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著者	LIU Chuan, LIN Shou, LUDWIG Lewis D.
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SPACES WITH A σ-POINT-DISCRETE WEAK BASE

By

Chuan LIU, Shou LIN, and Lewis D. LUDWIG

Abstract. In this paper σ-point-discrete weak bases are considered. Three necessary conditions that individually ensure that a space with a σ-point-discrete weak base has a σ-compact-finite weak base are given. We show that σ-compact-finite weak bases are preserved by closed sequence-covering maps. It is shown that a space X is metrizable if and only if X^{ω} has a σ-point-discrete weak base. Conditions are given to ensure when a paratopological group with σ-point-discrete weak base is metrizable. Several open questions are posed.

1. Introduction

Metrization theorems have played a key role in the study of general topology. Many now classic metrization theorems involve the use of different types of bases. For example,

THEOREM 1.1. *The following are equivalent for a regular space X*:
(1) *X is a metrizable space*;
(2) *X has a σ-locally finite base* [18] [23];
(3) *X has a σ-compact-finite base* [2];
(4) *X has a σ-hereditarily closure-preserving base* [3].

Besides these results, much more is known. For instance, there is a non-metrizable space with a σ-point-discrete base [3]. On the other hand, it was shown

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that a g-metrizable space (i.e., a regular space with a σ-locally finite weak base) if and only if it has a σ-hereditarily closure-preserving weak base [11]. It is still an open problem whether a regular space with a σ-compact-finite weak base is g-metrizable [17]. It was proved that a space has a σ-compact-finite weak base if and only if it is a k-space with a σ-point-discrete weak base. It is still unknown whether a separable space with a σ-point-discrete weak base has a countable weak base [12]. Thus some relations among special point-discrete families, compact-finite families and locally finite families are interesting.

In this paper, we continue this work by considering spaces with σ-point-discrete weak bases. Spaces with σ-compact-finite weak bases play an important role in this study, so in Section 3 we give three necessary conditions that individually ensure that a space with a σ-point-discrete weak base has a σ-compact-finite weak base (Theorem 3.1). In section 4, we show how σ-point-discrete weak bases behave under certain types of mappings. In particular, we show that σ-compact-finite weak bases are preserved by closed sequence-covering maps (Theorem 4.1). We provide a characterization of metrizable spaces by σ-point-discrete weak bases (Theorem 5.4). In Section 6 we use σ-point-discrete weak bases to provide necessary conditions to ensure that certain paratopological groups are metrizable. We close with several open questions in Section 7.

2. Necessary Preliminaries

We begin with some basic definitions. In this paper, all spaces are regular T_1, and all maps are continuous and onto. Readers may refer to Engelking [4] for unstated definitions and terminology.

DEFINITION 2.1. Let $\mathscr{B} = \{B_\alpha : \alpha \in I\}$ be a family of subsets of a space X.

(1) $\mathscr{B}$ is *point-discrete* (or *weakly hereditarily closure-preserving* [3]) if $\{x_\alpha : \alpha \in I\}$ is closed discrete in X, whenever $x_\alpha \in B_\alpha$ for each $\alpha \in I$.

(2) $\mathscr{B}$ is *compact-finite* if any compact subset of X meets at most finitely many members of $\mathscr{B}$.

It is easy to see that each compact-finite family is point-discrete in a k-space.

DEFINITION 2.2. Let X be a topological space. For every $x \in X$ let $\mathscr{T}_x$ be a family of subsets of X containing x. If the collection satisfies

(1) for every $x \in X$ the intersections of finitely many members of $\mathscr{T}_x$ belong to $\mathscr{T}_x$ and

(2) $U \subset X$ is open in X if and only if $x \in U$ implies $x \in T \subset U$ for some $T \in \mathcal{T}_x$

then it is called a *weak base* for X.

A topological space X is *weakly first-countable* if it has a weak base $\{\mathcal{T}_x : x \in X\}$ such that each $\mathcal{T}_x$ is countable. Each weakly first countable space is a sequential space [22] and each sequential space is a k-space [4]. A space is said to be a *g-metrizable* space if it has a σ-locally finite weak base [22].

Definition 2.3. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) $\mathcal{P}$ is called a *network* for X if for every $x \in X$ and any neighborhood U of x there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

(2) $\mathcal{P}$ is called a *cs-network* for X [6] if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then $\{x_n : n \geq m\} \cup \{x\} \subset P \subset U$ for some $m \in \mathbf{N}$ and some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is called a *k-network* for X [5] if whenever $K \subset U$ with K compact and U open in X, there exists $\mathcal{P}'$ of finitely many members of $\mathcal{P}$ such that $K \subset \bigcup \mathcal{P}' \subset U$.

If a space has a weak base, this also forms a *cs*-network for the space [22]. Each point-countable *cs*-network of a sequential space is a k-network [8]. A weak base for a space need not form k-network [11].

Definition 2.4. The *tightness* of a point x in a space X is the smallest cardinal number $\mathbf{m} \geq \omega$ with the property that if $x \in \overline{C}$, then there is $C_0 \subset C$ such that $|C_0| \leq \mathbf{m}$ and $x \in \overline{C}_0$; this cardinal number is denoted by $t(x, X)$. The tightness of a space X is the supremum of all numbers $t(x, X)$ for $x \in X$; this cardinal number is denoted by $t(X)$ [4].

Each sequential space has countable tightness.

3. Spaces with σ-compact-finite Weak Bases

In this section we consider under what conditions a space with a σ-point-discrete weak base has a σ-compact-finite weak base. Before we present the main results of the section we recall the following two results [9].

Lemma 3.1. *The following are equivalent for a space* X:

(1) *X has a σ-compact-finite weak base*;
(2) *X is a k-space with a σ-point-discrete weak base*;
(3) *X is a weakly first countable space with a σ-point-discrete weak base*;
(4) *$X \times S$ has a σ-point-discrete weak base, here S is a non-trivial convergent sequence.*

Lemma 3.2. *Let $\mathscr{P}$ be a point-discrete family of a space X. If $\mathscr{P}$ is a subset of a weak base at some $x \in X$ and there is a non-trivial sequence converging to x in X, then $\mathscr{P}$ is finite.*

Theorem 3.1. *Let X be a space with a σ-point-discrete weak base. Then X has a σ-compact-finite weak base if one of the following conditions holds*:
(1) **(CH)** $t(X) \le \omega$;
(2) *Each point of X is a G_δ-set and $t(X) \le \omega$*;
(3) $|X| < \aleph_\omega$.

Proof. By Lemma 3.1, we only need to show that X is weakly first countable. Let $\mathscr{B} = \bigcup\{\mathscr{B}_x(n) : x \in X, n \in \mathbf{N}\}$ be a weak base for X. Here $\bigcup\{\mathscr{B}_x(n) : x \in X\}$ is point-discrete for each $n \in \mathbf{N}$ and $\bigcup\{\mathscr{B}_x(n) : n \in \mathbf{N}\}$ is a weak base at x in X for each $x \in X$. Fix a non-isolated point $x_0 \in X$.

(1) Assume **CH** and $t(X) \le \omega$. Since $x_0 \in \overline{X\backslash\{x_0\}}$, there is a countable subset $B \subset X\backslash\{x_0\}$ with $x_0 \in \bar{B}$. $\bar{B}$ is a separable subspace with a σ-point-discrete weak base, hence $\bar{B}$ is g-metrizable [12, Theorem 2.8] under **CH**. Therefore, there is a non-trivial sequence in $\bar{B}\backslash\{x_0\}$ converging to x_0. By Lemma 3.2, $\mathscr{B}_{x_0}(n)$ is finite for each $n \in \mathbf{N}$. Thus, X is weakly first countable at x_0.

(2) Assume that each point of X is a G_δ-set and $t(X) \le \omega$. Let $\{U_n\}$ be a sequence of open neighborhoods of x_0 in X with $\{x_0\} = \bigcap\{U_n : n \in \mathbf{N}\}$ and each $\overline{U}_{n+1} \subset U_n$. For each $n \in \mathbf{N}$ and $P \in \mathscr{B}_{x_0}(n)$, since x is a non-isolated point in X and P is a weak neighborhood of x, $(P\backslash\{x_0\}) \cap U_n \ne \emptyset$. Thus pick $x(P,n) \in (P\backslash\{x_0\}) \cap U_n$. Let $Y = \{x_0\} \cup \{x(P,n) : P \in \mathscr{B}_{x_0}(n), n \in \mathbf{N}\}$. Then Y is a closed subset of X and x_0 is the unique non-isolated point of Y. So $\mathscr{B}|_Y$ is not only is a weak base for Y, but also is a base for Y [11]. Hence Y has a σ-point-discrete base and $t(Y) \le \omega$, and Y is metrizable [14, Theorem 2.1]. Thus there is a non-trivial sequence $\{x_n\} \subset Y\backslash\{x_0\}$ converging to x. By Lemma 3.2, $\mathscr{B}_{x_0}(n)$ is finite for each $n \in \mathbf{N}$. Therefore, X is weakly first-countable at x_0.

(3) Assume $|X| < \aleph_\omega$. Let $\mathscr{B}_{x_0} = \bigcup\{\mathscr{B}_{x_0}(n) : n \in \mathbf{N}\}$. We only need to show that $|\mathscr{B}_{x_0}| \le \aleph_0$.

First, prove that $|\mathscr{B}_{x_0}| < \aleph_\omega$. Suppose not, $|\mathscr{B}_{x_0}| \geq \aleph_\omega$. We write $\mathscr{B}_{x_0}(n) = \{B_\alpha(n) : \alpha \in I_n\}$ with I_n well-order for each $n \in \mathbf{N}$. Choose $x_\alpha(n) \in B_\alpha(n)$ for each $n \in \mathbf{N}$, $\alpha \in I_n$ by inductive method as follows. First, take a point $x_0(1) \in B_0(1)$. Assume $x_\alpha(n) \in B_\alpha(n)$ have been selected for each $n < k$, $\alpha \in I_n$ or $n = k$, $\alpha < \gamma$, where $x_\alpha(n) \neq x_\beta(n)$ if $\alpha \neq \beta$; $x_\alpha(i) \neq x_\beta(j)$ if $i \neq j$; $x_\alpha(n) \neq x_0$ for $\alpha \in I_n$, $n < k$ or $\alpha \in I_k$, $\alpha < \gamma$. Let $U = X \setminus (\bigcup\{x_\alpha(n) : n < k, \alpha \in I_n\} \cup \{x_\alpha(k) : \alpha < \gamma\})$. Then U is an open neighborhood of x_0, and we can pick $x_\gamma(k) \in U \cap B_\gamma(k) \setminus \{x_0\}$. This completes the inductive choice. Next, let $A = \{x_\alpha(n) : n \in \mathbf{N}, \alpha \in I_n\}$. Then $|A| = |\mathscr{B}_{x_0}| \geq \aleph_\omega$. On the other hand, $|A| \leq |X| < \aleph_\omega$, this is a contradiction. Hence $|\mathscr{B}_{x_0}| < \aleph_\omega$.

Now we prove that $|\mathscr{B}_{x_0}| \leq \aleph_0$. Suppose $|\mathscr{B}_{x_0}| = \aleph_n$ for some $n \in \mathbf{N}$, then $|\mathscr{B}_{x_0}(m)| = \aleph_n$ for some $m \in \mathbf{N}$. We rewrite that $\mathscr{B}_{x_0}(m) = \{B_\alpha : \alpha < \aleph_n\}$, $\mathscr{B}_{x_0} = \{C_\alpha : \alpha < \aleph_n\}$. Since x_0 is a non-isolated point, $B_\alpha \cap C_\alpha \neq \{x_0\}$ for each $B_\alpha \in \mathscr{B}_{x_0}(m)$ and $C_\alpha \in \mathscr{B}_{x_0}$. Thus pick $x_\alpha \in B_\alpha \cap C_\alpha \setminus \{x_0\}$ for each $\alpha < \aleph_n$. Then $\{x_\alpha : \alpha < \aleph_n\}$ is a closed discrete subset in X since $\mathscr{B}_{x_0}(m)$ is point-discrete. On other other hand, $x_0 \in \overline{\{x_\alpha : \alpha < \aleph_n\}}$ because $\mathscr{B}_{x_0}$ is a weak base at x_0. This is a contradiction. Hence X is weakly first-countable. $\square$

We now improve part (3) of Lemma 3.1.

THEOREM 3.2. *A space X has a σ-compact-finite weak base if and only if it is a weakly first countable space with a σ-point-discrete cs-network.*

PROOF. We only need to show sufficiency. Let $\mathscr{B} = \bigcup\{\mathscr{B}_n : n \in \mathbf{N}\}$ be a *cs*-network for X. Here $\mathscr{B}_n$ is point-discrete for each $n \in \mathbf{N}$. Since X is sequential, we may assume $\mathscr{B}_n \subset \mathscr{B}_{n+1}$ and each $\mathscr{B}_n$ is closed under finite intersection, in fact, if $\mathscr{B}_n$ is point-discrete, then $\{\bigcap \mathscr{A} : \mathscr{A} \in [\mathscr{B}_n]^{<\omega}\}$ is point-discrete if X is sequential. For each $x \in X$, if x is an isolated point, then $\{x\} \in \mathscr{B}$. If x is not an isolated point, let $\mathscr{B}_n(x) = \{B \in \mathscr{B}_n : x \in B$, and B contains a non-trivial sequence converging to $x\}$. Then $|\mathscr{B}_n(x)| < \omega$ for $n \in \mathbf{N}$.

Suppose not, then consider an infinite subset $\{B_k : k \in \mathbf{N}\} \subset \mathscr{B}_n(x)$. For each $m \in \mathbf{N}$, let $\{x_i(m)\}_i \subset B_m$ be a non-trivial sequence converging to x. There is $i_m \in \mathbf{N}$ such that $\{x_i(m) : i \geq i_k\}$ only meets finitely many B_k's. Otherwise, there is a subsequence $\{x_{i_j}\}_j$ of $\{x_i(m)\}_i$ such that each distinct x_{i_j} belong to a distinct B_k. Since $\mathscr{B}_n(x)$ is point-discrete, $\{x_{i_j} : j \in \mathbf{N}\}$ is closed discrete, hence X has a closed copy of the sequential fan S_ω, this is a contradiction since S_ω is not weakly first countable. Thus $|\mathscr{B}_n(x)| < \omega$.

Let $\mathscr{B}_x = \bigcup\{\mathscr{B}_n(x) : n \in \mathbf{N}\}$. Then $\mathscr{B}_x$ is a countable *cs*-network at x for X. Since X is weakly first-countable, there is a subfamily $\mathscr{P}_x \subset \mathscr{B}_x$ such that $\mathscr{P}_x$ is a weak base at x [8, Lemma 7(3)]. Hence $\bigcup\{\mathscr{P}_x : x \in X, x$ is not an isolated point$\} \cup \{\{x\} : x$ is an isolated point$\} \subset \mathscr{B}$ is a σ-point-discrete weak base of X. □

Example 3.1. There exists a weakly first countable space X with a σ-point-discrete k-network such that X has not any σ-point-discrete *cs*-network.

Proof. A space X having the properties was constructed Burke, Engelking, and Lutzer [3, Example 9.8]. Let Z be the topological sum of the closed unit interval $[0,1] = \mathbf{I}$ and the family $\{S(x) : x \in \mathbf{I}\}$ of 2^ω non-trivial convergent sequence $S(x)$. Let X be the space obtained from Z by identifying the limit point of $S(x)$ with $x \in \mathbf{I}$ for each $x \in \mathbf{I}$. Then X is a quotient and compact image of a metric space, hence X is a weakly first countable space.

Next, a σ-point-discrete k-network for X is given as follows. Assume that $S(x) = \{x\} \cup \{(x, 1/n) : n \in \mathbf{N}\}$ for each $x \in \mathbf{I}$, and denote X by $\mathbf{I} \cup \{(x, 1/n) : x \in \mathbf{I}, n \in \mathbf{N}\}$. Let $S_n(x) = \{(x, 1/i) : i \geq n\}$ for each $x \in X$, $n \in \mathbf{N}$. Let $\mathscr{P}_1$ be a countable base for $\mathbf{I}$ with respect to the usual topology, $\mathscr{P}_2 = \{\{x\} : x \in X \backslash \mathbf{I}\}$, and $\mathscr{P}_n = \{S_{n-2}(x) : x \in \mathbf{I}\}$ for each $n > 2$. Then $\bigcup\{\mathscr{P}_n : n \in \mathbf{N}\}$ is a σ-point-discrete k-network for X.

It was shown that X has no point-countable weak base [8, Remark 14(2)]. Then X has no σ-point-discrete *cs*-network by Theorem 3.2. □

4. Some Mapping Theorems

In this section we discuss some mapping properties of spaces with σ-point-discrete weak bases. It is known that spaces with a σ-compact-finite weak base are not preserved by perfect maps. For example, let S_2 be the Arens' space and S_ω the sequential fan. There is a perfect map $f : S_2 \to S_\omega$, however S_2 is a g-metrizable space and S_ω is not weakly first countable [24].

A map $f : X \to Y$ is called a *sequence-covering map* if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_n\}$ in X with $x_n \in f^{-1}(y_n)$ for each $n \in \mathbf{N}$ [21]. f is called a *compact-covering map* if whenever L is compact in Y there is a compact subset K in X such that $f(K) = L$ [4]. Recently, the first author proved that closed, irreducible and sequence-covering maps preserve σ-compact-finite weak bases [11]. This result is sharpened by the following.

Theorem 4.1. *σ-compact-finite weak bases are preserved by closed sequence-covering maps.*

Proof. Let $f : X \to Y$ be a closed sequence-covering map and X have a σ-compact-finite weak base. Since each image of a space with a point-countable weak base under a closed and sequence-covering map is weakly first countable [11, Lemma 3.1], Y is weakly first countable. It is easy to check that spaces with a σ-point-discrete *cs*-network are preserved by closed sequence-covering maps. Hence Y is a space with a σ-compact-finite weak base by Theorem 3.2. □

Theorem 4.2. *Each closed map on a space with a σ-point-discrete weak base is compact-covering under* **CH**.

Proof. Under **CH**, let $f : X \to Y$ be a closed map and X have a σ-point-discrete weak base. Assume that L is a compact subset in Y. We first show that L is metrizable. Since f is a closed map and X has a σ-point-discrete network, then Y also has a σ-point-discrete network. Let $\mathscr{P} = \bigcup\{\mathscr{P}_n : n \in \mathbf{N}\}$ be a network for Y, where each $\mathscr{P}_n$ is point-discrete. For each $n \in \mathbf{N}$, put $D_n = \{y \in Y : \mathscr{P}_n \text{ is not point finite at } y\}$ and $\mathscr{F}_n = \{P \backslash D_n : P \in \mathscr{P}_n\} \cup \{\{y\} : y \in D_n\}$. Then $\mathscr{F}_n$ is compact-finite in Y, and $\bigcup\{\mathscr{F}_n : n \in \mathbf{N}\}$ is a network for Y by the work of the first author [10, Proposition 2]. Thus L has a countable network and L is metrizable [4].

Now, L is a compact metric space, so it is separable. Take $D \subset L$ such that $|D| \leq \omega$ and $\bar{D} = L$. For each $y \in D$, pick $x_y \in f^{-1}(y)$. Let $E = \{x_y : y \in D\}$. Then $|E| \leq \omega$, and $\bar{E}$ is separable. By the work of the first author [12, Theorem 2.8], $\bar{E}$ has a countable weak base under **CH**, so $\bar{E}$ is a paracompact space. Since f is a closed map, we have

$$f(\bar{E}) = \overline{f(E)} = \bar{D} = L.$$

It is well known that a closed map on a paracompact space is a compact-covering [4], hence there exists a compact subset K with $K \subset \bar{E}$ such that $f(K) = L$. □

Example 4.1. A closed map on spaces with a σ-compact-finite *cs*-network need not be compact-covering.

Proof. There is a closed map f from a space X onto the one-point compactification of the discrete space of cardinality ω_1 such that each compact subset of X is finite. Then X has a σ-compact-finite *cs*-network, but f is not compact-covering [20, Example 1]. □

5. Metrization Theorems

In this section some metrization theorems are given for spaces with a σ-point-discrete base. A space X is called a *κ-Fréchet-Urysohn space* if for every $x \in \overline{U}$ with U open in X there exists a sequence $\{x_n\} \subset U$ converging to x in X [15]. Each Fréchet-Urysohn space is κ-Fréchet-Urysohn. But a κ-Fréchet-Urysohn space need not be a k-space or a space with a countable tightness [15]. The next result shows that a κ-Fréchet-Urysohn space is metrizable if it has a σ-point-discrete weak base.

Theorem 5.1. *A space X is metrizable if and only if it is a κ-Fréchet-Urysohn space with a σ-point-discrete weak base.*

Proof. Let X be a κ-Fréchet-Urysohn space with a σ-point-discrete weak base. First, we prove that X is weakly first-countable. Let $\mathscr{B} = \bigcup\{\mathscr{B}_x(n) : x \in X, n \in \mathbf{N}\}$ be a σ-point-discrete weak base as the proof of Theorem 3.1. Fix a non-isolated point $x \in X$, then $x \in \overline{X\backslash\{x\}}$. Since X is κ-Fréchet-Urysohn, there is a sequence $\{x_n\} \subset X\backslash\{x\}$ converging to x. $\mathscr{B}_x(n)$ is finite for each $n \in \mathbf{N}$ by Lemma 3.2. Thus $\bigcup\{\mathscr{B}_x(n) : n \in \mathbf{N}\}$ is a countable weak base of x.

It is straightforward to prove that a κ-Fréchet-Urysohn, weakly first-countable space is first-countable [15]. Thus X is first-countable. By Lemma 3.1, X has a σ-compact-finite weak base. It is not difficult to show that a compact-finite family in a first-countable space is locally-finite [2]. Thus X has a σ-locally-finite weak base. A first-countable space with a σ-locally-finite weak base is metrizable [22, Theorem 1.13]. □

We can prove the following by the similar method in Theorem 3.1(3).

Theorem 5.2. *If X has a σ-point-discrete base, then $\chi(X)$*[1] *$\leq |X|$.*

Theorem 5.3. *Let X be a space with a σ-point-discrete base. If for any non-isolated point $x \in X$, there is a subset $A \subset X$ such that $|A| < \aleph_\omega$ and $x \in \overline{A\backslash\{x\}}$, then X is metrizable.*

[1] $\chi(X)$ denotes the character of X.

Proof. Considering the proof of Theorem 3.1(3), X is first-countable. Then X is metrizable by Theorem 5.1. □

Corollary 5.1. *Let X have a σ-point-discrete base. Then X is metrizable if $t(X) < \aleph_\omega$, in particular, $|X| < \aleph_\omega$.*

Theorem 5.4. *If X^ω has a σ-point-discrete weak base, then X is metrizable.*

Proof. We can assume that $|X| > 1$. X contains $D = \{0, 1\}$ as a closed copy. Since $X^\omega \times X^\omega$ has a σ-point-discrete weak base and $X^\omega \times S$ is a closed subset of $X^\omega \times X^\omega$, where S is a non-trivial convergent sequence in D^ω, $X^\omega \times S$ has a σ-point-discrete weak base. By Lemma 3.1, X^ω has a σ-compact-finite weak base, hence it is a sequential space. Since every point-countable weak base is a point-countable k-network [8], X has a point-countable k-network and X^ω is a sequential space, then X has a point-countable base [16, Theorem 3.8], hence X is metrizable by Theorem 5.1. □

It is natural to ask the following question: If X^n has a σ-point-discrete weak base for each $n \in \mathbf{N}$, is X metrizable? The answer is no.

Example 5.1. There exists a non-metrizable space X such that X^n has a σ-point-discrete base for each $n \in \mathbf{N}$.

Proof. Burke, Engelking, and Lutzer created a space X which is non-metrizable with a σ-point-discrete base as follows [3, Eaxmple 9]. Let A be the set of all ordinals having cardinality less than $\aleph_\omega$ and let $Z = \{0, 1\}^A$. For each $z \in Z$ and $\alpha < \aleph_\omega$, put $z(\alpha) = \pi_\alpha(z)$, here $\pi_\alpha : Z \to \{0, 1\}$ is the projection onto the α-th coordinate. Let $X = \{s\} \cup \{z \in Z : \{\alpha \in A : z(\alpha) = 0\} \in A^{<\omega}\}$, here $s \in Z$ with $s(\alpha) = 0$ for each $\alpha < \aleph_\omega$. We now endow X with a topology. Let each point in $X \backslash \{s\}$ is isolated. The basic neighborhood of s is of form $\{B \cap X : B \in \mathscr{B}\}$, where $\mathscr{B}$ is a basic neighborhood base at s in the product space Z.

For each $n \in \mathbf{N}$, let

$$\mathscr{B}_1(n) = \{\{z\} : z \in X \backslash \{s\}, |\{\alpha \in A : z(\alpha) = 0\}| = n\}, \quad \text{and}$$

$$\mathscr{B}_2(n) = \{B \cap X : B \in \mathscr{B}, \Gamma(B) \subset [0, \omega_n]\},$$

$$\text{where } \Gamma(B) = \{\alpha \in A : \pi_\alpha(B) = \{0\}\}.$$

It was shown that $\mathscr{B}' = \bigcup_{n \in \mathbf{N}} (\mathscr{B}_1(n) \cup \mathscr{B}_2(n))$ is a σ-point-discrete base for X [3].

Fix $k \in \mathbf{N}$. We prove that $\{\prod_{i \le k} \mathscr{B}_m(j_i) : m = 1, 2, j_i \in \mathbf{N}\}$ is a σ-point-discrete base for X^k. If $p \in \overline{B \backslash \{p\}}$ for a subset $B \subset X^k$, there exists $m \le k$ such that $\pi_m(p) = s$. Since $s \in \overline{\pi_m(B) \backslash \{s\}}$, then $|\pi_m(B)| = \aleph_\omega$ by Theorem 5.3. Note that $|\mathscr{B}_2(n)| < \aleph_\omega$ for each $n \in \mathbf{N}$, then $\prod_{i \le k} \mathscr{B}_m(j_i)$ is a point-discrete family. □

It is known that $(S_2)^n$ has a countable weak base for each $n \in \mathbf{N}$.

6. Paratopological Groups with A σ-point-discrete Weak Base

Next, an application for paratopological groups is given. We first recall some compact-type properties. Let X be a Tychonoff space. X is called *Ohio complete* if in every Hausdorff compactification bX of X there exists a G_δ-subset Z such that $X \subset Z$ and every $y \in Z \backslash X$ is separated from X by a G_δ-subset of Z. X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ that has a countable base of open neighborhoods of X. It is known that locally compact spaces are p-spaces and p-spaces are Ohio complete spaces [1, 4].

A *paratopological group* G is a group G with a topology such that the product mapping of $G \times G$ into G is jointly continuous.

Theorem 6.1. *Let G be a paratopological group with a σ-point-discrete weak base and $bX \backslash X$ be Ohio complete. Then G is a metrizable space.*

Proof. If G is locally compact, then G has a σ-compact-finite weak base by Lemma 3.1, thus each compact subspace of G is metrizable. Hence G is first-countable, and it is metrizable by Theorem 5.1.

If G is not locally compact, then G is nowhere locally compact because G is homogeneous. It follows that the remainder, $X = bG \backslash G$, is dense in bG. Hence bG is also a Hausdorff compactification of X. Since X is Ohio complete, there is a G_δ-subset Y of bG such that $X \subset Y$ and every $y \in Y \backslash X$ can be separated from X by a G_δ-subset of Y.

Case 1: $X = Y$.

X is a G_δ-set in bG. Then G is a σ-compact subspace of bG, thus it is a Lindelöf space with a σ-point-discrete weak base. Thus G is weakly first-countable [9, Corollary 2], hence it is first-countable [19]. So G is metrizable.

Case 2: $Y \backslash X \neq \varnothing$.

Fix $y \in Y\backslash X$. There exists a G_δ-subset P of Y such that $y \in P \subset Y\backslash X$. Since Y is a G_δ-subset of bG, P is a G_δ-subset of bG. There is a sequence $\{U_n\}$ of open subsets of bG with $y \in P = \bigcap_{n\in\mathbf{N}} U_n$. Thus, we can find a sequence $\{V_n\}$ of open subsets of bG such that $y \in V_n \subset U_n$ and $\overline{V}_{n+1} \subset V_n$ for each $n \in \mathbf{N}$ because bG is regular. Let $K = \bigcap_{n\in\mathbf{N}} \overline{V}_n$. Then K is a compact G_δ-subset in bG. Hence K is metrizable, and $\{V_n\}_{n\in\mathbf{N}}$ is a neighborhood base of K in bG. So G is first-countable at y [4, 3.1.E], thus G is metrizable. □

Corollary 6.1. *Let G be a non-locally compact paratopological group with a σ-point-discrete weak base and $bG\backslash G$ be a p-space. Then G is a separable metrizable space.*

Proof. Since each p-space is Ohio complete [1], G is metrizable. G is nowhere locally compact, and bG is a Hausdorff compactification of $bG\backslash G$. Also, a p-space is of countable type. G is Lindelöf by [7], hence G is a separable metrizable space. □

Theorem 6.2. *Let G be a paratopological group with a σ-point-discrete weak base. Then G is metrizable if $Y = bG\backslash G$ is not pseudocompact.*

Proof. If G is locally compact, then G is first-countable in view of the proof of Theorem 6.1, hence G is metrizable.

If G is a non-locally compact paratopological group, then $Y = bG\backslash G$ is dense in bG. Since Y is not pseudocompact, there exists an infinite disjoint family $\xi = \{U_n : n \in \mathbf{N}\}$ of non-empty open sets in Y such that ξ is discrete in Y. For each $n \in \mathbf{N}$, find an open subset V_n of bG such that $U_n = V_n \cap Y$. $\eta = \{V_n : n \in \mathbf{N}\}$ does not have limit points in Y since ξ is discrete and $\overline{V_n} = \overline{U_n}$ for $n \in \mathbf{N}$. Let K be the all limit points of the family η in bG. Then K is closed in bG and $K \subset G$. Since bG is compact, K is a nonempty compact subset of bG. G has a σ-point-discrete weak base, K is metrizable. Fix $x \in K$, K has a countable base at x. Pick a decreasing family $\{W_n : n \in \mathbf{N}\}$ of open neighborhood of x in bG such that $\bigcap\{\overline{W_n} : n \in \mathbf{N}\} \cap K = \{x\}$. Since $x \in K$, we may define an increasing sequence of integers n_k with $x_{n_k} \in W_n \cap V_{n_k}$. Since bG is compact, $\{x_{n_k} : k \in \mathbf{N}\}$ has a cluster point, clearly, x is the unique limit point of $\{x_{n_k} : k \in \mathbf{N}\}$, hence $x_{n_k} \to x$.

By Lemma 3.2, G is weakly first countable at x. Hence G is first countable since G is a paratopological group, and G is metrizable by Theorem 5.1. □

7. Questions

We close with several questions that are natural extensions of this work.

Question 7.1. Let X have a σ-point-discrete (weak) base, is every point of X a G_δ-set?

Question 7.2. Let X be a pseudocompct (or ccc) space with a σ-point-discrete (weak) base, is X metrizable?

Remark: If the answer for Question 7.1 is positive, then a pseudocompact space with a σ-point-discrete weak base is metrizable.

Question 7.3. Let X have a σ-point-discrete (weak) base, is X normal, meta-Lindelöf?

Question 7.4. Let G be a topological (or paratopological) group with a σ-point-discrete (weak) base, is G metrizable?

Question 7.5. Are spaces with a σ-compact-finite weak base preserved by open and closed maps?

Remark: If each point of the domain of the map is a G_δ-set, then the answer to the above question is positive.

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(Liu)Department of Mathematics
Ohio University Zanesville Campus
Zanesville, OH 43701, USA
E-mail address: liuc1@ohio.edu

(Lin)Department of Mathematics
Zhangzhou Normal University
Zhangzhou 363000, P. R. China
Institute of Mathematics
Ningde Teachers' College
Ningde, Fujian 352100, P. R. China
E-mail address: linshou@public.ndptt.fj.cn

(Ludwig)Department of Mathematics and Computer Science
Denison University
Granville, OH 43023, USA
E-mail address: ludwigl@denison.edu