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$\mathcal{PT}$-symmetry and its spontaneous breakdown explained by anti-linearity

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Abstract

The impact of an anti-unitary symmetry on the spectrum of non-hermitean operators is studied. Wigner's normal form of an anti-unitary operator is shown to account for the spectral properties of non-hermitean, $\mathcal{PT}$-symmetric Hamiltonians. Both the occurrence of single real or complex conjugate pairs of eigenvalues follows from this theory. The corresponding energy eigenstates span either one- or two-dimensional irreducible representations of the symmetry $\mathcal{PT}$. In this framework, the concept of a spontaneously broken $\mathcal{PT}$-symmetry is not needed.

Deep in their hearts, many quantum physicists will renounce hermiticity of operators only reluctantly. However, non-hermitean Hamiltonians are applied successfully in nuclear physics, biology and condensed matter, often modelling the interaction of a quantum system with its environment in a phenomenological way. Since 1998, non-hermitean Hamiltonians continue to attract interest from a conceptual point of view [1]: surprisingly, the eigenvalues of a one-dimensional harmonic oscillator Hamiltonian remain *real* when the *complex* potential $\hat{V} = i\hat{x}^3$ is added to it. Numerical, semiclassical, and analytic evidence [2] has been accumulated confirming that bound states with *real* eigenvalues exist for the vast class of *complex* potentials satisfying $V^\dagger(\hat{x}) = V(-\hat{x})$. In addition, pairs of complex conjugate eigenvalues occur systematically.

$\mathcal{PT}$-symmetry has been put forward to explain the observed energy spectra. The Hamiltonian operators $\hat{H}$ under scrutiny are invariant under the combined action of parity $\mathcal{P}$ and time reversal $\mathcal{T}$,

$$[\hat{H}, \mathcal{PT}] = 0\,. \tag{1}$$

They act on the fundamental observables according to

$$\mathcal{P}: \begin{cases} \hat{x} \to -\hat{x}\,, \\ \hat{p} \to -\hat{p}\,, \end{cases} \qquad \mathcal{T}: \begin{cases} \hat{x} \to \hat{x}\,, \\ \hat{p} \to -\hat{p}\,, \end{cases} \tag{2}$$

and $\mathcal{T}$ *anti-commutes* with the imaginary unit,

$$\mathcal{T} i = i^* \mathcal{T} \equiv -i \mathcal{T} \, . \tag{3}$$

Whenever a $\mathcal{PT}$-symmetric Hamiltonian has a *real* eigenvalue E, the associated eigenstate $|E\rangle$ is found to be an eigenstate of the symmetry $\mathcal{PT}$,

$$E = E^* : \quad \hat{H}|E\rangle = E|E\rangle \, , \quad \mathcal{PT}|E\rangle = +|E\rangle \, . \tag{4}$$

Occasionally, $\mathcal{PT}|E\rangle = -|E\rangle$ occurs [3] which is equivalent to (4) upon redefining the phase of the state: $\mathcal{PT}(i|E\rangle) = +(i|E\rangle)$. There is no difference between symmetry and anti-symmetry under $\mathcal{PT}$.

However, if the eigenvalue E is *complex*, the operator $\mathcal{PT}$ does *not* map the corresponding eigenstate of $\hat{H}$ to itself,

$$E \neq E^* : \quad \hat{H}|E\rangle = E|E\rangle \, , \quad \mathcal{PT}|E\rangle \neq \lambda|E\rangle \, , \text{ any } \lambda \, . \tag{5}$$

This situation is described as a 'spontaneous breakdown' of $\mathcal{PT}$-symmetry. No mechanism has been identified which would explain this breaking of the symmetry.

The $\mathcal{PT}$-symmetric square-well model provides a simple example for this behavior [4]. It describes a particle moving between reflecting boundaries at $x = \pm 1$, in the presence of a piecewise constant complex potential,

$$V_Z(x) = \begin{cases} iZ \, , & x < 0 \, , \\ -iZ \, , & x > 0 \, , \end{cases} \qquad Z \in \mathbb{R} \, . \tag{6}$$

Acceptable solutions of Schrödinger's equation must satisfy both the boundary conditions, $\psi(\pm 1) = 0$, and continuity conditions at the origin. As long as the value of the parameter Z is below a critical value, $Z < Z_0^c$, the eigenvalues E_n of the non-hermitean Hamiltonian $\hat{H} = -\partial_{xx} + V_Z(x)$ are real, and each eigenstate $|\psi_n\rangle$ satisfies the relations (4), with eigenvalues E_n and $+1$, respectively. Above the threshold, $Z > Z_0^c$, at least one pair of complex conjugate eigenvalues E_0 and E_0^* develops. One of the corresponding eigenstates has the form [4]

$$\psi_0(x) = \begin{cases} K_p \sinh \kappa(1 - x) \, , & x > 0 \, , \\ K_n \sinh \lambda^*(1 + x) \, , & x < 0 \, , \end{cases} \tag{7}$$

the complex parameters κ, λ, K_n, and K_p being determined by the boundary and continuity conditions. The state $\psi_0(x)$ is not invariant under $\mathcal{PT}$, i.e. (5) holds.

The purpose of the present contribution is a group-theoretical analysis of $\mathcal{PT}$-symmetry. The properties of $\mathcal{PT}$-symmetric systems are explained in a natural way by taking into account that $\mathcal{PT}$ is not a unitary but an *anti-unitary* symmetry of a *non-hermitean* operator. The argument proceeds in three steps. First, Wigner's normal form of anti-unitary operators is reviewed, i.e. their (irreducible) representations are identified. Second, the properties of non-hermitean operators with anti-unitary symmetry are derived. These results are then shown to account for the characteristic features of $\mathcal{PT}$-symmetric systems.

Wigner develops a normal form of anti-unitary operators $\hat{A}$ in [5]. Anti-unitarity of $\hat{A}$ is defined by the relation

$$\langle \hat{A}\chi | \hat{A}\psi \rangle = \langle \psi | \chi \rangle \tag{8}$$

and it implies anti-linearity,

$$\hat{A}(\alpha|\psi\rangle + \beta|\chi\rangle) = \alpha^* \hat{A}|\psi\rangle + \beta^* \hat{A}|\chi\rangle\,. \tag{9}$$

which is equivalent to (3). The representation theory of $\hat{A}$ relies on the fact that the square of an anti-unitary operator is *unitary*:

$$\langle \hat{A}^2\chi | \hat{A}^2\psi \rangle = \langle \hat{A}\psi | \hat{A}\chi \rangle = \langle \chi | \psi \rangle\,. \tag{10}$$

Therefore, the operator $\hat{A}^2$ has a complete, orthonormal set of eigenvectors $|\Omega\rangle$ with eigenvalues Ω of modulus one,

$$\hat{A}^2|\Omega\rangle = \Omega|\Omega\rangle\,, \quad |\Omega| = 1\,. \tag{11}$$

It plays the role of a Casimir-type operator labelling different representations of $\hat{A}$. Wigner distinguishes three different types of representations corresponding to the eigenvalues of $\hat{A}^2$: complex Ω ($\neq \Omega$), $\Omega = +1$, or $\Omega = -1$, summarized in Table (1).

1. An eigenstate $|\Omega\rangle$ of $\hat{A}^2$ with eigenvalue Ω ($\neq \Omega^*$) is not invariant under $\hat{A}$. Instead, the states $|\Omega\rangle$ and $|\Omega^*\rangle \equiv \hat{A}|\Omega\rangle$ constitute a 'flipping pair' with complex 'flipping value' ω (and ω^*), where $\omega^2 = \Omega$. They span a two-dimensional space which is closed under the action of $\hat{A}$. Therefore, it carries a two-dimensional representation of $\hat{A}$, denoted by Γ_*, which is *irreducible*: due to the anti-linearity of $\hat{A}$, no (non-zero) linear combination of the flipping states exist which is invariant under $\hat{A}$.

2. Similarly, if $\hat{A}^2$ has an eigenvalue $\Omega = -1$, then the operator $\hat{A}$ flips the states $|-\rangle$ and $|-^*\rangle \equiv \hat{A}|-\rangle$. The flipping value is i, and the associated two-dimensional representation Γ_- is not reducible.

3. Two different situations arise if there is an eigenstate $|1\rangle$ of $\hat{A}^2$ with eigenvalue $+1$. The state $\hat{A}|1\rangle$ is either a multiple of itself or not. In the first case, the space spanned by $|1\rangle$ is invariant under $\hat{A}$ and hence carries a one-dimensional representation γ_+ of $\hat{A}$. When redefining the phase of the state appropriately, one obtains an eigenstate $|1\rangle$ of $\hat{A}$ with eigenvalue $+1$. In the second case, the two states $|+\rangle \equiv |1\rangle$ and $|+^*\rangle \equiv \hat{A}|1\rangle$ provide a flipping pair with flipping value $\omega = +1$, and hence a representation Γ_+. This representation, however, is *reducible* due to the reality of the flipping value: the linear combinations $|1_r\rangle = |+\rangle + |+^*\rangle$ and $|1_i\rangle = i(|+\rangle - |+^*\rangle)$ are both eigenstates of $\hat{A}$ with eigenvalue $+1$.

$\Omega \equiv \omega^2$	Γ	action of $\hat{A}$	dim Γ
$\Omega \neq \Omega^*$	Γ_*	$\hat{A}\lvert\Omega\rangle = \omega^*\lvert\Omega^*\rangle$ $\hat{A}\lvert\Omega^*\rangle = \omega\lvert\Omega\rangle$	2
-1	Γ_-	$\hat{A}\lvert-\rangle = -i\lvert-^*\rangle$ $\hat{A}\lvert-^*\rangle = +i\lvert-\rangle$	2
$+1$	Γ_+	$\hat{A}\lvert+\rangle = +\lvert+^*\rangle$ $\hat{A}\lvert+^*\rangle = +\lvert+\rangle$	2
$+1$	γ_+	$\hat{A}\lvert 1\rangle = +\lvert 1\rangle$	1

Table 1: Representations Γ of the operator $\hat{A}$

Consequently, a Hilbert space $\mathcal{H}$ naturally decomposes into a direct product of invariant subspaces, each invariant under the action of the anti-unitary operator $\hat{A}$,

$$\mathcal{H} = {\Gamma_*}^{\otimes N_*} \otimes {\Gamma_-}^{\otimes N_-} \otimes {\Gamma_+}^{\otimes N_+} \otimes {\gamma_+}^{\otimes n_+}\,; \tag{12}$$

the nonnegative integers N_*, $N_\pm$ and n_+ are related to the degeneracies of the eigenvalues $\Omega\,(\neq \Omega^*)$ and $\Omega = \pm 1$ of the operator $\hat{A}^2$. The corresponding decomposition of a vector $|\psi\rangle \in \mathcal{H}$ is the closest analog of an expansion into the eigenstates of a hermitean (or unitary) operator. Surprisingly, *two*-dimensional irreducible representations of $\hat{A}$ exist although there is only one generator, $\hat{A}$. No 'good quantum number' exists which would label the states spanning these representations.

A (diagonalizable) *non-hermitean* Hamiltonian $\hat{H}$ with a discrete spectrum [6] and its adjoint $\hat{H}^\dagger$ each have a complete set of eigenstates:

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle\,, \quad \hat{H}^\dagger|\psi^n\rangle = E^n|\psi^n\rangle\,, \tag{13}$$

with complex conjugate eigenvalues related by $E^n = E_n^*$. They form a *bi-orthonormal* basis in $\mathcal{H}$, as they provide two resolutions of unity,

$$\sum_n |\psi^n\rangle\langle\psi_n| = \sum_n |\psi_n\rangle\langle\psi^n| = \hat{I}\,, \tag{14}$$

and satisfy orthogonality relations,

$$\langle\psi_m|\psi^n\rangle = \delta_m^n\,. \tag{15}$$

Let the non-hermitean operator $\hat{H}$ have an anti-unitary symmetry $\hat{A}$,

$$[\hat{H}, \hat{A}] = 0\,. \tag{16}$$

Then the unitary operator $\hat{A}^2$ commutes with $\hat{H}$, and it has eigenvalues Ω of modulus one. Consequently, there are simultaneous eigenstates $|n,\Omega\rangle$ of $\hat{H}$ and $\hat{A}^2$:

$$\hat{H}|n,\Omega\rangle = E_n|n,\Omega\rangle\,, \quad \hat{A}^2|n,\Omega\rangle = \Omega|n,\Omega\rangle\,, \quad E_n \in \mathsf{C}\,. \tag{17}$$

For simplicity, the eigenvalues Ω are assumed discrete and not degenerate. Wigner's normal form of anti-unitary operators suggests to consider three cases separately: complex $\Omega\,(\neq \Omega^*)$ and $\Omega = \pm 1$.

$\underline{\Omega \neq \Omega^*}$ The state

$$|n,\Omega^*\rangle \equiv \omega\hat{A}|n,\Omega\rangle\,, \quad \omega^2 = \Omega\,, \tag{18}$$

is a second eigenstate of $\hat{A}^2$, with eigenvalue Ω^*. The states $\{|n,\Omega\rangle, |n,\Omega^*\rangle\}$ provide a *flipping pair* under the action of the operator $\hat{A}$,

$$\hat{A}|n,\Omega\rangle = \omega^*|n,\Omega^*\rangle\,, \quad \hat{A}|n,\Omega^*\rangle = \omega|n,\Omega\rangle\,, \tag{19}$$

carrying the representation Γ_*. No degeneracy of the eigenvalue E_n is implied by the anti-unitary $\hat{A}$-symmetry of $\hat{H}$. However, the non-hermitean Hamiltonian has a second eigenstate $|n,\Omega^*\rangle$ with eigenvalue E_n^*,

$$\hat{H}|n,\Omega^*\rangle = E_n^*|n,\Omega^*\rangle\,, \tag{20}$$

as follows from multiplying the first equation of (17) with $\hat{A}$ and ω.

$\underline{\Omega = -1}$ Formally, the results for the representation Γ_- are obtained from the previous case by setting $\omega = i$. Again, a pair of complex conjugate eigenvalues is found, and the associated flipping pair spans a two-dimensional representation space.

$\underline{\Omega = +1}$ This case is conceptually different from the previous ones as two possibilities arise. Consider the state $|n,+\rangle$, an eigenvector of both $\hat{H}$ and $\hat{A}^2$ with eigenvalues E_n and $+1$, respectively. It satisfies Eqs. (17) with $\Omega \to +$. If, on the one hand, applying $\hat{A}$ to $|n,+\rangle$ results in $e^{i\phi}|n,+\rangle$, then the state $|n,1\rangle \equiv e^{-i\phi/2}|n,+\rangle$ is an eigenstate of $\hat{A}$ with eigenvalue $+1$,

$$\hat{A}|n,1\rangle = |n,1\rangle\,. \tag{21}$$

This occurrence of the one-dimensional representation γ_+ forces the associated eigenvalue E_n of $\hat{H}$ to be real since

$$E_n|n,1\rangle = \hat{H}\hat{A}|n,1\rangle = \hat{A}\hat{H}|n,1\rangle = E_n^*|n,1\rangle\,. \tag{22}$$

If, on the other hand, $|n,+^*\rangle \equiv \hat{A}|n,+\rangle$ is *not* a multiple of $|n,+\rangle$, then these states combine to form the representation Γ_+, the flipping value being $+1$. Further, the state $|n,+^*\rangle$ is an eigenstate of the Hamiltonian with eigenvalue E_n^*. As the flipping number is real, linear combinations of $|n,+\rangle$ and $|n,+^*\rangle$ do exist which are eigenstates of $\hat{A}$—however, they are not eigenstates of $\hat{H}$. Consequently, the anti-unitary symmetry of the Hamiltonian makes itself felt (on a subspace with $(\mathcal{PT})^2 = +\hat{I}$) by either a single real eigenvalue or a pair of two complex conjugate eigenvalues.

If any of the two-dimensional representations Γ_* or $\Gamma_\pm$ occurs and the associated eigenvalue happens to be real, the anti-unitary symmetry implies a twofold degeneracy of the energy eigenvalue. Again, the symmetry provides no additional label, and simultaneous eigenstates of $\hat{H}$ and $\hat{A}$ can be constructed for Γ_+ only. These cases will be denoted by Γ_*^d or $\Gamma_\pm^d$.

It will be shown now that the properties of $\mathcal{PT}$-symmetric quantum systems are consistent with the representation theory of non-hermitean Hamiltonians possessing an anti-unitary symmetry. Upon identifying

$$\hat{A} = \mathcal{PT} \, , \tag{23}$$

one needs to check the value of $(\mathcal{PT})^2$ when applied to eigenstates of the Hamiltonian in order to decide which of the representations Γ_*, $\Gamma_\pm$, or γ_+, is realized. Various explicit examples will be given now.

For parameters $Z < Z_0^c$, the eigenvalues of the $\mathcal{PT}$-symmetric square-well are real throughout, and the operators $\hat{H}$ and $\mathcal{PT}$ have common eigenstates. Thus, the relations (4) correspond to a multiple occurrence of the representation γ_+, compatible with $(\mathcal{PT})^2 = +\hat{I}$.

For $Z > Z_0^C$, the energy eigenstate $\psi_0(x) \equiv \langle x|E_0,+\rangle$ in (7) satisfies $(\mathcal{PT})^2|E_0,+\rangle$ $= +|E_0,+\rangle$. Therefore, the states $|E_0,+\rangle$ and $|E_0,+^*\rangle \equiv \mathcal{PT}|E_0,+\rangle$ carry a representation Γ_+, and the presence of two complex energy eigenvalues, E_0 and E_0^* is justified. Eqs. (5) can be completed to read:

$$E \neq E^* : \quad \begin{array}{ll} \hat{H}|E_0,+\rangle = E_0|E_0,+\rangle \, , & \mathcal{PT}|E_0,+\rangle = +|E_0,+^*\rangle \, , \\ \hat{H}|E_0,+^*\rangle = E_0^*|E_0,+^*\rangle \, , & \mathcal{PT}|E_0,^*\rangle = +|E_0,+\rangle \, . \end{array} \tag{24}$$

Consequently, $\mathcal{PT}$-symmetry is not broken but at $Z = Z_0^c$ the system switches between the representations Γ_+ and γ_+, with a corresponding change of the energy spectrum.

The following examples are taken from a discrete family of non-hermitean operators [7],

$$\hat{H}_M = \hat{p}^2 - (\zeta \cosh 2x - iM)^2 \, , \quad \zeta \in I\!R \, , \tag{25}$$

M taking positive integer values. Each operator $\hat{H}_M$ is invariant under the combined action of $\mathcal{PT}$ where $\mathcal{P}$ is parity about the point $a = i\pi/2$: $x \to i\pi/2 - x$. Due to the reflection about a point off the real axis, the operators $\mathcal{P}$ and $\mathcal{T}$ do not commute as has been pointed out in [8]. However, this fact is not essential here since only the anti-unitary character of the symmetry $\mathcal{PT}$ is essential.

For $M = 2$, two complex conjugate eigenvalues E_+ and $E_- = E_+^*$ of $\hat{H}_2$ exist, with associated eigenstates

$$\psi_+(x) = \Psi(x) \cosh x \equiv \langle x|E_+,-\rangle \, , \quad \psi_-(x) = \Psi(x) \sinh x \equiv \langle x|E_+,-^*\rangle \, , \tag{26}$$

and a $\mathcal{PT}$-invariant function $\Psi(x) = \exp[(i/2)\zeta \cosh 2x]$. These states are a flipping pair with flipping value i,

$$\mathcal{PT}\psi_+(x) = -i\psi_-(x) \, , \quad \mathcal{PT}\psi_-(x) = i\psi_+(x) \, , \tag{27}$$

and the twofold application of $\mathcal{PT}$ gives (-1). Hence, the representation Γ_- is realized. Similarly, for $M = 4$, four eigenstates form two flipping pairs, i.e. two representations Γ_-, each being associated with a pair of complex conjugate eigenvalues.

For $M = 3$, three different real eigenvalues of the Hamiltonian $\hat{H}_3$ have been obtained analytically if $\zeta^2 < 1/4$. The corresponding eigenfunctions are given by

$$\psi(x) = \Psi(x) \sinh 2x \,, \quad \psi_\pm(x) = \Psi(x)(A \cosh 2x \pm iB) \,, \tag{28}$$

with real coefficients A and B. Under the action of $\mathcal{PT}$, the state $\psi(x)$ is mapped to itself, while $\psi_\pm(x)$ each acquire an additional minus sign. Therefore, the states $\psi(x) \equiv \langle x|E, +\rangle$ and $i\psi_\pm(x) \equiv \langle x|E_\pm\rangle$ are simultaneous eigenstates of $\hat{H}$ and $\mathcal{PT}$ with eigenvalues $+1$. The part of Hilbert space spanned by these three states transforms according to three copies of the representation γ_+. If $\zeta = 1/2$, the eigenvalues $E_\pm$ turn degenerate, and the eigenstates given in (28) merge, $i\psi_+(x) = i\psi_-(x) \equiv \varphi(x)$. However, a second, independent $\mathcal{PT}$-invariant solution of Schrödinger's equation can be found,

$$\phi(x) = \Psi(x) \int_{x_0}^{x} dy \frac{e^{-i\varphi(y)/2}}{\varphi^2(y)} \,. \tag{29}$$

The solutions $\{\varphi, \phi\}$ transform according to $\gamma_+ \otimes \gamma_+ \equiv \Gamma_+^d$. So far, the representation Γ_* has apparently not been realized in $\mathcal{PT}$-symmetric quantum systems—a possible explanation is the constraint $\mathcal{T}^2 = \pm 1$ for time reversal [9].

In summary, the representation theory of anti-unitary symmetries of non-hermitean 'Hamiltonians' has been developed on the basis of Wigner's normal form of anti-unitary operators. Typically, energy eigenvalues come in complex conjugate pairs, and the associated eigenstates of the Hamiltonian span a two-dimensional space carrying one of the two-dimensional representations Γ_*, or $\Gamma_\pm$. Furthermore, a single real eigenvalue may occur, related to a one-dimensional representation γ_+. In this case a single $\hat{A}$-invariant energy eigenstate state exists while there are no simultaneous eigenstates of the Hamiltonian and the symmetry operator in the two-dimensional $\hat{A}$-invariant subspaces. Instead, flipping pairs of states can be identified. Generally, the symmetry does not imply the existence of degenerate eigenvalues—only if the Hamiltonian happens to have a real eigenvalue, a two-dimensional degenerate subspace may exist occasionally. These results naturally explain the properties of eigenstates and eigenvalues of $\mathcal{PT}$-symmetric quantum systems. In particular, it is not necessary to invoke the concept of a *spontaneously broken* $\mathcal{PT}$-symmetry. Contrary to a unitary or hermitean symmetry, the presence of an anti-unitary symmetry does not imply the existence of a set of simultaneous eigenstates of $\hat{H}$ and $\mathcal{PT}$—simply because an anti-linear operator is not guaranteed to have a complete set of eigenstates. Finally, the present approach provides a new perspective on the suggested modification of the scalar product in Hilbert space [10] which will be presented elsewhere [11] in detail.

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