n}^{\rm opt}\underset{n\to\infty}{\longrightarrow}C_2^{\rm opt}$ in the BIBO gap topology.

So, the optimal robust stabilization problem in this context does not need further investigation beyond the work done in the standard feedback scheme case.

Let us now state the tracking problem we want to solve.

Consider here that we want to optimally track in L^∞ the output reference r defined by $r(t)=1$ for $t\ge 0$. In applications, it is difficult to be sure that we will be able to produce the precise reference output; we might as well produce a signal which is only "near" the reference signal. So, it is more realistic to try to track a family of reference signals. Here we choose to track the family $\{r\in L^\infty,\|r\|_{L^\infty=1}\}$.

We have $e_2-r=(I-NU)\,r+(Y+QN)Dd$.

So the optimal tracking problem can be stated as

$$\inf_{U\in\hat{\mathcal{A}}}\ \max_{\|r\|_{L^\infty}=1}\ \|(I-NU)\,r\|_{L^\infty}$$

but we have

$$\begin{aligned}\max_{\|r\|_{L^\infty}=1}\ \|(I-NU)\,r\|_{L^\infty}&=\|I-NU\|_{\langle 1\rangle}\\ &=\|I-NU\|_{\hat{\mathcal{A}}}\\ &=\|\delta-nu\|_{\mathcal{A}}\end{aligned}$$

and the optimal tracking problem mathematically reduces to an $\hat{\mathcal{A}}$ (or an $\mathcal{A}$)-optimization problem

$$\inf_{U\in\hat{\mathcal{A}}}\ \|I-NU\|_{\hat{\mathcal{A}}}.$$

It will turn out that a more realistic problem is the general one

$$\inf_{U\in\hat{\mathcal{A}}}\ \|W(I-NU)\|_{\hat{\mathcal{A}}}$$

where W is a weight function.

IV. Coprime Factorizations and Bézout Factors for a Class of Delay Systems

A. Generalities on Delay Systems

We consider in this paper linear systems with a finite number of delays in the state, the input, and output. Such systems are described by the following equations:

$$(S)\quad\begin{cases}\dot{x}(t)=\sum_{i=0}^{k}A_ix(t-t_i)+\sum_{i=0}^{m}B_iu(t-\tau_i)\\ y(t)=\sum_{i=0}^{l}C_ix(t-\sigma_i)+\sum_{i=0}^{p}d_iu(t-\nu_i)\end{cases}$$

where $x(t)\in\mathbb{R}^n$, $u(t)$, $y(t)\in\mathbb{R}$, A_i, B_i, C_i are $n\times n$, $n\times 1$ and $1\times n$ matrices and $d_i\in\mathbb{R}$.

The transfer function G of (S) is given by

$$G(s)=\left(\sum_{i=0}^{m}B_ie^{-s\tau_i}\right)\left(sI-\sum_{i=0}^{k}A_ie^{-st_i}\right)^{-1}\cdot\left(\sum_{i=0}^{l}C_ie^{-s\sigma_i}\right)+\sum_{i=0}^{p}d_ie^{-s\nu_i}.$$

We denote (S_0) the system without delay

$$(S_0)\quad\begin{cases}\dot{x}(t)=A_0x(t)+B_0u(t)\\ y(t)=C_0x(t)+d_0u(t)\end{cases}$$

with G_0 the transfer function and h_0 the nonatomic part of the impulse response of the system (S_0).

Suppose that (S_0) is BIBO-stable, that is $h_0+d_0\delta\in L^1+\mathbb{C}\delta$.

Let us recall, for simplicity in the case $n=1$ (A_i,B_i, C_i are then denoted a_i,b_i,c_i), how the different delays act on the BIBO-stability of (S).

Suppose there is no delay in the state ($a_i=0,i=1,\cdots,k$). The impulse response g of G is of the form

$$g=\sum_{i=0}^{m+l}\alpha_i\delta_{\beta_i}*h_0+\sum_{i=1}^{p}d_i\delta_{\nu_i}$$

with $\alpha_i,\beta_i\in\mathbb{R}$.

Obviously $g\in L^1+\Sigma_{i=0}^{\infty}\,\mathbb{C}\delta_{t-t_i}$: the delays τ_i,σ_i and ν_i do not change the BIBO stability of (S_0).

To see the effect of the delays t_i on the impulse response, we suppose that $d_i=0,i=1,\cdots,p$.

Let $y \in \mathbb{R}$

$$|G(iy)| \leq \frac{\left(\sum_{i=0}^{m} |b_i|\right)\left(\sum_{i=0}^{l} |c_i|\right)}{\left| |y| - \sum_{i=0}^{k} a_i \right|}.$$

We have $|G(iy)| \underset{y\to\pm\infty}{\longrightarrow} 0$ and this means that there is no impulse term in the impulse response. So the delays t_i do not contribute impulse terms to the impulse response. However, they modify the L^1-part of the impulse response and can make it fail to be in L^1.

We are looking here at robustness of stability relatively to coprime factor perturbations; we recall that for strictly proper systems a change in the delay (in the input, output, or state) corresponds to a variation in the BIBO gap topology.

Proposition 3.1: Let $G_0 = G^0$ be the transfer function of the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

and G_T (respectively, G^τ) be the transfer function of the delayed system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t-T) \\ y(t) = Cx(t) \end{cases}$$

or

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t-T) \end{cases}$$

$$\left(\text{respectively, } \begin{cases} \dot{x}(t) = Ax(t-\tau) + Bu(t) \\ y(t) = Cx(t) \end{cases}\right)$$

then $G_T - G_0 \underset{T\to 0}{\longrightarrow} 0$ and $G^\tau - G^0 \underset{\tau\to 0}{\longrightarrow} 0$ in the BIBO gap topology.

Proof: Note that the transfer functions in question are

$$G_T(s) = C(sI - A)^{-1}Be^{-sT}$$

and

$$G^\tau(s) = C(sI - Ae^{-s\tau})^{-1}B.$$

Let (N, D) and $(e^{-sT}N, D)$ be coprime factorizations over $\hat{\mathcal{A}}$ of G_0 and G_T, respectively. Note that N and D are rational, N is strictly proper, and D is proper but not strictly proper. Writing h_N for the impulse response of N, we have

$$\|N - e^{-sT}N\|_{\hat{\mathcal{A}}} = \int_0^\infty |h_N(t) - h_N(t-T)|dt.$$

Using the fact that $C_c(\mathbb{C})$ is dense in $L^1(\mathbb{C})$ it is easy to prove that

$$\|N - e^{-sT}N\|_{\hat{\mathcal{A}}} \underset{T\to 0}{\longrightarrow} 0.$$

The transfer function of the second class of system is

$$\begin{aligned} G^\tau(s) &= e^{s\tau}G^0(se^{s\tau}) \\ &= e^{s\tau}N(se^{s\tau})/D(se^{s\tau}) \\ &= N^\tau(s)/D^\tau(s), \quad \text{say.} \end{aligned}$$

It is easily verified that $N^\tau \to N$ and $D^\tau \to D$, as $\tau \to 0$. □

More complicated results can be proven similarly, taking a collection of different delays, all tending to zero.

B. Coprime Factorizations and Bézout Factors for a Class of Delay Systems

We consider the class of retarded delay systems with scalar transfer function given by $G(s) = h_2(s)/h_1(s)$ where

$$h_1(s) = \sum_{i=0}^{n_1} p_i(s)e^{-\gamma_i s} \quad \text{and} \quad h_2(s) = \sum_{i=0}^{n_2} q_i(s)e^{-\beta_i s}$$

with $0 = \gamma_0 < \gamma_1 \cdots < \gamma_{n_1}$, $0 \leq \beta_0 < \beta_1 \cdots < \beta_{n_2}$, the p_i being polynomials of degree δ_i and $\delta_i < \delta_0$ for $i \neq 0$ and the q_i being polynomials of degree $d_i < \delta_0$ for each i.

This class of systems was first analyzed by Bellman and Cooke [1]. They proved that these systems possess only finitely many poles in any right half-plane.

Normally one assumes that h_1 and h_2 have no common zeroes, but this is not necessary.

Proposition 3.2: Let $G(s) = h_2(s)/h_1(s)$ where $h_1(s)$ and $h_2(s)$ are defined as above. There exists a rational function $r(s)$ such that $(h_2(s)/r(s), h_1(s)/r(s))$ is a coprime factorization of G over $\hat{\mathcal{A}}$. If h_1 and h_2 have no more than δ_0 common unstable zeroes, then r can be taken to be a polynomial.

Proof: The idea beyond the proof is to take $r(s) = (s+1)^{\delta_0}$ suitably modified in order to deal with common unstable zeroes.

We have

$$\frac{h_2(s)}{(s+1)^{\delta_0}} = \sum_{i=0}^{n_2} \frac{q_i(s)e^{-\beta_i s}}{(s+1)^{\delta_0}}$$

with $\deg q_i < \delta_0$, so clearly $h_2(s)/(s+1)^{\delta_0} \in \hat{\mathcal{A}}_-(0)$. Now

$$\frac{h_1(s)}{(s+1)^{\delta_0}} = \sum_{i=0}^{n_1} \frac{p_i(s)e^{-\gamma_i s}}{(s+1)^{\delta_0}}$$

with $\deg\, p_0 = \delta_0$ and $\deg\, p_i < \delta_0$ for $i = 1, \cdots, n_1$, so $h_2(s)/(s+1)^{\delta_0} \in \hat{\mathcal{A}}_\infty(0)$.

As h_1 has a finite number of unstable zeroes we can test if there are common zeroes ξ_i ($i = 1, \cdots, l$ with multiplicity m_i) between h_2 and h_1. If so, then consider $(h_2(s)r(s))/(h_1(s)r(s))$ with

$$r(s) = \left[\prod_1^l (s - \xi_i)^{m_i}\right](s+1)^\lambda$$

where

$$\lambda + \sum_{i=1}^{l} m_i = \delta_0.$$

If the number of common unstable zeroes is less than δ_0, then r is a polynomial. Clearly, we have

$$\frac{h_2(s)}{\left[\prod_1^l (s - \xi_i)^{m_i}\right](s+1)^\lambda} \in \hat{\mathcal{A}}_-(0)$$

and

$$\frac{h_1(s)}{\left[\prod_1^l (s - \xi_i)^{m_i}\right](s+1)^\lambda} \in \hat{\mathcal{A}}_\infty(0).$$

Now, as $h_2(s)/r(s)$ and $h_1(s)/r(s)$ have no common zero in $\{\Re(s) \geq 0\}$, we deduce that $(h_2(s)/r(s),(h_1(s)/r(s))$ is a coprime factorization of G over $\hat{\mathcal{A}}_-(0)$. We recall that if $h_2(s)/r(s)$ and $h_1(s)/r(s)$ were not coprime over $\hat{\mathcal{A}}_-(0)$ they would necessarily have a common sequence of zeroes in $\{\Re(s) \geq 0\}$ and as $h_1(s)/r(s) \in \hat{\mathcal{A}}_\infty(0)$ the only possibility would be a finite common zero in $\{\Re(s) \geq 0\}$. □

Remark 3.1: Instead of $(s+1)^{\delta_0}$ in the above proof, we could begin with any polynomial of degree δ_0 without unstable zeroes. In fact, it might be possible to chose $r(s)$ such that $(h_2(s)/r(s), h_1(s)/r(s))$ is a normalized coprime factorization over $\hat{\mathcal{A}}$. For example, in the case when there is just one delay in the input or output, that is G is of the form $G(s) = e^{-sT}A(s)/B(s)$ where A and B are polynomials, we can deduce from the finite-dimensional case [17] that there exists a polynomial r such that $(e^{-sT}A(s)/r(s), B(s)/r(s))$ is a normalized coprime factorization of G over $\hat{\mathcal{A}}$.

We now give a formula for the Bézout factors in a coprime factorization of a retarded delay system. Recall that less explicit formulas have been given by Brethe and Loiseau [3] in the case of delay systems with commensurate time delays.

Theorem 3.1: Let m be the number of unstable zeroes σ of $h_1(s)$ (which are not zeroes of h_2) counted with their multiplicity.

Let

$$X(s) = \frac{-\mu(s)}{u(s)} \quad \text{and} \quad Y(s) = \frac{r(s) - \dfrac{\mu(s)h_2(s)}{u(s)}}{h_1(s)}$$

where μ is a polynomial of degree $m-1$ chosen such that

$$\left(r(s) - \frac{\mu(s)h_2(s)}{u(s)}\right)^{(k)} = 0$$

at $s=\sigma$ for $k = 0, \cdots, m_i - 1$ if σ is a zero of multiplicity $m_i \geq 1$, and u is a polynomial chosen such that its inverse is in $\hat{\mathcal{A}}_-(0)$ and X is proper ($\deg u \geq \deg \mu$).

Then X and Y are Bézout factors corresponding to the coprime factorizations $(h_2(s)/r(s), h_1(s)/r(s))$ of G over $\hat{\mathcal{A}}$.

To prove this result we need a lemma of [18] which we recall here.

Lemma 3.1 [18]: Let $\hat{g}$ be a holomorphic function on $\mathbb{C}_\alpha^+$ such that $s\hat{g}(s)$ is bounded on $\mathbb{C}_\alpha^+$. Then for any $\xi > 0$ there exists an h such that $\hat{g} = \hat{h}$ on $\mathbb{C}_{\alpha+\xi}^+$ and $\int_0^\infty e^{-(\alpha+\xi)t}|h(t)|\, dt < \infty$.

Proof: Clearly, $X(s) = (-\mu(s)/u(s)) \in \hat{\mathcal{A}}_-(0)$. Let $(\sigma_i)_{i=1,\cdots,l}$ be the unstable zeroes of h_1 of multiplicity m_i ($\Sigma_{i=1}^l m_i = m$), and μ is an interpolation polynomial of degree $m-1$ defined by the following m equations.

For $i = 1, \cdots, l$

$$\left(r(s) - \frac{\mu(s)h_2(s)}{u(s)}\right)^{(k)} = 0, \quad \text{for} \quad k = 0, \cdots, m_i - 1 \text{ at } s = \sigma.$$

Note that as σ is not a zero of h_2, for each i the m_i equations are solvable and have a unique solution.

Finally, each unstable zero of h_1 of given multiplicity is a zero of the function $(r(s) - (\mu(s)h_2(s)/u(s)))$ with the same multiplicity. So, there exists $\epsilon > 0$ such that $Y(s)$ is analytic in $\{\Re(s) > -\epsilon\}$. Now, clearly $Y(s) = C + Y_0(s)$ where $C \in \mathbb{R}$ and $sY_0(s)$ is bounded on $\{\Re(s) > -\epsilon\}$. So we can deduce from Lemma 3.1 that $Y \in \hat{\mathcal{A}}_-(0)$. Obviously, X and Y satisfy the Bézout equation $-XN + YD = 1$. □

Example 3.1: Let $G(s) = e^{-sT}/(s-\sigma)$ with $\sigma \in \mathbb{R}$ and $\gamma = \sqrt{1+\sigma^2}$ we know that $(e^{-sT}/(s+\gamma), (s-\sigma)/(s+\gamma))$ is a normalized coprime factorization of G over $\hat{\mathcal{A}}$. Corresponding Bézout factors are given by

$$X(s) = e^{T\sigma}(\sigma+\gamma)$$
$$Y(s) = 1 + (\sigma+\gamma)\frac{1-e^{-T(s-\sigma)}}{s-\sigma}.$$

Those factors depend on the delay but is this dependence continuous in the BIBO gap topology as it is the case for the normalized coprime factorizations? This continuity would be useful to establish convergence in the BIBO gap topology for controllers of the system with delays to controllers of the delay free system.

The next result proves more generally that Theorem 3.1 produces Bézout factors which are continuous respectively to variations of the coprime factors in the BIBO gap topology.

Proposition 3.3: Suppose h_1 has no zero on the imaginary axis and let

$$G_\Delta = \left(\frac{h_2^\Delta(s)}{r(s)}, \frac{h_1^\Delta(s)}{r(s)}\right)$$

such that

$$\left(\frac{h_2^\Delta(s)}{r(s)}, \frac{h_1^\Delta(s)}{r(s)}\right) \underset{\Delta\to 0}{\longrightarrow} \left(\frac{h_2(s)}{r(s)}, \frac{h_1(s)}{r(s)}\right)$$

in the BIBO gap topology. Then the Bézout factors X_Δ and Y_Δ given by Theorem 3.1 depend continuously on Δ, in the BIBO gap topology.

Proof: Let $\Psi_\Delta : \mathbb{C}^m \to \mathbb{C}^m$ be the linear mapping defined by

$$\Psi_\Delta(\mu_0, \cdots, \mu_{m-1})(k) = \left(\frac{\mu^\Delta(s)h_2^\Delta(s)}{u(s)}\right)^{(n(k))}(\sigma_k), \quad (k = 1, \cdots, m)$$

where $\mu(s) = \Sigma_{j=0}^{m-1} \mu_j s^j$, $(\sigma_1, \cdots, \sigma_m)$ denotes the unstable zeroes of $h_1(s)$ taken with multiplicity, and $n(k)$ is the kth term of the finite sequence $(0, \cdots, m_1 - 1, 0, \cdots, m_2 - 1, \cdots, m_l - 1)$, where l is the total number of distinct zeroes. The mapping Ψ_Δ is continuous, linear, and one–one and depends continuously on Δ. Thus, it has a continuous inverse. The polynomial μ^Δ which performs the interpolation is given by

$$\mu^\Delta = \Psi_\Delta^{-1}(r(\sigma_1), \cdots, r^{(m_l-1)}(\sigma_m))$$

and this also depends continuously on Δ.

It is now easy to see that the corresponding Bézout factors $X_\Delta = -\mu^\Delta/u$ and $Y_\Delta = (r - \mu^\Delta(h_2^\Delta)/u)/h_1^\Delta$ depend continuously on Δ. □

Remark 3.2: If h_1 has zeroes on the imaginary axis, they may become stable zeroes of h_1^Δ for Δ arbitrarily small. To obtain Bézout factors converging in this case, it is necessary to modify the construction of Theorem 3.1, interpolating at the corresponding stable zeroes of h_1^Δ, as well as the unstable ones.

Example 3.2: Consider $G_T(s) = h_2^T(s)/h_1^T(s)$ with $h_2^T(s) = e^{-sT}h_2(s)$ and $h_1^T(s) = h_1(s)$ where $h_2(s)$ and $h_1(s)$ are polynomials as in Remark 3.1 such that $G_T(s)$ is strictly proper. As a variation in the delay is a variation in the BIBO gap topology, we can deduce from the above proposition that the Bézout factors X_T and Y_T of G_T depend in a continuous fashion on the delay T.

V. Some L^1-Optimization Problems

The problem of optimal tracking is set as an optimization problem of the type

$$\inf_{r\in\hat{\mathcal{A}}} \|f - gr\|_{\hat{\mathcal{A}}}, \qquad \text{where} \quad f, g \in \hat{\mathcal{A}}. \tag{1}$$

We have

$$\inf_{r\in\hat{\mathcal{A}}} \|f - gr\|_{\hat{\mathcal{A}}} = d(f, \overline{(g)})$$

where (g) denotes the principal ideal $\{gr\colon r \in \hat{\mathcal{A}}\}$.

Moreover, $d(f, \overline{(g)}) = 0$ for all f if and only if $(g) = \hat{\mathcal{A}}$, since otherwise (g) is contained in a proper maximal ideal, which is necessarily closed. (We refer to [22] for general background on Banach algebras.) Now, clearly $(g) = \hat{\mathcal{A}}$ if and only if g is invertible in $\hat{\mathcal{A}}$.

Hille and Phillips [14] give the following invertibility condition for elements in $\hat{\mathcal{A}}$.

Let $f \in \hat{\mathcal{A}}$. Then f is invertible in $\hat{\mathcal{A}} \Leftrightarrow \inf_{\{\Re(s)\ge 0\}} |f(s)| > 0$.

In the particular case of commensurate time delays, that is, g is of the form $g = g_a + \Sigma_{i=0}^{\infty} g_i\delta_{t-Ti}$ with $g_a \in L^1$ (and we write $g \in L^1 \oplus \ell^1$), we can give a more explicit condition of invertibility.

Theorem 4.1: Let $g = g_a + \Sigma_{i=0}^{\infty} g_i\delta_{t-Ti}$ with $g_a \in L^1$, $\Sigma_{i=0}^{\infty} |g_i| < \infty$.

Then g is invertible in

$$L^1 \oplus \ell^1 \Leftrightarrow \begin{cases} \hat{g}(s) \neq 0 \quad \text{on} \quad \{\Re(s) \ge 0\} \\ \text{and} \\ \displaystyle\sum_{i=0}^{\infty} g_i z^i \neq 0 \text{ on } \{|z| \le 1\}. \end{cases}$$

Proof: We have that

$$g \text{ is invertible in } L^1 \oplus \ell^1 \Leftrightarrow \Phi(g) \neq 0 \quad \forall \Phi \in \Delta$$

where Δ is the set of all characters on $L^1 \oplus \ell^1$.

Recall that a character Φ is a complex homomorphism $\Phi\colon L^1 \oplus \ell^1 \longrightarrow \mathbb{C}$ with $\Phi(1) = 1$, $\Phi(f+g) = \Phi(f) + \Phi(g)$, $\Phi(f * g) = \Phi(f)\Phi(g)$.

It is well known (cf. [22]) that any character on L^1 that does not vanish there takes the form $\Phi(f) = \hat{f}(s_0)$ for some $s_0 \in \{\Re(s) \ge 0\}$. Then $\Phi(f * \delta_{t-nT}) = (f * \delta_{t-nT})\hat{\ }(s_0) = f(s_0)e^{-s_0 nT}$ so that $\Phi(\delta_{t-nT}) = e^{-s_0 nT}$ and so $\Phi(f) = \hat{f}(s_0)$ for all $f \in L^1 \oplus \ell^1$.

If Φ vanishes on L^1 then it acts as a character on ℓ^1 and thus takes the form $\Phi(\delta_{t-nT}) = w^n$ for some w with $|w| \le 1$.

So finally, $\Delta = \{\delta_\omega, \omega \in \{\Re(s) \ge 0\} \cup \{|z| \le 1\}\}$ where

$$\begin{aligned} &\text{for } \omega \in \{\Re(s) \ge 0\} && \delta_\omega(g) = \hat{g}(\omega) \\ &\text{for } \omega \in \{|z| \le 1\} && \delta_\omega(g_a) = 0 \text{ if } g \in L^1 \\ & && \delta_\omega(\delta_{t-Tn}) = \omega^n \text{ for } n \ge 0. \end{aligned}$$

So we have

$$g \text{ is invertible in } L^1 \oplus \ell^1 \Leftrightarrow \begin{cases} \hat{g}(s) \neq 0 \text{ on } \{\Re(s) \ge 0\} \\ \text{and} \\ \displaystyle\sum_{i=0}^{\infty} g_i z^i \neq 0 \text{ on } \{|z| \le 1\}. \end{cases}$$

Remark 4.1: In the case of the simpler algebra $L^1 + \mathbb{C}\delta$ we have $g_a + g_0\delta$ is invertible in $L^1 + \mathbb{C}\delta$ if and only if $\hat{g}_a(s) + g_0 \neq 0$ on $\{\Re(s) \ge 0\}$ and $g_0 \neq 0$ that is $\hat{g}(s) \neq 0$ on the extended right half plane.

Remark 4.2: In the case where g is not invertible in $L^1 \oplus \ell^1$, the necessary and sufficient condition of Theorem 4.1 being not satisfied, we can be able to give a lower bound for $\inf_{r\in\hat{\mathcal{A}}} \|f - gr\|_{\hat{\mathcal{A}}}$. If there is a character Φ such that $\Phi(g) = 0$, then $\Phi(gr) = 0$ also and so $\Phi(f - gr) = \Phi(f)$. Hence $\|f - gr\|_{\hat{\mathcal{A}}} \ge |\Phi(f - gr)| = |\Phi(f)|$, i.e., $\|f - gr\|_{\hat{\mathcal{A}}} \ge \sup_{\{\Phi\colon \Phi(g)=0\}} |\Phi(f)|$.

We consider now the resolution of equations of the type (1) firstly in the case which motivated our study, the delay integrator, and so consider in the next paragraph delay systems of the type $G(s) = e^{-sT}/(s - \sigma)$. We will consider more general systems in Section IV-B.

A. The Case When $G(s) = e^{-sT}/(s - \sigma)$

Let $G(s) = e^{-sT}/(s - \sigma), T > 0$, $\sigma \in \mathbb{R}$.

A normalized coprime factorization of G is given by

$$N(s) = \frac{e^{-sT}}{s + \gamma}, \qquad D(s) = \frac{s}{s + \gamma}$$

where $\gamma = \sqrt{1 + \sigma^2}$.

The optimal tracking problem is

$$\inf_{U\in\hat{\mathcal{A}}} \|I - NU\|_{\hat{\mathcal{A}}} = \inf_{u\in\mathcal{A}} \|\delta - un\|_{\mathcal{A}} = d(\delta, (e^{-\gamma t} * \delta_{t-T})).$$

By a convenient abuse of notation, we will write $d(I, e^{-sT}/(s + \gamma))$ instead of $d(\delta, (e^{-\gamma t} * \delta_{t-T}))$. This can be easily solved using the finite-dimensional results of Vidyasagar. We recall here two useful lemmas of [26].

Lemma 4.1 [26]: Suppose $k \ge 1$ is an integer, and let $\hat{g}(s) = (1/(s + 1))^k$.

Then $\overline{(g)} = L^1$.

Lemma 4.2 [26]: Suppose $g \in \mathcal{A}$, that $\hat{g}$ is rational, and construct a rational function $\hat{h} \in \hat{\mathcal{A}}$ as follows: g is a multiple of h, every zero of $\hat{g}$ in the open right half-plane is also a zero of $\hat{h}$ with the same multiplicity, and every zero of $\hat{g}$ on the extended $j\omega$ axis is a simple zero of $\hat{h}$. Then $\overline{(g)} = \overline{(h)}$.

Proposition 4.1: Let $N(s) = e^{-sT}/(s+\gamma)$. Then we have $\overline{(N)} = L^1[T,\infty)$ and $\inf_{U\in\hat{\mathcal{A}}} \|I - NU\|_{\hat{\mathcal{A}}} = 1$, the infimum being attained for $U^{\text{opt}} = 0$.

Proof: By Lemmas 4.1 and 4.2, we have $\overline{(1/(s+\gamma))} = L^1$ and obviously $\overline{(e^{-sT}/(s+\gamma))} = L^1[T,\infty)$. Now, $d(\delta, L^1[T,\infty)) = 1$ and $U = 0$ gives $\|I - NU\|_{\hat{\mathcal{A}}} = 1$. □

Those results on the infimum and optimal function are not surprising as tracking in L^∞ is very demanding. So the idea is to consider here a weighted optimization problem of the form $\inf_{U\in\hat{\mathcal{A}}} \|W(I-NU)\|_{\hat{\mathcal{A}}}$ where W is a weight which will help minimize the tracking error particularly at low frequencies rather than high frequencies. We can consider for example $W_1(s) = 1/(s+1)$, $W_2(s) = (s+a)/(s+b)$ with $a > 0$, $b > 0$, or $W_3(s) = 1/(\rho s + 1)$.

Theorem 4.2: For weights W_1, W_2, and W_3, the optimal tracking error of the weighted problem is given by

$$\begin{aligned}\inf_{U\in\hat{\mathcal{A}}} \|W_1(I-NU)\|_{\hat{\mathcal{A}}} &= 1 - e^{-T}\\ \inf_{U\in\hat{\mathcal{A}}} \|W_2(I-NU)\|_{\hat{\mathcal{A}}} &= 1 + |a-b|(1-e^{-bT})\\ \inf_{U\in\hat{\mathcal{A}}} \|W_3(I-NU)\|_{\hat{\mathcal{A}}} &= 1 - e^{-T/\rho}.\end{aligned}$$

Proof: For $i = 1,3$, we have $\inf_{U\in\hat{\mathcal{A}}} \|W_i(I - NU)\|_{\hat{\mathcal{A}}} = d(W_i, (W_iN))$. Applying Lemma 4.2 we get

$$\overline{\left(W_i\frac{1}{s+\gamma}\right)} = L^1 \quad \text{and} \quad \overline{\left(W_i e^{-sT}\ \frac{1}{s+\gamma}\right)} = L^1[T,\infty).$$

Now

$$\begin{aligned}d(W_1,(W_1N)) &= d(e^{-T}, L^1[T,\infty))\\ &= \int_0^T e^{-t}\,dt = 1 - e^{-T}\\ d(W_2,(W_2N)) &= d(\delta + (a-b)e^{-bT}, L^1[T,\infty))\\ &= 1 + |a-b|(1-e^{-bT})\\ d(W_3,(W_3N)) &= d(\delta + (a-b)e^{-bT}, L^1[T,\infty))\\ &= 1 - e^{-T/\rho}.\end{aligned}$$

□

Remark 4.3: As the infinimum is not attained, we cannot define U^{opt}, however it is possible to construct a sequence $(U_\epsilon)_\epsilon$ such that

$$\|W_1(I - NU_\epsilon)\|_{\hat{\mathcal{A}}} \underset{\epsilon\to 0}{\longrightarrow} \inf_{U\in\hat{\mathcal{A}}} \|W_1(I-NU)\|_{\hat{\mathcal{A}}}.$$

For example, the sequence

$$U_\epsilon(s) = e^{-T}\ \frac{s+\gamma}{\epsilon s + 1}$$

will achieve that.

B. A More General Case

In this section, we determine the ideal in $\hat{\mathcal{A}}$ generated by L^1 functions whose transforms are not necessarily rationals.

The case of L^1 functions whose Laplace transform does not vanish on $\{\Re(s) \ge 0\}$ has been considered by Nymen [19] (a simpler and more accessible proof of this result can be found in [9]).

Theorem 4.3 [9]: Let $f \in L^1$. Then $\overline{(f)} = L^1$ if and only if $\alpha(f) = 0$ and $\hat{f}(s) \neq 0$ in $\{\Re(s) \ge 0\}$.

Now, we want to consider the more general case where f has a finite number of zeroes in $\{\Re(s) \ge 0\}$ to be able to cover for example the case of delay systems of the type $G(s) = e^{-sT}R(s)$ where $R(s)$ is a rational function having zeroes in $\{\Re(s) \ge 0\}$. The next theorem shows that the result of Nymen extends naturally to the case of zeroes in $\{\Re(s) > 0\}$.

Theorem 4.4: Suppose $g \in L^1$, $\alpha(g) = T$, and $\hat{g}$ has a finite number of zeroes a_i, $i = 1,\cdots,l$, in $\{\Re(s) > 0\}$ and no zero on the imaginary axis. Then $\overline{(g)} = \{k \in L^1[0,\infty), \hat{k}(a_i) = 0, i = 1,\cdots,l, \alpha(k) \ge T\}$.

Proof: We prove this result in the case where $T = 0$ and there is only one zero in $\{\Re(s) > 0\}$; the extension to the case of several zeroes is straightforward by induction. The case $T > 0$ is immediately deduced.

Let $g \in L^1$, $\alpha(g) = 0$, and $\hat{g}(a_0) = 0$ for a_0 in $\{\Re(s) > 0\}$; otherwise $\hat{g}(s) \neq 0$ on $\{\Re(s) \ge 0\}$.

Now, let $\hat{g}_1(s) = \hat{g}(s)/(s - a_0)$, and we show first that $g_1 \in L^1[0,\infty)$.

For we can write $g_1 = g * \epsilon$ where

$$\epsilon(t) = \begin{cases} e^{a_0 t} \text{ on } (-\infty, 0] \\ 0 \text{ on } (0,\infty).\end{cases}$$

Clearly, $g \in L^1(-\infty,\infty)$ as $\|g_1\|_{L^1(-\infty,\infty)} \le \|g\|_{L^1[0,\infty)}\|\epsilon\|_{L^1(-\infty,0)}$.

Now, we have

$$g_1(t) = \int_{-\infty}^{\infty} \epsilon(t-s)g(s)\,ds = \int_0^{\infty} \epsilon(t-s)g(s)\,ds.$$

For

$$t \le 0, g_1(t) = \int_0^\infty e^{a_0(t-s)}g(s)\,ds = \hat{g}(a_0) = 0$$

so g_1 is in $L^1[0,\infty)$.

Now, let us prove that $\{k \in L^1, \hat{k}(a_0) = 0, \alpha(k) \ge 0\} \subset \overline{(g)}$ (the other inclusion is obvious).

Let $k \in L^1$ with $k(a_0) = 0$ and $k_1(s) = k(s)/(s - a_0)$. We have

$$k\frac{s+a_0}{s-a_0} = k\left(1 - \frac{2a_0}{s-a_0}\right) = k - \frac{2a_0 k}{s-a_0}.$$

So $k(s + a_0)/(s - a_0) \in L^1[0,\infty)$ and has no zero in $\{\Re(s) \ge 0\}$. From Theorem 4.3 we know there exists a sequence $(\phi_n)_{n=1}^\infty$ in $\hat{\mathcal{A}}$ such that

$$k\frac{s+a_0}{s-a_0} = \lim_{n\to\infty} g\frac{s+a_0}{s-a_0}\phi_n$$

that is $k = \lim_{n\to\infty} g\phi_n$ and $k \in \overline{(g)}$. □

The case of L^1 functions which have zeroes on the imaginary axis is difficult unless those functions are of the type $e^{-sT}R(s)$ with R rational in which case we can determine $\overline{(g)}$ using the finite-dimensional results of Vidyasagar.

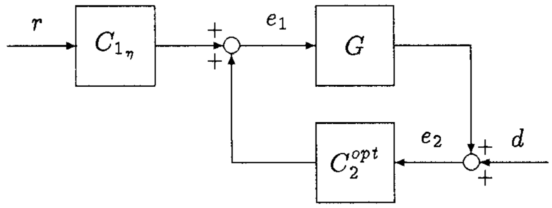

Fig. 4. Simplified two-parameter compensator scheme.

VI. Algorithm and Example

For delay systems which have a coprime factorization (N, D) over $\hat{\mathcal{A}}$ with $N \in L^1$ and N has only a finite number of zeroes in $\{\Re(s) > 0\}$, the robust control problem set in Section III can be solved using the following algorithm.

Step 1: Find $\{U_\epsilon\}_{\epsilon\geq 0}$ such that $\|W(I - NU_\epsilon)\|_{\hat{\mathcal{A}}} \underset{\epsilon\to 0}{\longrightarrow} \inf_{U\in\hat{\mathcal{A}}} \|W(I - NU)\|_{\hat{\mathcal{A}}}$.

Step 2: Find $(G_n)_{n\geq 0}$ a sequence of finite-dimensional transfer function such that $G_n \underset{n\to\infty}{\longrightarrow} G$ in the BIBO gap topology.

Step 3: Find Q_n^{opt} such that

$$\left\| \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} D_n \\ N_n \end{pmatrix} Q_n^{\text{opt}} \right\|_\infty = \inf_{Q_n\in H_\infty} \left\| \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} D_n \\ N_n \end{pmatrix} Q_n \right\|_\infty.$$

Choose ϵ small, define $C_1^\epsilon = U^\epsilon/(Y + NQ^{\text{opt}})$, and construct the controllers

$$C_{1_n}^\epsilon = \frac{U^\epsilon}{Y_n + N_n Q_n^{\text{opt}}} \quad \text{and} \quad C_{2_n}^0 = \frac{X_n + D_n Q_n^{\text{opt}}}{Y_n + N_n Q_n^{\text{opt}}}.$$

$C_{2_n}^0$ is a finite-dimensional controller, and $C_{1_n}^\epsilon$ can be an infinite-dimensional controller.

Proposition 5.1: If the greatest singular value of $D^*X + N^*Y$ is of multiplicity one, then:

1) $b^{\text{opt}}(G_n) \underset{n\to\infty}{\longrightarrow} b^{\text{opt}}(G)$;
2) $C_{1_n}^\epsilon \underset{n\to\infty}{\longrightarrow} C_1^\epsilon$ in the BIBO gap topology;
3) $C_{2_n}^0 \underset{n\to\infty}{\longrightarrow} C_2^{\text{opt}}$ in the BIBO gap topology.

For systems of the type $G(s) = e^{-sT}R(s)$ we might have a more direct implementation. If for $\eta \in \mathbb{N}$, η big, Q_η^{opt} is such that $Y_\eta + N_\eta Q_\eta^{\text{opt}}$ is invertible in $\hat{\mathcal{A}}$, then $C_{1_\eta}^0$ is stable and we can use the two-parameter compensator scheme of Fig. 4 where for C_2^{opt} we can take the closed form formula of the optimal robust controller determined by Dym *et al.* [10] so that we get in this scheme an optimal robust controller and suboptimal tracking controller.

We illustrate this by continuing our analysis of Example 3.1.

Example 5.1: Let $G(s) = e^{-sT}/(s - \sigma)$ with $\sigma \in \mathbb{R}$, $\gamma = \sqrt{1 + \sigma^2}$ and $W(s) = 1/(s + 1)$.

As pointed out before, a normalized coprime factorization (N, D) of G is given by

$$N(s) = \frac{e^{-sT}}{s + \gamma} \quad \text{and} \quad D(s) = \frac{s - \sigma}{s + \gamma}.$$

- *Step 1:* Find $\{U_\epsilon\}_{\epsilon\geq 0}$ such that
$$\left\| \frac{1}{s+1}\left(I - \frac{e^{-sT}}{s+\gamma}U_\epsilon\right)\right\|_{\hat{\mathcal{A}}} \underset{\epsilon\to\infty}{\longrightarrow} \inf_{U\in\hat{\mathcal{A}}} \left\| \frac{1}{s+1}\left(I - \frac{e^{-sT}}{s+\gamma}U\right)\right\|_{\hat{\mathcal{A}}}.$$
This is done in Remark 4.3 which gives $U_\epsilon(s) = e^{-T}(s + \gamma)/(\epsilon s + 1)$.
- *Step 2:* Find $(G_n)_{n\geq 0}$ a sequence of finite-dimensional transfer functions such that $G_n \underset{n\to\infty}{\longrightarrow} (e^{-sT}/(s - \sigma))$ in the BIBO gap topology.

Note that such a system as $e^{-sT}/(s + \gamma)$ is badly approximated by rational functions (in many other examples one can achieve faster convergence) and the best rate for approximation in $\hat{\mathcal{A}}$-norm known so far is $O(\log\ n/n)$ (see [12]).

From [12] and the recent work of Partington and Mäkilä [21] on approximation of such systems in $\hat{\mathcal{A}}$, we get that

$$N_n(s) = \left(\frac{1 - \dfrac{sT}{2n}}{1 + \dfrac{sT}{2n}}\right)^n \frac{1}{(s + \gamma)\left(1 + \dfrac{s}{n}\right)}$$

and

$$D_n(s) = D(s)$$

are approximations of N and D in $\hat{\mathcal{A}}$ with error of order $O(n^{-2/3})$. These approximations are suboptimal, but easy to calculate.

- *Step 3:* With
$$N_n(s) = \left(\frac{1 - \dfrac{sT}{2n}}{1 + \dfrac{sT}{2n}}\right)^n \frac{1}{(s + \gamma)\left(1 + \dfrac{s}{n}\right)}$$
$$D_n(s) = D(s)$$
the Bézout factors X_n and Y_n satisfying $-N_nX_n + D_nY_n = 1$ can be determined using standard algorithms (available in Matlab or Scilab).

The finite-dimensional H_∞-optimization which follows is also standard (see [27], available in Matlab or Scilab).

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