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Bifurcations of periodic orbits with spatio-temporal symmetries

A M Rucklidge†§ and M Silber‡‖

† Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, UK

‡ Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, IL 60208, USA

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Abstract. Motivated by recent analytical and numerical work on two- and three-dimensional convection with imposed spatial periodicity, we analyse three examples of bifurcations from a continuous group orbit of spatio-temporally symmetric periodic solutions of partial differential equations. Our approach is based on centre manifold reduction for maps, and is in the spirit of earlier work by Iooss (1986) on bifurcations of group orbits of spatially symmetric equilibria. Two examples, two-dimensional PWs and three-dimensional alternating pulsating waves (APW), have discrete spatio-temporal symmetries characterized by the cyclic groups Z_n, $n = 2$ (PW) and $n = 4$ (APW). These symmetries force the Poincaré return map $\mathcal{M}$ to be the nth iterate of a map $\tilde{\mathcal{G}}$: $\mathcal{M} = \tilde{\mathcal{G}}^n$. The group orbits of PW and APW are generated by translations in the horizontal directions and correspond to a circle and a two-torus, respectively. An instability of PWs can lead to solutions that drift along the group orbit, while bifurcations with Floquet multiplier (FM) $+1$ of APWs do not lead to drifting solutions. The third example we consider, alternating rolls, has the spatio-temporal symmetry of APWs as well as being invariant under reflections in two vertical planes. This leads to the possibility of a doubling of the marginal FM and of bifurcation to two distinct types of drifting solutions. We conclude by proposing a systematic way of analysing steady-state bifurcations of periodic orbits with discrete spatio-temporal symmetries, based on applying the equivariant branching lemma to the irreducible representations of the spatio-temporal symmetry group of the periodic orbit, and on the normal form results of Lamb J S W (1996 Local bifurcations in k-symmetric dynamical systems *Nonlinearity* **9** 537–57). This general approach is relevant to other pattern formation problems, and contributes to our understanding of the transition from ordered to disordered behaviour in pattern-forming systems.

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1. Introduction

Techniques for analysing symmetry-breaking bifurcations of Γ-invariant equilibria of Γ-equivariant differential equations are well developed in the case of compact Lie groups Γ (Golubitsky *et al* 1988). The motivation for developing these methods comes, in large

§ E-mail: A.M.Rucklidge@damtp.cam.ac.uk
‖ E-mail: silber@nimbus.esam.nwu.edu

part, from problems of pattern formation in fluid dynamics (see, for example, Crawford and Knobloch 1991). In the simplest cases, the symmetry-breaking bifurcation corresponds to a pattern-forming instability of a basic state that is both time independent and fully symmetric, for example, a spatially uniform equilibrium solution of the governing equations. A symmetry-breaking Hopf bifurcation of this spatially uniform state often leads to time-periodic solutions that break the translation invariance of the governing equations and that have spatio-temporal and spatial symmetries. In this paper we address bifurcations of such periodic orbits, which have broken the translation invariance but have retained a discrete group of spatio-temporal symmetries.

This work contributes an approach to analysing certain transitions from order towards spatial disorder that occur as pattern-forming systems are driven harder. Here we analyse symmetry-breaking bifurcations from nontrivial time-periodic solutions of pattern-forming partial differential equations (PDEs). Since the exact form of these solutions may only be known numerically, we will not in general be able to predict which bifurcations will occur at what parameter values. However, the symmetry properties of the solutions may be known; we exploit this qualitative information to determine the possibilities for bifurcation. This knowledge should prove especially useful in interpreting the results of numerical simulations or experiments.

We consider problems posed with periodic boundary conditions, for which there is an S^1 symmetry associated with each direction of imposed periodicity. If this symmetry is broken by an equilibrium solution, then the solution is not isolated; there is a continuous family of equilibria related through the translations. An instability of this solution can excite the neutral translation mode(s) and lead to new solutions that drift along the translation group orbit. This is the case, for example, in the 'parity-breaking bifurcation': a reflection-symmetric steady state undergoes a symmetry-breaking bifurcation to a uniformly translating solution. Another example of a bifurcation leading to drift has been observed in two-dimensional convection: when the vertical mirror plane of symmetry that separates steady counter-rotating rolls is broken in a Hopf bifurcation, the resulting solution, called a direction-reversing travelling wave or pulsating wave (PW), drifts to and fro (Landsberg and Knobloch 1991, Matthews *et al* 1993). This periodic orbit is invariant under the combination of advance of half the period in time with a reflection; any drift in one direction in the first half of the oscillation is exactly balanced by a drift in the other direction in the second half, so there is no net drift during the oscillation. Similarly in three-dimensional convection with imposed spatial periodicity, a symmetry-breaking Hopf bifurcation from steady convection in a square pattern can lead to alternating pulsating waves (APW), which are invariant under the combination of advance of one quarter the period and rotation by 90° (Rucklidge 1997). These solutions drift alternately along the two horizontal coordinate directions, but again have no net drift over the whole period of the oscillation.

There have been a number of studies of bifurcations of compact group orbits of (relative) equilibria. Iooss (1986) developed an approach based on centre manifold reduction to investigate bifurcations of Taylor vortices in the Taylor–Couette problem. Specifically, he analysed bifurcations in directions orthogonal to the tangent space to the group orbit of equilibria, with the neutral translation mode incorporated explicitly in the bifurcation problem. Krupa (1990) provided a general setting for investigating bifurcations of relative equilibria that focuses on the local dynamics in directions orthogonal to the tangent space to the group orbit. He showed that the resulting bifurcation problem is Σ-equivariant, where Σ is the isotropy subgroup of symmetries of the relative equilibrium, and, building on work of Field (1980), provided a group theoretic method for determining whether or not the bifurcating solutions drift. Ashwin and Melbourne (1997) have recently generalized this to the case of noncompact Lie groups. Aston *et al* (1992), and Amdjadi *et al* (1997) develop a technique for numerically investigating bifurcations of relative equilibria in $O(2)$-equivariant partial differential equations, and apply their method to the Kuramoto–Sivashinsky equation. Their approach isolates one solution on a group orbit, while still keeping track of any constant drift along the group orbit.

In this paper we investigate bifurcations of time-periodic solutions that are not isolated as they have broken the translation invariance, but that do possess a discrete group of spatio-temporal symmetries. Our approach is similar to that of Iooss (1986). However, we are interested in instabilities of periodic solutions, so we use centre manifold reduction for Poincaré maps. We are particularly interested in determining whether the symmetries of the basic state place any restrictions on the types of bifurcations that occur, and whether the bifurcating solutions drift along the underlying group orbit or not. We consider three examples that are motivated by numerical studies of convection with periodic boundary conditions in the horizontal direction(s). First we investigate bifurcations of the PWs and APWs described above. These solutions have discrete spatio-temporal symmetries Z_2 and Z_4, respectively. The group orbit of the PWs is S^1, while the group orbit of the APWs is a two-torus, due to imposed periodicity in two horizontal directions. The third example we treat in this paper is alternating rolls (ARs), which have the same spatio-temporal symmetry as APW but are also invariant under reflection in two orthogonal vertical planes (Silber and Knobloch 1991). After considering these three examples, we discuss how to treat more general problems.

The Z_n ($n = 2, 4$) spatio-temporal symmetry of the basic state places restrictions on the Poincaré return map $\mathcal{M}$; specifically, $\mathcal{M}$ is the nth iterate of a map $\tilde{\mathcal{G}}$, which is determined by the spatio-temporal symmetry. A direct consequence of this is that period-doubling bifurcations are nongeneric (Swift and Wiesenfeld 1984). Throughout the paper we restrict our analysis to bifurcation with Floquet multiplier (FM) $+1$; we do not consider Hopf bifurcations. We also restrict attention to bifurcations that preserve the spatial periodicity of the basic state.

Our paper is organized as follows. In the next section we lay the framework for our analysis in the setting of a simple example, namely bifurcation of PWs. We show how the spatio-temporal symmetry is manifest in the Poincaré return map. Section 3 considers bifurcations of the three-dimensional analogue of PWs, namely APWs. Section 4 considers bifurcations of ARs. For this problem we need to consider six different cases, which we classify by the degree to which the spatial and spatio-temporal symmetries are broken. In the case that the spatial reflection symmetries are fully broken by the marginal modes, the FM $+1$ is forced to have multiplicity two, and more than one solution branch bifurcates from the basic AR state. In one case we find a bifurcation of the AR state leading to two distinct drifting solutions. We present an example of one of the drifting patterns, obtained by numerically integrating the equations of three-dimensional compressible magnetoconvection. In the course of the analysis of bifurcations of ARs, we make contact with the work on k-symmetries of Lamb and Quispel (1994) and Lamb (1996, 1998). In section 5, we outline a group-theoretic approach to the analysis of steady-state bifurcations of periodic orbits with spatio-temporal symmetries, based on the equivariant branching lemma and the irreducible representations of the spatio-temporal symmetry group that leaves the periodic orbit invariant, making use of normal form results from Lamb (1996). Section 6 contains a summary and indicates some directions for future work.

2. Two dimensions: PWs

We write the PDEs for two-dimensional convection symbolically as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{F}(U;\mu), \tag{1}$$

where U represents velocity, temperature, density, etc as functions of the horizontal coordinate x, the vertical coordinate z and time t; μ represents a parameter of the problem; and $\mathcal{F}$ is a nonlinear operator between suitably chosen function spaces. We assume periodic boundary conditions, with spatial period ℓ, in the x-direction.

The symmetry group of the problem is $O(2)$, which is the semidirect product of Z_2, generated by a reflection κ_x, and an $SO(2)$ group of translations τ_a, which act as

$$\kappa_x\colon x \to -x, \qquad \tau_a\colon x \to x + a \pmod{\ell}, \tag{2}$$

where τ_ℓ is the identity and $\tau_a\kappa_x = \kappa_x\tau_{-a}$. The PDEs (1) are equivariant under the action of these symmetry operators, so $\mathcal{F}(\tau_a U;\mu) = \tau_a\mathcal{F}(U;\mu)$ and $\mathcal{F}(\kappa_x U;\mu) = \kappa_x\mathcal{F}(U;\mu)$, where τ_a and κ_x act on the functions as follows:

$$\tau_a U(x,z,t) \equiv U(x-a,z,t), \qquad \kappa_x U(x,z,t) \equiv M_{\kappa_x} U(-x,z,t). \tag{3}$$

Here M_{κ_x} is a matrix representing κ_x; it reverses the sign of the horizontal component of velocity and leaves all other fields in U unchanged.

Suppose that when the parameter $\mu = 0$, there is a known PW solution $U_0(x,z,t)$ of (1) with temporal period T and spatial period $\lambda = \ell/N$, where N specifies the number of PWs that fit into the periodic box. The symmetries of U_0 are summarized as follows:

$$U_0(x,z,t) = \kappa_x U_0(x,z,t+\tfrac{1}{2}T) = U_0(x,z,t+T) = \tau_\lambda U_0(x,z,t). \tag{4}$$

There is a continuous group orbit of PWs generated by translations: $U_a = \tau_a U_0$. We are interested in bifurcations from this group orbit. Following the approach developed by Iooss (1986) and Chossat and Iooss (1994) for studying instabilities of continuous group orbits of steady solutions, we expand about the group orbit of periodic solutions as follows:

$$U(x,z,t) = \tau_{c(t)}(U_0(x,z,t) + A(x,z,t)). \tag{5}$$

Here translation along the group orbit is given by $\tau_{c(t)}$, where c is a coordinate parametrizing the group orbit. Small perturbations, orthogonal to the tangent direction of the group orbit, are specified by $A(x,z,t)$. The expansion (5) is substituted into the PDEs (1) and, after suitable projection that separates translations along the group orbit from the evolution of the perturbation orthogonal to it, we obtain equations of the form (see Chossat and Iooss 1994):

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \mathcal{G}(A,U_0;\mu), \qquad \frac{\mathrm{d}c}{\mathrm{d}t} = h(A,U_0;\mu), \tag{6}$$

where $\mathcal{G}$ and h satisfy $\mathcal{G}(0,U_0;0) = 0$ and $h(0,U_0;0) = 0$. An important consequence of the translation invariance of the original PDEs is that $\mathcal{G}$ and h do not depend on the position c along the group orbit; the equation for the drift c is decoupled from the equation for the amplitude of the perturbation A. Here we find it convenient to keep track of the explicit time dependence of $\mathcal{G}$ and h, which enters through their dependence on the basic state U_0, by listing U_0 as one of the arguments of $\mathcal{G}$ and h. We determine how the spatio-temporal reflection symmetry of U_0 is manifest in the equations for c and A by noting that if $\tau_{c(t)}(U_0(x,z,t) + A(x,z,t))$ is a solution of the PDEs (1), then so is

$$\kappa_x\tau_{c(t)}(U_0(x,z,t) + A(x,z,t)) = \tau_{-c(t)}(\kappa_x U_0(x,z,t) + \kappa_x A(x,z,t)). \tag{7}$$

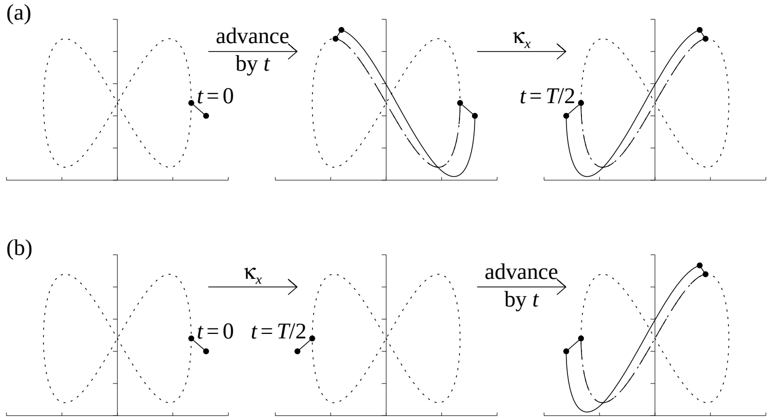

Figure 1. Illustration of $\kappa_x \mathcal{M}_0^t = \mathcal{M}_{T/2}^{T/2+t}\kappa_x$. In this example, the reflection κ_x changes the sign of the horizontal coordinate. The PW periodic solution is shown as a dotted curve. (*a*) A perturbation at $t = 0$ is advanced in time by an amount t (the full curve, which stays close to the broken curve on the periodic orbit), then the system is reflected. (*b*) We arrive at the same final position if we reflect (so now the perturbation is about the PW at $t = \frac{1}{2}T$) and then advance in time by the same amount.

Hence

$$\begin{aligned}\mathcal{G}(\kappa_x A, \kappa_x U_0; \mu) &= \kappa_x \mathcal{G}(A, U_0; \mu),\\ h(\kappa_x A, \kappa_x U_0; \mu) &= -h(A, U_0; \mu).\end{aligned} \tag{8}$$

Since our basic state U_0 is T-periodic, we seek a map that gives the perturbation $A(T)$ at time $t = T$ given a perturbation $A(0)$ at some initial time $t = 0$. Specifically, we define a time advance map $\mathcal{M}_0^t$ acting on the perturbation $A(0)$ by $A(t) = \mathcal{M}_0^t(A(0))$. We adopt the approach of Swift and Wiesenfeld (1984) and split the time interval from 0 to T into two stages using the symmetry property of the underlying PWs. Specifically, since $\kappa_x A(t)$ satisfies $\frac{\mathrm{d}(\kappa_x A)}{\mathrm{d}t} = \mathcal{G}(\kappa_x A, \kappa_x U_0; \mu)$ and $\kappa_x U_0(x, z, t) = U_0(x, z, t + \frac{T}{2})$, we have $\kappa_x A(t) = \mathcal{M}_{T/2}^{t+T/2}(\kappa_x A(0))$; hence

$$\kappa_x \mathcal{M}_0^t = \mathcal{M}_{T/2}^{T/2+t}\kappa_x. \tag{9}$$

Advancing the perturbation by a time t starting from time 0 and then reflecting the whole system is equivalent to reflecting the whole system then advancing by a time t starting from time $\frac{1}{2}T$ (see figure 1). It follows immediately that the full period map $\mathcal{M}_0^T$ can be written as the second iterate of a map $\tilde{\mathcal{G}}$:

$$\mathcal{M}_0^T = \mathcal{M}_{T/2}^T \kappa_x^2 \mathcal{M}_0^{T/2} = (\kappa_x \mathcal{M}_0^{T/2})^2 \equiv \tilde{\mathcal{G}}^2. \tag{10}$$

Rather than consider the full period map $\mathcal{M}_0^T$, we will consider the map $\tilde{\mathcal{G}} \equiv \kappa_x \mathcal{M}_0^{T/2}$. The map $\tilde{\mathcal{G}}$ has no special property under reflections, but it commutes with translations τ_λ, which leave the underlying PWs invariant: $\tilde{\mathcal{G}}\tau_\lambda = \tau_\lambda \tilde{\mathcal{G}}$. The underlying PW periodic orbit is a fixed point of $\tilde{\mathcal{G}}$ as a consequence of its spatio-temporal symmetry.

The dynamics of the perturbation is now given by the map $\tilde{\mathcal{G}}$: $\mathcal{A}_{n+1} = \tilde{\mathcal{G}}(\mathcal{A}_n; \mu)$, where each iterate corresponds to advancing in time by $\frac{1}{2}T$ and reflecting; thus $A(\frac{1}{2}T) = \kappa_x \mathcal{A}_1$, starting from $\mathcal{A}_0$ at time 0. In order to compute the drift c_1 of the solution at time $\frac{1}{2}T$, we integrate the $\mathrm{d}c/\mathrm{d}t$ equation (6) for a time $\frac{1}{2}T$, starting at a position c_0 and with initial perturbation $A(0) = \mathcal{A}_0$:

$$c_1 = c_0 + \int_0^{T/2} h(\mathcal{M}_0^t(\mathcal{A}_0), U_0(t); \mu)\,\mathrm{d}t \equiv c_0 + \tilde{h}(\mathcal{A}_0; \mu). \tag{11}$$

Then, after a second half-period,

$$\begin{aligned} c_2 = c_1 + \int_{T/2}^{T} h(\mathcal{M}_{T/2}^t(A(\tfrac{1}{2}T)), U_0(t); \mu)\,\mathrm{d}t &= c_1 + \int_{T/2}^{T} h(\mathcal{M}_{T/2}^t(\kappa_x \mathcal{A}_1), U_0(t); \mu)\,\mathrm{d}t \\ &= c_1 + \int_{T/2}^{T} h(\kappa_x \mathcal{M}_0^{t-T/2}(\mathcal{A}_1), \kappa_x U_0(t - T/2); \mu)\,\mathrm{d}t \\ &= c_1 - \int_0^{T/2} h(\mathcal{M}_0^{t'}(\mathcal{A}_1), U_0(t'); \mu)\,\mathrm{d}t' = c_1 - \tilde{h}(\mathcal{A}_1; \mu). \end{aligned} \tag{12}$$

Thus the combined dynamics of the perturbation and translation can be written as

$$\mathcal{A}_{n+1} = \tilde{\mathcal{G}}(\mathcal{A}_n; \mu), \qquad c_{n+1} = c_n + (-1)^n \tilde{h}(\mathcal{A}_n; \mu). \tag{13}$$

Since the unperturbed PW is a nondrifting solution of the problem at $\mu = 0$ we have $\tilde{\mathcal{G}}(0; 0) = 0$ and $\tilde{h}(0; 0) = 0$. Moreover, the spatial periodicity of U_0 places some symmetry restrictions on $\tilde{\mathcal{G}}$ and $\tilde{h}$; specifically, $\tilde{\mathcal{G}}(\tau_\lambda \mathcal{A}; \mu) = \tau_\lambda \tilde{\mathcal{G}}(\mathcal{A}; \mu)$ and $\tilde{h}(\tau_\lambda \mathcal{A}; \mu) = \tilde{h}(\mathcal{A}; \mu)$.

We turn now to the codimension-one bifurcations of the PW, which are the trivial fixed points $\mathcal{A} = 0$, $c = c_0$ of (13) when $\mu = 0$. The map (13) always has one FM equal to one because of the translation invariance of the c part of the map. Bifurcations occur when a FM of the linearization of $\tilde{\mathcal{G}}$ crosses the unit circle: either a FM = 1, or a FM = −1, or there is a pair of complex conjugate FMs with unit modulus (we do not consider the last case in this paper). Because we have assumed periodic boundary conditions in the original PDEs, we expect the spectrum of the linearization to be discrete and the centre manifold theorem for maps to apply. (See Chossat and Iooss (1994) for a discussion of the centre manifold reduction in the similar problem of bifurcations from Taylor vortices.) Let ζ be the eigenfunction associated with the critical FM, so that on the centre manifold, we can write

$$\mathcal{A}_n = a_n \zeta + \Phi(a_n), \tag{14}$$

where Φ is the graph of the centre manifold. The unfolded dynamics takes the form

$$a_{n+1} = \hat{g}(a_n; \mu), \qquad c_{n+1} = c_n + (-1)^n \hat{h}(a_n; \mu), \tag{15}$$

where $\hat{g}$ and $\hat{h}$ are the maps $\tilde{\mathcal{G}}$ and $\tilde{h}$ reduced to the centre manifold; $\hat{g}$ and $\hat{h}$ share the same symmetry properties as $\tilde{\mathcal{G}}$ and $\tilde{h}$.

In this paper, we only consider the case where τ_λ acts trivially. We therefore expect only generic bifurcations in the map $\hat{g}$: saddle-node when FM = 1, period-doubling when FM = −1 and Hopf when there are a pair of complex FMs. The FMs for the full period map $\mathcal{M}_0^T$, which are the squares of the FMs of $\hat{g}$, will generically be either one or come in complex conjugate pairs. In particular, we do not expect $\mathcal{M}_0^T$ to have a FM = −1; this mechanism for suppressing period-doubling bifurcations was discussed by Swift and Wiesenfeld (1984).

Here we consider only the cases where $\hat{g}$ has a FM = +1 or −1. The normal form in the case FM = 1 is

$$a_{n+1} = \mu + a_n - a_n^2, \qquad c_{n+1} = c_n + (-1)^n \hat{h}(a_n; \mu), \tag{16}$$

to within a rescaling and a change of sign. The parameter μ is zero at the bifurcation point, and the fixed points of the a part of the map are $a = \pm\sqrt{\mu}$ when μ is positive. The spatial translations are

$$c_0, \qquad c_1 = c_0 + \hat{h}(a; \mu), \qquad c_2 = c_0, \ldots. \tag{17}$$

We therefore have a c_0-parametrized family of solutions that vanish, in pairs, as μ is decreased through $\mu = 0$. We interpret this bifurcation by considering the solutions with $c_0 = -\frac{1}{2}\hat{h}(a; \mu)$, $a = \pm\sqrt{\mu}$. In this case, we have a pair of PW solutions, translated with respect to the original PW by $\frac{1}{2}\hat{h}(\pm\sqrt{\mu}; \mu)$, which collide in a saddle-node bifurcation at $\mu = 0$. The remainder of the family of solutions is obtained by translating this pair.

The case FM $= -1$ is more interesting. The normal form in the supercritical case is

$$a_{n+1} = (-1 + \mu)a_n - a_n^3, \qquad c_{n+1} = c_n + (-1)^n \hat{h}(a_n; \mu), \tag{18}$$

with a fixed point $a = 0$ and a period-two orbit $a_n = (-1)^n\sqrt{\mu}$. The dynamics of the spatial translations are

$$\begin{aligned} &c_0, \qquad c_1 = c_0 + \hat{h}(a_0; \mu), \qquad c_2 = c_0 + \hat{h}(a_0; \mu) - \hat{h}(-a_0; \mu), \\ &\qquad\quad c_3 = c_0 + 2\hat{h}(a_0; \mu) - \hat{h}(-a_0; \mu), \ldots. \end{aligned} \tag{19}$$

Since $\hat{h}(0; 0) = 0$, and generically $\frac{\partial \hat{h}}{\partial a}(0; 0) \neq 0$, $\hat{h}(a_0; \mu)$ and $\hat{h}(-a_0; \mu)$ have opposite sign for small μ; this represents a symmetry-breaking bifurcation that leads to a solution that drifts along the group orbit of the PW.

The main points of interest in this section are the approach that we have taken in analysing the instabilities of the group orbit of the spatio-temporally symmetric periodic orbit, and the observation that an instability of the PW with FM $= 1$ in the full-period map can lead to drifting solutions or not. Whether solutions drift can only be determined by examining the half-period map. In the next two sections, we apply our method to three-dimensional APWs and to ARs, the latter having spatial as well as spatio-temporal symmetries.

3. Three dimensions: APWs

APWs are the simplest three-dimensional analogue of the PWs discussed in the previous section. These periodic oscillations have been observed in numerical simulations of three-dimensional compressible magnetoconvection with periodic boundary conditions in the two horizontal directions (Matthews *et al* 1995). They appear either after a series of global bifurcations (Rucklidge and Matthews 1995, Matthews *et al* 1996) or in a Hopf bifurcation from convection in a square pattern (Rucklidge 1997), and are invariant under the combined operation of advancing one quarter period in time and rotating 90° in space.

The full symmetry group of the problem is the semidirect product of the D_4 symmetry group of the square lattice and a two-torus T^2 of translations in the two horizontal directions, x and y. D_4 is generated by a reflection κ_x and a clockwise rotation by 90° ρ:

$$\begin{aligned} &\kappa_x\colon (x, y) \to (-x, y), \qquad \rho\colon (x, y) \to (y, -x), \\ &\tau_{a,b}\colon (x, y) \to (x + a \pmod{\ell}, \; y + b \pmod{\ell}), \end{aligned} \tag{20}$$

where $\rho\tau_{a,b} = \tau_{b,-a}\rho$.

As before, we assume that at $\mu = 0$, we have a known APW solution $U_{0,0}(x, y, z, t)$ with spatial period λ in each direction and temporal period T; then $U_{0,0}$ satisfies

$$\begin{aligned} U_{0,0}(x, y, z, t) &= \rho U_{0,0}(x, y, z, t + \tfrac{1}{4}T) = U_{0,0}(x, y, z, t + T) = \tau_{\lambda,0}U_{0,0}(x, y, z, t) \\ &= \tau_{0,\lambda}U_{0,0}(x, y, z, t). \end{aligned} \tag{21}$$

We consider only the case where $\tau_{\lambda,0}$ and $\tau_{0,\lambda}$ act trivially. There is a two-parameter continuous group orbit of APWs generated by translations: $U_{a,b} = \tau_{a,b}U_{0,0}$. We expand about this group orbit:

$$U(x, y, z, t) = \tau_{c_x(t),c_y(t)}(U_{0,0}(x, y, z, t) + A(x, y, z, t)), \tag{22}$$

where (c_x, c_y) is a time-dependent translation around the group orbit and A is the perturbation orthogonal to the tangent plane to the group orbit. As before, we separate the evolution of the translations from that of the perturbation:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \mathcal{G}(A, U_{0,0}; \mu), \qquad \frac{\mathrm{d}c_x}{\mathrm{d}t} = h_x(A, U_{0,0}; \mu), \qquad \frac{\mathrm{d}c_y}{\mathrm{d}t} = h_y(A, U_{0,0}; \mu), \tag{23}$$

where we keep track of the explicit time dependence of $\mathcal{G}$, h_x, and h_y through the argument $U_{0,0}$. The spatio-temporal symmetry of the basic state $U_{0,0}$ is manifest in $\mathcal{G}$, h_x and h_y as follows:

$$\begin{aligned}
&\mathcal{G}(\rho A, \rho U_{0,0}; \mu) = \rho\mathcal{G}(A, U_{0,0}; \mu),\\
&h_x(\rho A, \rho U_{0,0}; \mu) = h_y(A, U_{0,0}; \mu),\\
&h_y(\rho A, \rho U_{0,0}; \mu) = -h_x(A, U_{0,0}; \mu),
\end{aligned} \tag{24}$$

where $\rho U_{0,0}(t + \frac{1}{4}T) = U_{0,0}(t)$. It is convenient to introduce a complex translation $c \equiv c_x + \mathrm{i}c_y$ and a corresponding $h \equiv h_x + \mathrm{i}h_y$, so $\rho\tau_c = \tau_{-\mathrm{i}c}\rho$.

As before, we define a time advance map acting on the perturbation so $A(t) = \mathcal{M}_0^t(A(0))$; this has the property

$$\mathcal{M}_0^t\rho = \rho\mathcal{M}_{T/4}^{T/4+t}, \qquad \mathcal{M}_0^t\rho^2 = \rho^2\mathcal{M}_{T/2}^{T/2+t}, \qquad \mathcal{M}_0^t\rho^3 = \rho^3\mathcal{M}_{3T/4}^{3T/4+t} \tag{25}$$

because of the underlying spatio-temporal symmetry of the APW. The full period map $\mathcal{M}_0^T$ is then the fourth iterate of a map $\tilde{\mathcal{G}}$:

$$\mathcal{M}_0^T = \rho^4\mathcal{M}_{3T/4}^T\mathcal{M}_0^{3T/4} = \rho\mathcal{M}_0^{T/4}\rho^3\mathcal{M}_0^{3T/4} = (\rho\mathcal{M}_0^{T/4})^4 \equiv \tilde{\mathcal{G}}^4. \tag{26}$$

Instead of $\mathcal{M}_0^T$, we consider $\tilde{\mathcal{G}} \equiv \rho\mathcal{M}_0^{T/4}$, which has no special properties under reflections and rotations.

The dynamics of the perturbation is given by $\mathcal{A}_{n+1} = \tilde{\mathcal{G}}(\mathcal{A}_n)$, where $A(\frac{1}{4}T) = \rho^3\mathcal{A}_1$, etc. Then the position of the pattern at time $\frac{1}{4}T$ is

$$c_1 = c_0 + \int_0^{T/4} h(\mathcal{M}_0^t(\mathcal{A}_0), U_{0,0}(t); \mu)\,\mathrm{d}t \equiv c_0 + \tilde{h}(\mathcal{A}_0; \mu), \tag{27}$$

where the map $\tilde{h} = \tilde{h}_x + \mathrm{i}\tilde{h}_y$. After the next quarter period, we find

$$\begin{aligned}
c_2 = c_1 &+ \int_{T/4}^{T/2} h(\mathcal{M}_{T/4}^t(A(\tfrac{1}{4}T)), U_{0,0}(t); \mu)\,\mathrm{d}t\\
&= c_1 + \int_{T/4}^{T/2} h(\mathcal{M}_{T/4}^t(\rho^3\mathcal{A}_1), U_{0,0}(t); \mu)\,\mathrm{d}t\\
&= c_1 + \int_{T/4}^{T/2} h(\rho^3\mathcal{M}_0^{t-T/4}(\mathcal{A}_1), \rho^3U_{0,0}(t - T/4); \mu)\,\mathrm{d}t\\
&= c_1 + \mathrm{i}\tilde{h}(\mathcal{A}_1; \mu).
\end{aligned} \tag{28}$$

So the combined dynamics of the perturbation and the translation can be written as

$$\mathcal{A}_{n+1} = \tilde{\mathcal{G}}(\mathcal{A}_n; \mu), \qquad c_{n+1} = c_n + \mathrm{i}^n\tilde{h}(\mathcal{A}_n; \mu), \tag{29}$$

where $\tilde{\mathcal{G}}(0; 0) = \tilde{h}(0; 0) = 0$.

Note that, as in the case of PWs, the generic bifurcations of APWs are either steady-state (FM $= +1$) or Hopf, since $\mathcal{M}_0^T = \tilde{\mathcal{G}}^4$. We consider bifurcations with FM $= +1$ of $\mathcal{M}_0^T$ only; generically, these occur when the linearization of $\tilde{\mathcal{G}}$ has a FM of $+1$ or -1.

Near a bifurcation point we reduce the dynamics onto the centre manifold

$$a_{n+1} = \hat{g}(a_n; \mu), \qquad c_{n+1} = c_n + \mathrm{i}^n \hat{h}(a_n; \mu). \tag{30}$$

When a FM $= 1$, once again we have a saddle-node bifurcation, this time involving pairs of APWs that are translated relative to each other. If a FM is -1, we have $a_n = (-1)^n \sqrt{\mu}$, and the spatial translations are:

$$\begin{aligned} c_0, \quad & c_1 = c_0 + \hat{h}(a_0; \mu), \qquad c_2 = c_0 + \hat{h}(a_0; \mu) + \mathrm{i}\hat{h}(-a_0; \mu), \\ & c_3 = c_0 + \mathrm{i}\hat{h}(-a_0; \mu), \quad c_4 = c_0, \ldots. \end{aligned} \tag{31}$$

This solution has no net drift (unlike in the two-dimensional problem), but travels back and forth different amounts in the two horizontal directions since, generically, $\hat{h}_x(a_0; \mu) \neq \hat{h}_y(a_0; \mu)$. The solution remains invariant under advance of half its period in time combined with a rotation of 180°. To see this, we construct the solution $U(x, y, z, t)$ at $t = 0$ and $t = \frac{1}{2}T$ using the solution in the c_0-parametrized family that satisfies $c_0 = -c_2$. Specifically, we insert the centre manifold solution $A(0) = \mathcal{A}_0 = a_0\zeta + \Phi(a_0)$, $A(\frac{1}{2}T) = \rho^2\mathcal{A}_2 = \rho^2(a_0\zeta + \Phi(a_0))$ in (22). We obtain

$$\begin{aligned} U(0) &= \tau_{c_0}(U_{0,0}(0) + a_0\zeta + \Phi(a_0)) \\ U(\tfrac{1}{2}T) &= \tau_{-c_0}(U_{0,0}(\tfrac{1}{2}T) + \rho^2 a_0\zeta + \rho^2\Phi(a_0)) = \tau_{-c_0}\rho^2(U_{0,0}(0) + a_0\zeta + \Phi(a_0)) \\ &= \rho^2 U(0) \end{aligned} \tag{32}$$

where we have suppressed the (x, y, z)-dependence of U, retaining only its t-dependence.

Thus, in the simple case of APW, we cannot get drifting solutions in a bifurcation with FM $= 1$ for the time-T return map. We next consider the same bifurcation for the more complicated example of ARs. This solution has the same spatio-temporal symmetry as APW but has extra spatial reflection symmetries. We shall see that in this case a particular symmetry-breaking bifurcation leads to two distinct types of drifting solutions.

4. Additional spatial symmetries: ARs

ARs are created in a primary Hopf bifurcation from a $D_4 \ltimes T^2$ invariant trivial solution (Silber and Knobloch 1991). Like APWs, ARs are invariant under the spatio-temporal symmetry of advancing one-quarter period in time and rotating 90° in space, but have the additional property of being invariant under reflections in two orthogonal vertical planes. ARs have been observed to be stable near the initial Hopf bifurcation over a wide range of parameter values in three-dimensional incompressible and compressible magnetoconvection (Clune and Knobloch 1994, Matthews *et al* 1995).

For convenience in this section, we define $\tilde{\rho}$ to be the combined advance of one quarter period in time followed by a 90° clockwise rotation about the line $(x, y) = (0, 0)$. Reflecting in the planes $x = \frac{1}{4}\lambda$ or $y = \frac{1}{4}\lambda$ leaves ARs unchanged at all times, so the 16-element spatio-temporal symmetry group that leaves the AR invariant is generated by κ_x', κ_y' and $\tilde{\rho}$, where

$$\begin{aligned} &\kappa_x' : (x, y, z, t) \to (\tfrac{1}{2}\lambda - x, y, z, t), \\ &\kappa_y' : (x, y, z, t) \to (x, \tfrac{1}{2}\lambda - y, z, t), \\ &\tilde{\rho} : (x, y, z, t) \to (y, -x, z, t + \tfrac{1}{4}T). \end{aligned} \tag{33}$$

Table 1. Summary of six types of bifurcations of ARs, distinguished by the action of κ'_x and κ'_y on the critical modes, and by the critical FMs of $\tilde{\mathcal{G}}$. Generators of isotropy subgroups (up to conjugacy) of the bifurcating solution branches are indicated; in cases B+ and B−, there are two distinct solution branches.

Case	Action of κ'_x, κ'_y on marginal modes	Floquet multiplier(s)	Bifurcation (drift or not)	Isotropy subgroup
A+(+1)	$\kappa'_x\kappa'_y\zeta = \zeta$ $\kappa'_x\zeta = \kappa'_y\zeta = \zeta$	FM = +1	Saddle-node (no drift)	$\langle\kappa'_x, \kappa'_y, \tilde{\rho}\rangle$
A+(−1)	as A+(+1)	FM = −1	Symmetry-breaking (no drift)	$\langle\kappa'_x, \kappa'_y, \tilde{\rho}^2\rangle$
A−(+1)	$\kappa'_x\kappa'_y\zeta = \zeta$ $\kappa'_x\zeta = \kappa'_y\zeta = -\zeta$	FM = +1	Symmetry-breaking (no drift)	$\langle\kappa'_x\kappa'_y, \tilde{\rho}\rangle$
A−(−1)	as A−(+1)	FM = −1	Symmetry-breaking (no drift)	$\langle\kappa'_x\kappa'_y, \kappa'_x\tilde{\rho}\rangle$
B+	$\kappa'_x\kappa'_y\zeta_\pm = -\zeta_\pm$ $\zeta_- = \kappa'_x\zeta_+ = -\kappa'_y\zeta_+$	FM = ±1	Symmetry-breaking (no net drift)	$\langle\kappa'_x, \tilde{\rho}^2\rangle$ $\langle\tilde{\rho}\rangle$
B−	as B+	FM = ±i	Symmetry-breaking (drift)	$\langle\kappa'_y, \tau_{c_0-c_2}\kappa'_x\kappa'_y\tilde{\rho}^2\rangle$ $\langle\tau_{c_0-\mathrm{i}c_1^*}\kappa'_y\tilde{\rho}\rangle$

The basic AR solution $U_{0,0}(x, y, z, t)$ exists at $\mu = 0$ and satisfies

$$U_{0,0}(x, y, z, t) = \rho U_{0,0}(x, y, z, t + \tfrac{1}{4}T) = U_{0,0}(x, y, z, t + T) = \kappa'_x U_{0,0}(x, y, z, t)$$
$$= \kappa'_y U_{0,0}(x, y, z, t) = \tau_{\lambda,0} U_{0,0}(x, y, z, t) = \tau_{0,\lambda} U_{0,0}(x, y, z, t). \quad (34)$$

As in section 3, we expand about this basic solution and recover the map (29). The presence of extra reflection symmetries of the underlying solution manifests itself in the following way:

$$\tilde{\mathcal{G}}(\kappa'_x\mathcal{A}) = \kappa'_y\tilde{\mathcal{G}}(\mathcal{A}), \qquad \tilde{\mathcal{G}}(\kappa'_y\mathcal{A}) = \kappa'_x\tilde{\mathcal{G}}(\mathcal{A}),$$
$$\tilde{h}_x(\kappa'_x\mathcal{A}) = -\tilde{h}_x(\mathcal{A}), \qquad \tilde{h}_x(\kappa'_y\mathcal{A}) = \tilde{h}_x(\mathcal{A}), \quad (35)$$
$$\tilde{h}_y(\kappa'_x\mathcal{A}) = \tilde{h}_y(\mathcal{A}), \qquad \tilde{h}_y(\kappa'_y\mathcal{A}) = -\tilde{h}_y(\mathcal{A}).$$

The first of these equations deserves some explanation: $\tilde{\mathcal{G}}(\kappa'_x\mathcal{A}) = \rho\mathcal{M}_0^{T/4}\kappa'_x\mathcal{A} = \rho\kappa'_x\mathcal{M}_0^{T/4}\mathcal{A}$, since κ'_x commutes with the time advance map, as the underlying periodic orbit is invariant under κ'_x. Now $\rho\kappa'_x = \kappa'_y\rho$, so $\rho\kappa'_x\mathcal{M}_0^{T/4}\mathcal{A} = \kappa'_y\rho\mathcal{M}_0^{T/4}\mathcal{A} = \kappa'_y\tilde{\mathcal{G}}(\mathcal{A})$. Note that it is the rotation in the definition of $\tilde{\mathcal{G}}$ (26) that implies that reflecting with κ'_x then applying $\tilde{\mathcal{G}}$ is equivalent to applying $\tilde{\mathcal{G}}$ then reflecting with κ'_y—not κ'_x. In the terminology of Lamb and Quispel (1994), κ'_x and κ'_y are 2-symmetries of $\tilde{\mathcal{G}}$, that is, $\tilde{\mathcal{G}}^2(\kappa'_x\mathcal{A}) = \kappa'_x\tilde{\mathcal{G}}^2(\mathcal{A})$. In general, k-symmetries arise when the spatial part of the spatio-temporal symmetry of a time-periodic solution does not commute with its purely spatial symmetries (Lamb 1998). We discuss this point in more detail in section 5.

The remainder of this section is devoted to the discussion of the codimension-one steady-state bifurcations of this problem. We do not consider bifurcations that break the spatial periodicity, so $\tau_{\lambda,0}$ and $\tau_{0,\lambda}$ act trivially, nor do we consider Hopf bifurcations. The results are summarized in table 1.

We begin by noting that $\tilde{\mathcal{G}}(\kappa'_x\kappa'_y\mathcal{A}) = \kappa'_x\kappa'_y\tilde{\mathcal{G}}(\mathcal{A})$, so $\kappa'_x\kappa'_y$ commutes with the linearization $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{G}}$, whereas $\kappa'_x\tilde{\mathcal{L}} = \tilde{\mathcal{L}}\kappa'_y$. The eigenspaces of $\tilde{\mathcal{L}}$ are invariant under the rotation $\kappa'_x\kappa'_y$. We assume the generic situation of one-dimensional eigenspaces; then each eigenfunction ζ must be either even or odd under the rotation $\kappa'_x\kappa'_y$, i.e. $\kappa'_x\kappa'_y\zeta = \zeta$

(case A) or $\kappa'_x\kappa'_y\zeta = -\zeta$ (case B), since $(\kappa'_x\kappa'_y)^2$ is the identity. In case A, if $\zeta = \kappa'_x\kappa'_y\zeta$ is an eigenfunction of $\tilde{\mathcal{L}}$ with FM $= s$, then $\kappa'_x\zeta = \kappa'_y\zeta$ has the same FM:

$$\tilde{\mathcal{L}}\kappa'_x\zeta = \kappa'_y\tilde{\mathcal{L}}\zeta = s\kappa'_y\zeta = s\kappa'_x\zeta. \tag{36}$$

Since ${\kappa'_x}^2$ is the identity, it follows that ζ and $\kappa'_x\zeta$ are linearly dependent and that either $\kappa'_x\zeta = \zeta$ (case A+) or $\kappa'_x\zeta = -\zeta$ (case A−). Finally, these two cases are subdivided according to the value of the critical FM of $\tilde{\mathcal{L}}$ (either +1 or −1) at the bifurcation point.

Case B is rather different. Here we have $\kappa'_x\zeta = -\kappa'_y\zeta$, so

$$\tilde{\mathcal{L}}\kappa'_x\zeta = \kappa'_y\tilde{\mathcal{L}}\zeta = s\kappa'_y\zeta = -s\kappa'_x\zeta. \tag{37}$$

Thus $\kappa'_x\zeta$ has FM $= -s$ and is linearly independent of ζ, which has FM $= s$. We define ζ_+ to be the eigenfunction of s and ζ_- to be the eigenfunction of $-s$, with $\zeta_- = \kappa'_x\zeta_+ = -\kappa'_y\zeta_+$. There are two ways in which two FMs s and $-s$ can cross the unit circle: either at +1 and −1 (case B+) or at +i and −i (case B−). Note that in the absence of the reflection symmetries these bifurcations would be codimension-two; here they occur as generic bifurcations. Since the FMs of the time-T map $\mathcal{M}_0^T$ are the fourth power of the FMs of $\tilde{\mathcal{G}}$, the effect of the symmetry in case B is to force a repeated FM $= +1$ in the map $\mathcal{M}_0^T$.

In case A, we write

$$\mathcal{A}_n = a_n\zeta + \Phi(a_n), \tag{38}$$

near the bifurcation point, where Φ is the graph of the centre manifold. On the centre manifold we have $\mathcal{A} = \kappa'_x\kappa'_y\mathcal{A}$, so

$$\tilde{h}_x(\mathcal{A}) = \tilde{h}_x(\kappa'_x\kappa'_y\mathcal{A}) = -\tilde{h}_x(\kappa'_y\mathcal{A}) = -\tilde{h}_x(\mathcal{A}) = 0, \tag{39}$$

where we have used (35). Thus in case A, $\tilde{h}_x$ and $\tilde{h}_y$ are identically zero, and no bifurcation will lead to drift along the group orbit of ARs.

The reflections κ'_x and κ'_y act trivially in case A+. A FM = +1 leads to a saddle-node bifurcation of ARs. The normal form in the case FM = −1 gives $a_n = (-1)^n a_0$, from which the bifurcating solution $U(t)$ can be reconstructed. Choosing the initial translation c_0 to be zero, and suppressing the (x, y, z)-dependence of U, we have

$$\begin{aligned} &U(0) = U_{0,0}(0) + a_0\zeta + \Phi(a_0), && U(\tfrac{1}{4}T) = \rho^3(U_{0,0}(0) - a_0\zeta + \Phi(-a_0)),\\ &U(\tfrac{1}{2}T) = \rho^2(U_{0,0}(0) + a_0\zeta + \Phi(a_0)), && U(\tfrac{3}{4}T) = \rho(U_{0,0}(0) - a_0\zeta + \Phi(-a_0)). \end{aligned} \tag{40}$$

Here it should be recalled that $U_{0,0}(\frac{1}{4}T) = \rho^3 U_{0,0}(0)$, and that on the centre manifold

$$A(\tfrac{1}{4}T) = \rho^3\mathcal{A}_1 = \rho^3(a_1\zeta + \Phi(a_1)) = \rho^3(-a_0\zeta + \Phi(-a_0)). \tag{41}$$

This solution satisfies

$$U(t) = \kappa'_x U(t) = \kappa'_y U(t) = \rho^2 U(t + \tfrac{1}{2}T), \tag{42}$$

and thus has the same symmetries as 'standing cross-rolls', described by Silber and Knobloch (1991).

In case A−, κ'_x and κ'_y act nontrivially, so the behaviour on the centre manifold is governed by a pitchfork normal form ($a_n = a_0$) when the FM = +1 and by a period-doubling normal form ($a_n = (-1)^n a_0$) when the FM = −1. At leading order in a_0, the bifurcating solutions $U(t)$ in the two cases are

$$\begin{aligned} &U(0) = U_{0,0}(0) + a_0\zeta, && U(\tfrac{1}{4}T) = \rho^3(U_{0,0}(0) \pm a_0\zeta),\\ &U(\tfrac{1}{2}T) = \rho^2(U_{0,0}(0) + a_0\zeta), && U(\tfrac{3}{4}T) = \rho(U_{0,0}(0) \pm a_0\zeta). \end{aligned} \tag{43}$$

These solutions are not invariant under κ_x' or κ_y' (since these change the sign of ζ), but are invariant under the product $\kappa_x'\kappa_y'$. In addition, $U(t) = \rho U(t + \frac{1}{4}T)$ in the case FM $= +1$ and $U(t) = \kappa_x' \rho U(t + \frac{1}{4}T)$ in the case FM $= -1$.

Case B is more interesting. On the two-dimensional centre manifold, we write

$$\mathcal{A}_n = (-a_n + b_n)\zeta_+ + (a_n + b_n)\zeta_- + \Phi(a_n, b_n); \tag{44}$$

the form of this expression is chosen for later convenience. The map (29) reduces to

$$(a_{n+1}, b_{n+1}) = \hat{g}(a_n, b_n; \mu), \qquad c_{n+1} = c_n + \mathrm{i}^n(\hat{h}_x(a_n, b_n; \mu) + \mathrm{i}\hat{h}_y(a_n, b_n; \mu)). \tag{45}$$

Since $\zeta_- = \kappa_x'\zeta_+ = -\kappa_y'\zeta_+$, we have

$$\begin{aligned} \kappa_x'\mathcal{A}_n &= (a_n + b_n)\zeta_+ + (-a_n + b_n)\zeta_- + \kappa_x'\Phi(a_n, b_n), \\ \kappa_y'\mathcal{A}_n &= (-a_n - b_n)\zeta_+ + (a_n - b_n)\zeta_- + \kappa_y'\Phi(a_n, b_n); \end{aligned} \tag{46}$$

thus

$$\kappa_x'(a_n, b_n) = (-a_n, b_n), \qquad \kappa_y'(a_n, b_n) = (a_n, -b_n). \tag{47}$$

From this and from (35), we deduce that on the centre manifold

$$\begin{aligned} \hat{h}_x(a_n, b_n) &= -\hat{h}_x(-a_n, b_n) = \hat{h}_x(a_n, -b_n), \\ \hat{h}_y(a_n, b_n) &= -\hat{h}_y(a_n, -b_n) = \hat{h}_y(-a_n, b_n), \end{aligned} \tag{48}$$

implying that $\hat{h}_x(0, b; \mu) = 0$ and $\hat{h}_y(a, 0; \mu) = 0$. Moreover, $\hat{g}$ inherits the symmetries (35) of $\tilde{\mathcal{G}}$:

$$\kappa_y'\hat{g}(a_n, b_n) = \hat{g}(\kappa_x'(a_n, b_n)), \qquad \kappa_x'\hat{g}(a_n, b_n) = \hat{g}(\kappa_y'(a_n, b_n)). \tag{49}$$

Thus the linearization $\hat{\mathcal{L}}$ of $\hat{g}$ satisfies

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\mathcal{L}} = \hat{\mathcal{L}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{50}$$

which forces $\hat{\mathcal{L}}$ to be of the form

$$\hat{\mathcal{L}} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \tag{51}$$

where a_n, b_n can be scaled so that $\alpha = |\beta|$. There is a bifurcation when $\beta = +1$ or $\beta = -1$, yielding FMs ± 1 (case B+) or $\pm\mathrm{i}$ (case B−), respectively.

In order to analyse the dynamics near the bifurcation point, we compute the normal form of the bifurcation problems, expanding $\hat{g}$ as a Taylor series in a and b. The symmetry $\kappa_x'\kappa_y'$ prohibits quadratic terms, and all but two of the cubic terms can be removed by near-identity transformations. For reasons to be explained in section 5, we choose the transformation so that the normal form commutes with the linearization of the map (51) at the bifurcation point. (That this can be done follows from general results on normal forms; see, for example, Crawford (1991) or Lamb (1996).) We thus have the unfolded normal form, truncated at cubic order, in the two cases B+ and B−:

$$\begin{aligned} a_{n+1} &= (1 + \mu + P a_n^2 + Q b_n^2) b_n, \\ b_{n+1} &= \pm(1 + \mu + P b_n^2 + Q a_n^2) a_n, \\ c_{n+1} &= c_n + \mathrm{i}^n(\hat{h}_x(a_n, b_n; \mu) + \mathrm{i}\hat{h}_y(a_n, b_n; \mu)), \end{aligned} \tag{52}$$

where $\mu = 0$ at the bifurcation point and P and Q are constants. In case B+ the second iterate is

$$\begin{aligned} a_{n+2} &= (1 + 2\mu + 2Qa_n^2 + 2Pb_n^2)a_n, \\ b_{n+2} &= (1 + 2\mu + 2Pa_n^2 + 2Qb_n^2)b_n, \end{aligned} \tag{53}$$

while in case B− the fourth iterate is

$$\begin{aligned} a_{n+4} &= (1 + 4\mu + 4Qa_n^2 + 4Pb_n^2)a_n, \\ b_{n+4} &= (1 + 4\mu + 4Pa_n^2 + 4Qb_n^2)b_n. \end{aligned} \tag{54}$$

The maps (53) and (54) have the same form as the generic D_4-equivariant steady-state bifurcation problem (see, for example, Silber and Knobloch (1989)). There are two group orbits of solution branches; one associated with solutions on the a and b coordinate axes, and the other associated with solutions on the diagonal lines $a = \pm b$. We now interpret these in terms of period-one, -two and -four solutions of the original map (52). We note that our results are the same as those of Lamb (1996) who analysed a different normal form for this bifurcation problem.

In case B+, there are two distinct types of orbits created. The first is a period-two orbit $(a_0, 0) \leftrightarrow (0, a_0)$, with $0 = \mu + Qa_0^2$. From this and the symmetries (48) of $\hat{h}$, we deduce the drift of the solution at each iterate:

$$\begin{aligned} &c_0, \qquad c_1 = c_0 + \hat{h}_x(a_0, 0; \mu), \qquad c_2 = c_0 + \hat{h}_x(a_0, 0; \mu) - \hat{h}_y(0, a_0; \mu), \\ &\qquad\quad c_3 = c_0 - \hat{h}_y(0, a_0; \mu), \qquad c_4 = c_0, \ldots. \end{aligned} \tag{55}$$

There is no net drift along the group orbit in this case. Moreover, the c_0-parametrized family of solutions drift to and fro in the x-direction only since $c_n - c_0$ is real. Consider $c_0 = \frac{1}{2}(-\hat{h}_x(a_0, 0; \mu) + \hat{h}_y(0, a_0; \mu)) = -c_2$, where c_0 is real and thus corresponds to a translation in the x-direction. The reconstructed solution $U(t)$, at leading order in a_0, satisfies

$$\begin{aligned} U(0) &= \tau_{c_0}(U_{0,0}(0) - a_0\zeta_+ + a_0\zeta_-), \\ U(\tfrac{1}{4}T) &= \tau_{c_1}\rho^3(U_{0,0}(0) + a_0\zeta_+ + a_0\zeta_-), \\ U(\tfrac{1}{2}T) &= \tau_{-c_0}\rho^2(U_{0,0}(0) - a_0\zeta_+ + a_0\zeta_-), \\ U(\tfrac{3}{4}T) &= \tau_{-c_1}\rho(U_{0,0}(0) + a_0\zeta_+ + a_0\zeta_-), \end{aligned} \tag{56}$$

so we have $U(t) = \kappa_y' U(t) = \rho^2 U(t + \frac{1}{2}T)$. The conjugate orbit, $(0, a_0) \leftrightarrow (a_0, 0)$, has symmetry $\langle \kappa_x', \tilde{\rho}^2 \rangle$ and does not drift at all in the x-direction. This orbit bifurcates stably if $P < Q < 0$ in (52).

The second type of orbit created in case B+ is a period-one orbit (a_0, a_0), with $0 = \mu + (P + Q)a_0^2$. The translations at each iterate are

$$\begin{aligned} &c_0, \qquad c_1 = c_0 + \hat{h}(a_0, a_0; \mu), \qquad c_2 = c_0 + \hat{h}(a_0, a_0; \mu) + \mathrm{i}\hat{h}(a_0, a_0; \mu), \\ &\qquad\quad c_3 = c_0 + \mathrm{i}\hat{h}(a_0, a_0; \mu), \qquad c_4 = c_0, \ldots. \end{aligned} \tag{57}$$

This orbit also has no net drift, and by choosing $c_0 = -\frac{1}{2}(1 + \mathrm{i})\hat{h}(a_0, a_0; \mu)$, we have $c_1 = \mathrm{i}c_0$. The reconstructed solution $U(t)$, at leading order in a_0, satisfies

$$\begin{aligned} &U(0) = \tau_{c_0}(U_{0,0}(0) + 2a_0\zeta_-), \qquad U(\tfrac{1}{4}T) = \tau_{\mathrm{i}c_0}\rho^3(U_{0,0}(0) + 2a_0\zeta_-), \\ &U(\tfrac{1}{2}T) = \tau_{-c_0}\rho^2(U_{0,0}(0) + 2a_0\zeta_-), \qquad U(\tfrac{3}{4}T) = \tau_{-\mathrm{i}c_0}\rho(U_{0,0}(0) + 2a_0\zeta_-), \end{aligned} \tag{58}$$

so $U(t) = \rho U(t + \frac{1}{4}T)$, and the isotropy subgroup is $\langle \tilde{\rho} \rangle$. This solution has the same symmetries as the APWs described in section 3, so APWs may be created in a symmetry-breaking bifurcation of ARs. There is also a period-two orbit $(a_0, -a_0) \leftrightarrow (-a_0, a_0)$ that has the conjugate isotropy subgroup $\langle\kappa_x'\kappa_y'\tilde{\rho}\rangle$. These solution branches are stable provided $Q < -|P|$ in (52).

Finally, we turn to case B−. Here, there are two types of periodic orbit created in the bifurcation at $\mu = 0$, and in this case they are both of period four. The first orbit is $(a_0, 0) \to (0, -a_0) \to (-a_0, 0) \to (0, a_0)$, with $0 = -\mu + Qa_0^2$ in (52). Again, this orbit is stable if $P < Q < 0$. The translations are

$$\begin{aligned}
&c_0,\\
&c_1 = c_0 + \hat{h}_x(a_0, 0; \mu),\\
&c_2 = c_0 + \hat{h}_x(a_0, 0; \mu) + \hat{h}_y(0, a_0; \mu),\\
&c_3 = c_0 + 2\hat{h}_x(a_0, 0; \mu) + \hat{h}_y(0, a_0; \mu),\\
&c_4 = c_0 + 2\hat{h}_x(a_0, 0; \mu) + 2\hat{h}_y(0, a_0; \mu), \ldots.
\end{aligned} \tag{59}$$

Note that $c_n - c_0$ is real so there is no drift at all in the y direction, but there is a systematic drift in the x-direction. The reconstructed solution $U(t)$ satisfies

$$\begin{aligned}
&U(0) = \tau_{c_0}(U_{0,0}(0) - a_0\zeta_+ + a_0\zeta_-),\\
&U(\tfrac{1}{4}T) = \tau_{c_1}\rho^3(U_{0,0}(0) - a_0\zeta_+ - a_0\zeta_-),\\
&U(\tfrac{1}{2}T) = \tau_{c_2}\rho^2(U_{0,0}(0) + a_0\zeta_+ - a_0\zeta_-),\\
&U(\tfrac{3}{4}T) = \tau_{c_3}\rho(U_{0,0}(0) + a_0\zeta_+ + a_0\zeta_-),
\end{aligned} \tag{60}$$

so we have

$$U(t) = \kappa_y' U(t) = \tau_{c_0-c_2}\kappa_x'\kappa_y'\rho^2 U(t + \tfrac{1}{2}T). \tag{61}$$

The isotropy subgroup of this solution is $\langle\kappa_y', \tau_{c_0-c_2}\kappa_x'\kappa_y'\tilde{\rho}^2\rangle$, which is homomorphic (modulo translations) to D_2. A conjugate orbit, started a quarter period later, has isotropy subgroup $\langle\kappa_x', \tau_{c_0-c_2}\kappa_x'\kappa_y'\tilde{\rho}^2\rangle$ and drifts systematically in the y-direction.

The second type of orbit created in case B− is $(a_0, a_0) \to (a_0, -a_0) \to (-a_0, -a_0) \to (-a_0, a_0)$, with $0 = -\mu + (P + Q)a_0^2$. It is stable provided $Q < -|P|$ in (52). The translations are

$$\begin{aligned}
&c_0,\\
&c_1 = c_0 + \hat{h}_x(a_0, a_0; \mu) + \mathrm{i}\hat{h}_y(a_0, a_0; \mu),\\
&c_2 = c_0 + (1 + \mathrm{i})(\hat{h}_x(a_0, a_0; \mu) + \hat{h}_y(a_0, a_0; \mu)),\\
&c_3 = c_0 + (2 + \mathrm{i})\hat{h}_x(a_0, a_0; \mu) + (1 + 2\mathrm{i})\hat{h}_y(a_0, a_0; \mu),\\
&c_4 = c_0 + (2 + 2\mathrm{i})(\hat{h}_x(a_0, a_0; \mu) + \hat{h}_y(a_0, a_0; \mu)), \ldots.
\end{aligned} \tag{62}$$

This corresponds to a solution that drifts along the diagonal, with a wobble from side to side as it goes. At leading order in a_0, the reconstructed solution $U(t)$ satisfies

$$\begin{aligned}
&U(0) = \tau_{c_0}(U_{0,0}(0) + 2a_0\zeta_-), && U(\tfrac{1}{4}T) = \tau_{c_1}\rho^3(U_{0,0}(0) - 2a_0\zeta_+),\\
&U(\tfrac{1}{2}T) = \tau_{c_2}\rho^2(U_{0,0}(0) - 2a_0\zeta_-), && U(\tfrac{3}{4}T) = \tau_{c_3}\rho(U_{0,0}(0) + 2a_0\zeta_+),
\end{aligned} \tag{63}$$

which has fully broken the spatial symmetries. This solution has the spatio-temporal symmetry

$$U(t) = \tau_{c_0 - ic_1^*}\kappa_y'\rho U(t + \tfrac{1}{4}T), \tag{64}$$

which generates the isotropy subgroup $\langle\tau_{c_0 - ic_1^*}\kappa_y'\tilde{\rho}\rangle$, homomorphic (modulo translations) to Z_4.

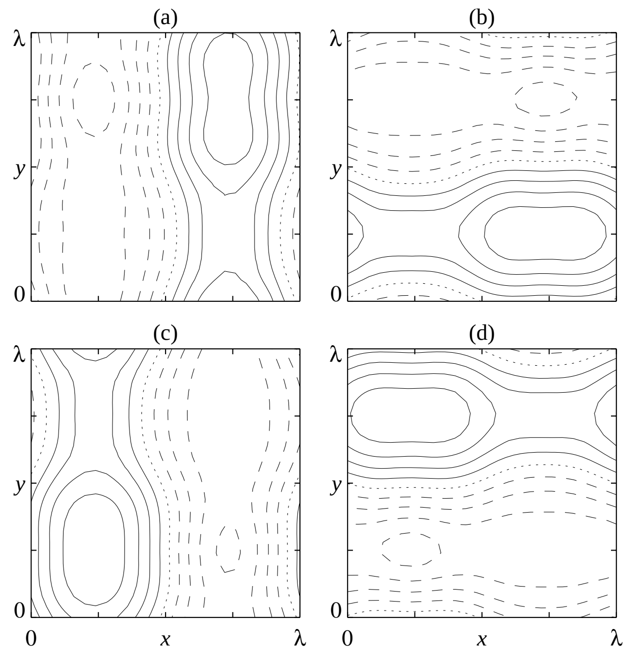

Figure 2. ARs in three-dimensional compressible magnetoconvection, starting with parameter values from Matthews *et al* (1995). The four frames are (approximately) at times (*a*) $t = 0$, (*b*) $t = \frac{1}{4}T$, (*c*) $t = \frac{1}{2}T$ and (*d*) $t = \frac{3}{4}T$. The frames show contours of the vertical velocity in a horizontal plane in the middle of the layer: full curves denote fluid travelling upwards, broken curves denote fluid travelling downwards, and the dotted curves denote zero vertical velocity. The spatial symmetries κ_x' and κ_y' are manifest, as is the spatio-temporal symmetry of advancing a quarter period in time followed by a 90° rotation (counter-clockwise in this example). The dimensionless parameters are: the mid-layer Rayleigh number (proportional to the temperature difference across the layer) $R = 2324$; the Chandrasekhar number (proportional to the square of the imposed magnetic field) $Q = 1033$; the Prandtl number $\sigma = 0.1$; the mid-layer magnetic diffusivity ratio $\zeta = 0.1$; the adiabatic exponent $\gamma = \frac{5}{3}$; the polytropic index $m = \frac{1}{4}$; the thermal stratification $\theta = 6$; the mid-layer plasma beta $\beta = 32$; and the horizontal wavelengths $\lambda = 2$ in units of the layer depth.

M An MPEG movie of this figure is available from the article's abstract page in the online journal; see `http://www.iop.org`.

In summary, we have examined the six different cases in which ARs undergo a bifurcation with FM $= +1$ in the full period map. All six bifurcations preserve the underlying spatial periodicity of the ARs, but may break the spatial and spatio-temporal symmetries. The 2-symmetry present in the B cases forces two FMs to cross the unit circle together, and we find two branches of bifurcating solutions, with distinct symmetry properties. In these cases, we find that if both solution branches bifurcate supercritically, then one and only one of the two solutions will be stable. It is only in case B−, with FMs $\pm$i in the map $\tilde{\mathcal{G}}$, that the bifurcation leads to systematically drifting solutions: one solution drifts along a coordinate axis, while the other drifts along a diagonal.

We finish this section by presenting examples of ARs and drifting ARs, which we interpret as an instance of a B− bifurcation. We have solved the PDEs for three-dimensional compressible magnetoconvection in a periodic $2\times2\times1$ box, using the code of Matthews *et al*

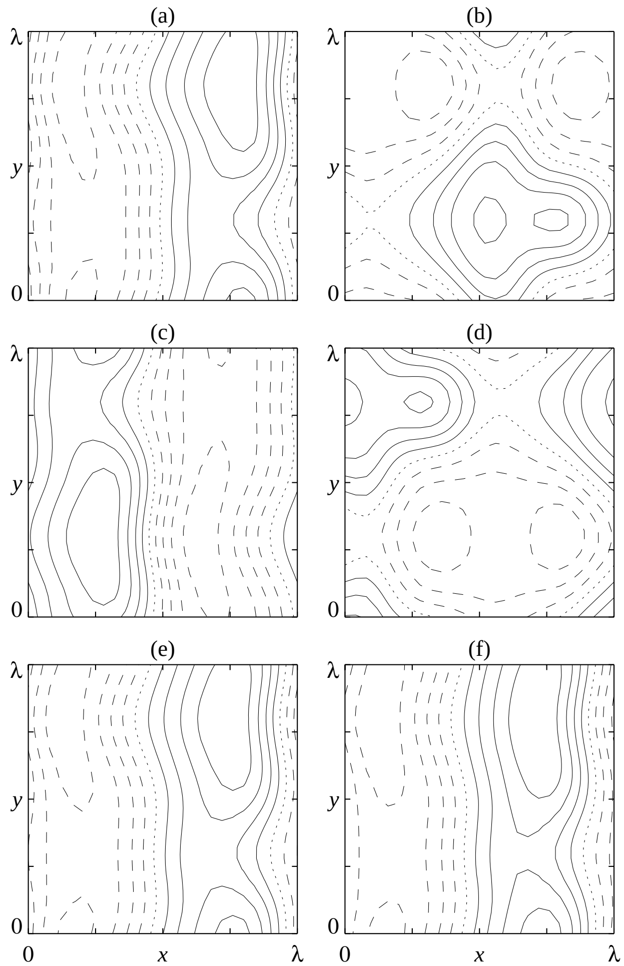

Figure 3. After a bifurcation of type B−, the ARs begin to drift. The parameter values are as in figure 2, but with a higher thermal forcing: $R = 3000$ and $Q = 1333$. The frames are (approximately) at times (*a*) $t = 0$, (*b*) $t = \frac{1}{4}T$, (*c*) $t = \frac{1}{2}T$, (*d*) $t = \frac{3}{4}T$, (*e*) $t = T$ and (*f*) $t = 2T$. Note how all spatial and spatio-temporal symmetries have been broken, with the exception of κ'_y, a reflection in the plane $y = \frac{1}{4}\lambda$ (modulo a slight shift in the periodic box). The slow leftward drift of the pattern can be seen by comparing frames (*a*), (*e*) and (*f*), In addition, a drift symmetry $\kappa'_x\kappa'_y\tilde{\rho}^2\tau_{c_0-c_2}$, conjugate to (61), can be seen by comparing frames (*a*) and (*c*) or (*b*) and (*d*).

M An MPEG movie of this figure is available from the article's abstract page in the online journal; see `http://www.iop.org`.

(1995). The PDEs and description of the parameters and numerical method can be found in that paper. Figure 2 shows an example of an AR at times approximately 0, $\frac{1}{4}T$, $\frac{1}{2}T$ and $\frac{3}{4}T$; the two reflection symmetries κ'_x and κ'_y in planes $x = \frac{1}{4}\lambda$ and $y = \frac{1}{4}\lambda$ and the spatio-temporal symmetry of advancing a quarter period in time followed by a 90° rotation about the centre of the box are manifest. Increasing the controlling parameter, the temperature difference across the layer, leads to the solution in figure 3: the data are shown at times 0, $\frac{1}{4}T$, $\frac{1}{2}T$, $\frac{3}{4}T$, T and $2T$. The only spatial symmetry remaining is the invariance under κ'_y, and the spatio-temporal symmetry has been broken. By comparing frames (a), (e) and (f) at times 0, T and $2T$, it can be seen that the solution is drifting slowly leftwards along the x-axis. Moveover, a drift symmetry $\kappa'_x\kappa'_y\tilde{\rho}^2\tau_{c_0-c_2}$ conjugate to (61) can be seen by comparing frames (a) and (c). This evidence points to there being a bifurcation of type B− at an intermediate parameter value. Closer inspection reveals that for the chosen values of the other parameters, the first instability of ARs as the temperature difference is increased is in fact a Hopf bifurcation leading to patterns that drift in an oscillatory fashion, with no net drift, on a timescale that is much longer than the timescale of the ARs (so it is not a bifurcation of type B+). Subsequently, stability is transferred to the drifting solution shown in figure 3, which we conjecture was created in a bifurcation of type B− from ARs, after their Hopf bifurcation. This scenario, of a Hopf bifurcation to oscillations followed closely by bifurcation to steady solutions, with a transfer of stability between the two branches, is familiar from studies of the Takens–Bogdanov bifurcation (Guckenheimer and Holmes 1983, Rucklidge 1994), but further investigation is beyond the scope of this paper.

5. Remarks on a group theoretic approach

In our analysis of the AR example above, we made use of the observation that $\kappa'_x\kappa'_y$ is a symmetry of $\tilde{\mathcal{G}}$; the fact that $(\kappa'_x\kappa'_y)^2 =$ identity, which implies that $\kappa'_x\kappa'_y$ acts either as $+1$ or as -1 on marginal eigenfunctions, enabled us to compute all the different bifurcations that are possible. We do not expect that all problems can be tackled in this way, so a more general and systematic approach is desirable. Here we outline such an approach based on the irreducible representations of the spatio-temporal symmetry group that leaves the periodic orbit invariant.

Suppose $\tilde{\Gamma}$ is the compact group of spatial and spatio-temporal symmetries that leave a time-periodic solution of the PDEs invariant: for example, in the case of ARs, $\tilde{\Gamma}$ is generated by the spatial reflections κ'_x and κ'_y and by the spatio-temporal symmetry $\tilde{\rho}$, and $|\tilde{\Gamma}| = 16$. (Here we call a symmetry 'spatio-temporal' if it involves a nontrivial time-advance along the periodic orbit; otherwise we refer to it as 'spatial'.) In the more general case considered here, we assume that $\tilde{\Gamma}$ can be generated by spatial symmetries and by a single spatio-temporal symmetry group element $\tilde{\gamma}_t \in \tilde{\Gamma}$. In particular, we assume that $\tilde{\Gamma}$ is the (semi)direct product of the spatial symmetry group and the cyclic spatio-temporal group Z_p generated by the element $\tilde{\gamma}_t$. Let $\gamma_t \notin \tilde{\Gamma}$ denote the spatial part of $\tilde{\gamma}_t$ and let p be the smallest positive integer with $\gamma_t^p =$ identity.

The aim is to answer the question: given a periodic orbit with spatio-temporal symmetry group $\tilde{\Gamma}$, what steady-state bifurcations are possible, and what are the symmetries of the bifurcating solutions? We will find that different irreducible representations of $\tilde{\Gamma}$ lead to different normal forms for the weakly nonlinear behaviour near the bifurcation point, with the number of marginally stable FMs equal to the dimension of the representation. Some but not all of these maps will have k-symmetries ($k > 1$), as in cases B described above. The bifurcation problem can be formulated in terms of a normal form that is equivariant with respect to the irreducible representation under consideration. Our approach is complementary to that of Lamb (1998), who starts from k-symmetric maps and demonstrates that these may appear as normal forms for bifurcations from spatio-temporally symmetric periodic orbits.

The full-period map $\mathcal{M}_0^T$ will factorize as in the previous examples (Lamb 1998):

$$\mathcal{M}_0^T = (\gamma_t\mathcal{M}_0^{T/p})^p \equiv \tilde{\mathcal{G}}^p, \tag{65}$$

where $\tilde{\mathcal{G}}$, defined as $\gamma_t \mathcal{M}_0^{T/p}$, acts on perturbations $\mathcal{A}_n$ of the underlying periodic orbit, with time measured in units of T/p. Due to its spatio-temporal symmetry, the underlying periodic orbit is a fixed-point of the map $\tilde{\mathcal{G}}$. Similarly, c_n, now regarded as a vector whose two components give the position of the pattern, evolves as

$$c_{n+1} = c_n + \{M_{\gamma_t}^{-1}\}^n \tilde{h}(\mathcal{A}_n; \mu), \tag{66}$$

where M_γ is the two by two real matrix that represents the standard action of the symmetry γ on the plane. In the AR example, we had $\gamma_t = \rho$ and $M_{\gamma_t} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, or equivalently $-\mathrm{i}$ when c_n is regarded as a complex coordinate, so $M_{\gamma_t}^{-1} = \mathrm{i}$.

Spatial symmetries do not necessarily commute with $\tilde{\mathcal{G}}$, but we do have

$$\tilde{\mathcal{G}}(\gamma_s \mathcal{A}) = \gamma_t \mathcal{M}_0^{T/p} \gamma_s \mathcal{A} = \gamma_t \gamma_s \mathcal{M}_0^{T/p} \mathcal{A} = \gamma_t \gamma_s \gamma_t^{-1} \tilde{\mathcal{G}}(\mathcal{A}), \tag{67}$$

where $\gamma_t \gamma_s \gamma_t^{-1}$ is a spatial symmetry of the underlying periodic orbit. (However, the spatial symmetries do commute with $\tilde{\mathcal{G}}^p$.) Lamb's (1996) approach is based on the purely spatial symmetry group and the mapping from γ_s to $\gamma_t \gamma_s \gamma_t^{-1}$, while we use the full spatio-temporal symmetry group $\tilde{\Gamma}$, which includes these relations.

Similarly, the rate of drift of $\gamma_s \mathcal{A}$ can be given in terms of the rate of drift of $\mathcal{A}$:

$$\tilde{h}(\gamma_s \mathcal{A}) = M_{\gamma_s} \tilde{h}(\mathcal{A}). \tag{68}$$

The relations (67) and (68) are the generalizations of (35).

Now suppose there is a bifurcation at $\mu = 0$, and assume that there are finitely many linearly independent, marginally stable eigenfunctions $\zeta_1, \zeta_2, \ldots, \zeta_m$ of the linearization $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{G}}$ with FMs $s_1, s_2, \ldots, s_m$, so $\tilde{\mathcal{L}}\zeta_i = s_i \zeta_i$ with $|s_i| = 1$. On the centre manifold, we write

$$\mathcal{A} = \sum_{i=1}^{m} a_i \zeta_i + \Phi(a_1, a_2, \ldots, a_m). \tag{69}$$

The spatial symmetries γ_s take the subspace spanned by these eigenfunctions to itself, which implies that the spatial symmetries act linearly on the mode amplitudes: $\gamma_s(\boldsymbol{a}) = R_{\gamma_s} \boldsymbol{a}$, where each R_{γ_s} is an $m \times m$ matrix, and $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)$. Similarly, the action of advancing the linearized flow in time by T/p and operating with γ_t takes the centre eigenspace to itself; this operation takes $\boldsymbol{a}$ to $\hat{\mathcal{L}}\boldsymbol{a}$, where $\hat{\mathcal{L}}$ is $\tilde{\mathcal{L}}$ restricted to the centre eigenspace. We consider only the case of steady-state bifurcations with $\hat{\mathcal{L}}^p$ being the identity, for which there are m FMs of the full-period map equal to one. From this it follows that the matrices R_{γ_s} and $\hat{\mathcal{L}}$ form a representation of the group $\tilde{\Gamma}$: the purely spatial symmetries are represented by the R_{γ_s} matrices; the Z_p spatio-temporal symmetry group generated by $\tilde{\gamma}_t$ is represented by $\hat{\mathcal{L}}$ and its powers; and we observe that if $\tilde{\gamma}_t \gamma_s' = \gamma_s \tilde{\gamma}_t$, then $\hat{\mathcal{L}} R_{\gamma_s'} = R_{\gamma_s} \hat{\mathcal{L}}$ (cf (50) and (67)). Thus every element $\gamma \in \tilde{\Gamma}$ is represented by a matrix R_γ, with $\gamma(\boldsymbol{a}) = R_\gamma \boldsymbol{a}$. There is a basis in which the matrices R_γ are orthogonal (Miller 1972).

On the centre manifold, the unfolded dynamics is given by the normal form

$$\boldsymbol{a}_{n+1} = \hat{g}(\boldsymbol{a}_n; \mu), \qquad c_{n+1} = c_n + \{M_{\gamma_t}^{-1}\}^n \hat{h}(\boldsymbol{a}_n; \mu). \tag{70}$$

Since $R_{\tilde{\gamma}_t}$ is an orthogonal matrix, the normal form $\hat{g}$ can be constructed to commute with its linearization at the bifurcation point (which is $R_{\tilde{\gamma}_t}$) at any order (Crawford 1991, Lamb 1996). Moreover, the spatial symmetries transform $\hat{g}$ in the same way that they transform $\tilde{\mathcal{G}}$ (67). Thus we have

$$\hat{g}(R_{\gamma_s}\boldsymbol{a}; \mu) = R_{\tilde{\gamma}_t} R_{\gamma_s} R_{\tilde{\gamma}_t}^{-1} \hat{g}(\boldsymbol{a}; \mu), \qquad \hat{g}(R_{\tilde{\gamma}_t}\boldsymbol{a}; \mu) = R_{\tilde{\gamma}_t} \hat{g}(\boldsymbol{a}; \mu), \tag{71}$$

for the symmetry properties of $\hat{g}$. With the translations we can only say that

$$\hat{h}(R_{\gamma_s}\boldsymbol{a}; \mu) = M_{\gamma_s} \hat{h}(\boldsymbol{a}; \mu), \tag{72}$$

where the matrices R_{γ_s} and M_{γ_s} represent the action of the spatial symmetry γ_s on the spaces of perturbations $\boldsymbol{a}$ and translations c, respectively.

At the bifurcation point $\mu = 0$, the linearization of the normal form is $R_{\tilde{\gamma}_t}$; unfolding the normal form leads to linear terms that also obey the relations (71), and with the assumption that the representation is absolutely irreducible, we have

$$\hat{g}(\boldsymbol{a}; \mu) = (1+\mu) R_{\tilde{\gamma}_t} \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2). \tag{73}$$

Small-amplitude bifurcating branches can be found by seeking solutions of

$$\hat{g}(\boldsymbol{a}; \mu) = R_{\tilde{\gamma}_t} \boldsymbol{a}, \tag{74}$$

which will be periodic points of the map with period q, where q is the smallest positive integer such that $R^q_{\tilde{\gamma}_t}$ is the identity. Such periodic points are zeros of the function $f(\boldsymbol{a}; \mu)$ defined by (cf Lamb 1996)

$$f(\boldsymbol{a}; \mu) = R^{-1}_{\tilde{\gamma}_t} \hat{g}(\boldsymbol{a}; \mu) - \boldsymbol{a}. \tag{75}$$

We note that since we have chosen the normal form $\hat{g}$ to commute with $R_{\tilde{\gamma}_t}$, $f(\boldsymbol{a}; \mu)$ is equivariant under the full spatio-temporal symmetry group $\tilde{\Gamma}$ as represented by the matrices R_γ: $f(R_\gamma \boldsymbol{a}; \mu) = R_\gamma f(\boldsymbol{a}; \mu)$ for all $\gamma \in \tilde{\Gamma}$. With this, we can apply the equivariant branching lemma (see Golubitsky *et al* 1988), which implies that, under certain nondegeneracy conditions, we are guaranteed a unique branch of bifurcating solutions for all isotropy subgroups of $\tilde{\Gamma}$ with one-dimensional fixed point subspaces. These isotropy subgroups characterize the spatio-temporal symmetries of the bifurcating periodic points of $\hat{g}$ (Lamb 1998).

Finally, we use the approach described above to verify the results of the previous section on ARs. Here the 16 element spatio-temporal symmetry group $\tilde{\Gamma}$ has 10 irreducible representations, of which four are one-dimensional and real (corresponding to the four cases A in table 1), four are one-dimensional and complex (and so are not absolutely irreducible: these will not arise in the steady-state bifurcations we are considering here) and two are two-dimensional and real (corresponding to the two cases B). In these last two representations, we have $R_{\kappa'_x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_{\kappa'_y} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and either $R_{\tilde{\rho}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (case B+) or $R_{\tilde{\rho}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (case B−). With $q = 2$ in B+ and $q = 4$ in B−, we recover the two bifurcating branches in each case, together with their respective isotropy subgroups (modulo translations), as given in table 1.

In this way, the problem of determining which steady-state bifurcations from a given periodic orbit are possible reduces to determining the absolutely irreducible representations of the spatio-temporal symmetry group that leaves the periodic orbit invariant, and then applying the equivariant branching lemma with an appropriate interpretation. This general approach complements the specific calculations we have performed for ARs, and is the natural way to progress to more complicated situations. The k-symmetries that arose in the cases B are now seen as a natural consequence of the structure of the two-dimensional representations of $\tilde{\Gamma}$.

6. Conclusion

We have developed a technique for investigating the possible steady-state instabilities from continuous group orbits of spatio-temporally symmetric time-periodic solutions of partial differential equations in periodic domains. Our approach is based on centre manifold reduction and symmetry arguments. It is in the spirit of earlier work by Iooss (1986) on bifurcations from continuous group orbits of spatially symmetric steady solutions of partial differential equations. We have treated three examples that arise in convection problems: PWs in two dimensions, and APWs and ARs in three dimensions. A simple bifurcation can lead to drifting solutions in the case of PWs but not APWs. The additional spatial symmetries of ARs can force two FMs to cross the unit circle together; this degeneracy can lead to drifting solutions, as in the numerical example presented in the section 4. We have related our work to the theory of k-symmetries developed by Lamb and Quispel (1994). The relevance that our work has for convection (and other pattern-forming) problems lies not only in the analysis of the specific examples we have considered; rather, it is the possibility of treating bifurcations of a wide variety of symmetric time-dependent patterns that is most interesting and will be most useful in interpreting the results of numerical simulations and experiments.

We have outlined an approach to the steady-state bifurcation problem, based on computing the absolutely irreducible representations of the spatio-temporal symmetry group that leaves a periodic solution of the PDEs invariant. This approach can readily be applied to other problems. A more general group-theoretic approach, which can also treat period-doubling and Hopf bifurcations of spatio-temporally symmetric periodic orbits, is being developed by Lamb and Melbourne (1998).

In the future, we plan to tackle spatial period doubling and multiplying, where the τ_λ symmetries do not act trivially; such instabilities are relevant to simulations of convection carried out in larger boxes (Weiss *et al* 1996), and will be related to the study of the long-wavelength instabilities of ARs (Hoyle 1994). We also plan to examine the case of the hexagonal lattice: a Hopf bifurcation on a hexagonal lattice leads to a wide variety of periodic orbits with different spatio-temporal symmetries (Roberts *et al* 1986). Moreover, in recent experiments on parametrically excited surface waves (Kudrolli *et al* 1998), a transition from a hexagonal standing wave to a state possessing discrete spatio-temporal symmetries has been observed. This transition is also accompanied by a change in the spatial periodicity of the pattern. Finally, we plan to investigate the effect of including the extra Z_2 mid-layer reflection symmetry that arises when making the Boussinesq approximation for incompressible fluids.

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