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UPPER BOUND ON THE CHARACTERS OF THE SYMMETRIC GROUPS FOR BALANCED YOUNG DIAGRAMS AND A GENERALIZED FROBENIUS FORMULA

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ABSTRACT. We study asymptotics of an irreducible representation of the symmetric group S_n corresponding to a balanced Young diagram λ (a Young diagram with at most $C\sqrt{n}$ rows and columns for some fixed constant C) in the limit as n tends to infinity. We show that there exists a constant D (which depends only on C) with a property that

$$|\chi^\lambda(\pi)| = \left| \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\operatorname{Tr} \rho^\lambda(e)} \right| \leq \left(\frac{D \max(1, \frac{|\pi|^2}{n})}{\sqrt{n}} \right)^{|\pi|},$$

where $|\pi|$ denotes the length of a permutation (the minimal number of factors necessary to write π as a product of transpositions). Our main tool is an analogue of the Frobenius character formula which holds true not only for cycles but for arbitrary permutations.

1. INTRODUCTION

1.1. **Asymptotics of characters of symmetric groups.** The character of an irreducible representation corresponding to a Young diagram λ can theoretically be calculated with the help of the Murnaghan–Nakayama rule. Unfortunately, this combinatorial algorithm quickly becomes intractable when the number of boxes of λ tends to infinity. For this reason, one is in need of more analytic (possibly approximate) methods which would give meaningful answers asymptotically.

In this article we will concentrate on a scaling in which the number of rows and columns of a Young diagram λ having n boxes is bounded from above by $C\sqrt{n}$ for some fixed constant C. Such Young diagrams will be called *balanced*. The investigation of this limit was initiated by Vershik and Kerov [VK77] and by Logan and Shepp [LS77], and was continued by Kerov [Ker93, Ker99]. It was further developed by Biane [Bia98, Bia01], who proved that

$$|\chi^\lambda(\pi)| = O(n^{-\frac{|\pi|}{2}}) \tag{1}$$

holds true asymptotically for any fixed permutation π in the limit $n \to \infty$ for balanced Young diagrams λ. Here, by $|\pi|$ we denote the minimal number of factors necessary to write π as a product of transpositions. The quantity χ^λ appearing in (1) is defined as

$$\chi^\lambda(\pi) = \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\operatorname{Tr} \rho^\lambda(e)},$$

where $\rho^\lambda(\pi)$ is the representation of the symmetric group indexed by λ and evaluated at the conjugacy class indexed by π and, as usual, $\rho^\lambda(e)$ is the degree of the character ρ^λ. Since this normalized character will appear repeatedly in this article, for simplicity we will refer to it as the *character* and the name *normalized character* will be reserved for the quantity which will be introduced in (12). It should be stressed that the estimate (1) was proved by Biane only asymptotically for $n \gg |\pi|$. Biane, in fact, described the asymptotics of (1) very precisely in terms of *free cumulants*, which can be regarded as certain functionals of the shape of a Young diagram.

The main result of this article is that the inequality (1) holds true for $|\pi| = O(\sqrt{n})$, and for larger values of $|\pi|$ we still get some meaningful estimates. We state it formally in the following theorem and give the proof in Section 4.

Theorem 1 (Main result)**.** *For every $C > 0$ there exists a constant D with the following property. If λ is a Young diagram with n boxes which has at most $C\sqrt{n}$ rows and columns and $\pi \in S_n$ is a permutation then*

$$|\chi^\lambda(\pi)| < \left(\frac{D \max(1, \frac{|\pi|^2}{n})}{\sqrt{n}} \right)^{|\pi|}. \tag{2}$$

It should be stressed that for permutations π such that $|\pi| = o(n^{3/4})$, the inequality (2) is significantly stronger than the best previous estimate for the values of characters on long permutations, given in [Roi96]. There, the author shows that

$$|\chi^\lambda(\pi)| < q^{|\pi|}$$

holds true for balanced Young diagrams, where $0 < q < 1$ is a constant.

In this introduction, we present the context in which (2) appeared, its applications, and the main tools used in the proof.

1.2. **Generalized Frobenius character formula.** Our strategy in proving estimate (2) is to use a generalized version of Frobenius' formula for characters, which relates the value of a normalized character evaluated at some Young diagram with the residue at infinity of a certain series described by the shape of that Young diagram. The original Frobenius formula gives the values of the characters only on cycles, and in Theorem 5 we show its generalization to arbitrary permutations. This result is interesting in its own right and this article seems to be the first in the literature to contain such an expression.

1.3. **Expansion of characters in terms of Boolean cumulants.** For asymptotic problems, it is convenient to encode the shape of a Young diagram in terms of its *shifted Boolean cumulants*. The generalized Frobenius formula introduced in this article allows us to express normalized characters as polynomials in shifted Boolean cumulants by expanding all power series involved. It seems that inequality (2) could be proved directly in this way by crude estimation of each of the summands, but here we concentrate on a more elegant approach which we present in the following.

In Theorem 7 we will show that all coefficients in this expression of characters in terms of shifted Boolean cumulants are non-negative integers.

1.4. **Bound for characters.** Let Young diagrams λ, μ be given with the shifted Boolean cumulants $\tilde{B}^\lambda_2, \tilde{B}^\lambda_3, \dots$ and $\tilde{B}^\mu_2, \tilde{B}^\mu_3, \dots$ respectively. Suppose that

$$|\tilde{B}^\lambda_i| \leq \tilde{B}^\mu_i \tag{3}$$

holds true for any $i \geq 2$. The positivity of the coefficients in the expansion of characters in terms of shifted Boolean cumulants implies that

$$|\chi^\lambda(\pi)| \leq \chi^\mu(\pi) \tag{4}$$

holds true for any permutation π. Inequality (4) shows that in order to prove (2) it is enough to prove it for Young diagrams μ with sufficiently big shifted Boolean cumulants and we have some freedom of choosing μ for which the calculation would take a particularly nice form. In this article as the reference Young diagram μ we take a rectangular Young diagram $p \times q$, since there are very simple formulae of Stanley for their characters [Sta04].

Unfortunately, we encounter some difficulty when trying to follow the above method, namely for every Young diagram μ with n boxes the second shifted Boolean cumulant is given by $\tilde{B}^\mu_2 = -n < 0$, and hence the inequalities (3) cannot by fulfilled. A solution to this problem is to take as μ a rectangular Young diagram $p \times q$ with $p < 0$ and $q > 0$. Clearly, μ does not make any sense as a Young diagram, nevertheless we show that for such an object the Stanley character formula still holds true.

We finish the proof of (2) with the help of Stanley's character formulae.

1.5. **Application: asymptotics of Kronecker tensor products and quantum computations.** Since this section is not directly connected to the rest of the article we allow ourselves to be less formal in the following.

The results of this article were motivated by recent work of Moore and Russell [MR06]. They explore quantum algorithms for solving the graph isomorphism problem, which are analogous to Kuperberg's algorithm for the dihedral group. Namely, their algorithm uses repeated adaptive tensoring of irreducible representations of the symmetric group S_n. The probability of success of the algorithm in each iteration depends on the solution to the following problem.

Problem 2. *Let λ, μ be balanced Young diagrams, each having n boxes. Can we find some upper bound for the (relative) multiplicities in the decomposition of the Kronecker tensor product $[\lambda] \otimes [\mu]$ into irreducible components in the limit $n \to \infty$? In particular, how far is this tensor product away (in some suitable distance) from the left-regular representation (Plancherel measure)?*

Moore and Russell conjectured the following stronger version of the estimate (2) and proved that this conjecture would give an upper bound for the multiplicities in Problem 2, implying that the expected time to completion of the considered quantum algorithm for the graph isomorphism problem would not be better than the time to completion of the best known classical algorithms.

Conjecture 3. *For every $C > 0$ there exists a constant D with the following property. If λ is a Young diagram with n boxes which has at most $C\sqrt{n}$ rows and columns and $\pi \in S_n$ is a permutation then*

$$|\chi^\lambda(\pi)| < \left(\frac{D}{\sqrt{n}}\right)^{|\pi|}. \tag{5}$$

In a subsequent paper [MRŚ07], we show how inequality (2) (despite being weaker than Conjecture 3) is sufficient to fill the gap in the work of Moore and Russell [MR06].

1.6. **Application: random walks on symmetric groups.**

Diaconis and Shahshahani [DS81] initiated the use of representation theory in the study of random walks on groups. A typical problem studied in this context is the speed of convergence of the convolution powers of a given measure towards the uniform distribution. This method was used with success in numerous publications (for a review article see [Dia96]).

Our improved upper bounds for the characters have an immediate application in this context in the case of a random walk on the symmetric group with steps given by random long permutations; the details will be presented in a forthcoming paper [ŚU06].

1.7. **Open problems.**

1.7.1. *Optimality of the bound.* The main result of this article, inequality (2), opens many questions. One problem is to determine how optimal is this bound. The exact formulae of Stanley [Sta04] show that for rectangular Young diagrams and $|\pi| \leq O(\sqrt{n})$ the right-hand side of (2) is also a lower bound (with a different value of D) and, therefore, this estimate cannot be significantly improved. A possibility of improvement for $|\pi| \geq O(\sqrt{n})$ remains open, nevertheless some partial results suggest that the estimates as strong as Conjecture 3 should not be true for general balanced Young diagrams and $|\pi| = O(n)$.

1.7.2. *Free cumulants and Kerov polynomials.* In this article we consider the expansion of the normalized characters in terms of shifted Boolean cumulants, while in the asymptotic theory of representations it is much more common [Bia98, Bia01, Śni06a, Śni06b] to consider analogous expansions in terms of *free cumulants*. Such expansions of characters in terms of free cumulants are called *Kerov polynomials*. The main reason for the popularity of free cumulants is that the expansions of characters in terms of free cumulants have a much simpler structure compared to expansions in terms of Boolean cumulants. For this reason it would be very interesting to find the analogues of the equalities and estimates presented in this article in which the shifted Boolean cumulants would be replaced by free cumulants.

For many Young diagrams appearing in the asymptotic theory of representations, we may find much better estimates on free cumulants than on Boolean cumulants. For example, a typical Young diagram contributing to the Plancherel measure or to a tensor product of balanced Young diagrams has very small free cumulants (except for the second one), while its Boolean cumulants cannot be bounded better than for a general balanced Young diagram. It seems plausible that the analogues of (2) obtained by analysis of free cumulants expansions of characters might give much better estimates for such Young diagrams with small free cumulants. In particular, if the shapes of the Young diagrams μ and λ are very close to the shape of a typical Young diagram [LS77, VK77], then it should be possible to show that their tensor product $[\mu] \otimes [\lambda]$ in Problem 2 is indeed very close to the Plancherel measure.

For the method of the proof presented in this article to work with free cumulants, we would need to show that the coefficients of Kerov polynomials are non-negative integers. This statement is known as *Kerov's conjecture*. Unfortunately, despite many results in this direction [Bia98, Śni06a, GR07, Bia05], Kerov's conjecture remains open until today.

It should be stressed that Kerov polynomials arose to compute the value of Σ_k as defined in (12), i.e. the normalized character evaluated on a single cycle. For the purposes of this article, such characters are not sufficient and we need to generalize the definition of Kerov polynomials to cover the expansions for general permutations. The problem of positivity of more general Kerov polynomials has not been studied before. In particular, it seems that in order to have positive coefficients in generalized Kerov polynomials when expanded in terms of free cumulants, one should replace the characters by some kind of cumulants, such as the ones considered in [Śni06b]. We expect that the generalized Frobenius formula presented in this article will shed some light into this topic.

In Section 5 we introduce some of the main ideas concerning generalized Kerov's polynomials and give some examples.

1.7.3. *Combinatorial interpretation.* The positivity of coefficients in the expansion of characters into shifted Boolean cumulants and the (conjectured) positivity of coefficients of Kerov polynomials are rather unexpected and immediately raise questions about some natural combinatorial interpretation to these coefficients. Such interpretations were hinted by Biane [Bia03] but until now there are no concrete results in this direction.

1.7.4. *Approximate factorization of characters.* Biane [Bia01] proved *approximate factorization property* for characters of irreducible representations which can be informally stated as

$$\chi^\lambda(\pi_1\pi_2) \approx \chi^\lambda(\pi_1)\chi^\lambda(\pi_2)$$

for permutations π_1, π_2 with disjoint supports. Using the notation introduced in [Śni06b], this property can be formulated formally as the special case $l = 2$ of more general asymptotic estimates on cumulants, given by

$$k_l(\pi_1, \dots, \pi_l) = O(n^{\frac{-|\pi_1|-\cdots-|\pi_l|-2(l-1)}{2}}). \tag{6}$$

The formula (6) holds for balanced Young diagrams λ, and $|\pi_1|, \dots, |\pi_l| \ll n$. Since Biane's estimate (1) holds true in a more general case of $|\pi| = O(\sqrt{n})$ (Theorem 1), it raises the question if estimates (6) holds true for $|\pi_1|, \dots, |\pi_l| \leq O(\sqrt{n})$.

Biane [Bia98] and Śniady [Śni06b] found not only the asymptotic bound of the right-hand side (1) and (6) for short lengths of the permutations $\pi, \pi_1, \dots, \pi_l$ but also computed explicitly the leading order terms which turn out to be relatively simple functions of free cumulants. It would be very interesting to check if these formulae for leading order terms hold true without assumptions on $|\pi|, |\pi_1|, \dots, |\pi_l|$ (our conjecture is that the answer for this question is negative) and how these leading terms change in the general case.

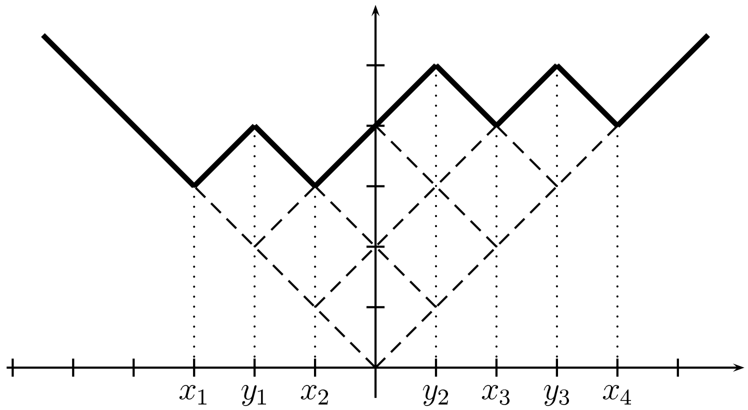

FIGURE 1. Young diagram $(4,3,1)$ drawn according to the Russian convention; the ragged bold line represents the profile of a Young diagram. In this case $(x_1,\dots,x_4) = (-3,-1,2,4)$ and $(y_1,\dots,y_3) = (-2,1,3)$.

1.7.5. *Asymptotics of cumulants.* It seems plausible that the generalized Frobenius formula presented in this article can be used to simplify the calculation of some of the cumulants in the paper [Śni06b].

2. CONTINUOUS YOUNG DIAGRAMS AND BOOLEAN CUMULANTS

Figure 1 presents our preferred convention for drawing Young diagrams which is—as opposed to French and English conventions—sometimes referred to as the *Russian convention*. According to this convention a Young diagram is represented by its *profile* which is a piecewise affine function on the real line with the sequence of local minima $x_1 < \cdots < x_s$ and the sequence of local maxima $y_1 < \cdots < y_{s-1}$. The modern approach to representation theory of symmetric groups [OV96] suggests that the Russian convention is the most natural one. The use of the Russian convention originates in the papers of Kerov [Ker93, Ker98].

To a Young diagram λ we associate its *Cauchy transform* $G(z)$ given by

$$G(z) = \frac{(z-y_1)\cdots(z-y_{s-1})}{(z-x_1)\cdots(z-x_s)}.$$

In the following we represent a Young diagram with n boxes as a tuple $(\lambda_1,\dots,\lambda_m)$ of integers such that $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$, in this case we can also write

$$G(z) = \frac{(z-\lambda_1+1)(z-\lambda_2+2)\cdots(z-\lambda_m+m)}{(z-\lambda_1)(z-\lambda_2+1)\cdots(z-\lambda_m+m-1)(z+m)}.$$

In this article we will be concerned mainly with the reciprocal of the Cauchy transform

$$H(z) = \frac{1}{G(z)}. \tag{7}$$

For any fixed constant ζ, we consider the power series expansion of H aroung ζ in decreasing powers of z:

$$H(z+\zeta) = \frac{1}{G(z+\zeta)} = z + \zeta + \tilde{B}_1 + \tilde{B}_2 z^{-1} + \tilde{B}_3 z^{-2} + \cdots .$$

In fact, one can show that $x_1 + \cdots + x_s = y_1 + \cdots + y_{s-1}$, and therefore $\tilde{B}_1 = 0$ and

$$H(z+\zeta) = \frac{1}{G(z+\zeta)} = z + \zeta + \tilde{B}_2 z^{-1} + \tilde{B}_3 z^{-2} + \cdots . \tag{8}$$

The coefficients $\tilde{B}_2, \tilde{B}_3, \dots$ will be called *shifted Boolean cumulants*. This name is coined after *Boolean cumulants* introduced by Speicher and Woroudi [SW97] as coefficients in the expansion of H around 0:

$$H(z) = \frac{1}{G(z)} = z - B_2 z^{-1} - B_3 z^{-2} - \cdots .$$

It follows that if $\zeta = 0$ shifted Boolean cumulants coincide (up to sign) with the Boolean cumulants of Speicher and Woroudi. We will call these cumulants *twisted* Boolean cumulants. That is, if

$$H(z) = \frac{1}{G(z)} = z + \hat{B}_2 z^{-1} + \hat{B}_3 z^{-2} - \cdots , \tag{9}$$

then $\hat{B}_i$ are called twisted Boolean cumulants, and $\hat{B}_i = -B_i$. Hereafter, we will always assume that $\zeta \geq 0$.

Lemma 4. *For any Young diagram whose number of rows and columns are both smaller than A, the corresponding shifted Boolean cumulant satisfies*

$$|\tilde{B}_k| \leq \frac{[2(A+\zeta)]^k}{2}$$

for each integer $k \geq 2$.

Proof. Since the sequences of local minima and maxima interlace $x_1 < y_1 < x_2 < \cdots < y_{s-1} < x_s$, the function G has positive residues in all of its poles, and it is the Cauchy transform of a certain probability measure ν called *transition measure* of the Young diagram λ:

$$G(z) = \int_{\mathbb{R}} \frac{1}{z-x} d\nu(x).$$

The Cauchy transform can be expanded around ζ as a power series in decreasing powers of z, which gives

$$G(z+\zeta) = z^{-1} + M_1 z^{-2} + M_2 z^{-3} + M_3 z^{-4} + \cdots .$$

The coefficients M_k are the moments of the shifted transition measure

$$M_k = \int (x-\zeta)^k d\nu(x).$$

Since the support of the transition measure ν is equal to $\{x_1, \dots, x_s\} \subset [-A, A]$, we have

$$|M_k| \leq (A+\zeta)^k. \tag{10}$$

We observe that maximal possible absolute value of the shifted Boolean cumulants for the sequence (M_k) fulfilling (10) is obtained for

$$H(z+\zeta) = \frac{1}{z^{-1} - (A+\zeta)z^{-2} - (A+\zeta)^2 z^{-3} - (A+\zeta)^3 z^{-4} - \cdots} \tag{11}$$

and a direct calculation of the power series expansion of the right hand side of (11) finishes the proof. □

The constant in the above lemma is not optimal (like all constants in the following), since in this article we prefer the simplicity of arguments over the optimality of the constants.

3. Normalized characters and generalized Frobenius formula

If $k_1, \dots, k_l$ are positive integers such that $k_1 + \cdots + k_l = n$, we identify $(k_1, \dots, k_l)$ with some permutation in S_n with the corresponding cycle decomposition.

Let λ be a fixed Young diagram with n boxes. For integers $k_1, \dots, k_l \geq 1$ we define the *normalized character of a conjugacy class*

$$\Sigma_{k_1,\dots,k_l} = \frac{\operatorname{Tr} \rho^\lambda(k_1, \dots, k_l, 1^{n-k_1-\cdots-k_l})}{\operatorname{Tr} \rho^\lambda(e)} (n)_{k_1+\cdots+k_l}, \tag{12}$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ denotes the falling factorial. We use the convention that $\Sigma_{k_1,\dots,k_l} = 0$ whenever $k_1 + \cdots + k_l > n$. Normalized characters of conjugacy classes arise naturally in the study of asymptotics of representations of symmetric groups [IK99]. The following theorem gives a method of computing them.

Theorem 5 (Generalized Frobenius formula)**.** *For any integers $k_1, \dots, k_l \geq 1$*

$$(13)\quad (-1)^l k_1 \cdots k_l \Sigma_{k_1,\dots,k_l} = [z_1^{-1}] \cdots [z_l^{-1}] \Bigg[\left(\prod_{1 \leq r \leq l} H(z_r) H(z_r - 1) \cdots H(z_r - k_r + 1) \right) \prod_{1 \leq s < t \leq l} \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} \Bigg].$$

The right-hand side of (13) *should be understood as follows: we expand the expression appearing there as a power series in decreasing powers of z_l with coefficients being functions of $z_1, \dots, z_{l-1}$ and select the appropriate coefficient. We repeat this procedure with respect to z_{l-1}, z_{l-2},$\dots$, z_1.*

Before presenting the proof, we will introduce some notation. For sequences of non-negative integers $\mu = (\mu_1, \dots, \mu_m)$ and $k = (k_1, \dots, k_l)$ let $\tilde{\chi}_k^\mu$ denote the coefficient of $x^\mu = x_1^{\mu_1} \cdots x_m^{\mu_m}$ in

$$\left(\sum_{1 \leq i \leq m} x_i^{k_1} \right) \cdots \left(\sum_{1 \leq i \leq m} x_i^{k_l} \right) \sum_{w \in S_m} (-1)^w x^{w\delta},$$

where $\delta = (m-1, m-2, \dots, 1, 0)$. As seen in [Mac95], if $\lambda = (\lambda_1, \dots, \lambda_m)$ is a Young diagram with n boxes, $\mu = \lambda + \delta$ and $k_1 + \cdots + k_l = n$, then the value of the unnormalized character is

$$\operatorname{Tr} \rho^\lambda(k) = \tilde{\chi}_k^\mu.$$

If $\mu = (\mu_1, \dots, \mu_m)$ is a sequence of integers which are not all non-negative, we define $\tilde{\chi}_k^\mu = 0$.

For $\mu = (\mu_1, \dots, \mu_m)$, we consider the corresponding function

$$\tilde{H}_\mu(z) = \frac{(z - \mu_1 - 1)(z - \mu_2 - 1) \cdots (z - \mu_m - 1) z}{(z - \mu_1)(z - \mu_2) \cdots (z - \mu_m)}.$$

In other words, if $\mu = \lambda + \delta$, then

$$\tilde{H}_\mu(z + m) = H(z),$$

where H denotes the reciprocal (7) of the Cauchy transform of the Young diagram λ.

It becomes clear that Theorem 5 will follow from Lemma 6 below.

Lemma 6. *For any tuple $\mu = (\mu_1, \dots, \mu_m)$ of integers and a non-negative integer n such that $\mu_1 + \dots + \mu_m = n + \binom{m}{2}$ and $k_1, \dots, k_l \geq 1$, we have*

$$(-1)^l \, (n)_{k_1+\dots+k_l} \, k_1 \cdots k_l \, \tilde{\chi}^{\mu}_{(k_1,\dots,k_l,1^{n-k_1-\dots-k_l})} =$$

$$\tilde{\chi}^{\mu}_{(1^n)} \, [z_1^{-1}] \cdots [z_l^{-1}] \Bigg[\left(\prod_{1 \leq r \leq l} \tilde{H}_{\mu}(z_r) \tilde{H}_{\mu}(z_r - 1) \cdots \tilde{H}_{\mu}(z_r - k_r + 1) \right)$$

$$\prod_{1 \leq s < t \leq l} \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} \Bigg].$$

Proof. It is enough to prove the lemma under assumption that $\mu = (\mu_1, \dots, \mu_m)$ is a sequence of non-negative integers; otherwise both sides of the equality are trivially equal to zero.

We will use induction with respect to l. The case $l = 0$ holds trivially.

For $l \geq 1$ we clearly have

$$\tilde{\chi}^{\mu}_{(k_1,\dots,k_l,1^{n-k_1-\dots-k_l})} = \sum_i \tilde{\chi}^{\mu^{(i)}}_{\left(k_1,\dots,k_{l-1},1^{(n-k_l)-k_1-\dots-k_{l-1}}\right)},$$

where $\mu^{(i)} = (\mu_1, \dots, \mu_{i-1}, \mu_i - k_l, \mu_{i+1}, \dots, \mu_m)$. Therefore, the induction hypothesis gives

$$(-1)^l (n)_{k_1+\dots+k_l} k_1 \cdots k_l \tilde{\chi}^{\mu}_{(k_1,\dots,k_l,1^{n-k_1-\dots-k_l})} = (-k_l)(n)_{k_l} \sum_i \tilde{\chi}^{\mu^{(i)}}_{(1^{n-k_l})} \times$$

$$[z_1^{-1}] \cdots [z_{l-1}^{-1}] \Bigg[\left(\prod_{1 \leq r \leq l-1} \tilde{H}_{\mu^{(i)}}(z_r) \tilde{H}_{\mu^{(i)}}(z_r - 1) \cdots \tilde{H}_{\mu^{(i)}}(z_r - k_r + 1) \right)$$

$$\prod_{1 \leq s < t \leq l-1} \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} \Bigg].$$

Since

$$\tilde{H}_{\mu^{(i)}}(z) = \tilde{H}_{\mu}(z) \frac{(z - \mu_i)(z - \mu_i + k_l - 1)}{(z - \mu_i - 1)(z - \mu_i + k_l)}$$

and ([Mac95], page 118)

$$(n)_{k_l} \tilde{\chi}^{\mu^{(i)}}_{(1^{n-k_l})} = \tilde{\chi}^{\mu}_{(1^n)} (\mu_i)_{k_l} \prod_{j \neq i} \frac{\mu_i - \mu_j - k_l}{\mu_i - \mu_j},$$

we have

$$(-1)^l (n)_{k_1+\cdots+k_l} k_1 \cdots k_l \tilde{\chi}^{\mu}_{(k_1,\dots,k_l,1^{n-k_1-\cdots-k_l})} = (-k_l)\tilde{\chi}^{\mu}_{(1^n)} \times \tag{14}$$

$$[z_1^{-1}] \cdots [z_{l-1}^{-1}] \sum_i \Bigg[(\mu_i)_{k_l} \left(\prod_{j \neq i} \frac{\mu_i - \mu_j - k_l}{\mu_i - \mu_j} \right)$$

$$\Bigg(\prod_{1 \leq r \leq l-1} \tilde{H}_\mu(z_r) \tilde{H}_\mu(z_r - 1) \cdots \tilde{H}_\mu(z_r - k_r + 1)$$

$$\frac{(z_r - \mu_i)(z_r - \mu_i + k_l - k_r)}{(z_r - k_r - \mu_i)(z_r - \mu_i + k_l)} \Bigg) \prod_{1 \leq s < t \leq l-1} \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} \Bigg].$$

It is easy to check that each summand of (14) is equal to the residue for $z_l = \mu_i$ of the function

$$\tilde{\chi}^{\mu}_{(1^n)} [z_1^{-1}] \cdots [z_{l-1}^{-1}] \Bigg[\left((z_l)_{k_l} \prod_j \frac{z_l - \mu_j - k_l}{z_l - \mu_j} \right) \tag{15}$$

$$\Bigg(\prod_{1 \leq r \leq l-1} \tilde{H}_\mu(z_r) \tilde{H}_\mu(z_r - 1) \cdots \tilde{H}_\mu(z_r - k_r + 1)$$

$$\frac{(z_r - z_l)(z_r - z_l + k_l - k_r)}{(z_r - k_r - z_l)(z_r - z_l + k_l)} \Bigg) \prod_{1 \leq s < t \leq l-1} \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} \Bigg] =$$

$$\tilde{\chi}^{\mu}_{(1^n)} [z_1^{-1}] \cdots [z_{l-1}^{-1}] \Bigg[\left(\prod_{1 \leq r \leq l} \tilde{H}_\mu(z_r) \tilde{H}_\mu(z_r - 1) \cdots \tilde{H}_\mu(z_r - k_r + 1) \right)$$

$$\prod_{1 \leq s < t \leq l} \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} \Bigg].$$

Notice that apart from these residues with respect to z_l, for each value of $r \in \{1, \dots, l-1\}$ the series (15) also has residues for $z_l = z_r - k_r$ and $z_l = z_r + k_l$. The residues, however, at these two points sum to zero, completing the proof. □

Theorem 7. *Suppose that positive integers $k_1, \dots, k_l$ are such that*

$$k_1, \dots, k_l \leq \zeta. \tag{16}$$

Then the normalized character $(-1)^l \Sigma_{k_1,\dots,k_l}$ is a polynomial in shifted Boolean cumulants $\tilde{B}_2, \tilde{B}_3, \dots$ with non-negative coefficients.

Proof. From (8) it follows that the expansion of

$$H(z_r) H(z_r - 1) \cdots H(z_r - k_r + 1)$$

as a power series in falling powers of z_r and shifted Boolean cumulants has all non-negative coefficients.

Without loss of generality we may assume that $k_1 \leq \cdots \leq k_l$. It follows that the factors contributing to (13) can be written as

$$\frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} = 1 + \frac{k_s k_t}{k_s + k_t} \sum_{i \geq 0} \frac{(z_s + k_t)^i - (z_s - k_s)^i}{z_t^{1+i}}$$

and are, therefore, power series in descending powers of z_t, the coefficients of which are polynomials in z_s with non-negative coefficients. □

It is an interesting question if the condition (16) is really necessary; in particular if the above theorem holds true for $\zeta = 0$, when the shifted Boolean cumulants coincide with the twisted Boolean cumulants introduced in (9). Indeed, the following examples are the generalized Frobenius formula expanded in terms of twisted Boolean cumulants.

$$\begin{aligned} (-1)^2 \Sigma_{3,2} &= \hat{B}_4 \hat{B}_3 + 13\, \hat{B}_2 \hat{B}_3 + \hat{B}_2^2 \hat{B}_3 + 6\, \hat{B}_5 + 18\, \hat{B}_3 \\ (-1)^2 \Sigma_{4,2} &= \hat{B}_5 \hat{B}_3 + 32\, \hat{B}_2 + 80\, \hat{B}_4 + 8\, \hat{B}_6 + 17\, \hat{B}_3^2 + 40\, \hat{B}_2^2 \\ &\qquad + 3\, \hat{B}_2 \hat{B}_3^2 + 8\, \hat{B}_2^3 + 24\, \hat{B}_2 \hat{B}_4 \end{aligned}$$

We conjecture that these expressions are always positive.

Conjecture 8. *The normalized character $\Sigma_{k_1,\dots,k_l}$ can be expressed as a polynomial in twisted Boolean cumulants $\hat{B}_i$ with coefficients being non-negative integers.*

4. Asymptotics of characters

In the case of a rectangular Young diagram $\lambda = p \times q$ (i.e. p parts, all equal to q), we have

$$H(z) = \frac{(z-q)(z+p)}{z+p-q}, \tag{17}$$

which corresponds to the sequence of shifted Boolean cumulants

$$\tilde{B}_i = -pq(q-p-\zeta)^{i-2} \qquad \text{for } i \geq 2. \tag{18}$$

Theorem 9 (Stanley [Sta04])**.** *Let $\pi \in S_{k_1+\cdots+k_l}$ be a permutation of type $(k_1, \dots, k_l)$ and let $\lambda = p \times q$ be a rectangular Young diagram. Then the corresponding normalized character is given by*

$$\Sigma^{p\times q}_{k_1,\dots,k_l} = (-1)^{k_1+\cdots+k_l} \sum_{\sigma_1 \sigma_2 = \pi} p^{\kappa(\sigma_1)} (-q)^{\kappa(\sigma_2)}, \tag{19}$$

where the sum runs over all permutations $\sigma_1, \sigma_2 \in S_{k_1+\cdots+k_l}$ such that $\sigma_1\sigma_2 = \pi$ and $\kappa(\sigma)$ denotes the number of cycles of permutation σ.

The following lemma was proved by Féray (private communication).

Lemma 10. *Let $\pi, \sigma_1, \sigma_2 \in S_n$ be such that $\pi = \sigma_1\sigma_2$. There exist permutations $\sigma_1', \sigma_2' \in S_n$ such that $\pi = \sigma_1'\sigma_2'$, $|\sigma_1'| + |\sigma_2'| = |\sigma_1| + |\sigma_2|$, $|\sigma_2'| = |\sigma_2'\sigma_2^{-1}| + |\sigma_2|$ and every cycle of σ_1' is contained in some cycle of σ_2'. Furthermore, $\kappa(\sigma_2') \leq \kappa(\pi)$.*

Proof. If every cycle of σ_1 is contained in some cycle of σ_2 then $\sigma_1' = \sigma_1$ and $\sigma_2' = \sigma_2$ have the required property. Otherwise, there exist $a, b \in \{1, \dots, n\}$ such that a and b belong to the same cycle of σ_1 but not the same cycle of σ_2. We define $\sigma_1' = \sigma_1(a, b)$, $\sigma_2' = (a, b)\sigma_2$ and we iterate this procedure if necessary. Notice that $|\sigma_1'| = |\sigma_1| - 1$, $|\sigma_2'| = |\sigma_2| + 1$, so this procedure will finish after a finite number of steps. It remains to prove that $|\sigma_2'| \geq |\sigma_2'\sigma_2^{-1}| + |\sigma_2|$ (the opposite inequality follows from the triangle inequality); notice that $|\sigma_2'| - |\sigma_2|$ is equal to k (where k is the number of steps after which the procedure has terminated), and $\sigma_2'\sigma_2^{-1}$ is a product of k transpositions, implying that $|\sigma_2'\sigma_2^{-1}| \leq k$.

For any $a \in \{1, \dots, n\}$ the elements $\sigma_2'(a)$ and $\sigma_1'(\sigma_2'(a)) = \pi(a)$ belong to the same cycle of σ_1' and, hence, to the same cycle of σ_2'. It follows that the elements a and $\pi(a)$ belong to the same cycle of σ_2', and therefore every cycle of π is contained in some cycle of σ_2'. The result now follows. □

Lemma 11. *For any integers $n \geq 1$ and $i \geq 0$ and for any $\pi \in S_n$, we have*

$$\#\{\sigma \in S_n : |\sigma| = i\} \leq \frac{n^{2i}}{i!}. \tag{20}$$

Proof. Since every permutation in S_n appears exactly once in the product

$$[1 + (12)][1 + (13) + (23)] \cdots [1 + (1n) + \cdots + (n-1, n)],$$

from which it follows

$$\sum_i x^i \, \#\{\sigma \in S_n : |\sigma| = i\} = (1 + x)(1 + 2x) \cdots (1 + (n-1)x).$$

Each of the coefficients of x^k on the right-hand side is bounded from above by the corresponding coefficient of $e^x e^{2x} \cdots e^{(n-1)x} = e^{\frac{n(n-1)x}{2}}$, finishing the proof. □

Theorem 12. *For any Young diagram λ with at most A rows and columns and for any integers $k_1, \dots, k_l \geq 1$, we have*

$$|\Sigma_{k_1,\dots,k_l}| < \begin{cases} (16e^2 A)^{K+l} & \text{for } K \leq 8A, \\ (4eK)^K (4A)^l & \text{for } K \geq 8A, \end{cases}$$

where $K = k_1 + \cdots + k_l$.

Proof. Before beginning the proof, notice that the Murnaghan-Nakayama rule implies that we can assume $k_1, \dots, k_l \leq 2A$, since otherwise $\Sigma_{k_1,\dots,k_l} = 0$ holds trivially. By setting $\zeta = 2A$, the assumptions of Theorem 7 are satisfied.

On a purely formal level $\Sigma_{k_1,\dots,k_l}$ is a polynomial in the indeterminates $\tilde{B}_2, \tilde{B}_3, \dots$, and hence it is well-defined for shifted Boolean cumulants given by (18). This allows us to define $\Sigma^{p\times q}_{k_1,\dots,k_l}$ as a polynomial in p and q.

From Lemma 4 it follows that

$$|\tilde{B}^{\lambda}_i| \leq \tilde{B}^{p\times q}_i, \tag{21}$$

where $\tilde{B}^{\lambda}_i$ denotes the shifted Boolean cumulant of λ and $\tilde{B}^{p\times q}_i$ is the shifted Boolean cumulant given by (18) for

$$p = -4A, \qquad q = 4A. \tag{22}$$

Theorem 7 together with (21) imply that

$$|\Sigma^{\lambda}_{k_1,\dots,k_l}| \leq |\Sigma^{p\times q}_{k_1,\dots,k_l}|.$$

We regard $\Sigma^{p\times q}_{k_1,\dots,k_l}$ as a polynomial in indeterminates p, q and we know that (19) holds true whenever p, q are positive integers; it follows that (19) holds as equality between polynomials in p, q. In particular, (19) holds true for (22) and

$$|\Sigma^{\lambda}_{k_1,\dots,k_l}| \leq \sum_{\sigma_1\sigma_2=\pi} (4A)^{\kappa(\sigma_1)+\kappa(\sigma_2)}.$$

Here π has cycle type $(k_1, \dots, k_l)$.

We consider a map which to a pair (σ_1, σ_2) associates any pair (σ'_1, σ'_2) as prescribed by Lemma 10. For any fixed σ'_2 the permutations σ_2 such that $|\sigma'_2| = |\sigma'_2\sigma_2^{-1}| + |\sigma_2|$ can be identified with non-crossing partitions of the cycles of σ'_2 (see [Bia97, Section 1.3]). It follows that the number of such permutations σ_2 is equal to the product of appropriate Catalan numbers and, hence, this product is bounded from above by 4^K, where $K = k_1 + \cdots + k_l$. Therefore

$$\sum_{\substack{\sigma_1,\sigma_2\in S_K,\\ \sigma_1\sigma_2=\pi}} (4A)^{\kappa(\sigma_1)+\kappa(\sigma_2)} \leq 4^K \sum_{\substack{\sigma'_1,\sigma'_2\in S_K,\\ \sigma'_1\sigma'_2=\pi,\\ \kappa(\sigma'_2)\leq\kappa(\pi)}} (4A)^{\kappa(\sigma'_1)+\kappa(\sigma'_2)} \leq \tag{23}$$

$$4^K \sum_{\sigma'_1\in S_K} (4A)^{\kappa(\sigma'_1)+l} \leq 4^K \sum_{0\leq i\leq K-1} \frac{K^{2i}}{i!} (4A)^{K-i+l},$$

where the last inequality follows from Lemma 11.

If $K \leq 8A$ we use the estimate

$$4^K \sum_{0 \leq i \leq K-1} \frac{K^{2i}}{i!} (4A)^{K-i+l} < 4^K (4A)^{K+l} \exp\left(\frac{K^2}{4A}\right) \leq (16e^2 A)^{K+l}.$$

If $K \geq 8A$ then in the sum on the right-hand-side of (23) the quotient of consecutive summands is greater than 2. We can, therefore, bound the sum by the sum of an appropriate geometric series and

$$4^K \sum_{0 \leq i \leq K-1} \frac{K^{2i}}{i!} (4A)^{K-i+l} < 4^K \frac{K^{2K}}{K!} (4A)^l < (4eK)^K (4A)^l,$$

where the second last inequality follows from $K! \geq \left(\frac{K}{e}\right)^K$. □

Theorem 13. *Let λ be a Young diagram with n boxes and at most $C\sqrt{n}$ rows and columns. Then*

$$|\chi^\lambda(\pi)| \leq \left(\frac{\max\left[(16e^3C)^3, (32e^2C)^2 \frac{|\pi|^2}{n}\right]}{\sqrt{n}} \right)^{|\pi|}. \tag{24}$$

Proof. Let $k_1, \dots, k_l, 1^{n-K}$ be the cycle decomposition of π with $k_1, \dots, k_l \geq 2$ and $K = k_1 + \cdots + k_l$. Notice that $|\pi| = K - l$, $K + l \leq 3|\pi|$, and $K \leq 2|\pi|$.

The inequality $(n)_K \geq \left(\frac{n}{e}\right)^K$ shows that for $K \leq 8C\sqrt{n}$

$$|\chi^\lambda(\pi)| = \frac{|\Sigma_{k_1,\dots,k_l}|}{(n)_K} \leq \frac{(16e^3C)^{K+l}}{(\sqrt{n})^{K-l}} \leq \left(\frac{(16e^3C)^3}{\sqrt{n}}\right)^{|\pi|} \tag{25}$$

and for $K \geq 8C\sqrt{n}$

$$|\chi^\lambda(\pi)| = \frac{|\Sigma_{k_1,\dots,k_l}|}{(n)_K} \leq \left(\frac{16e^2CK}{\sqrt{n}}\right)^K \frac{1}{(\sqrt{n})^{|\pi|}} \leq \left(\frac{(16e^2CK)^2}{n^{3/2}}\right)^{|\pi|} < \left(\frac{(32e^2C|\pi|)^2}{n^{3/2}}\right)^{|\pi|}. \tag{26}$$

□

We are now ready to prove our main theorem.

Proof of Theorem 1. The inequality (24) shows that Theorem 1 holds true for $D = \max\left[(16e^3C)^3, (32e^2C)^2\right]$. □

The following result will be useful in the study of some quantum algorithms [MRŚ07].

Corollary 14. *For every C there exist constants $A > 0$ and B such that if a Young diagram λ with n boxes has at most $C\sqrt{n}$ boxes in each row and column then*

$$\sum_{\substack{\pi \in S_n,\\ |\pi| \leq An^{4/7}}} |\chi^\lambda(\pi)|^4 \leq B. \tag{27}$$

Proof. We shall use the notation introduced in the proof of Theorem 13. By (25) the contribution of the permutations π with support size K less than or equal to $8C\sqrt{n}$ is bounded from above by

$$\sum_{i \geq 0} \frac{n^{2i}}{i!} \left(\frac{(16e^3C)^3}{\sqrt{n}} \right)^{4i} < \exp\left[(16e^3C)^{12} \right].$$

By (26) the contribution of the permutations π with support size K greater than or equal to $8C\sqrt{n}$ is bounded from above by

$$\sum_{0 \leq i \leq An^{4/7}} \frac{n^{2i}}{i!} \left(\frac{(32e^2Ci)^2}{n^{3/2}} \right)^{4i} < \sum_{0 \leq i \leq An^{4/7}} \frac{n^{2i}e^i}{i^i} \left(\frac{(32e^2Ci)^2}{n^{3/2}} \right)^{4i} =$$
$$\sum_{0 \leq i \leq An^{4/7}} \left(\frac{e(32e^2C)^8 i^7}{n^4} \right)^i < 1$$

for a suitably chosen constant $A > 0$. □

5. Generalized Kerov Polynomials

Theorem 5 expresses the normalized character $\Sigma_{k_1,\dots,k_l}$ in terms of the boolean cumulants B_i. It is, however, desirable to express normalized characters in terms of free cumulants, as discussed in Section 1.7.2. We call the expressions of normalized characters in terms of free cumulants *generalized Kerov polynomials*. We use a change of variables so as to allow ourselves to use the Lagrange Inversion Theorem (see [GJ04, Theorem 1.2.4]). Set $\phi(z) = zH(z^{-1})$. Then $\phi(z) = 1 - \sum_{i \geq 1} B_i z^i$. If $K(z) = z^{-1} + \sum_{k \geq 1} R_k z^{k-1}$, where the R_k are free cumulants, then by definition

$$K(z) = (1/H(z))^{\langle -1 \rangle} \tag{28}$$

where $\langle -1 \rangle$ denotes compositional inverse. Set $R(z) = zK(z)$. One can show easily that (28) implies

$$\left(\frac{z}{R(z)} \right)^{\langle -1 \rangle} = \frac{z}{\phi(z)},$$

and therefore it follows by Lagrange Inversion that

$$R_{k+1} = -\frac{1}{k}[z^{k+1}]\ \phi(z)^k \tag{29}$$

and

$$B_{k+1} = \frac{1}{k}[z^{k+1}]\ R(z)^k.$$

With this notation, Theorem 5 becomes

$$(-1)^l k_1 k_2 \cdots k_l \Sigma_{k_1,k_2,\dots,k_l} = [z_1^{k_1+1}]\ \cdots [z_l^{k_l+1}]$$
$$\prod_{r=1}^{l} \left((1-z_r)\cdots(1-(k-1)z_r)\phi(z_r)\cdots\phi\left(\frac{z_r}{1-(k_r-1)z_r}\right)\right)$$
$$\prod_{1\le s<t\le l} \frac{(z_t-z_s)(z_t-z_s+(k_t-k_s)z_s z_t)}{(z_t-z_s-k_t z_s z_t)(z_t-z_s+k_s z_s z_t)}.$$

At this point it may be useful to give some examples. In [Bia03, Page 2], one can find example of the original Kerov polynomials. The main conjecture concerning the original Kerov polynomials is that they are positive in free cumulants. Some progress has been made concerning this conjecture in [Bia03, GR07, Śni06a], but the conjecture remains open. The following are two examples of generalized Kerov polynomials.

$$\begin{aligned} \Sigma_{3,2} &= R_3R_4 - 5R_2R_3 - 6R_5 - 18R_3 \\ \Sigma_{2,2,2} &= R_3^3 - 12R_3R_4 + 58R_3R_2 + 40R_5 + 80R_3 - 6R_3R_2^2 \end{aligned} \tag{30}$$

We use a grading on generalized Kerov polynomials like the one used in [GR07]. Namely, we consider the new series

$$\begin{aligned} \Sigma_{k_1,k_2,\dots,k_l;2n} &= [u^{\left(\sum_i k_i + l\right)-2n}]\Sigma_{k_1,k_2,\dots,k_l}(u) \\ &= [u^{\left(\sum_i k_i + l\right)-2n}]\Sigma_{k_1,k_2,\dots,k_l}(R_2u^2, R^3u^3,\dots). \end{aligned}$$

That is, if the *weight* of a monomial $R_{i_1}^{j_1}\cdots R_{i_t}^{j_t}$ is given by $\sum_s i_s j_s$, then $\Sigma_{k_1,k_2,\dots,k_l;2n}$ consists of the terms of weight $\sum_i k_i + l - 2n$ in $\Sigma_{k_1,k_2,\dots,k_l}$. Define

$$\Phi(x,u) = \sum_{i\ge 0}\Phi_i(x)u^i = (1-ux)\phi\left(\frac{x}{1-ux}\right). \tag{31}$$

Proposition 15. *The following equations hold.*

a. For $k_1, k_2, \dots, k_l \geq 1$,

$$(-1)^l k_1 \cdots k_l \Sigma_{k_1,\dots,k_l} = [z_1^{k_1+1}] \cdots [z_l^{k_l+1}] \prod_{r=1}^{l} \prod_{j=0}^{k_r-1} \Phi(z_r, j) \prod_{1 \leq s < t \leq l} \frac{(z_t - z_s)(z_t - z_s + (k_t - k_s) z_s z_t)}{(z_t - z_s - k_s z_s z_t)(z_t - z_s + k_t z_s z_t)}.$$

b. For $k_1, k_2, \dots, k_l \geq 1$,

$$(-1)^l k_1 \cdots k_l \Sigma_{k_1,\dots,k_l;2n} = [u^{2n}][z_1^{k_1+1}] \cdots [z_l^{k_l+1}] \prod_{r=1}^{l} \prod_{j=0}^{k_r-1} \Phi(z_r, ju) \prod_{1 \leq s < t \leq l} \frac{(z_t - z_s)(z_t - z_s + (k_t - k_s) z_s z_t u)}{(z_t - z_s - k_s z_s z_t u)(z_t - z_s + k_t z_s z_t u)}.$$

Proof. The first part of the proposition is a trivial substitution, and the proof of the second part is similar to the proof of [GR07, Proposition 4.1]. □

The following proposition also appears in [Śni06a, Theorem 4.9], and is obtained with ease below.

Proposition 16. *There is only one term of highest weight in* $\Sigma_{k_1,\dots,k_l}$ *and it is* $R_{k_1} \cdots R_{k_l}$*, i.e.*

$$\Sigma_{k_1,\dots,k_l;0} = R_{k_1} \cdots R_{k_l}.$$

Proof. Note that the constant term in Proposition 15.b can be obtained by substituting $u = 0$ into the equation. Doing so we obtain

$$(-1)^l k_1 \cdots k_l \Sigma_{k_1,\dots,k_l;0} = [z_1^{k_1+1}] \cdots [z_l^{k_l+1}] \prod_{r=1}^{l} \prod_{j=0}^{k_r-1} \Phi(z_r, 0).$$

From the definition of $\Phi(x, u)$ given in (31), we see that $\Phi(x, 0) = \phi(x)$. Therefore, we have

$$(-1)^l k_1 \cdots k_l \Sigma_{k_1,\dots,k_l;0} = [z_1^{k_1+1}] \cdots [z_l^{k_l+1}] \prod_{r=1}^{l} \phi(z_r)^{k_r}.$$

The result now follows from (29). □

We focus on a special case in what follows.

5.1. **Special case:** $l = 2$**.** Specializing Proposition 15.a to the case of two indeterminates, we obtain the formula

$$\Sigma_{r,s} = \Sigma_r\Sigma_s - [x^{r+1}][y^{s+1}]\prod_{j=0}^{r-1}\Phi(x,j)\prod_{j=0}^{s-1}\Phi(y,j) \cdot \frac{(xy)^2}{(y-x-rxy)(x-y-sxy)},$$

and from Proposition 15.b it follows

$$\Sigma_{r,s}(u) = \Sigma_r(u)\Sigma_s(u) - [x^{r+1}][y^{s+1}]\prod_{j=0}^{r-1}\Phi(x,ju)\prod_{j=0}^{s-1}\Phi(y,ju) \cdot \frac{(xy)^2u^2}{(y-x-rxyu)(x-y-sxyu)}.$$

From numerical evidence, we have the following conjecture concerning the second term on the right hand side of the previous equation.

Conjecture 17. *The following is positive in free cumulants.*

$$[x^{r+1}][y^{s+1}]\prod_{j=0}^{r-1}\Phi(x,j)\prod_{j=0}^{s-1}\Phi(y,j)\frac{(xy)^2}{(y-x-rxy)(x-y-sxy)}$$

As noted earlier positivity of the original Kerov polynomials is still open and it appears that the positivity in Conjecture 17 is, likewise, difficult to prove (in fact, the expression in Conjecture 17 is more complicated). Below we give an explicit form for the case $l = 2$. We also assume that both parts are the same; that is, if our two parts are k_1 and k_2, then $k_1 = k_2$. We are not, however, able to show positivity from our explicit expression.

Before giving our explicit expression, we prove a lemma. For any ring R, the ring $R[[\lambda]]_1$ is the ring of power series with invertible constant term.

Lemma 18. *Suppose that $\phi(\lambda) \in R[[\lambda]]_1$, $\omega = t\phi(\omega)$ and $F(\lambda) = \lambda^{-k}G(\lambda)$ with $G(\lambda) \in R[[\lambda]]_1$. Then,*

$$\sum_{n\geq -k} a_n t^n = \frac{F(\omega)}{1-t\phi'(\omega)}, \qquad \text{where } a_n = [\lambda^n]F(\lambda)\phi^n(\lambda).$$

Proof. Set $L(\lambda) = \frac{G(\lambda)}{\phi^k(\lambda)}$. Since $\phi(\lambda), G(\lambda) \in R[[\lambda]]_1$, the series $L(\lambda)$ is a formal power series. Thus, we have by [GJ04, Theorem 1.2.4, Part 2]

$$\sum_{n\geq 0} c_n t^n = \frac{L(\omega)}{1-t\phi'(\omega)}, \qquad \text{where } c_n = [\lambda^n]L(\lambda)\phi^n(\lambda).$$

However, we have

$$\begin{aligned}\sum_{n\geq 0} c_n t^n &= \frac{L(\omega)}{1-t\phi'(\omega)} \\ &= \frac{\omega^k F(\omega)}{\phi^k(\omega)\left(1-t\phi'(\omega)\right)} \\ &= t^k \frac{F(\omega)}{1-t\phi'(\omega)}.\end{aligned}$$

Thus,

$$\sum_{n\geq 0} c_n t^{n-k} = \frac{F(\omega)}{1-t\phi'(\omega)}$$

where

$$\begin{aligned}c_n &= [\lambda^n]L(\lambda)\phi^n(\lambda) \\ &= [\lambda^{n-k}]F(\lambda)\phi^{n-k}(\lambda)\end{aligned}$$

for $n \geq 0$. Setting for $n \geq -k$

$$a_n = c_{n+k} = [\lambda^n]F(\lambda)\phi^n(\lambda)$$

gives the result. □

We now give an explicit form for the terms of weight $k+2-2n$ in the expression in Conjecture 17, which is given by

$$[u^{2n}][x^{r+1}][y^{r+1}]\prod_{j=0}^{r-1}\Phi(x,ju)\prod_{j=0}^{r-1}\Phi(y,ju)\frac{(xy)^2u^2}{((y-x)^2-(rxyu)^2)}. \tag{32}$$

Setting

$$a(x,y) = \frac{1}{r^2}\frac{(rxyu)^2}{(y-x)^2}$$

and noting that

$$\begin{aligned}\prod_{j=0}^{r-1}\Phi(x,ju) &= \prod_{j=0}^{r-1}\left(\phi(x)+\sum_{i\geq 1}\Phi_i(x)(ju)^i\right) \\ &= \sum_{\lambda}\hat{m}_\lambda\Phi_\lambda(x)\phi^{r-\ell(\lambda)}(x)u^{|\lambda|}\end{aligned}$$

where $\hat{m}_\lambda$ is the monomial symmetric function with the substitution $x_i = i$ for $i \leq r-1$ and $x_i = 0$ for $i \geq r$ and $\Phi_\lambda(x) = \prod_i \Phi_{\lambda_i}(x)$ for $\lambda =$

$\lambda_1, \lambda_2, \dots$, we expand (32) in power series in u to obtain

$$[u^{2n}][x^{r+1}][y^{r+1}]\prod_{j=0}^{r-1}\Phi(x,ju)\prod_{j=0}^{r-1}\Phi(y,ju)\frac{1}{r^2}\frac{a(x,y)u^2}{(1-a(x,y)u^2)}$$
$$= [u^{2n}][x^{r+1}][y^{r+1}]\sum_{\lambda_1,\lambda_2}\hat{m}_{\lambda_1}\hat{m}_{\lambda_2}\Phi_{\lambda_1}(x)\Phi_{\lambda_2}(y)\phi^{r-\ell(\lambda_1)}(x)\phi^{r-\ell(\lambda_2)}(y)$$
$$\cdot u^{|\lambda_1|}u^{|\lambda_2|}\frac{1}{r^2}\sum_{i\geq 1}\left(a(x,y)u^2\right)^i$$
$$= [x^{r+1}][y^{r+1}]\sum_{i\geq 1}\frac{1}{r^2}a(x,y)^i$$
$$\cdot \sum_{\substack{\lambda_1,\lambda_2\\ |\lambda_1|+|\lambda_2|=2n-2i}}\hat{m}_{\lambda_1}\hat{m}_{\lambda_2}\Phi_{\lambda_1}(x)\Phi_{\lambda_2}(y)\frac{\phi^{r+1}(x)}{\phi^{\ell(\lambda_1)+1}(x)}\frac{\phi^{r+2}(y)}{\phi^{\ell(\lambda_2)+1}(y)}$$
$$= \frac{1}{r^2}[x^{r+1}][y^{r+1}]\sum_{i\geq 1}a(w_1,w_2)^i$$

(33)

$$\cdot \sum_{\substack{\lambda_1,\lambda_2\\ |\lambda_1|+|\lambda_2|=2n-2i}}\hat{m}_{\lambda_1}\hat{m}_{\lambda_2}\frac{\Phi_{\lambda_1}(w_1)\Phi_{\lambda_2}(w_2)}{C(x)C(y)\phi^{\ell(\lambda_1)}(w_1)\phi^{\ell(\lambda_2)}(w_2)}$$

where the second last equality follows from two applications of Lemma 18 and where

$$C(x) = \frac{1}{1-\sum_{i\geq 2}(i-1)R_i x^i}$$

and

$$w_1 = \frac{x}{R(x)}, \qquad w_2 = \frac{y}{R(y)}.$$

In (33) we have an explicit expression for the terms of weight $k+2-2n$ in $\Sigma_{r,r}$. For $n=1$, that is for the terms of weight k in $\Sigma_{r,r}$, we have

$$\frac{1}{r^2}[x^{r+1}][y^{r+1}]\frac{a(w_1,w_2)}{C(x)C(y)}$$

(34)

$$= \frac{1}{r^2}[x^{r+1}][y^{r+1}]\frac{\left(1-\sum_{i\geq 2}(i-1)R_i x^i\right)\left(1-\sum_{i\geq 2}(i-1)R_i y^i\right)x^2y^2}{\left(xR(y)-yR(x)\right)^2}.$$

For the original Kerov polynomials, some considerable amount of work was needed to obtain positivity for terms of weight $k-1$ (see [Śni06a] and [GR07]). We see that Conjecture 17 seems at least as difficult.

5.2. **Linear Terms.** Note that Conjecture 17 can be expressed as $(-1)^{l+1}k^{\mathrm{id}}(\Sigma_{i_1},\ldots,\Sigma_{i_l})$ is positive in free cumulants, where $k^{\mathrm{id}}(\Sigma_{i_1},\ldots,\Sigma_{i_l})$ is the l^{th} cumulant (see [Śni06b]). For example, for $\mu = (r,s)$, we have

$$(-1)k^{\mathrm{id}}(\Sigma_r,\Sigma_s) = \Sigma_r\Sigma_s - \Sigma_{r,s}$$

and for $\mu = (r,s,t)$, we have

$$k^{\mathrm{id}}(\Sigma_r,\Sigma_s,\Sigma_t) = \Sigma_{r,s,t} - \Sigma_r\Sigma_{s,t} - \Sigma_s\Sigma_{r,t} - \Sigma_t\Sigma_{r,s} + 2\,\Sigma_r\Sigma_s\Sigma_t.$$

Note that all the linear terms are expressed in $k^{\mathrm{id}}(\Sigma_{i_1},\ldots,\Sigma_{i_l})$. That is, note that in the expansion

$$\begin{aligned}\Sigma_{r,s,t} &= \Sigma_r\Sigma_{s,t} + \Sigma_s\Sigma_{r,t} + \Sigma_t\Sigma_{r,s} - 2\,\Sigma_r\Sigma_s\Sigma_t \\ &\quad + [x^{r+1}]\,[y^{s+1}]\,[z^{t+1}]\,\prod_{j=0}^{r-1}\Phi(x,j)\prod_{j=0}^{s-1}\Phi(y,j)\prod_{j=0}^{t-1}\Phi(z,j) \\ &\quad \cdot \frac{rs(xy)^2}{(y-x-rxy)(y-x+sxy)}\cdot\frac{rt(xz)^2}{(z-x-rxz)(z-x+txz)} \\ &\quad \cdot \frac{st(yz)^2}{(z-y-syz)(z-y+tzy)}\end{aligned}$$

that the linear terms of $\Sigma_{r,s,t}$ do not occur in the expression $\Sigma_r\Sigma_{s,t}+\Sigma_s\Sigma_{r,t}+\Sigma_t\Sigma_{r,s} - 2\,\Sigma_r\Sigma_s\Sigma_t$. We can prove that the linear terms in $(-1)^{l+1}k^{\mathrm{id}}(\Sigma_{i_1},\ldots,\Sigma_{i_l})$ have positive coefficients.

Theorem 19. *Suppose that $\mu = (k_1,k_2,\ldots,k_l) \vdash k \le n$ and let ω_μ be a fixed element in the conjugacy class μ in S_k. Then the coefficient of R_{b+1} in $(-1)^{l+1}k^{\mathrm{id}}(\Sigma_{i_1},\ldots,\Sigma_{i_l})$ is the number of k-cycles c such that $c\cdot\omega_\mu$ has b cycles.*

Proof. For the rectangular shape $p\times q$, the series $H(z)$ is given (17). From [Sta04, Theorem 1 and Proposition 2] and [Rat07, Proposition 4.1] the free cumulants for this shape are given by

$$R_{b+1} = (-1)^b \sum_{\substack{u,v\in S_b\\ u\cdot v=(b)\\ \kappa(u)+\kappa(v)=b+1}} p^{\kappa(u)}(-q)^{\kappa(v)} \tag{35}$$

where $(b) = (1\ 2\ \cdots b)$ and $\kappa(u)$ is the number of cycles in u. However, [Sta04, Theorem 1] states

$$(-1)^k\Sigma^{p\times q}_{k_1,\ldots,k_l} = \sum_{\substack{u,v\in S_k\\ u\cdot v=\omega_\mu}} p^{\kappa(u)}(-q)^{\kappa(v)}. \tag{36}$$

Expanding $\Sigma_{k_1,\ldots,k_l}$ in terms of the free cumulants given in (35) should give us the same equation as (36).

The coefficient of $p(-q)^b$ in (36) is clearly the number of k-cycles c such that $c \cdot \omega_\mu$ has b cycles. The coefficient of $p(-q)^b$ in (35), however, is clearly the number of b-cycles c such that $c \cdot (b)$ has b cycles. But the latter number is clearly 1. Since R_{b+1} accounts for all occurrences of $p(-q)^b$ in $(-1)^{l+1} k^{\mathrm{id}}(\Sigma_{i_1}, \dots, \Sigma_{i_l})$, comparing coefficients give us the result. □

We remark this theorem implies that that if $b \geq k$, then the coefficient of R_{b+1} in $(-1)^{l+1} k^{\mathrm{id}}(\Sigma_{i_1}, \dots, \Sigma_{i_l})$ is zero. Also note that when l is even (odd), the coefficient of R_{b+1} is negative (positive) in $\Sigma_{k_1,\dots,k_l}$. Finally, we see that this last theorem also implies that the sum of the coefficients of all linear terms of $\Sigma_{k_1,\dots,k_l}$ must be the number of long cycles in $S_{k_1+\cdots+k_l}$, namely $(k_1 + \cdots + k_l - 1)!$. One can compare these observations to the normalized characters given in (30).

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