Semi-infinite Lakshmibai-Seshadri path model for level-zero extremal weight modules over quantum affine algebras

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<td>学位授与年度</td>
<td>2014年</td>
</tr>
<tr>
<td>報告番号</td>
<td>データ第117号</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2241/00123234">http://hdl.handle.net/2241/00123234</a></td>
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Motohiro Ishii

February 2014
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Motohiro Ishii
Doctoral Program in Mathematics

Submitted to the Graduate School of
Pure and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Science
at the
University of Tsukuba
1 Introduction

The quantized universal enveloping algebra (or, quantum group) $U_q(g)$ associated with a (symmetrizable) Kac–Moody Lie algebra $g$ was introduced independently by Drinfeld ([Dri85]) and Jimbo ([Jim85]) in their study of solvable lattice models in statistic physics. This is a non-commutative and non-cocommutative Hopf algebra defined over the field $C(q)$, where $q$ is transcendental over $C$ or $q \in C^\times$, and is defined by deforming the defining relations of the universal enveloping algebra $U(g)$ of $g$; the classical $U(g)$ can be obtained from $U_q(g)$ by taking the limit $q \to 1$. Since then, the quantized universal enveloping algebras have provided many applications to mathematics such as the representation theory of algebraic groups and Hecke algebras, quantum invariants of knots and 3-manifolds ([RT91]), and so forth. Also, it is known that the quantized universal enveloping algebras naturally arise in mathematics. For example, Ringel ([Rin90]) gave a realization of the negative part $U_q^-(g)$ of $U_q(g)$ as the Hall algebra of quiver representations (or equivalently, representations of a path algebra) over a finite field; recently, Bridgeland ([Bri13]) described the whole of $U_q(g)$ via the Hall algebra of $(\mathbb{Z}/2\mathbb{Z})$-graded complexes of quiver representations. Also, Khovanov–Lauda ([KL09, KL11]) and Rouquier ([Rou08]) provided a categorification of $U_q(g)$ by use of representations of certain generalizations of affine Hecke algebras, called Khovanov–Lauda–Rouquier (or, quiver Hecke) algebras.

Motivated by Ringel’s description of quantized universal enveloping algebras, Lusztig ([Lus90, Lus91]: see also [Lus93]) gave a geometric realization of $U_q^-(g)$ by using the theory of perverse sheaves ([BBD82]) on moduli spaces of quiver representations, and then he discovered remarkable bases of $U_q^-(g)$ and the integrable (irreducible) highest weight $U_q(g)$-module $V(\lambda)$ of highest weight $\lambda$, called canonical bases. Also, in an algebraic viewpoint, Kashiwara ([Kas90, Kas91]) independently developed the theory of (global) crystal bases of $U_q^-(g)$ and $V(\lambda)$; in fact, it is known from [GL93] that the global crystal bases are identical to the canonical bases. The crystal basis $B(\infty)$ (resp., $B(\lambda)$) of $U_q^-(g)$ (resp., $V(\lambda)$) can be regarded as a basis at the crystal limit $q = 0$, which has a combinatorial structure, called a crystal ([Kas93]; see also [HK02, Kas95, Kas02a, Lus93]). At the crystal limit, many of the problems in representation theory of $U_q(g)$ can be reduced to combinatorial ones. So, in order to obtain further applications of crystal bases to the study of representation theory of $U_q(g)$ and other areas of mathematics, it is a fundamental problem to give explicit realizations of crystal bases. For integrable highest weight $U_q(g)$-modules, there are many useful realizations of their crystal bases. For example, Littelmann ([Lit94, Lit95]) introduced the Lakshmibai–Seshadri (LS for short) path model for $V(\lambda)$ in his study of the standard monomial theory ([LMS79, LS86]); soon after, Kashiwara ([Kas96]) and Joseph ([Jos95]) independently proved that the crystal $B(\lambda)$ of LS paths of shape $\lambda$ is isomorphic to the crystal basis $B(\lambda)$. In addition, we know the following realizations of the crystal basis $B(\lambda)$ of the integrable highest weight $U_q(g)$-module $V(\lambda)$:

- polyhedral realization, due to Nakashima–Zelevinsky ([NZ97]);
- Lagrangian construction via quiver varieties, due to Kashiwara–Saito ([KS97, Sai02]);
- monomial realization, due to Nakajima ([Nak03]; see also [HN06]);
- categorification via Khovanov–Lauda–Rouquier algebras, due to Lauda–Vazirani ([LV11]);

for $g$ of affine type.
• Kyoto path realization via perfect crystals, due to Kang–Kashiwara–Misra–Miwa–Nakashima–Nakayashiki ([KKMMNN92]);
• Young wall realization, due to Kang ([Kan03]);

and for \( \mathfrak{g} \) of finite type,

• tableaux realization, due to Kashiwara–Nakashima ([KN94]);
• realization in terms of Mirković–Vilonen cycles and polytopes, due to Braverman–Finkelberg–Gaitsgory, Anderson, and Kamnitzer ([And03, BG01, BFG06, Kam07, Kam10]).

In his study of the canonical basis of the tensor product of an integrable highest weight \( U_q(\mathfrak{g}) \)-module and an integrable lowest weight \( U_q(\mathfrak{g}) \)-module, Lusztig ([Lus92]) introduced the modified version of the quantized universal enveloping algebra, and showed that this algebra has the canonical basis (or global crystal basis). Also, Kashiwara ([Kas94]) studied the crystal basis of the modified quantized universal enveloping algebra, and proved that the structure of it is essentially reduced to that of the crystal bases of extremal weight modules over \( U_q(\mathfrak{g}) \). Here, for each (not necessarily dominant) integral weight \( \lambda \) of \( \mathfrak{g} \), the extremal weight \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) of extremal weight \( \lambda \) is a natural generalization of an integrable highest or lowest weight \( U_q(\mathfrak{g}) \)-module; in fact, if \( \lambda \) is a dominant (resp., anti-dominant) integral weight for \( \mathfrak{g} \), then \( V(\lambda) \) is isomorphic to the integrable highest (resp., lowest) weight \( U_q(\mathfrak{g}) \)-module of highest (resp., lowest) weight \( \lambda \). However, the structures of a general extremal weight module and its crystal basis are more complicated than those of integrable highest (or, lowest) weight modules and their crystal bases; for example, general extremal weight modules are not necessarily semisimple, and there are no explicit realizations of crystal bases of extremal weight modules except in some very special cases.

Extremal weight modules over the quantized universal enveloping algebra associated with an affine Lie algebra are especially important, and we will give an explanation about them in detail. Let \( \mathfrak{g}_{af} \) be an (untwisted) affine Lie algebra (see [Kac90]); we call the quantized universal enveloping algebra \( U_q(\mathfrak{g}_{af}) \) associated with \( \mathfrak{g}_{af} \) the quantum affine algebra for short. Let \( I_{af} \) be the set of indices of simple roots of \( \mathfrak{g}_{af} \) with a special index 0 \( \in I_{af} \); we set \( I := I_{af} \setminus \{0\} \) and let \( \mathfrak{g} \) be the finite-dimensional semisimple Lie subalgebra of \( \mathfrak{g}_{af} \) corresponding to \( I \). The level of an integral weight \( \lambda \) of \( \mathfrak{g}_{af} \) is defined as the pairing \( \langle c, \lambda \rangle \in \mathbb{Z} \) with the canonical central element \( c \) of \( \mathfrak{g}_{af} \). It is known that if the level of \( \lambda \) is positive (resp., negative), then the extremal weight module \( V(\lambda) \) of extremal weight \( \lambda \) is a highest (resp., lowest) weight module, and the structure of \( V(\lambda) \) is well-known in this case. So, it is essential to consider the level-zero extremal weight modules. In [Kas02a] (see also [AK97, Kas05]), Kashiwara studied level-zero extremal weight modules over quantum affine algebras, and showed that any finite-dimensional irreducible representation of the subalgebra \( U'_q(\mathfrak{g}_{af}) := U_q([\mathfrak{g}_{af}, \mathfrak{g}_{af}]) \) of \( U_q(\mathfrak{g}_{af}) \) can be obtained as a quotient of tensor product of level-zero extremal weight modules; finite-dimensional representation theory of \( U'_q(\mathfrak{g}_{af}) \) is studied by many people in connection with integrable systems, geometric, categorical, and combinatorial representation theory including Kirillov–Reshetikhin modules and crystals, cluster algebras, Khovanov–Lauda–Rouquier algebras, and so forth (see e.g., [KR90, HKOTY99, HKOTT02, FZ02, FZ03, BFZ05, FZ07, KKK13a, KKK13b, KKK13c]). Also, it is known from [Nak04, Remark 2.15] (see also [CP01, Proposition 4.5]) that the extremal weight module \( V(\lambda) \)
of level-zero extremal weight $\lambda$ is isomorphic to the quantum (or global) Weyl module $W_q(\lambda)$, introduced by Chari–Pressley ([CP01]). Notice that the level-zero extremal weight module is also equipped with an action of the quantum loop algebra $U_q(\mathcal{L}_q)$, where $\mathcal{L}_q := g \otimes \mathbb{C}[t, t^{-1}]$ is the loop algebra associated with $g$; remark that $U_q(\mathcal{L}_q)$ (resp., $\mathcal{L}_q$) is not a subalgebra of $U_q(g_{af})$ (resp., $g_{af}$). Then, when $g$ is of type $A, D, E$, level-zero $V(\lambda)$ is isomorphic to the universal standard module $M(\lambda)$, introduced by Nakajima ([Nak01]), which is defined as the Grothendieck group of equivariant coherent sheaves on the (Lagrangian) Nakajima quiver variety $\mathcal{L}(\lambda)$, with a level-zero dominant integral weight $\lambda$, equipped with the geometrically defined action of the quantum loop algebra $U_q(\mathcal{L}_q)$.

In [Kas02a], Kashiwara stated some conjectural structures on level-zero extremal weight modules over $U_q(g_{af})$ and their crystal bases. His conjecture was proved by Beck–Nakajima ([BN04]; see also [Bec02, Nak04]), and in particular, it was shown that

$$B(\lambda) \cong \bigotimes_{i \in I} B(m_i \varpi_i)$$  \hspace{1cm} (1.0.1)

with $\lambda = \sum_{i \in I} m_i \varpi_i$, $m_i \in \mathbb{Z}_{\geq 0}$, where $\varpi_i$, $i \in I$, denotes an $i$-th level-zero fundamental weight of $g_{af}$. Also, Naito–Sagaki ([NS03, NS06]) gave an explicit realization of the crystal basis $B(m_i \varpi_i)$ for each $i \in I$ in terms of LS paths of shape $m_i \varpi_i$. Namely, they proved that the set $B(m_i \varpi_i)$ of LS paths of shape $m_i \varpi_i$ is isomorphic as a crystal to $B(m_i \varpi_i)$, though neither of (the crystal graphs of) these two crystals is connected if $m_i > 1$. However, it turned out in [NS08] that the crystals $B(\lambda)$ and $B(\lambda)$ are not necessarily isomorphic for a general level-zero dominant integral weight $\lambda$; for example, if $\lambda$ is of the form $\sum_{i \in K} \varpi_i$ for $K \subseteq I$ with $\# K \geq 2$, then the crystals $B(\lambda)$ and $B(\lambda)$ are never isomorphic, though both of these crystals are connected (for details, see [NS08, Appendix]). This is mainly because each connected component of the crystal $B(\lambda)$ has fewer extremal elements than that of the crystal $B(\lambda)$.

Let us explain the situation above more precisely. As an easy consequence of the isomorphism (1.0.1), we see that the extremal elements in the connected component $B_0(\lambda)$ of $B(\lambda)$ containing the extremal element $u_\lambda$ of extremal weight $\lambda$ is given as $\{u_x := S_x u_\lambda \mid x \in W_{af}\} \cong W_{af}/(W_J)_{af}$, where $W_{af} = W \ltimes \{\xi \mid \xi \in Q^+\}$ is the (affine) Weyl group of $g_{af}$, and $(W_J)_{af} := W_J \ltimes \{\xi \mid \xi \in Q^+_J\}$, with $J := \{i \in I \mid m_i = 0\}$, is equal to the stabilizer $\{x \in W_{af} \mid S_x u_\lambda = u_\lambda\}$ of $u_\lambda$ in $W_{af}$ (see Proposition 5.1.1); here $S_x$, $x \in W_{af}$, denote the action of $W_{af}$ on $B(\lambda)$. In contrast, the extremal elements in the connected component $B_0(\lambda)$ of $B(\lambda)$ containing the straight line path $\pi_\lambda := (\lambda; 0, 1)$ of weight $\lambda$ is easily seen to be the straight lines $\{\pi_{x \lambda} := (x \lambda; 0, 1) \mid x \in W_{af}\} \cong W_{af}/(W_J)_{af}$, where $(W_J)_{af}$ is the stabilizer of $\lambda$ in $W_{af}$, and equals $W_J \ltimes \{t_\xi \mid \{\xi, \lambda\} = 0\}$. Here we have $(W_J)_{af} \subset (W_J)_{af}$ in general, with equality if and only if $\lambda$ is a nonnegative integer multiple of $\varpi_i$ for some $i \in I$ modulo the null root $\delta$.

In order to overcome the difficulty above, we introduce the notion of semi-infinite Lakshmibai–Seshadri ($\frac{\infty}{\infty}$-LS for short) paths of shape $\lambda$. A $\frac{\infty}{\infty}$-LS path of shape $\lambda$ is, by definition, a pair $(x; a)$ of a decreasing sequence $x : x_1 \geq \frac{\infty}{\infty} x_2 \geq \frac{\infty}{\infty} \cdots \geq \frac{\infty}{\infty} x_s$ in the set $(W_J)_{af}$ of Peterson’s coset representatives for the cosets in $W_{af}/(W_J)_{af}$, equipped with the semi-infinite Bruhat order $\geq_{\frac{\infty}{\infty}}$, and an increasing sequence $a : 0 < a_0 < a_1 < \cdots < a_s = 1$ of rational numbers, while a (usual) LS path of shape $\lambda$ is a pair $(\lambda; a)$ of a decreasing sequence $\lambda : \lambda_1 > \lambda_2 > \cdots > \lambda_s$ of elements in the affine Weyl group orbit $W_{af} \lambda$, equipped with the partial order $\geq$ that Littelmann defined in [Lit95], and an increasing sequence $a$ of rational numbers as above.
The coset representatives \((W^J)_\text{af}\) were originally introduced by Peterson ([Pet97]; see also [LS10]) in his study of the relationship between the \(T\)-equivariant homology (ring) \(H^T_\ast(G_G)\) of the affine Grassmannian \(G_G := G(\mathbb{C}(t))/G(\mathbb{C}[t])\) and the \(T\)-equivariant (small) quantum cohomology ring \(\mathcal{QH}^T_\ast(G/P)\) of the partial flag variety \(G/P\), where \(G\) denotes a simply-connected simple algebraic group over \(\mathbb{C}\), \(P \subset G\) a parabolic subgroup, and \(T \subset G\) a maximal torus; notice that \(G_G\) is weakly homotopy equivalent to the space \(\Omega K\) of based loops into the maximal compact subgroup \(K \subset G\) ([GR75, Mit88]), and the ring structure of the \(T\)-equivariant homology \(H^T_\ast(G_G) \cong H^T_\ast(\Omega K)\) comes from the group structure of \(\Omega K\). Also, we see from [Soe97, Claim 4.14] that the semi-infinite Bruhat order on the affine Weyl group \(W_{\text{af}}\) is essentially the same one as Lusztig’s generic Bruhat order on \(W_{\text{af}}\) (for details, see Appendix A.2); the generic Bruhat order was originally introduced by Lusztig ([Lus80]) in his study of the conjectural character formula for the irreducible quotient of the Weyl module of a simply-connected almost simple algebraic group over an algebraically closed field of (sufficiently large) positive characteristic. As for the geometric meaning of the semi-infinite Bruhat order, it is known ([FFKM99, §5]; see also [FF90, §4]) that the semi-infinite Bruhat order describes the closure relation among the fine Schubert strata, parametrized by \(W_{\text{af}}\), of the Drinfeld compactification \(\mathcal{QM}^\alpha\), called the space of quasi-maps, of the variety of algebraic maps of a fixed degree \(\alpha \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i\) from the complex projective line \(\mathbb{P}^1\) to the flag variety \(G/B\). In addition, we remark that in Peterson’s lecture note ([Pet97]), the semi-infinite Bruhat order (or, stable Bruhat order in his terminology) plays an important role, and that some of our arguments in the study of the \(\frac{\omega}{2}\)-LS paths use (parabolic) quantum Bruhat graphs ([BFP99, LNSS13a]), which appear in the equivariant quantum Chevalley formula for \(\mathcal{QH}^T_\ast(G/P)\) ([Mih07]).

Now we are ready to state our main result. First, we can show that the natural surjection from the poset \((W^J)_\text{af}\) (equipped with the semi-infinite Bruhat order) onto the poset \(W_{\text{af}}\lambda\) (equipped with Littelmann’s partial order) is order-preserving, and hence that there exists a surjection from the set \(B^\infty(\lambda)\) of \(\frac{\omega}{2}\)-LS paths of shape \(\lambda\) onto the set \(B(\lambda)\) of \(LS\) paths of shape \(\lambda\). We define a crystal structure on \(B^\infty(\lambda)\) for \(U_q(\mathfrak{g}_{\text{af}})\) in such a way that this surjection becomes a morphism of crystals. Then we obtain the following theorem.

**Theorem 1.0.1.** Let \(\mathfrak{g}_{\text{af}}\) be an untwisted affine Lie algebra, and \(\lambda = \sum_{i \in I} m_i \varpi_i\) a level-zero dominant integral weight, with \(m_i \in \mathbb{Z}_{\geq 0}\) for all \(i \in I\). Let \(B(\lambda)\) denote the crystal basis of the extremal weight module \(V(\lambda)\) over \(U_q(\mathfrak{g}_{\text{af}})\), and let \(B^\infty(\lambda)\) denote the set of \(\frac{\omega}{2}\)-LS paths of shape \(\lambda\), equipped with the \(U_q(\mathfrak{g}_{\text{af}})\)-crystal structure as above. Then, we have an isomorphism of crystals

\[
B(\lambda) \cong B^\infty(\lambda).
\]

We remark that for each \(i \in I\), we have a natural identification \(B^\infty(m_i \varpi_i) = B(m_i \varpi_i)\), since the equality \((W_{\text{af}})_i = (W_{\text{af}})_{m_i \varpi_i}\), holds. Hence we recover the results obtained in [NS05, NS06] by Naito–Sagaki.

Also, we should mention that Hernandez and Nakajima ([HN06]) gave a monomial realization of a connected component of \(B(\lambda)\) for a general Kac–Moody Lie algebra; however, their realization is given in a recursive way, and hence it is difficult to determine all the elements in \(B(\lambda)\) explicitly in this realization.

This paper is organized as follows. In §2, we fix our notation for affine root data and review Peterson’s coset representatives and semi-infinite Bruhat order on them. In §3, we introduce the
notion of \( \frac{\Xi}{\Xi} \)-LS paths and define a crystal structure on \( \mathbb{B}^\Xi(\lambda) \), deferring until §4 the proof of the stability property of \( \frac{\Xi}{\Xi} \)-LS paths under Kashiwara operators. Also, we state our main result, i.e., the isomorphism theorem above between \( B(\lambda) \) and \( \mathbb{B}^\Xi(\lambda) \). In §4, we study the relation among quantum Bruhat graphs, Littelmann’s level-zero weight posets, and semi-infinite Bruhat graphs, and then show that there exists a surjective morphism of crystals from \( \mathbb{B}^\Xi(\lambda) \) to \( B(\lambda) \). Also, we give the deferred proof of the fact that the set \( \mathbb{B}^\Xi(\lambda) \) is stable under the Kashiwara operators \( e_i, f_i, i \in I_{af} \), defined in §3. In §5, we show that the connected component \( B_0(\lambda) \) of \( B(\lambda) \) containing \( u_\lambda \) is isomorphic, as a crystal, to the connected component \( \mathbb{B}^\Xi_0(\lambda) \) of \( \mathbb{B}^\Xi(\lambda) \) containing the element \( \eta_{e} := (e; 0, 1) \). In §6, we first study certain directed paths in (parabolic) quantum Bruhat graphs in order to give a bijection between all the connected components of \( B(\lambda) \) and those of \( \mathbb{B}^\Xi(\lambda) \). By combining this result with the results in §4 and §5, we finally obtain the desired isomorphism \( B(\lambda) \cong \mathbb{B}^\Xi(\lambda) \). In Appendix A, we give a new description of the semi-infinite Bruhat order on Peterson’s coset representatives. Also, we mention the relation between the semi-infinite Bruhat order and Lusztig’s generic Bruhat order.

Acknowledgements. This paper is based on a joint work with Professor Satoshi Naito and Professor Daisuke Sagaki. I would like to thank Professor Masahiko Miyamoto, Professor Satoshi Naito, Professor Daisuke Sagaki, Professor Hiroki Shimakura, Professor Yusuke Arike, and Professor Scott Carnahan for their mathematical supports and warm encouragements.

2 Preliminaries

2.1 Untwisted affine root data

Let \( g_{af} \) be an untwisted affine Lie algebra over \( \mathbb{C} \) with Cartan subalgebra \( h_{af} \). Let \( \{ \alpha_i \}_{i \in I_{af}} \subset h_{af}^* := \text{Hom}_\mathbb{C}(h_{af}, \mathbb{C}) \) and \( \{ \alpha_i^\vee \}_{i \in I_{af}} \subset h_{af} \) be the sets of simple roots and simple coroots, respectively. Let \( \langle - , - \rangle : h_{af} \times h_{af}^* \to \mathbb{C} \) denote the canonical pairing. Throughout this paper, we take and fix a weight lattice \( P_{af} \subset h_{af}^* \) satisfying

\[
\begin{align*}
\{ & \alpha_i \in P_{af} \text{ and } \alpha_i^\vee \in \text{Hom}_\mathbb{Z}(P_{af}, \mathbb{Z}) \text{ for all } i \in I_{af}, \\
& \text{for each } i \in I_{af}, \text{ there exists } \lambda_i \in P_{af} \text{ such that } (\alpha_j^\vee, \lambda_i) = \delta_{ij} \text{ for } j \in I_{af}. \nonumber
\end{align*}
\]

Let \( \delta = \sum_{i \in I_{af}} a_i \alpha_i \) and \( c = \sum_{i \in I_{af}} a_i^\vee \alpha_i^\vee \) be the null root and the canonical central element, respectively. We take and fix \( \delta \in I_{af} \) such that \( \delta_0 = a_0^\vee = 1 \), and set \( I := I_{af} \setminus \{0\} \). For each \( i \in I \), we define \( \varpi_i := \lambda_i - \langle e, \lambda_i \rangle \Lambda_0 \); note that \( \langle c, \varpi_i \rangle = 0 \) for all \( i \in I \). Set

\[
\begin{align*}
Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i, & \quad Q^\vee := \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee, & \quad P^+ := \sum_{i \in I} \mathbb{Z}_{\ge 0} \varpi_i. \quad (2.1.1)
\end{align*}
\]

Let \( W_{af} := \langle r_i \mid i \in I_{af} \rangle \) be the (affine) Weyl group of \( g_{af} \), where \( r_i \) denotes the simple reflection with respect to \( \alpha_i \), and set \( W := \langle r_i \mid i \in I \rangle \subset W_{af} \). Let \( e \in W_{af} \) be the unit element, and \( \ell : W_{af} \to \mathbb{Z}_{\ge 0} \) the length function. Denote by \( \le \) the Bruhat order on \( W_{af} \).

Denote by \( \Delta_{af} \) the set of real roots of \( g_{af} \), and \( \Delta_{af}^+ \) the set of positive real roots of \( g_{af} \); we know from [Kac90, Proposition 6.3] that

\[
\begin{align*}
\Delta_{af} &= \{ \alpha + n \delta \mid \alpha \in \Delta, \ n \in \mathbb{Z} \}, \nonumber \\
\Delta_{af}^+ &= \Delta^+ \cup \{ \alpha + n \delta \mid \alpha \in \Delta, \ n \in \mathbb{Z}_{> 0} \}. \quad (2.1.2)
\end{align*}
\]
where $\Delta := \Delta_{af} \cap Q$ is the (finite) root system corresponding to $I$, and $\Delta^+ := \Delta \cap \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. For each $x \in W_{af}$, we set $\text{Inv}(x) := \{ \beta \in \Delta^+_I \mid x \beta = -\Delta^+_I \}$; note that \( f(x) = \# \text{Inv}(x) \).

For $\beta \in \Delta_{af}$, denote by $\beta^\vee$ the coroot of $\beta$, and $r_\beta \in W_{af}$ the reflection with respect to $\beta$. For $\xi \in Q^\vee$, denote by $t_\xi \in W_{af}$ the translation with respect to $\xi$. We know from [Kac90, Proposition 6.5] that $\{ t_\xi \mid \xi \in Q^\vee \}$ is an abelian normal subgroup of $W_{af}$, with $t_\xi t_\zeta = t_{\xi + \zeta}$ for $\xi, \zeta \in Q^\vee$, and $W_{af} = W \ltimes \{ t_\xi \mid \xi \in Q^\vee \}$; remark that if $\beta \in \Delta_{af}$ is of the form $\beta = \alpha + n \delta$ with $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then

$$r_\beta = r_\alpha t_{n \alpha^\vee}.$$

For $w \in W$ and $\xi \in Q^\vee$, we have

$$wt_\xi w = w \mu - \langle \xi, \mu \rangle \delta \quad \text{if } \mu \text{ satisfies } \langle c, \mu \rangle = 0.$$

For a subset $J$ of $I$, set

$$Q_J := \bigoplus_{\alpha \in J} \mathbb{Z} \alpha, \quad Q_J^\vee := \bigoplus_{\alpha \in J} \mathbb{Z} \alpha^\vee, \quad \Delta_J := \Delta \cap Q_J,$$

$$\Delta_J^+ := \Delta_J \cap \sum_{\alpha \in J} \mathbb{Z}_{\geq 0} \alpha, \quad W_J := \langle r_j \mid j \in J \rangle, \quad \rho_J := \frac{1}{2} \sum_{\alpha \in \Delta_J^+} \alpha.$$ (2.1.5)

Set $\rho := \rho_I = (1/2) \sum_{\alpha \in \Delta^+} \alpha$. Since $\langle \alpha_j^\vee, \rho \rangle = (\alpha_j^\vee, \rho) = 1$ for all $j \in J$, we have

$$\langle \xi, \rho - \rho_J \rangle = 0 \quad \text{for all } \xi \in Q_J^\vee.$$

Denote by $W^J$ the set of minimal coset representatives of $W/W_J$; we see from [BB05, §2.4] that

$$W^J = \{ w \in W \mid w a \in \Delta^+ \text{ for all } a \in \Delta_J^+ \}.$$ (2.1.8)

For $w \in W$, we denote by $[w] \in W^J$ the minimal coset representative for the coset $wW_J$ in $W/W_J$.

### 2.2 Peterson’s coset representatives ($W^J_{af}$)

Let $J$ be a subset of $I$. Following [Pet97] (see also [LS10, §10]), we define

$$(\Delta_J)_{af} := \{ \alpha + n \delta \mid \alpha \in \Delta_J, n \in \mathbb{Z} \} \subset \Delta_{af},$$

$$(\Delta_J)^+_{af} := (\Delta_J)_{af} \cap \Delta^+_{af} = \Delta_J^+ \cup \{ \alpha + n \delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}_{\geq 0} \} \subset \Delta^+_{af},$$

$$(W_J)_{af} := W_{af} \ltimes \{ t_\xi \mid \xi \in Q_J^\vee \},$$

$$(W_J)_{af} := \{ x \in W_{af} \mid x \beta \in \Delta^+_{af} \text{ for all } \beta \in (\Delta_J)^+_{af} \}. $$ (2.2.1)

Remark 2.2.1. We see that $(W_J)_{af} = \langle r_\beta \mid \beta \in (\Delta_J)^+_{af} \rangle$. Indeed, we have $r_\beta = r_\alpha t_{n \alpha^\vee} \in (W_J)_{af}$ for all $\beta = \alpha + n \delta$ in $(\Delta_J)^+_{af}$, which implies that $(W_J)_{af} \supset \langle r_\beta \mid \beta \in (\Delta_J)^+_{af} \rangle$. Also, we have $W_J = \langle r_j \mid j \in J \rangle \subset \langle r_\beta \mid \beta \in (\Delta_J)^+_{af} \rangle$, and $t_{\alpha_j^\vee} = r_\alpha r_{\alpha_j + \delta} \in \langle r_\beta \mid \beta \in (\Delta_J)^+_{af} \rangle$ for $j \in J$, which implies that $t_\xi \in \langle r_\beta \mid \beta \in (\Delta_J)^+_{af} \rangle$ for all $\xi \in Q_J^\vee$. Thus we have proved $(W_J)_{af} \subset \langle r_\beta \mid \beta \in (\Delta_J)^+_{af} \rangle$, and hence $(W_J)_{af} = \langle r_\beta \mid \beta \in (\Delta_J)^+_{af} \rangle$.

Lemma 2.2.2 ([Pet97]; see also [LS10, Lemma 10.6]). For every $x \in W_{af}$, there exists a unique factorization $x = x_1 x_2$ with $x_1 \in (W_J)_{af}$ and $x_2 \in (W_J)_{af}$. 

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Definition 2.2.3. Define a (surjective) map $\Pi^J : W_{af} \to (W^J)_{af}$ by $\Pi^J(x) := x_1$ if $x = x_1x_2$ with $x_1 \in (W^J)_{af}$ and $x_2 \in (W_J)_{af}$.

Lemma 2.2.4 ([Pet97]; see also [LS10, Proposition 10.10]). (1) $\Pi^J(w) = |w|$ for $w \in W$.

(2) $\Pi^J(xt_\xi) = \Pi^J(x)\Pi^J(t_\xi)$ for $x \in W_{af}$ and $\xi \in Q^\vee$.

Definition 2.2.5 ([LNSSS13a, §3.4]). An element $\xi \in Q^\vee$ is said to be $J$-adjusted if $(\xi, \gamma) \in \{-1, 0\}$ for all $\gamma \in \Delta^+_J$. Let $Q^\vee_{J, adj}$ denote the set of $J$-adjusted elements.

Lemma 2.2.6 ([LNSSS13a, (3.7) and Lemma 3.7]). (1) For each $\xi \in Q^\vee$, there exists a unique $\phi_J(\xi) \in Q^\vee_{J, adj}$.

(2) For each $\xi \in Q^\vee$, there exists a unique $z_\xi \in W_J$ such that $\Pi^J(t_\xi) = z_\xi t_{\xi + \phi_J(\xi)}$.

(3) For $w \in W$ and $\xi \in Q^\vee$, we have $\Pi^J(wt_\xi) = |w|z_\xi t_{\xi + \phi_J(\xi)}$. In particular,

$$(W^J)_{af} = \{ wz_\xi t_\xi \mid w \in W^J, \xi \in Q^\vee_{J, adj} \}. \quad (2.2.5)$$

Let us show some technical lemmas, which are needed later.

Lemma 2.2.7. Let $x \in (W^J)_{af}$ and $i \in I_{af}$. If $x^{-1}\alpha_i \notin (\Delta_J)_{af}$, then $r_i x \in (W^J)_{af}$.

Proof. Following the definition (2.2.4) of $(W^J)_{af}$, we show that $r_i x \beta \in \Delta^+_af$ for all $\beta \in (\Delta_J)_{af}$. Let $\beta \in (\Delta_J)_{af}$. Because $x \in (W^J)_{af}$, we have $x \beta \in \Delta^+_af$. Since $x^{-1}\alpha_i \notin (\Delta_J)_{af}$ by assumption, it follows that $x^{-1}\alpha_i \neq \beta$, and hence $\alpha_i \neq x \beta$. Therefore, $r_i x \beta \in \Delta^+_af$ since $\text{Inv}(r_i) = \{\alpha_i\}$. Thus we have proved the lemma.

Lemma 2.2.8. Let $x \in W_{af}$, and $\xi \in Q^\vee_{J, adj}$. Then, $xz_\xi t_\xi \in (W^J)_{af}$ if and only if $x \in (W^J)_{af}$.

Proof. First we remark that

$$\Pi^J(xz_\xi t_\xi) = \Pi^J(xz_\xi)\Pi^J(t_\xi) \quad (\text{by Lemma 2.2.4 (2)})$$

$$= \Pi^J(x)z_\xi t_\xi \quad (\text{by Lemmas 2.2.4 (1) and 2.2.6 (2)}). \quad (2.2.6)$$

Now, let us show the “only if” part. Assume that $xz_\xi t_\xi \in (W^J)_{af}$; note that $\Pi^J(xz_\xi t_\xi) = xz_\xi t_\xi$. Combining this and (2.2.6), we obtain $\Pi^J(x)z_\xi t_\xi = xz_\xi t_\xi$, and hence $\Pi^J(x) = x$, which implies that $x \in (W^J)_{af}$. Next, let us show the “if” part. Assume that $x \in (W^J)_{af}$; note that $\Pi^J(x) = x$. Combining this and (2.2.6), we obtain $\Pi^J(xz_\xi t_\xi) = \Pi^J(x)z_\xi t_\xi = xz_\xi t_\xi$, which implies that $xz_\xi t_\xi \in (W^J)_{af}$. Thus we have proved the lemma.

2.3 Semi-infinite Bruhat order on $(W^J)_{af}$

Definition 2.3.1 ([Pet97]). Let $x \in W_{af}$, and write it as $x = vt_\xi$ with $v \in W$ and $\xi \in Q^\vee$. Then we define

$$\ell^\pm(x) := \ell(v) + 2(\zeta, \rho). \quad (2.3.1)$$

Definition 2.3.2. (1) Let $J$ be a subset of $I$. Define the (parabolic) semi-infinite Bruhat graph $SB^J$ to be the $\Delta^+_af$-colored, oriented graph, with $(W^J)_{af}$ the set of vertices, whose edges are drawn as follows: for $x \in (W^J)_{af}$ and $\beta \in \Delta^+_af$, we write $x \xrightarrow{\beta} y_\beta x$ if the following two conditions hold:
(i) \( r_\beta x \in (W^J)_{af} \),
(ii) \( \ell^{\infty}_J(r_\beta x) = \ell^{\infty}_J(x) + 1 \).

(2) Define a partial order \( \leq^{\infty}_J \) on \((W^J)_{af}\), called the \textit{semi-infinite Bruhat order}, as follows: for \( x, y \in (W^J)_{af} \), we write \( x \leq^{\infty}_J y \) if there exists a directed path from \( x \) to \( y \) in \( SB^J \).

3 Main results

Throughout this section, we fix \( \lambda \in P^+ \), and set \( J := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \subset I \).

3.1 Semi-infinite Lakshmibai–Seshadri paths

Definition 3.1.1. For a rational number \( 0 < a \leq 1 \), define \( SB^J(\lambda ; a) \) to be the subgraph of \( SB^J \) consisting of the same vertices and only the edges
\[
X \xrightarrow{\beta} Y \text{ with } a(\beta^\vee, x) \in \mathbb{Z}.
\]  

(3.1.1)

Note that \( SB^J(\lambda ; 1) = SB^J \).

Definition 3.1.2. A \textit{semi-infinite Lakshmibai–Seshadri} (\( \frac{\infty}{2} \)-LS for short) \textit{path} of shape \( \lambda \) is, by definition, a pair \( (x; a) \) of a decreasing sequence \( x : x_1 > \frac{\infty}{2} \cdots > \frac{\infty}{2} x_s \) in \( (W^J)_{af} \) and an increasing sequence \( a : 0 = a_0 < a_1 < \cdots < a_s = 1 \) of rational numbers satisfying the condition that there exists a directed path from \( x_{u+1} \) to \( x_u \) in \( SB^J(\lambda ; a_u) \) for each \( u = 1, 2, \ldots, s - 1 \). Denote by \( \mathbb{B}^J(\lambda) \) the set of \( \frac{\infty}{2} \)-LS paths of shape \( \lambda \).

For \( \eta = (x_1, \ldots, x_s; a_0, a_1, \cdots, a_s) \in \mathbb{B}^J(\lambda) \), we define a piecewise-linear, continuous map \( \pi_\eta : [0, 1] \to \mathbb{R} \otimes \mathbb{Z} P_{af} \) by
\[
\pi_\eta(t) := \sum_{u=1}^{k-1} (a_u - a_{u-1})x_u \lambda + (t - a_{k-1})x_k \lambda \quad \text{for } t \in [a_{k-1}, a_k], \quad 1 \leq k \leq s.
\]  

(3.1.2)

We will show the following proposition in §4.3; for the definition of a Lakshmibai–Seshadri path of shape \( \lambda \), see Definition 4.1.3.

Proposition 3.1.3. For every \( \eta \in \mathbb{B}^J(\lambda) \), \( \pi_\eta \) is a Lakshmibai–Seshadri path of shape \( \lambda \).

Hence we see from [Lit95, §4] that \( \pi_\eta(1) \in P_{af} \) for all \( \eta \in \mathbb{B}^J(\lambda) \). So, we define a map \( wt : \mathbb{B}^J(\lambda) \to P_{af} \) by \( wt(\eta) := \pi_\eta(1) \).

Remark 3.1.4. We see from [Lit95, Lemma 4.5 d)] that for each \( \eta \in \mathbb{B}^J(\lambda) \) and \( i \in I_{af} \), all local minimal values of the function
\[
H_i^\pi_\eta(t) := \langle \alpha_i^\vee, \pi_\eta(t) \rangle \quad \text{for } t \in [0, 1],
\]  

(3.1.3)

are integers.

Now, let us define operators \( e_i, f_i, i \in I_{af}, \) on \( \mathbb{B}^J(\lambda) \sqcup \{0\} \), where \( 0 \) is an additional element not contained in any crystal; we call these operators the root operators.
**Definition 3.1.5** (cf. [Lit94, Proposition 4.2]). Let $\eta = (x_1, x_2, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}^{\infty} (\lambda)$, and $i \in I_{af}$. Define $H_i^{\eta_n} (t), t \in [0, 1]$, as (3.1.3), and set

$$m_i^{\eta_n} := \min \{ H_i^{\eta_n}(t) \mid t \in [0, 1] \};$$

(3.1.4)

note that $m_i^{\eta_n} \in \mathbb{Z}_{\leq 0}$ and $H_i^{\eta_n} (1) - m_i^{\eta_n} \in \mathbb{Z}_{\geq 0}$ by Remark 3.1.4.

1. If $m_i^{\eta_n} = 0$, then we define $e_i \eta = 0$. If $m_i^{\eta_n} \leq -1$, then we set

$$t_1 := \min \{ t \in [0, 1] \mid H_i^{\eta_n}(t) = m_i^{\eta_n} \},
$$

$$t_0 := \max \{ t \in [0, t_1] \mid H_i^{\eta_n}(t) = m_i^{\eta_n} + 1 \};$$

(3.1.5)

we deduce from Remark 3.1.4 that $H_i^{\eta_n}(t)$ is strictly decreasing on $[t_0, t_1]$. Notice that there exists $1 \leq q \leq s$ such that $t_1 = a_q$. Let $1 \leq p \leq q$ be such that $a_{p-1} \leq t_0 < a_p$. Then we define

$$e_i \eta := (x_1, x_2, \ldots, x_p; r_i x_p, r_i x_{p+1}, \ldots, r_i x_q, x_{q+1}, x_{q+2}, \ldots, x_s; a_0, a_1, a_2, \ldots, a_{p-1}, t_0, a_p, a_{p+1}, \ldots, a_{q-1}, t_1, a_q, a_{q+1}, \ldots, a_{s-1}, a_s);$$

if $t_0 = a_{p-1}$, then we drop $x_p$ and $a_{p-1}$, and if $r_i x_q = x_{q+1}$, then we drop $x_{q+1}$ and $t_1$.

2. If $H_i^{\eta_n} (1) - m_i^{\eta_n} = 0$, then we define $f_i \eta := 0$. If $H_i^{\eta_n} (1) - m_i^{\eta_n} \geq 1$, then we set

$$t_0 := \max \{ t \in [0, 1] \mid H_i^{\eta_n}(t) = m_i^{\eta_n} \},
$$

$$t_1 := \min \{ t \in [t_0, 1] \mid H_i^{\eta_n}(t) = m_i^{\eta_n} + 1 \};$$

(3.1.6)

we deduce from Remark 3.1.4 that $H_i^{\eta_n}(t)$ is strictly increasing on $[t_0, t_1]$. Notice that there exists $1 \leq p \leq s$ such that $t_0 = a_{p-1}$. Let $p \leq q \leq s$ be such that $a_{q-1} < t_1 \leq a_q$. Then we define

$$f_i \eta := (x_1, x_2, \ldots, x_{p-1}; r_i x_p, r_i x_{p+1}, \ldots, r_i x_q, x_{q+1}, \ldots, x_s; a_0, a_1, a_2, \ldots, a_{p-2}, t_0, a_p, a_{p+1}, \ldots, a_{q-1}, t_1, a_q, a_{q+1}, \ldots, a_{s-1}, a_s);$$

if $t_1 = a_q$, then we drop $x_q$ and $a_q$, and if $x_{p-1} = r_i x_p$, then we drop $x_{p-1}$ and $t_0 = a_{p-1}$.

3. Define $e_i 0 = f_i 0 := 0$ for all $i \in I_{af}$.

We will prove the following theorem in §4.4; for the definition of crystals, see [Kas95, §7.2] and [HK02, Definition 4.5.1] for example.

**Theorem 3.1.6.** (1) The set $\mathbb{B}^{\infty}(\lambda) \cup \{0\}$ is stable under the action of the root operators $e_i$ and $f_i$, $i \in I_{af}$.

(2) For each $\eta \in \mathbb{B}^{\infty}(\lambda)$ and $i \in I_{af}$, we set

$$\varepsilon_i(\eta) := \max \{ k \geq 0 \mid e_i^k \eta \neq 0 \},
$$

$$\varphi_i(\eta) := \max \{ k \geq 0 \mid f_i^k \eta \neq 0 \}.$$

(3.1.7)

The set $\mathbb{B}^{\infty}(\lambda)$ together with the maps $\varepsilon_i, f_i, i \in I_{af},$ and $e_i, \varphi_i, i \in I_{af}$, is a crystal with weights in $P_{af}$.
3.2 Isomorphism theorem between $\mathcal{B}(\lambda)$ and $\mathbb{B}^\infty(\lambda)$

Denote by $V(\lambda)$ the extremal weight module of extremal weight $\lambda \in P_a$ over the quantized universal enveloping algebra $U_q(\mathfrak{g}_a)$ associated with $\mathfrak{g}_a$, which is an integrable $U_q(\mathfrak{g}_a)$-module generated by a single element $v_\lambda$ with the defining relation that “$v_\lambda$ is an extremal weight vector of weight $\lambda$” (see [Kas94, §8] and [Kas02b, §3]). We know from [Kas94, §8] that $V(\lambda)$ has a crystal basis $\mathcal{B}(\lambda)$. The main result of this paper is the following theorem.

**Theorem 3.2.1.** Let $\lambda \in P^+$. The crystal basis $\mathcal{B}(\lambda)$ of the extremal weight module $V(\lambda)$ of extremal weight $\lambda$ is isomorphic, as a crystal, to the crystal $\mathbb{B}^\infty(\lambda)$ of $\mathcal{B}^{\infty}$-LS paths of shape $\lambda$.

Let us give a sketch of the proof of Theorem 3.2.1. Let $\mathbb{B}^\infty_0(\lambda)$ be the connected component of $\mathbb{B}^\infty(\lambda)$ containing $\eta_\lambda := (e; 0, 1) \in \mathbb{B}^\infty(\lambda)$. Also, let $u_\lambda$ be the element of $\mathcal{B}(\lambda)$ corresponding to the generator $v_\lambda$ of $V(\lambda)$, and let $\mathcal{B}_0(\lambda)$ be the connected component of $\mathcal{B}(\lambda)$ containing $u_\lambda \in \mathcal{B}(\lambda)$.

We will prove the following proposition in §5.

**Proposition 3.2.2.** For $\lambda \in P^+$, there exists a unique isomorphism $\mathcal{B}_0(\lambda) \xrightarrow{\sim} \mathbb{B}^\infty_0(\lambda)$ of crystals that maps $u_\lambda$ to $\eta_\lambda$.

Write $\lambda \in P^+$ as $\lambda = \sum_{i \in I} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$, and set

$$\text{Par}(\lambda) := \left\{ \rho = (\rho(i))_{i \in I} \mid \rho(i) \text{ is a partition of length less than } m_i \text{ for } i \in I \right\};$$

(3.2.1)

we understand that $\rho(i)$ is the empty partition if $m_i = 0$. We give $\text{Par}(\lambda)$ a crystal structure as follows: for each $\rho = (\rho(i))_{i \in I} \in \text{Par}(\lambda)$, we set

$$\begin{cases} e_i \rho = f_i \rho := 0, & \varepsilon_i(\rho) = \varphi_i(\rho) := -\infty \quad \text{for } i \in I_a, \\ \text{wt}(\rho) := -\sum_{i \in I} |\rho(i)| \delta, \end{cases}$$

(3.2.2)

where $|\rho(i)| := \sum_{u=1}^{m_i-1} \rho_u(i)$ if $\rho(i) = (\rho_1(i) \geq \rho_2(i) \geq \cdots \geq \rho_{m_i-1}(i) \geq 0)$. By Proposition 3.2.2, we have an isomorphism

$$\text{Par}(\lambda) \otimes \mathcal{B}_0(\lambda) \cong \text{Par}(\lambda) \otimes \mathbb{B}^\infty_0(\lambda)$$

(3.2.3)

of crystals. So, let $\mathcal{B}$ be either $\mathcal{B}_0(\lambda)$ or $\mathbb{B}^\infty_0(\lambda)$. For each $\rho \in \text{Par}(\lambda)$, we set $\{\rho\} \otimes \mathcal{B} := \{\rho \otimes b \mid b \in \mathcal{B}\} \subset \text{Par}(\lambda) \otimes \mathcal{B}$; we see from the tensor product rule of crystals (see [Kas95, §7.2]) that $e_i(\rho \otimes b) = \rho \otimes e_i b$ and $f_i(\rho \otimes b) = \rho \otimes f_i b$ for all $b \in \mathcal{B}$ and $i \in I_a$. Therefore, $\{\rho\} \otimes \mathcal{B}$ is a connected subcrystal of $\text{Par}(\lambda) \otimes \mathcal{B}$, and

$$\text{Par}(\lambda) \otimes \mathcal{B} = \bigcup_{\rho \in \text{Par}(\lambda)} \{\rho\} \otimes \mathcal{B}.$$

Moreover, the map $\mathcal{B} \to \{\rho\} \otimes \mathcal{B}$ defined by $b \mapsto \rho \otimes b$ is bijective and commutes with the Kashiwara operators.

Now, we know the following proposition from [BN04, Theorem 4.16 (i)].

**Proposition 3.2.3.** For $\lambda \in P^+$, there exists an isomorphism $\mathcal{B}(\lambda) \xrightarrow{\sim} \text{Par}(\lambda) \otimes \mathcal{B}_0(\lambda)$ of crystals.

Also, we will show the following proposition in §6.
**Proposition 3.2.4.** For \( \lambda \in P^+ \), there exists an isomorphism \( \mathcal{B}(\lambda) \cong \text{Par}(\lambda) \otimes \mathbb{B}^\infty_0(\lambda) \) of crystals.

Thus we obtain
\[
\begin{align*}
\mathcal{B}(\lambda) &\cong \text{Par}(\lambda) \otimes \mathbb{B}^\infty_0(\lambda) \\
&\cong \text{Par}(\lambda) \otimes \mathbb{B}^\infty(\lambda) \\
&\cong \mathbb{B}^\infty(\lambda)
\end{align*}
\]
(by Proposition 3.2.3)
(by (3.2.3))
(by Proposition 3.2.4).

## 4 Proofs of Proposition 3.1.3 and Theorem 3.1.6

### 4.1 Lakshmibai–Seshadri paths

Throughout this subsection, we fix \( \lambda \in P^+ \).

**Definition 4.1.1 ([Lit95, §4]).** We define a partial order \( \leq \) on \( \mathcal{W}_{af} \lambda \) as follows: for \( \mu, \nu \in \mathcal{W}_{af} \lambda \), we write \( \mu \leq \nu \) if there exist a sequence \( \mu = \nu_0, \nu_1, \ldots, \nu_s = \nu \) of elements in \( \mathcal{W}_{af} \lambda \) and a sequence \( \beta_1, \ldots, \beta_s \) of elements in \( \Delta_{af}^+ \) such that \( \nu_u = r_{\beta_u} \nu_{u-1} \) and \( \langle \beta^\vee_u, \nu_{u-1} \rangle \in \mathbb{Z}_{>0} \) for \( u = 1, \ldots, s \). We call \( (\mathcal{W}_{af} \lambda, \leq) \) the level-zero weight poset of shape \( \lambda \).

**Definition 4.1.2.**

1. Define \( \text{LP}(\lambda) \) to be the \( \Delta_{af}^+ \)-colored, oriented graph, with \( \mathcal{W}_{af} \lambda \) the set of vertices, whose edges are drawn as follows: for \( \mu, \nu \in \mathcal{W}_{af} \lambda \), we write \( \mu \xrightarrow{\beta} \nu \) if \( \nu \) covers \( \mu \) in the poset \( \mathcal{W}_{af} \lambda \), where the label \( \beta \) of the edge is a unique positive real root \( \beta \in \Delta_{af}^+ \) such that \( \nu = r_{\beta} \mu \) and \( \langle \beta^\vee, \mu \rangle > 0 \).

2. Let \( 0 < a \leq 1 \) be a rational number. Define \( \text{LP}(\lambda; a) \) to be the subgraph of \( \text{LP}(\lambda) \) consisting of the same vertices and only the edges
\[
\mu \xrightarrow{\beta} \nu \text{ with } a\langle \beta^\vee, \mu \rangle \in \mathbb{Z}.
\]
(4.1.1)

Note that \( \text{LP}(\lambda; 1) = \text{LP}(\lambda) \).

**Definition 4.1.3 ([Lit95, §4]).** A Lakshmibai–Seshadri (LS for short) path of shape \( \lambda \) is, by definition, a pair \( (\nu; a) \) of a decreasing sequence \( \nu : \nu_1 > \cdots > \nu_s \) in \( \mathcal{W}_{af} \lambda \) and an increasing sequence \( a : 0 = a_0 < a_1 < \cdots < a_s = 1 \) of rational numbers satisfying the condition that there exists a directed path from \( \nu_{u+1} \) to \( \nu_u \) in \( \text{LP}(\lambda; a_u) \) for each \( u = 1, 2, \ldots, s \). Let \( \mathbb{B}(\lambda) \) denote the set of LS paths of shape \( \lambda \).

As in (3.1.2), we identify \( \pi = (\nu_1, \ldots, \nu_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}(\lambda) \) with the piecewise-linear, continuous map \( \pi : [0, 1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{af} \) defined by
\[
\pi(t) := \sum_{u=1}^{k-1} (a_u - a_{u-1}) \nu_u + (t - a_{k-1}) \nu_k \quad \text{for } t \in [a_{k-1}, a_k], \; 1 \leq k \leq s.
\]
(4.1.2)

Now, we give \( \mathbb{B}(\lambda) \) a crystal structure as follows. We define \( \text{wt} : \mathbb{B}(\lambda) \to P_{af} \) by \( \text{wt}(\pi) := \pi(1) \in P_{af} \) (see [Lit95, §4]). Following [Lit95, §1] (see also [NS06, §1]), we define the root operators \( e_i, f_i, \; i \in I_{af}, \) on \( \mathbb{B}(\lambda) \cup \{0\} \).

**Definition 4.1.4.** Let \( \pi = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}(\lambda) \), and \( i \in I_{af} \). Define \( H_i^\pi(t), \) \( t \in [0, 1], \) and \( m_i^\pi \) as (3.1.3) and (3.1.4), with \( \pi \eta \) replaced by \( \pi \), respectively.
(1) If \( m^\pi_i = 0 \), then we define \( e_i \pi := 0 \). If \( m^\pi_i \leq -1 \), then we define \( t_0, t_1 \in [0, 1] \) as (3.1.5), and set
\[
(e_i \pi)(t) := \begin{cases} 
\pi(t) & \text{if } t \in [0, t_0], \\
\pi(t_0) + r_i(\pi(t) - \pi(t_0)) & \text{if } t \in [t_0, t_1], \\
\pi(t) + \alpha_i & \text{if } t \in [t_1, 1], 
\end{cases}
\]
or equivalently,
\[
e_i \pi := (\nu_1, \nu_2, \ldots, \nu_p, r_i \nu_p, r_i \nu_{p+1}, \ldots, r_i \nu_q, \nu_{q+1}, \nu_{q+2}, \ldots, \nu_s; \\
a_0, a_1, a_2, \ldots, a_{p-1}, t_0, a_p, a_{p+1}, \ldots, a_{q-1}, t_1, a_q, a_{q+1}, \ldots, a_{s-1}, a_s),
\]
where \( 1 \leq p \leq q \leq s \) are such that \( a_{p-1} \leq t_0 < a_p \) and \( t_1 = a_q \); if \( t_0 = a_{p-1} \), then we drop \( \nu_p \) and \( a_{p-1} \), and if \( r_i \nu_q = \nu_{q+1} \), then we drop \( \nu_{q+1} \) and \( t_1 \).

(2) If \( H^\pi_i(1) - m^\pi_i = 0 \), then we define \( f_i \pi := 0 \). If \( H^\pi_i(1) - m^\pi_i \geq 1 \), then we define \( t_0, t_1 \in [0, 1] \) as (3.1.6), and set
\[
(f_i \pi)(t) := \begin{cases} 
\pi(t) & \text{if } t \in [0, t_0], \\
\pi(t_0) + r_i(\pi(t) - \pi(t_0)) & \text{if } t \in [t_0, t_1], \\
\pi(t) - \alpha_i & \text{if } t \in [t_1, 1], 
\end{cases}
\]
or equivalently,
\[
f_i \pi := (\nu_1, \nu_2, \ldots, \nu_{p-1}, r_i \nu_p, r_i \nu_{p+1}, \ldots, r_i \nu_q, \nu_{q+1}, \nu_{q+2}, \ldots, \nu_s; \\
a_0, a_1, a_2, \ldots, a_{p-2}, t_0, a_p, a_{p+1}, \ldots, a_{q-1}, t_1, a_q, a_{q+1}, \ldots, a_{s-1}, a_s),
\]
where \( 1 \leq p \leq q \leq s \) are such that \( t_0 = a_{p-1} \) and \( a_{q-1} < t_1 \leq a_q \); if \( t_1 = a_q \), then we drop \( \nu_q \) and \( a_q \), and if \( \nu_{p-1} = r_i \nu_p \), then we drop \( \nu_{p-1} \) and \( t_0 \).

(3) Define \( e_i 0 = f_i 0 = 0 \) for all \( i \in I_{af} \).

We know from [Lit95, Proposition 4.7] that the set \( \mathfrak{B}(\lambda) \sqcup \{0\} \) is stable under the root operators \( e_i, f_i, i \in I_{af} \). So, we define \( e_i(\pi) := \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k \pi \neq 0\} \), and \( \varphi_i(\pi) := \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k \pi \neq 0\} \) for \( \pi \in \mathfrak{B}(\lambda) \) and \( i \in I_{af} \). We know from [Lit95, §2 and §4] that the set \( \mathfrak{B}(\lambda) \) together with the maps \( e_i \), \( f_i \), \( i \in I_{af} \), and \( e_i, \varphi_i, i \in I \), is a crystal with weights in \( P_{af} \).

### 4.2 Quantum Bruhat graphs

**Definition 4.2.1** ([LNSSS13a, §4]; see also [BFP99, §6]). Let \( J \) be a subset of \( I \). Define the (parabolic) **quantum Bruhat graph** \( QB^J \) to be the \( (\Delta^+ \setminus \Delta^+_J) \)-colored oriented graph, with \( W^J \) the set of vertices, whose edges are drawn as follows: for \( w \in W^J \) and \( \alpha \in \Delta^+ \setminus \Delta^+_J \), we write \( w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor \) if either of the following holds:

(B) \( \ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 \), or

(Q) \( \ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 - 2\langle \alpha^\vee, \rho - \rho_J \rangle \).

We call an edge \( w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor \) satisfying condition (B) (resp., (Q)) a **Bruhat** (resp., **quantum**) edge, and write as \( w \xrightarrow{\alpha}_{B} \lfloor wr_\alpha \rfloor \) (resp., \( w \xrightarrow{\alpha}_{Q} \lfloor wr_\alpha \rfloor \)).
Throughout this subsection, we define the following:

- Proposition 4.3.3.
- Lemma 4.3.2 ([NS08, Lemma 2.11]).
- Definition 4.2.3.
- Remark between SB and QB.

For a rational number $0 < a \leq 1$, let $\text{QB}(\lambda; a)$ denote the subgraph of $\text{QB}^J$ consisting of the same vertices and only the edges $w \xrightarrow{a} [wr_a]$ with $a(\alpha^\vee, \lambda) \in \mathbb{Z}$.

Note that $\text{QB}(\lambda; 1) = \text{QB}^J$.

### 4.3 Proof of Proposition 3.1.3

Throughout this subsection, we fix $\lambda \in P^+$ and set $J := \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$. Let $\eta = (x_1, x_2, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}^>(\lambda)$, and define $\pi_\eta : [0, 1] \to \mathbb{R} \otimes \mathbb{F}_{af}$ as (3.1.2). By (4.1.2), this $\pi_\eta$ corresponds to

$$\pi_\eta = (x_1 \lambda, x_2 \lambda, \ldots, x_s \lambda; a_0, a_1, \ldots, a_s) \in \mathbb{B}(\lambda).$$

Thus it suffices to show that for each $1 \leq u \leq s - 1$, there exists a directed path from $x_{u+1} \lambda$ to $x_u \lambda$ in $\text{LP}(\lambda; a_u)$. This follows immediately from the next proposition, which gives a relation between $\text{SB}(\lambda; a)$ and $\text{LP}(\lambda; a)$.

**Proposition 4.3.1.** Let $0 < a \leq 1$ be a rational number, $x \in (W^J)_{af}$, and $\beta \in \Delta_{af}^+$. Then, $x \xrightarrow{\beta} r_\beta x$ in $\text{SB}(\lambda; a)$ if and only if $x \lambda \xrightarrow{\beta} r_\beta x \lambda$ in $\text{LP}(\lambda; a)$.

We will show Proposition 4.3.1 “via $\text{QB}(\lambda; a)$”; we will give a relation between $\text{QB}(\lambda; a)$ and $\text{LP}(\lambda; a)$ in Proposition 4.3.3, and a relation between $\text{QB}(\lambda; a)$ and $\text{SB}(\lambda; a)$ in Proposition 4.3.7.

**Lemma 4.3.2 ([NS08, Lemma 2.11]).** Let $\mu, \nu \in W_{af} \lambda$ and $\beta \in \Delta_{af}^+$. If $\mu \xrightarrow{\beta} r_\beta \mu$ in $\text{LP}(\lambda)$, then $\beta \in \Delta^+ \sqcup \{-\gamma + \delta \mid \gamma \in \Delta^+ \}$.

**Proposition 4.3.3.** Let $0 < a \leq 1$ be a rational number.

1. Let $w \in W^J$, and $\alpha \in \Delta^+$. Assume that $w \xrightarrow{\alpha} [wr_a]$ (resp., $w \xrightarrow{\alpha} [wr_a]$) in $\text{QB}(\lambda; a)$.

If we set $\beta := w\alpha$ (resp., $\beta := w\alpha + \delta$), then $\beta \in \Delta_{af}^+$ and $w_\xi t_\xi \lambda \xrightarrow{\beta} r_\beta w_\xi t_\xi \lambda$ in $\text{LP}(\lambda; a)$ for every $\xi \in Q_{\lambda, \text{adj}}$. 

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(2) Let \( w \in W_j \), and \( \xi \in Q'_j\text{-adj} \). Assume that \( wz_t \xi \lambda \overset{\beta}{\longrightarrow} rz_t \xi \lambda \) in \( \mathsf{LP}(\lambda; a) \); recall from Lemma 4.3.2 that \( \beta \in \Delta^+ \cup \{-\gamma + \delta \mid \gamma \in \Delta^+\} \). Set \( \alpha := w^{-1}\beta \) (resp., \( \alpha := w^{-1}(\beta - \delta) \)) if \( \beta \in \Delta^+ \) (resp., \( \beta \in \{-\gamma + \delta \mid \gamma \in \Delta^+\} \)). Then, \( \alpha \in \Delta^+ \setminus \Delta^+_j \), and \( w \overset{\alpha}{\longrightarrow} \lfloor [w\alpha] \rfloor \) in \( \mathsf{QB}(\lambda; a) \).

Proof. (1) By [LNSSS13a, Theorem 6.5], we see that \( wz_t \xi \lambda \overset{\beta}{\longrightarrow} rz_t \xi \lambda \) in \( \mathsf{LP}(\lambda) \) for every \( \xi \in Q'_j\text{-adj} \). Since \( a(\alpha^\vee, \lambda) \in \mathbb{Z} \) and \( z_t \xi \lambda \) is in \( \mathsf{LP}(\lambda) \), we get \( a(\alpha^\vee, wz_t \xi \lambda) \in \mathbb{Z} \), which implies that the edge \( wz_t \xi \lambda \overset{\beta}{\longrightarrow} rz_t \xi \lambda \) is in \( \mathsf{LP}(\lambda; a) \).

(2) By [LNSSS13a, Theorem 6.5], we see that \( w \overset{\alpha}{\longrightarrow} \lfloor [w\alpha] \rfloor \) in \( \mathsf{QB}^j \). Since \( a(\alpha^\vee, wz_t \xi \lambda) \in \mathbb{Z} \) and \( (\alpha^\vee, \lambda) = (\beta^\vee, wz_t \xi \lambda) \), we get \( a(\alpha^\vee, \lambda) \in \mathbb{Z} \), which implies that the edge \( w \overset{\alpha}{\longrightarrow} \lfloor [w\alpha] \rfloor \) is in \( \mathsf{QB}(\lambda; a) \).

We know the following lemma from [BFP99, Lemma 4.3].

Lemma 4.3.4. We have \( \ell(r_\alpha) \leq 2(\alpha^\vee, \rho) - 1 \) for all \( \alpha \in \Delta^+ \).

Lemma 4.3.5. Let \( x = wz_t \xi \in (W^j)_{af} \) with \( w \in W_j \) and \( \xi \in Q'_j\text{-adj} \), and let \( \beta \in \Delta^+_j \) be such that \( x \overset{\beta}{\longrightarrow} rx \) in \( \mathsf{SB}^j \). Write \( \beta \) as \( \beta = \gamma + \chi \delta \) with \( \gamma \in \Delta \) and \( \chi \in \mathbb{Z}_{\geq 0} \), and set \( \alpha := w^{-1}\gamma \); note that \( \beta = \omega + \chi \delta \). Then the following hold.

1. \( \alpha \in \Delta^+ \setminus \Delta^+_j \),
2. \( \ell(wz_t \xi) = \ell(wz_t \xi) + 1 - 2\chi(z_t^{-1}\alpha^\vee, \rho) \),
3. \( \chi \in \{0, 1\} \).

Proof. (1) We first show that \( \alpha \not\in \Delta_j \). Suppose that \( \alpha \in \Delta_j \). By simple computation, we see that

\[
x^{-1}\beta = t_{-\xi} z_t^{-1} w^{-1}(w \alpha + \chi \delta) = t_{-\xi} (z_t^{-1}\alpha + \chi \delta) = z_t^{-1}\alpha + n\delta
\]

for some \( n \in \mathbb{Z} \).

Because \( \alpha \in \Delta_j \) and \( z_t \in W_j \), it follows that \( x^{-1}\beta \in (\Delta_j)_{af} \), and hence \( r_{x^{-1}\beta} \in (W^j)_{af} \). Therefore, we deduce from Lemma 2.2.2 that \( r_{x^{-1}\beta} x \) is not contained in \( (W^j)_{af} \), which contradicts the assumption that \( x \overset{\beta}{\longrightarrow} rx \) in \( \mathsf{SB}^j \) (see condition (i) in Definition 2.3.2). Thus we get \( \alpha \not\in \Delta_j \).

We next show that \( \alpha \in \Delta^+ \). Suppose that \( -\alpha \in \Delta^+ \). Then, \( -z_t^{-1}\alpha \in \Delta^+ \) since \( \alpha \not\in \Delta_j \). We have

\[
\ell^\mathcal{F}(r_{\beta} x) = \ell^\mathcal{F}(wz_t \xi \in z_t^{-1}\alpha^\vee)
= \ell(wz_t \xi) + 2(\xi + \chi z_t^{-1}\alpha^\vee, \rho) \quad \text{(since } wz_t \xi \in z_t^{-1}\alpha^\vee \text{)}
\leq \ell(wz_t \xi) + \ell(r_{z_t^{-1}\alpha} x + 2(\xi + \chi z_t^{-1}\alpha^\vee, \rho) \quad \text{(see } [BB05, \text{Proposition 1.4.2 (v)}])
\leq \ell(wz_t \xi) + (2(-z_t^{-1}\alpha, \rho) - 1) + 2(\xi + \chi z_t^{-1}\alpha^\vee, \rho) \quad \text{(by Lemma 4.3.4)}
= \ell^\mathcal{F}(x) - 1 + 2(\chi - 1)(z_t^{-1}\alpha^\vee, \rho).
\]
Thus we have proved part (2).

(2) By simple computation, we have

\[
1 = \ell(\alpha) - \ell(x)
\]

which contradicts \(\ell(x) = 1\). Thus we get \(\alpha \in \Delta^+\). This proves part (1).

(3) First, we remark that \(z_\xi^{-1}\alpha \in \Delta^+\) since \(z_\xi \in W_J\) and \(\alpha \in \Delta^+ \setminus \Delta^+_J\) by (1). It follows that

\[
1 = \ell(w_{z_\xi} \alpha) - \ell(x)
\]

which implies that \(\chi \in \{0, 1\}\) since \(\langle z_\xi^{-1}\alpha, \chi \rangle > 0\). This completes the proof of Lemma 4.3.5. □

The next lemma follows from [LS10, Lemma 10.3] and [LNSSS13a, Lemma 3.10].

**Lemma 4.3.6.** For \(\xi \in Q^\vee_J, \text{adj}\), we have \(\text{Inv}(z_\xi) = \{\gamma \in \Delta^+_J \mid \langle \xi, \gamma \rangle = -1\}\) and \(\ell(z_\xi) = -2\langle \xi, r_J \rangle\).

**Proposition 4.3.7 (cf. [LNSSS13a, Theorem 5.2]).** Let \(0 < a \leq 1\) be a rational number.

(1) Let \(x = w z_{\xi}^a \in (W_J)^a\) with \(w \in W_J\) and \(\xi \in Q^\vee_J, \text{adj}\). Assume that \(x \rightarrow r_{\beta} x\) in \(\text{SB}(\lambda; a)\) for \(\beta \in \Delta^+_a\); by Lemma 4.3.5, \(\beta = \omega a + \chi\delta\) for some \(\alpha \in \Delta^+ \setminus \Delta^+_J\) and \(\chi \in \{0, 1\}\). If \(\chi = 0\) (resp., \(\chi = 1\)), then \(w \rightarrow \omega a + \alpha\) in \(\text{SB}(\lambda; a)\).

(2) Let \(w \in W_J\), and \(\alpha \in \Delta^+ \setminus \Delta^+_J\). Assume that \(w \rightarrow \omega a\) (resp., \(w \rightarrow \omega a + \delta\)) in \(\text{SB}(\lambda; a)\). Set \(\beta := \omega a\) (resp., \(\beta := \omega a + \delta\)). Then, \(\beta \in \Delta^+_a\), \(r_{\beta} w z_{\xi}^a \in (W_J)^a\), and \(w z_{\xi}^a \rightarrow r_{\beta} w z_{\xi}^a\) in \(\text{SB}(\lambda; a)\) for every \(\xi \in Q^\vee_J, \text{adj}\).

**Proof.** (1) We first assume that \(\chi = 0\), and hence \(\beta = \omega a\). Because \((W_J)^a \ni r_{\beta} x = w_{z_\xi} \alpha \in (W_J)^a\), it follows from (2.2.5) that \(w_{z_\xi} \in W_J\). Hence, \(\ell(w_{z_\xi}) = \ell(w_{z_\xi}) + \ell(z_\xi)\) since \(z_\xi \in W_J\). Also, we see from Lemma 4.3.5 (2) that \(\ell(w_{z_\xi}) = \ell(w_{z_\xi}) + 1 = \ell(w) + \ell(z_\xi) + 1\). Combining these, we get \(\ell(w_{z_\xi}) = \ell(w) + 1\). Thus, \(w \rightarrow \omega a\) in \(\text{SB}(\lambda; a)\); notice that \(\alpha(\alpha^\vee, \lambda) = a(\beta^\vee, x\lambda) \in \mathbb{Z}\).
We next assume that $\chi = 1$, and hence $\beta = w\alpha + \delta$. Since $(W^J)_{af} \supseteq r_\beta x = r_\beta w z t_\xi = wr_\alpha z_\xi t_{\xi+z_\xi^{-1}\alpha^\vee}$, we get $\xi + z_\xi^{-1}\alpha^\vee \in Q^J_+ \text{adj}$ by (2.2.5). We have

$$wr_\alpha z_\xi t_{\xi+z_\xi^{-1}\alpha^\vee} = \Pi^J(wr_\alpha z_\xi t_{\xi+z_\xi^{-1}\alpha^\vee})$$

which implies that $wr_\alpha z_\xi = [wr_\alpha]z_\xi z_\xi^{-1}\alpha^\vee$, and hence $\ell(wr_\alpha z_\xi) = \ell([wr_\alpha]) + \ell(z_\xi z_\xi^{-1}\alpha^\vee)$. Therefore,

$$\ell([wr_\alpha]) = \ell(wr_\alpha z_\xi) - \ell(z_\xi z_\xi^{-1}\alpha^\vee)$$

which implies that $w \frac{\alpha}{Q} \not\in [wr_\alpha]$ in QB($\lambda$; $a$); note that $a(\alpha^\vee, \lambda) = a(\beta^\vee, x\lambda) \in \mathbb{Z}$.

(2) Set $x := w z_\xi t_\xi$. We first assume that $w \frac{\alpha}{B} \not\in [wr_\alpha] = wr_\alpha \in W^J$ in QB($\lambda$; $a$) (see Remark 4.2.2 (1)); we see by condition (B) in Definition 4.2.1 that $\beta = w\alpha \in \Delta^+ \subset \Delta_{af}^+$. Since $r_\beta x = wr_\alpha z_\xi t_\xi$ and $wr_\alpha \in W^J$, it follows immediately from 2.2.5 that $r_\beta x \in (W^J)_{af}$. Also, we have

$$\ell^J(r_\beta x) = \ell(wr_\alpha z_\xi) + 2(\xi, \rho)$$

which implies $\beta \not\in (W^J)_{af}$ by Remark 4.2.2 (2), it follows from Lemma 2.2.8 that $r_\beta x \in (W^J)_{af}$. Because

$$r_\beta x = \Pi^J(r_\beta x)$$

we deduce that

$$\ell^J(r_\beta x) = \ell^J([wr_\alpha]z_\xi z_\xi^{-1}\alpha^\vee t_{\xi+z_\xi^{-1}\alpha^\vee})$$

$$= \ell([wr_\alpha]) + \ell(z_\xi z_\xi^{-1}\alpha^\vee) + 2(\xi, \rho)$$
Proposition 4.3.7 (2) that
\[ x \in \mathbb{L}(\lambda; a) \] and proved Proposition 4.3.7.

Proof of Theorem 3.1.6

Proof of Proposition 4.3.1. As mentioned in Definition 3.1.5 (1), the function \( f_\delta \) is strictly decreasing on \([t_0, t_1]\). Thus we see that \( \langle \alpha_i^\vee, x_u \lambda \rangle < 0 \) for every \( p \leq u \leq q \). In particular, \( x_u^{-1} \alpha_i \notin (\Delta_f)_{af} \). Hence it follows from Lemma 2.2.7 that \( r_i x_u \in (W^J)_{af} \).

4.4 Proof of Theorem 3.1.6

We show part (1) only for \( e_i \); the proof for \( f_i \) is similar. Let \( \eta = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in B_\pi^\lambda \) with \( m^\pi_{a_1} \leq -1 \). Define \( t_0, t_1 \in [0, 1] \) as (3.1.5), and let \( 1 \leq p \leq q \leq s \) be such that \( a_{p-1} \leq t_0 < a_q \) and \( t_1 = a_q \); then

\[
e_i \eta = \left( x_1, x_2, \ldots, x_p, r_i x_p, r_i x_{p+1}, \ldots, r_i x_q, x_{q+1}, x_{q+2}, \ldots, x_s; a_0, a_1, a_2, \ldots, a_{p-1}, t_0, a_p, a_{p+1}, \ldots, a_{q-1}, t_1, a_{q+1}, a_{q+2}, \ldots, a_{s-1}, a_s \right);
\]

if \( t_0 = a_{p-1} \), then we drop \( x_p \) and \( a_{p-1} \), and if \( r_i x_q = x_{q+1} \), then we drop \( x_{q+1} \) and \( t_1 \). We need to prove that

(i) \( r_i x_u \in (W^J)_{af} \) for every \( p \leq u \leq q \);

(ii) if \( t_0 \neq a_{p-1} \) (resp., \( t_0 = a_{p-1} \) and \( p > 1 \)), then there exists a directed path from \( r_i x_p \) to \( x_p \) (resp., \( x_{p-1} \)) in \( \mathbb{L}(\lambda; t_0) \);

(iii) for each \( p \leq u \leq q - 1 \), there exists a directed path from \( r_i x_{u+1} \) to \( r_i x_u \) in \( \mathbb{L}(\lambda; a_u) \);

(iv) if \( r_i x_q \neq x_{q+1} \), then there exists a directed path from \( x_{q+1} \) to \( r_i x_q \) in \( \mathbb{L}(\lambda; t_1) = \mathbb{L}(\lambda; a_q) \).

First, let us show (i). As mentioned in Definition 3.1.5 (1), the function \( H_i^{\pi_{a_1}}(t) \) is strictly decreasing on \([t_0, t_1]\). Thus we see that \( \langle \alpha_i^\vee, x_u \lambda \rangle < 0 \) for every \( p \leq u \leq q \). In particular, \( x_u^{-1} \alpha_i \notin (\Delta_f)_{af} \). Hence it follows from Lemma 2.2.7 that \( r_i x_u \in (W^J)_{af} \).
Here recall from Proposition 3.1.3 (see also (4.3.1)) that
\[ \pi_q = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}() \]
By Definition 4.1.4 (1), we see that
\[
\begin{align*}
\pi_q = (x_1, x_2, \ldots, x_p, \ldots, x_{p+1}, x_{p+2}, \ldots, x_s; a_0, a_1, a_2, \ldots, a_{p-1}, t_0, a_p, a_{p+1}, \ldots, a_{q-1}, t_1, a_{q+1}, a_{q+2}, \ldots, a_s) \in \mathbb{B}();
\end{align*}
\] (4.4.1)
if \( t_0 = a_{p-1} \), then we drop \( x_p \) and \( a_{p-1} \), and if \( r_i x_q \lambda = x_{q+1} \lambda \), then we drop \( x_{q+1} \lambda \) and \( t_1 \).
Now, let us show (ii). Since \( \langle \alpha_i \lambda x_p \rangle < 0 \), we have \( r_i x_p \lambda \to x_p \lambda \) in \( \mathbb{L}() \). Also, by applying [Lit95, Lemma 4.5 c)] to \( \tau_i \in \mathbb{L}() \) and \( t_0 \in [0, 1] \), we see that \( t_0 \langle \alpha_i \lambda, x_p \lambda \rangle \in \mathbb{Z}() \), which implies that the edge \( r_i x_p \lambda \to x_p \lambda \) is in \( \mathbb{L}() \). Hence it follows from Proposition 4.3.1 that \( r_i x_p \to x_p \) in \( \mathbb{L}() \). Thus we have shown (ii) in the case that \( t_0 \neq a_{p-1} \). Assume next that \( t_0 = a_{p-1} \) and \( p > 1 \). By assumption, there exists a directed path from \( x_p \) to \( x_{p-1} \) in \( \mathbb{L}() \). By concatenating this directed path and \( r_i x_p \to x_p \) obtained above, we get a directed path from \( r_i x_p \) to \( x_{p-1} \) in \( \mathbb{L}() \). Thus we have proved (ii).
Next, let us show (iii). Fix \( p \leq u \leq q - 1 \). Let
\[ x_{u+1} = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} y_l = x_u \]
be a directed path from \( x_{u+1} \) to \( x_u \) in \( \mathbb{S}() \). By using Proposition 4.3.1 repeatedly, we obtain a directed path
\[ x_{u+1} \lambda = y_0 \lambda \xrightarrow{\beta_1} y_1 \lambda \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} y_l \lambda = x_u \lambda \]
from \( x_{u+1} \lambda \) to \( x_u \lambda \) in \( \mathbb{L}() \). Then we deduce from [Lit95, Proof of Proposition 4.7] and (4.4.1) that \( r_i \beta_m \in \Delta^+_a \) for all \( 1 \leq m \leq l \), and
\[
\begin{align*}
r_i x_{u+1} \lambda = r_i y_0 \lambda \xrightarrow{r_i \beta_1} r_i y_1 \lambda \xrightarrow{r_i \beta_2} \cdots \xrightarrow{r_i \beta_l} r_i y_l \lambda = r_i x_u \lambda
\end{align*}
\]
is a directed path from \( r_i x_{u+1} \lambda \) to \( r_i x_u \lambda \) in \( \mathbb{L}() \). Again by using Proposition 4.3.1 repeatedly, we obtain a directed path
\[ r_i x_{u+1} = r_i y_0 \xrightarrow{r_i \beta_1} r_i y_1 \xrightarrow{r_i \beta_2} \cdots \xrightarrow{r_i \beta_l} r_i y_l = r_i x_u \]
from \( r_i x_{u+1} \) to \( r_i x_u \) in \( \mathbb{S}() \), as desired.
Finally we show (iv). As in the proof of (iii), let
\[ x_{q+1} = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} y_l = x_q \]
be a directed path from \( x_{q+1} \) to \( x_q \) in \( \mathbb{S}() \), and let
\[ x_{q+1} \lambda = y_0 \lambda \xrightarrow{\beta_1} y_1 \lambda \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} y_l \lambda = x_q \lambda \]
be the corresponding directed path from \( x_{q+1} \lambda \) to \( x_q \lambda \) in \( \mathbb{L}() \). We deduce from the definition of \( t_1 \) that \( \langle \alpha_i \lambda x_q \rangle \geq 0 \) and \( \langle \alpha_i \lambda x_q \lambda \rangle < 0 \). Set \( m := \max \{0 \leq k \leq l \mid \langle \alpha_i \lambda y_k \lambda \rangle \geq 0 \} \). Then we see from [Lit95, Lemmas 4.1 and 4.3] and their proofs that \( \beta_m = \alpha_i \) (and hence \( y_{m-1} = r_i y_m \)), \( \beta_k \neq \alpha_i \) (and hence \( r_i \beta_k \in \Delta^+_a \)) for all \( m + 1 \leq k \leq l \), and
\[
\begin{align*}
x_{q+1} \lambda = y_0 \lambda \xrightarrow{\beta_1} y_1 \lambda \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{m-1}} y_{m-1} \lambda = r_i y_m \lambda \xrightarrow{r_i \beta_{m+1}} \cdots \xrightarrow{r_i \beta_l} r_i y_l \lambda = r_i x_q \lambda.
\end{align*}
\]
is a directed path from $x_{q+1}\lambda$ to $r_ix_q\lambda$ in $\text{LP}(\lambda; a_q)$. Again by using Proposition 4.3.1 repeatedly, we obtain a directed path

$$x_{q+1} = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{m-1}} y_{m-1} = r_iy_m \xrightarrow{r_i\beta_{m+1}} \cdots \xrightarrow{r_i\beta_l} r_iy_l = r_ix_q$$

from $x_{q+1}$ to $r_ix_q$ in $\text{SB}(\lambda; a_q)$, as desired. This proves part (1).

(2) We see from the definitions of the root operators that for each $\eta \in \mathbb{B}_i^\infty(\lambda)$ and $i \in I_{af}$, $e_i\eta \neq 0$ (resp., $f_i\eta \neq 0$) if and only if $e_i\pi_\eta \neq 0$ (resp., $f_i\pi_\eta \neq 0$). Hence,

$$\varepsilon_i(\eta) = \varepsilon_i(\pi_\eta), \varphi_i(\eta) = \varphi_i(\pi_\eta) \text{ for all } \eta \in \mathbb{B}_i^\infty(\lambda) \text{ and } i \in I_{af}. \quad (4.4.2)$$

Also, it follows immediately from the definitions that

$$\text{wt}(\eta) = \text{wt}(\pi_\eta) \text{ for all } \eta \in \mathbb{B}_i^\infty(\lambda). \quad (4.4.3)$$

Thus we deduce that $\mathbb{B}_i^\infty(\lambda)$, together with the maps $\text{wt}$, $e_i$, $f_i$, $i \in I_{af}$, and $\varepsilon_i, \varphi_i, i \in I_{af}$, satisfies the axioms of crystals except that

$$\text{for } \eta_1, \eta_2 \in \mathbb{B}_i^\infty(\lambda) \text{ and } i \in I_{af}, \text{ } e_i\eta_1 = \eta_2 \text{ if and only if } \eta_1 = f_i\eta_2. \quad (4.4.4)$$

We give a proof only for the “only if” part; the proof for the “if” part is similar. Define $t_0, t_1 \in [0, 1]$ as (3.1.5) for $\eta_1$ and $i \in I_{af}$. Then we deduce that

$$t_0 = \max\{t \in [0, 1] \mid H_{i}^{\pi_{\eta_1}}(t) = m_i^{\pi_\eta_2}\},$$

$$t_1 = \min\{t \in [t_0, 1] \mid H_{i}^{\pi_{\eta_2}}(t) = m_i^{\pi_\eta_2} + 1\}.$$

Therefore, we see from the definition of the root operator $f_i$ that $f_i\eta_2 = f_i\varepsilon_i\eta_1 = \eta_1$. This proves part (2). Thus we have proved Theorem 3.1.6. \hfill $\square$

Remark 4.4.1. (1) We see from the definition of the root operators (see also (4.1.2)), and (4.4.2), (4.4.3) that the map $\mathbb{B}_i^\infty(\lambda) \rightarrow \mathbb{B}(\lambda)$, $\eta \mapsto \pi_\eta$, is a strict morphism of crystals in the sense of [Kas94, §1.5]; in fact, this map is surjective.

(2) By (4.4.2) and [Lit95, Lemma 2.1 c]), we have $\varepsilon_i(\eta) = -m_i^{\pi_\eta}$ and $\varphi_i(\eta) = H_{i}^{\pi_{\eta}}(1) - m_i^{\pi_\eta}$ for all $\eta \in \mathbb{B}_i^\infty(\lambda)$ and $i \in I_{af}$.

5 Proof of Proposition 3.2.2

Throughout this section, we fix $\lambda \in P^+$ and set $J := \{i \in I \mid \langle \alpha^\vee_i, \lambda \rangle = 0\}$.

5.1 Extremal elements in $\mathcal{B}_0(\lambda)$ and $\mathbb{B}_0^\infty(\lambda)$

We know from [Kas94, §7] that the affine Weyl group $W_{af}$ acts on $\mathcal{B}(\lambda)$ in such a way that

$$S_{\alpha_i, b} := \begin{cases} i^{-\langle \alpha_i^\vee, \text{wt}(b) \rangle} & \text{if } \langle \alpha_i^\vee, \text{wt}(b) \rangle \geq 0, \\ i^{\langle \alpha_i^\vee, \text{wt}(b) \rangle} & \text{if } \langle \alpha_i^\vee, \text{wt}(b) \rangle \leq 0 \end{cases} \quad (5.1.1)$$

for each $b \in \mathcal{B}(\lambda)$ and $i \in I_{af}$.
Proposition 5.1.1 (cf. [Kas02b, Conjecture 5.11]; see also Remark 2.2.1). It holds that

\[(W_J)^{af} = \{ x \in W_{af} \mid S_xu_\lambda = u_\lambda \}.\]

Therefore, the correspondence \((W_J)^{af} \ni x \mapsto S_xu_\lambda \in \{ S_yu_\lambda \mid y \in W_{af} \}\) is bijective.

**Proof.** Write \(\lambda \in P^+\) as \(\lambda = \sum_{i \in I} m_i \varpi_i\) with \(m_i \in \mathbb{Z}_{\geq 0}\), \(i \in I\). Then we know from [BN04, Remark 4.17] (see also [Kas02b, §13]) that there exists an embedding \(\Psi : B_0(\lambda) \hookrightarrow \bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i}\) of crystals such that \(\Psi(u_\lambda) = \bigotimes_{i \in I} u^{\otimes m_i}\). We can show by induction on \(\ell(x)\) that \(\Psi(S_xu_\lambda) = \bigotimes_{i \in I} (S_xu_{\varpi_i})^{\otimes m_i}\) for all \(x \in W_{af}\). Therefore,

\[
\{ x \in W_{af} \mid S_xu_\lambda = u_\lambda \} = \bigcap_{i \in I \setminus J} \{ x \in W_{af} \mid S_xu_{\varpi_i} = u_{\varpi_i} \}. \tag{5.1.2}
\]

We know from [Kas02b, Lemma 5.6] that

\[
\{ x \in W_{af} \mid S_xu_{\varpi_i} = u_{\varpi_i} \} = \{ r_\beta \mid \beta \in \Delta^+_{af} \text{ with } \langle \beta^\vee, \varpi_i \rangle = 0 \}. \tag{5.1.3}
\]

Notice that if \(\beta \in (\Delta_j)^{af}_{\lambda},\) then \(\langle \beta^\vee, \varpi_i \rangle = 0\) for every \(i \in I \setminus J\), and hence \(r_\beta \in \{ x \in W_{af} \mid S_xu_{\varpi_i} = u_{\varpi_i} \}\) for every \(i \in I \setminus J\). Therefore, \((W_J)^{af} = \{ r_\beta \mid \beta \in (\Delta_j)^{af}_{\lambda} \}\) by Remark 2.2.1, it follows immediately that \((W_J)^{af} \subset \{ x \in W_{af} \mid S_xu_\lambda = u_\lambda \}\).

Let us prove the opposite inclusion. Let \(x \in W_{af}\) be such that \(S_xu_\lambda = u_\lambda\). By (5.1.2), \(S_xu_{\varpi_i} = u_{\varpi_i}\) for all \(i \in I \setminus J\); in particular, \(x_{\varpi_i} = \varpi_i\) for all \(i \in I \setminus J\) since the weight of \(S_xu_{\varpi_i}\) is equal to \(x_{\varpi_i}\). Write \(x = \omega t_\xi\) with \(w \in W\) and \(\xi \in Q^\vee\). Then, for every \(i \in I \setminus J\), \(\varpi_i = x_{\varpi_i} = w_{\varpi_i} - \langle \xi, \varpi_i \rangle \delta\) by (2.1.4). Hence, \(w_{\varpi_i} = \varpi_i\) and \(\langle \xi, \varpi_i \rangle = 0\) for every \(i \in I \setminus J\). Therefore, we see that \(w \in W_J \subset (W_J)^{af}\), and \(\xi \in Q^\vee_J\), which implies that \(t_\xi \in \{ (t_\alpha^J \mid j \in J) \subset (W_J)^{af}\). Thus, we obtain \(x \in (W_J)^{af}\), and hence \((W_J)^{af} \supset \{ x \in W_{af} \mid S_xu_\lambda = u_\lambda \}\). This completes the proof of Proposition 5.1.1. \(\square\)

For \(x \in (W_J)^{af}\), we set \(u_x := S_xu_\lambda \in B_0(\lambda);\) note that \(S_yu_\lambda = u_{\Pi_J(y)}\) for \(y \in W_{af}\). We see from [Kas94, §8] that

\[
wt(u_x) = x_\lambda, \quad \epsilon_i(u_x) = \max\{0, -\langle \alpha_i^\vee, x_\lambda \rangle\}, \quad \varphi_i(u_x) = \max\{0, \langle \alpha_i^\vee, x_\lambda \rangle\} \tag{5.1.4}
\]

for all \(x \in (W_J)^{af}\) and \(i \in I_{af}\).

Now, \(\eta_x := (x; 0, 1)\) is an element of \(B_{\overline{T}}(\lambda)\) for all \(x \in (W_J)^{af}\). By Remark 4.4.1 (2),

\[
wt(\eta_x) = x_\lambda, \quad \epsilon_i(\eta_x) = \max\{0, -\langle \alpha_i^\vee, x_\lambda \rangle\}, \quad \varphi_i(\eta_x) = \max\{0, \langle \alpha_i^\vee, x_\lambda \rangle\} \tag{5.1.5}
\]

for all \(x \in (W_J)^{af}\) and \(i \in I_{af}\). For \(x \in (W_J)\) and \(i \in I_{af}\), we define \(S_{ri} \eta_x \neq 0\) by

\[
S_{ri} \eta_x := \begin{cases} 
    f_{i}^{(\alpha_i^\vee, x_\lambda)} \eta_x & \text{if } \langle \alpha_i^\vee, x_\lambda \rangle \geq 0, \\
    e_{i}^{(\alpha_i^\vee, x_\lambda)} \eta_x & \text{if } \langle \alpha_i^\vee, x_\lambda \rangle \leq 0.
\end{cases}
\]

Then,

\[
S_{ri} \eta_x = \eta_{\Pi_J(r, x)} = \begin{cases} 
    \eta_{r, x} & \text{if } \langle \alpha_i^\vee, x_\lambda \rangle \neq 0, \\
    \eta_x & \text{if } \langle \alpha_i^\vee, x_\lambda \rangle = 0.
\end{cases} \tag{5.1.6}
\]

Indeed, if \(\langle \alpha_i^\vee, x_\lambda \rangle = 0\), then it is obvious that \(S_{ri} \eta_x = \eta_x\). Also, we see that \(x^{-1} \alpha_i \in (\Delta_j)^{af}\) and hence \(r_x^{-1} \alpha_i \in (W_J)^{af}\). Thus, \(\Pi_J(r, x) = \Pi_J(x x^{-1} \alpha_i) = x\) since \(x \in (W_J)^{af}\). Assume that
$n := \langle \alpha_i^\vee, x \lambda \rangle > 0$; we see that $x^{-1} \alpha_i \not\in (\Delta_J)_{af}$, and hence $r_i x \in (W^J)_{af}$ by Lemma 2.2.7. It can be easily seen by induction on $k$ that

$$f_i^k \eta_x = (r_i x, x; 0, k/n, 1) \quad \text{for } 0 \leq k \leq n;$$

in particular, we get $f_i^n \eta_x = \eta_{r_i x} = \eta_{\Pi^J(r_i x)}$, as desired. The proof for the case that $\langle \alpha_i^\vee, x \lambda \rangle < 0$ is similar. Since $\Pi^J(x_1 \Pi^J(x_2)) = \Pi^J(x_1 x_2)$ for all $x_1, x_2 \in W_{af}$, we deduce, by using (5.1.6) repeatedly, that

$$S_{r_{i_1}} S_{r_{i_2}} \cdots S_{r_{i_l}} \eta_x = \eta_{\Pi^J(r_{i_1} r_{i_2} \cdots r_{i_l})}$$

(5.1.7)

for every $i_1, i_2, \ldots, i_l \in I_{af}$ and $x \in (W^J)_{af}$. For $y \in W_{af}$, we define $S_y := S_{r_{i_1}} S_{r_{i_2}} \cdots S_{r_{i_l}}$ if $y = r_{i_1} r_{i_2} \cdots r_{i_l}$; we see by (5.1.7) that $S_y$ does not depend on the choice of an expression $y = r_{i_1} r_{i_2} \cdots r_{i_l}$ of $y$. Thus we get an action of $W_{af}$ on the set $\{ \eta_x \mid x \in (W^J)_{af} \}$.

5.2 $N$-multiple maps

Proposition 5.2.1. Let $N \in \mathbb{Z}_{>0}$. There exists a unique injective map $\sigma_N : B_0(\lambda) \hookrightarrow B_0(\lambda)^{\otimes N}$ such that $\sigma_N(u_\lambda) = u_\lambda^{\otimes N}$, and

$$\begin{align*}
\text{wt}(\sigma_N(b)) &= N \text{wt}(b), & \varepsilon_i(\sigma_N(b)) &= N \varepsilon_i(b), & \varphi_i(\sigma_N(b)) &= N \varphi_i(b), \\
\sigma_N(e_i b) &= e_i^N \sigma_N(b), & \sigma_N(f_i b) &= f_i^N \sigma_N(b).
\end{align*}$$

(5.2.1)

(5.2.2)

for $b \in B_0(\lambda)$ and $i \in I_{af}$. Here, we understand that $\sigma_N(0) = 0$.

Proof. Write $\lambda = \sum_{i \in I} m_i \omega_i$, and let $N \in \mathbb{Z}_{>0}$. By [NS03, Theorem 3.7], there exists an injective map $t_1 : B_0(\lambda) \hookrightarrow B_0(N\lambda)$ such that $t_1(u_\lambda) = u_{N\lambda}$, and

$$\begin{align*}
\text{wt}(t_1(b)) &= N \text{wt}(b), & \varepsilon_i(t_1(b)) &= N \varepsilon_i(b), & \varphi_i(t_1(b)) &= N \varphi_i(b), \\
t_1(e_i b) &= e_i^N t_1(b), & t_1(f_i b) &= f_i^N t_1(b).
\end{align*}$$

for $b \in B_0(\lambda)$ and $i \in I_{af}$. Also, we deduce from the existence of combinatorial $R$-matrices (see [Kas02b, §10]) and [BN04, Remark 4.17] that there exist injective maps

$$t_2 : B_0(N\lambda) \hookrightarrow \bigotimes_{i \in I} B(\pi_i)^{\otimes N_{m_i}} \quad \text{and} \quad t_3 : B_0(\lambda)^{\otimes N} \hookrightarrow \bigotimes_{i \in I} B(\pi_i)^{\otimes N_{m_i}}$$

such that $t_2(u_{N\lambda}) = \bigotimes_{i \in I} u_{\pi_i}^{\otimes N_{m_i}}$. By the connectedness of $B_0(N\lambda)$, we deduce that the map $t_2$ factors through $t_3$, namely, there exists an (injective) map $t_4 : B_0(N\lambda) \to B_0(\lambda)^{\otimes N}$ such that $t_4(u_{N\lambda}) = u_{\lambda}^{\otimes N}$ and $t_2 = t_3 \circ t_4$. Now, we can easily see that the map $\sigma_N := t_4 \circ t_1 : B_0(\lambda) \to B_0(\lambda)^{\otimes N}$ satisfies the desired conditions. Thus we have proved Proposition 5.2.1.

We can prove the following proposition in exactly the same way as [NS03, Proposition 3.12] and [Kas02a, Proposition 8.3.2 (3)].

Proposition 5.2.2. Let $b \in B_0(\lambda)$. There exists $N_b \in \mathbb{Z}_{>0}$ such that for every multiple $N \in \mathbb{Z}_{>0}$ of $N_b$,

$$\sigma_N(b) = u_{x_1} \otimes u_{x_2} \otimes \cdots \otimes u_{x_N} \in B_0(\lambda)^{\otimes N}$$

for some $x_1, x_2, \ldots, x_N \in (W^J)_{af}$.  

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Let $N \in \mathbb{Z}_{>0}$. Notice that the set $\{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$ is identical to $J = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$. Hence, for every rational number $0 < \alpha \leq 1$, the set of vertices for $\text{SB}(N\lambda; a)$ is identical to that for $\text{SB}(N\lambda; a)$, i.e., $(W^J)_{af}$.

If $x \xrightarrow{\beta} y$ in $\text{SB}(N\lambda; a)$ for $x, y \in (W^J)_{af}$ and $\beta \in \Delta^+_af$, then it can be easily seen that $x \xrightarrow{\beta} y$ in $\text{SB}(N\lambda; a)$. Hence, $\text{SB}(N\lambda; a)$ is a subgraph of $\text{SB}(N\lambda; a)$.

Now, let $\eta = (x_1, \ldots, x_s; a_0, \ldots, a_s) \in \mathbb{B}^\infty(N\lambda)$. By the observation above, there exists a directed path from $x_{u+1}$ to $x_u$ in $\text{SB}(N\lambda; a_u)$ for each $1 \leq u \leq s - 1$, which implies that $\eta \in \mathbb{B}^\infty(N\lambda)$. Thus we obtain the canonical inclusion

$$\Phi_N : \mathbb{B}^\infty(N\lambda) \hookrightarrow \mathbb{B}^\infty(N\lambda), \quad \eta \mapsto \eta.$$ (5.2.3)

**Lemma 5.2.3** (cf. [Lit95, Lemma 2.4]). We have $\Phi_N(\eta_0) = \eta_e$, $\Phi_N(e_i \eta) = e_i^N \Phi_N(\eta)$, and $\Phi_N(f_i \eta) = i_i^N \Phi_N(\eta)$ for all $\eta \in \mathbb{B}^0_0(N\lambda)$ and $i \in I_{af}$. Here, we understand that $\Phi_N(0) = 0$.

**Proof.** It is obvious that $\Phi_N(\eta_0) = \eta_e$. Let us show $\Phi_N(e_i \eta) = e_i^N \Phi_N(\eta)$; the equality $\Phi_N(f_i \eta) = i_i^N \Phi_N(\eta)$ can be shown similarly. For simplicity of notation, we set $\pi = \pi_\eta \in \mathbb{B}(\lambda)$ and $\pi' = \pi_\Phi(\eta) \in \mathbb{B}(\lambda)$. We see by definition that

$$\pi'(t) = N \pi(t) \quad \text{for} \quad t \in [0, 1],$$

which implies that $H_1(t) = NH_1(t)$, and hence $m_i^{\pi'} = N m_i^{\pi}$.

If $e_i \eta = 0$, i.e., $m_i^{\pi} = 0$, then $m_i^{\pi'} = 0$, which implies that $e_i \Phi_N(\eta) = 0$, and hence $e_i^N \Phi_N(\eta) = 0$. Conversely, assume that $e_i^N \Phi_N(\eta) = 0$. Since $e_i(\Phi_N(\eta)) = -m_i^{\pi'}$ by Remark 4.4.1 (2), it follows that $-m_i^{\pi'} < N$. Combining this and $m_i^{\pi'} = N m_i^{\pi}$, we get $m_i^{\pi} = 0$, which implies that $e_i \eta = 0$.

Assume that $e_i \eta \neq 0$, or equivalently, $e_i^N \Phi_N(\eta) \neq 0$. For $\pi = \pi_\eta$, define $t_0, t_1 \in [0, 1]$ as in (3.1.5); recall that $H_1(t)$ is strictly decreasing on $[t_0, t_1]$. Also, by Remark 3.1.4, $H_1(t) \geq m_i^{\pi} + 1$ for $t \in [0, t_0]$. Since $H_1(t) = NH_1(t)$ for $t \in [0, 1]$, it follows immediately that

$$t_0 = \min\{t \in [0, 1] \mid H_1'(t) = m_i^{\pi'} = N m_i^{\pi}\},$$

$$t_1 = \max\{t \in [0, t_0] \mid H_1'(t) = N m_i^{\pi} + N\},$$

and that $H_1'(t)$ is strictly decreasing on $[t_0, t_1]$, and $H_1'(t) \geq N m_i^{\pi} + N$ for $t \in [0, t_0]$. Hence we deduce from the definition of the root operator $e_i$ that $\Phi_N(e_i \eta) = e_i^N \Phi_N(\eta)$, as desired. Thus we have proved the lemma. \hfill $\square$

Let $N \in \mathbb{Z}_{>2}$. We define an injective map $\psi_N : \mathbb{B}^\infty(N\lambda) \hookrightarrow \mathbb{B}^\infty_{\lambda}(\lambda) \otimes \mathbb{B}^\infty_{(N-1)\lambda}$ as follows: let $\eta = (x_1, x_2, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}^\infty(N\lambda)$. Let $0 \leq p \leq s - 1$ be such that $a_p \leq 1/N < a_{p+1}$, and set

$$\eta_1 := (x_1, x_2, \ldots, x_p, 0; a_0, a_1, \ldots, a_p, 1),$$

$$\eta_2 := (x_{p+1}, x_{p+2}, \ldots, x_s; 0, a_{p+1} - 1, a_{p+2} - 1, \ldots, a_s - 1, \frac{N a_{p+1} - 1}{N - 1}, \frac{N a_{p+2} - 1}{N - 1}, \ldots, \frac{N a_s - 1}{N - 1} = 1);$$

if $a_p = 1/N$, i.e., $N a_p = 1$, then we drop $x_{p+1}$ and 1 from $\eta_1$. It can be verified that $\eta_1 \in \mathbb{B}^\infty_{\lambda}(\lambda)$ and $\eta_2 \in \mathbb{B}^\infty_{((N-1)\lambda)}$. So we define

$$\psi_N(\eta) := \eta_1 \otimes \eta_2 \in \mathbb{B}^\infty_{\lambda}(\lambda) \otimes \mathbb{B}^\infty_{((N-1)\lambda)}.$$ 

By convention, we set $\psi_N(0) := 0$. 

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Remark 5.2.4. Keep the notation above. Then we see that $\pi_\eta \in B(N\lambda)$ is a “concatenation at $t = 1/N$” of $\pi_{\eta_1} \in B(\lambda)$ and $\pi_{\eta_2} \in B((N-1)\lambda)$, that is,

$$\pi_\eta(t) = \begin{cases} 
\pi_{\eta_1}(Nt) & \text{for } t \in [0, 1/N], \\
\pi_{\eta_1}(1) + \pi_{\eta_2}((N-1)^{-1}(Nt-1)) & \text{for } t \in [1/N, 1].
\end{cases}$$

Proposition 5.2.5. The map $\psi_N : B\widetilde{\mathcal{T}}(\lambda) \rightarrow B\tilde{\mathcal{T}}(\lambda) \otimes B\widetilde{\mathcal{T}}((N-1)\lambda)$ is a strict embedding of crystals in the sense of [Kas94, §1.5].

Proof. We only show that $\psi_N(f_\eta) = f_i \psi_N(\eta)$ for $\eta \in B\widetilde{\mathcal{T}}(N\lambda)$ and $i \in I_{af}$; the proofs for the other conditions are similar or easier. Let $\eta \in B\widetilde{\mathcal{T}}(N\lambda)$, and assume that $\psi_N(\eta) = \eta_1 \otimes \eta_2$ with $\eta_1 \in B\widetilde{\mathcal{T}}(\lambda)$ and $\eta_2 \in B\widetilde{\mathcal{T}}((N-1)\lambda)$. We deduce from the tensor product rule of crystals and Remark 4.4.1 (2) that

$$f_i(\eta_1 \otimes \eta_2) = \begin{cases} 
(f_i\eta_1) \otimes \eta_2 & \text{if } H_i^{\eta_1}(1) - m_i^{\eta_1} > -m_i^{\eta_2}, \\
\eta_1 \otimes (f_i\eta_2) & \text{otherwise}.
\end{cases}$$

Assume that $H_i^{\eta_1}(1) - m_i^{\eta_1} > -m_i^{\eta_2}$. Since $-m_i^{\eta_2} \geq 0$, we see by definition that $f_i\eta_1 \neq 0$. Also, since $m_i^{\eta_1} < H_i^{\eta_1}(1) + m_i^{\eta_2}$, it follow immediately from Remark 5.2.4 that $H_i^{\eta_1}(t) > m_i^{\eta_1}$ for $t \in [1/N, 1]$. Hence, $m_i^{\eta_1} = m_i^{\eta_2}$, and

$$H_i^{\eta_1}(1) - m_i^{\eta_2} \geq H_i^{\eta_1}(1) + H_i^{\eta_2}(1) - m_i^{\eta_1} = H_i^{\eta_1}(1) + H_i^{\eta_2}(1) - m_i^{\eta_1} \geq H_i^{\eta_1}(1) + m_i^{\eta_2} - m_i^{\eta_1} > 0.$$ 

Thus we get $f_i\eta \neq 0$ by definition. For the $\eta$, define $t_0, t_1 \in [0, 1]$ as (3.1.6). Because $H_i^{\eta}(t) \geq m_i^{\eta_1}$ for $t \in [1/N, 1]$ as seen above, it follows immediately from Remark 5.2.4 that $H_i^{\eta}(t) \geq m_i^{\eta_1} + 1$ for $t \in [1/N, 1]$. Hence we get $t_0, t_1 \in [0, 1/N]$. We deduce from the definitions of the root operator $f_i$ and the map $\psi_N$ that $\psi_N(f_i\eta) = (f_i\eta_1) \otimes \eta_2 = f_i(\eta_1 \otimes \eta_2)$. Similarly, we can verify that if $H_i^{\eta_1}(1) - m_i^{\eta_1} \leq -m_i^{\eta_2}$, then $\psi_N(f_i\eta) = \eta_1 \otimes (f_i\eta_2) = f_i(\eta_1 \otimes \eta_2)$. Thus we have proved that $\psi_N(f_i\eta) = f_i\psi_N(\eta)$ for $\eta \in B\widetilde{\mathcal{T}}(N\lambda)$ and $i \in I_{af}$, as desired. 

For each $N \in \mathbb{Z}_{>0}$, define a strict embedding $\Psi_N : B\widetilde{\mathcal{T}}(N\lambda) \rightarrow B\widetilde{\mathcal{T}}(\lambda)^{\otimes N}$ of crystals by

$$\Psi_1 := \text{id}_{B\widetilde{\mathcal{T}}(\lambda)}, \quad \Psi_N := (\text{id}_{B\widetilde{\mathcal{T}}(\lambda)} \otimes \Psi_{N-1}) \circ \psi_N$$

and then define

$$\sigma_N := \Psi_N \circ \Phi_N : B\widetilde{\mathcal{T}}(\lambda) \rightarrow B\widetilde{\mathcal{T}}(\lambda)^{\otimes N}. \quad (5.2.4)$$

We see that this map $\sigma_N$ has the following properties: $\sigma_N(\eta_\epsilon) = \eta_\epsilon^{\otimes N}$, and

$$\text{wt}(\sigma_N(\eta)) = N\text{wt}(\eta), \quad \varepsilon_i(\sigma_N(\eta)) = N\varepsilon_i(\eta), \quad \varphi_i(\sigma_N(\eta)) = N\varphi_i(\eta), \quad (5.2.5)$$

$$\sigma_N(e_i \eta) = e_i^N \sigma_N(\eta), \quad \sigma_N(f_i \eta) = f_i^N \sigma_N(\eta) \quad (5.2.6)$$

for $\eta \in B\widetilde{\mathcal{T}}(\lambda)$ and $i \in I_{af}$. Now, the following lemma can be easily shown by induction on $N \in \mathbb{Z}_{>0}$.
Lemma 5.2.6. Let \( \eta = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathcal{B}_Z(N\lambda) \). If \( k_u := Na_u \in \mathbb{Z} \) for all \( 0 \leq u \leq s \), then

\[
\Psi_N(\eta) = \eta_{x_1} \otimes \cdots \otimes \eta_{x_s} = \eta_{x_1} \otimes \cdots \otimes \eta_{x_s}.
\]

Since \( \{c, \lambda\} = 0 \), we see that

\[
\{\langle \beta^\vee, \lambda \rangle \mid \beta \in \Delta_{af}\} = \{\langle \alpha^\vee, \lambda \rangle \mid \alpha \in \Delta\}
\]

is a finite set. Define \( N_\lambda \in \mathbb{Z}_{>0} \) to be the least common multiple of the finite set \( \{\langle \beta^\vee, \lambda \rangle \mid \beta \in \Delta_{af}\} \setminus \{0\} \).

Lemma 5.2.7. Let \( N \in \mathbb{Z}_{>0} \) be a multiple of \( N_\lambda \). For every \( \eta = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathcal{B}_Z(N\lambda) \), it holds that \( k_u := Na_u \in \mathbb{Z} \) for all \( 0 \leq u \leq s \), and

\[
\sigma_N(\eta) = \eta_{x_1} \otimes \cdots \otimes \eta_{x_s} = \eta_{x_1} \otimes \cdots \otimes \eta_{x_s}.
\]

Proof. We show that \( k_u = Na_u \in \mathbb{Z} \) for all \( 0 \leq u \leq s \). If \( u = 0 \) or \( s \), then the assertion is obvious. Let \( 1 \leq u \leq s - 1 \); by the definition of a \( \mathcal{F} \)-LS path, there exists a directed path from \( x_u \) to \( x_{u+1} \) in \( \text{SB}(\lambda; a_u) \). Let \( x \xrightarrow{a} y \) be an edge in \( \text{SB}(\lambda; a_u) \). Then, \( a_u \langle x^{-1} \beta^\vee, \lambda \rangle \in \mathbb{Z} \setminus \{0\} \). Indeed, since the edge is in \( \text{SB}(\lambda; a_u) \), we have \( a_u \langle x^{-1} \beta^\vee, \lambda \rangle \in \mathbb{Z} \) by the definition. Suppose that \( \langle x^{-1} \beta^\vee, \lambda \rangle = 0 \). Then, \( x^{-1} \beta \in \Delta_{af} \), and hence \( r_{x^{-1} \beta} \in (W_J)_{af} \) by Remark 2.2.1. Since \( x \in (W_J)_{af} \), \( y = r_{x^{-1} \beta} x = x r_{x^{-1} \beta} \notin (W_J)_{af} \), which is a contradiction. Thus we have shown that \( a_u \langle x^{-1} \beta^\vee, \lambda \rangle \in \mathbb{Z} \setminus \{0\} \). Therefore we see by the assumption on \( N \) that \( k_u = Na_u \in \mathbb{Z} \). Since \( \sigma_N(\eta) = \Psi_N(\Phi_N(\eta)) \), the assertion of the lemma follows from Lemma 5.2.6.

5.3 Proof of Proposition 3.2.2

Lemma 5.3.1. Let \( X = g_0g_{p-1} \cdots g_2g_1 \) be a monomial in the Kashiwara operators, where \( g_q \in \{e_i, f_i \mid i \in I_{af}\} \) for each \( 1 \leq q \leq p \).

(1) Assume that \( Xu_\lambda \neq 0 \). Then, \( X\eta_e \neq 0 \). Further, take \( N \in \mathbb{Z}_{>0} \) such that the element \( \sigma_N(g_0g_1 \cdots g_{p-1}ux_\lambda) \) is a tensor product of \( N \) elements in \( \{ux \mid x \in (W_J)_{af}\} \) for each \( 0 \leq q \leq p \) (see Proposition 5.2.2). Write \( \sigma_N(Xu_\lambda) = \sigma_N(g_0g_{p-1} \cdots g_2g_1u_\lambda) \) as \( \sigma_N(Xu_\lambda) = u_{x_1} \otimes \cdots \otimes u_{x_N} \) with \( x_1, \ldots, x_N \in (W_J)_{af} \). Then, \( \sigma_N(\eta_e) = \eta_{x_1} \otimes \cdots \otimes \eta_{x_N} \).

(2) Assume that \( X\eta_e \neq 0 \). Then, \( Xu_\lambda \neq 0 \). Further, let \( N \in \mathbb{Z}_{>0} \) be a multiple of \( N_\lambda \) (see Lemma 5.2.7), and write \( \sigma_N(X\eta_e) \) as \( \sigma_N(X\eta_e) = \eta_{x_1} \otimes \cdots \otimes \eta_{x_N} \) with some \( x_1, \ldots, x_N \in (W_J)_{af} \). Then, \( \sigma_N(Xu_\lambda) = u_{x_1} \otimes \cdots \otimes u_{x_N} \).

Proof. We give a proof only for part (1); the proof for part (2) is similar. Let us show part (1) by induction on \( p \). If \( p = 0 \), then the assertion is obvious. Assume that \( p > 0 \). Set \( Y := g_{p-1} \cdots g_2g_1 \); note that \( Yux \neq 0 \). Write \( \sigma_N(Yu_\lambda) \) as \( \sigma_N(Yu_\lambda) = u_{y_1} \otimes \cdots \otimes u_{y_N} \) with some \( y_1, \ldots, y_N \in (W_J)_{af} \). Then, by the induction hypothesis, \( Y\eta_e \neq 0 \), and \( \sigma_N(Y\eta_e) = \eta_{y_1} \otimes \cdots \otimes \eta_{y_N} \). Here, we should recall from (5.1.4) and (5.1.5) that \( \text{wt}(ux) = \text{wt}(\eta_e), \varepsilon_i(ux) = \varepsilon_i(\eta_e), \phi_i(ux) = \phi_i(\eta_e) \) for all \( x \in (W_J)_{af} \) and \( i \in I_{af} \). Thus it follows from the tensor product rule of crystals that
wt(\(\sigma_N(Yu_\lambda)\)) = wt(\(\sigma_N(Y\eta_e)\)), \(\varepsilon_i(\sigma_N(Yu_\lambda)) = \varepsilon_i(\sigma_N(Y\eta_e))\), \(\varphi_i(\sigma_N(Yu_\lambda)) = \varphi_i(\sigma_N(Y\eta_e))\) for all \(i \in I_{af}\). Hence, by (5.2.1) and (5.2.5), we have \(\varepsilon_i(Yu_\lambda) = \varepsilon_i(Y\eta_e)\) and \(\varphi_i(Yu_\lambda) = \varphi_i(Y\eta_e)\) for all \(i \in I_{af}\). Thus, \(Xu_\lambda = gpYu_\lambda \neq 0\) implies that \(X\eta_e = gpY\eta_e \neq 0\). Moreover, \(\sigma_N(Xu_\lambda)\) is of the form

\[
\begin{align*}
\sigma_N(Xu_\lambda) &= \sigma_N(gpYu_\lambda) = g_p^N \sigma_N(Yu_\lambda) \\
&= g_p^n (u_{y_1} \otimes \cdots \otimes u_{y_N}) \\
&= g_p^n u_{y_1} \otimes \cdots \otimes g_p^n u_{y_N}
\end{align*}
\]

for some \(n_1, \ldots, n_N \in \mathbb{Z}_{\geq 0}\) with \(n_1 + \cdots + n_N = N\). Then we deduce from (5.2.1), (5.2.5), and the tensor product rule of crystals, together with the equalities \(wt(u_x) = wt(\eta_x)\), \(\varepsilon_i(u_x) = \varepsilon_i(\eta_x)\), \(\varphi_i(u_x) = \varphi_i(\eta_x)\) for all \(x \in (W^J)_{af}\) and \(i \in I\), that

\[
\begin{align*}
\sigma_N(X\eta_e) &= g_p^n (\eta_{y_1} \otimes \cdots \otimes \eta_{y_N}) \\
&= g_p^n \eta_{y_1} \otimes \cdots \otimes g_p^n \eta_{y_N}.
\end{align*}
\]

Because \(\sigma_N(Xu_\lambda) = g_p^n u_{y_1} \otimes \cdots \otimes g_p^n u_{y_N} = u_{x_1} \otimes \cdots \otimes u_{x_N}\) by assumption, we deduce that \(g_p^n \eta_{y_u} = \eta_{x_u}\) for each \(1 \leq u \leq N\). Thus we have proved part (1). 

**Proof of Proposition 3.2.2.** It suffices to show that the following hold for monomials \(X, Y\) in the Kashiwara operators (cf. [Kas96, Proof of Theorem 4.1] and [NS03, Proof of Theorem 5.1]):

1. \(Xu_\lambda \neq 0\) in \(\mathcal{B}_0(\lambda)\) if and only if \(X\eta_e \neq 0\) in \(\mathcal{E}_0^\geq(\lambda)\),
2. \(Xu_\lambda = Yu_\lambda\) in \(\mathcal{B}_0(\lambda)\) if and only if \(X\eta_e = Y\eta_e\) in \(\mathcal{E}_0^\geq(\lambda)\).

Assertion (1) has already been proved in Lemma 5.3.1. Let us show assertion (2). We first assume that \(Xu_\lambda = Yu_\lambda \neq 0\). By Lemma 5.3.1 (1), we have \(X\eta_e \neq 0\) and \(Y\eta_e \neq 0\). Take \(N \in \mathbb{Z}_{\geq 0}\) such that the assumption of Lemma 5.3.1 (1) holds for both of \(Xu_\lambda\) and \(Yu_\lambda\); write \(\sigma_N(Xu_\lambda)\) and \(\sigma_N(Yu_\lambda)\) as

\[
\begin{align*}
\sigma_N(Xu_\lambda) &= u_{x_1} \otimes u_{x_2} \otimes \cdots \otimes u_{x_N}, \\
\sigma_N(Yu_\lambda) &= u_{y_1} \otimes u_{y_2} \otimes \cdots \otimes u_{y_N}
\end{align*}
\]

with some \(x_1, \ldots, x_N \in (W^J)_{af}\) and \(y_1, \ldots, y_N \in (W^J)_{af}\). Then, by Lemma 5.3.1 (1),

\[
\begin{align*}
\sigma_N(X\eta_e) &= \eta_{x_1} \otimes \eta_{x_2} \otimes \cdots \otimes \eta_{x_N}, \\
\sigma_N(Y\eta_e) &= \eta_{y_1} \otimes \eta_{y_2} \otimes \cdots \otimes \eta_{y_N}.
\end{align*}
\]

Since \(Xu_\lambda = Yu_\lambda\), we have \(x_u = y_u\) for all \(1 \leq u \leq N\). Therefore, we obtain \(\sigma_N(X\eta_e) = \sigma_N(Y\eta_e)\), and hence \(X\eta_e = Y\eta_e\) by the injectivity of \(\sigma_N\). Thus we have proved the “only if” part of assertion (2). The “if” part of assertion (2) can be shown similarly; use Lemma 5.3.1 (2) instead of Lemma 5.3.1 (1). This completes the proof of Proposition 3.2.2. 

**6 Proof of Proposition 3.2.4**

Throughout this section, we fix \(\lambda \in P^+\) and set \(J := \{i \in I \mid \langle \alpha_i^+, \lambda \rangle = 0\} \subset I\).
6.1 Directed paths from $e$ in $\text{QB}(\lambda; a)$

For a rational number $0 < a \leq 1$, set

$$I(\lambda; a) := \{i \in I \mid a(\alpha_i^\vee, \lambda) \in \mathbb{Z}\} \subset I,$$  \hspace{1cm} (6.1.1)

$$[W_{I(\lambda; a)}] := \{[v] \mid v \in W_{I(\lambda; a)} = \langle r_i \mid i \in I(\lambda; a) \rangle \} \subset W^J;$$  \hspace{1cm} (6.1.2)

note that $J \subset I(\lambda; a)$. Also, we set

$$W_e^J(\lambda; a) := \{w \in W^J \mid \text{there exists a directed path from } e \text{ to } w \text{ in } \text{QB}(\lambda; a)\}. \hspace{1cm} (6.1.3)$$

**Proposition 6.1.1.** We have $[W_{I(\lambda; a)}] = W_e^J(\lambda; a)$.

In order to prove Proposition 6.1.1, we need some lemmas.

**Lemma 6.1.2.** Let $w \in W^J$ be an arbitrary element. Every shortest directed path from $e$ to $w$ in $\text{QB}^J$ consists only of Bruhat edges. In particular, the length of a shortest directed path from $e$ to $w$ in $\text{QB}^J$ is equal to $\ell(w)$.

**Proof.** Let $w \in W^J$. First, observe that there exists a directed path from $e$ to $w$ consisting only of Bruhat edges. Indeed, let $w = r_{i_1}r_{i_2} \cdots r_{i_{l-1}}r_{i_l}$ be a reduced expression of $w$. Then, $r_{i_k}r_{i_{k+1}} \cdots r_{i_{l-1}}r_{i_l} \in W^J$ for all $1 \leq k \leq l$, and

$$e \overset{\alpha_{i_1}}{\rightarrow}_{B} r_{i_1} \overset{r_{i_1}\alpha_{i_1-1}}{\rightarrow}_{B} r_{i_1-1}r_{i_1} \overset{r_{i_1}r_{i_{l-1}}\alpha_{i_{l-2}}}{\rightarrow}_{B} \cdots \overset{r_{i_1}r_{i_{l-1}}-r_{i_2}\alpha_{i_1}}{\rightarrow}_{B} r_{i_1} \cdots r_{i_{l-1}}r_{i_l} = w \text{ in } \text{QB}^J.$$  

Thus the length $p$ of a shortest directed path from $e$ to $w$ in $\text{QB}^J$ is less than or equal to $l = \ell(w)$.

Let $e = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_p = w$ be a shortest directed path from $e$ to $w$ in $\text{QB}^J$. By the definition of $\text{QB}^J$ (see also Remark 4.2.2 (3)), we have

$$\ell(w) = \sum_{q=1}^{p} (\ell(w_q) - \ell(w_{q-1})) \leq p. \hspace{1cm} (6.1.4)$$

Because $p \leq \ell(w)$ as seen above, we obtain $\ell(w) = p$. The equality in (6.1.4) holds if and only if $\ell(w_q) - \ell(w_{q-1}) = 1$ for all $1 \leq q \leq p$, or equivalently, all the edges are Bruhat edges. Thus we have proved the lemma. \hfill \Box

We know the following from [LNSSS13b, Lemma 4.1.8 (1)].

**Lemma 6.1.3.** Let $w, v \in W^J$, and assume that there exists a directed path

$$v = v_0 \overset{\gamma_1}{\rightarrow} v_1 \overset{\gamma_2}{\rightarrow} \cdots \overset{\gamma_k}{\rightarrow} v_k = w \text{ in } \text{QB}(\lambda; a).$$

Let $i \in I$. If there exists $0 \leq m \leq k - 1$ such that $\langle \alpha_i^\vee, v_m \lambda \rangle < 0$ for all $m + 1 \leq n \leq k$ and $\langle \alpha_i^\vee, v_m \lambda \rangle \geq 0$, then $[r_i v_{m+1}] = r_i v_{m+1} = v_m$ and there exists a directed path from $v$ to $[r_i w] = r_i w$ of the form

$$v = v_0 \overset{\gamma_1}{\rightarrow} \cdots \overset{\gamma_m}{\rightarrow} v_m = r_i v_{m+1} \overset{\gamma_{m+2}}{\rightarrow} r_i v_{m+2} \overset{\gamma_{m+3}}{\rightarrow} \cdots \overset{\gamma_k}{\rightarrow} r_i v_k = r_i w \text{ in } \text{QB}(\lambda; a).$$

**Lemma 6.1.4.** If there exists a directed path from $w_1 \in W^J$ to $w_2 \in W^J$ in $\text{QB}(\lambda; a)$, then $a(w_2 \lambda - w_1 \lambda) \in Q$.  

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Proof. It suffices to show that if \( w_1 \xrightarrow{\alpha} [w_1 r_\alpha] = w_2 \) in \( \text{QB}(\lambda; a) \), then \( a(w_2 \lambda - w_1 \lambda) \in Q \). Since \( a(\alpha^\vee, \lambda) \in Z \), we have \( a(w_2 \lambda - w_1 \lambda) = a(w_1 r_\alpha \lambda - w_1 \lambda) = a(\alpha^\vee, \lambda) w_1 \alpha \in Q \), as desired. \( \square \)

**Lemma 6.1.5.** If there exists a directed path from \( e \) to \( w \in W^J \) in \( \text{QB}(\lambda; a) \), then all shortest directed paths from \( e \) to \( w \) in \( \text{QB}^J \) are in \( \text{QB}(\lambda; a) \).

**Proof.** We show the assertion by induction on \( \ell(w) \). If \( \ell(w) = 0 \), i.e., \( w = e \), then the assertion is obvious. Assume that \( \ell(w) > 0 \), and let 
\[
e = v_0 \xrightarrow{\gamma_1} v_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_k} v_k = w
\]
be a directed path from \( e \) to \( w \) in \( \text{QB}(\lambda; a) \). Take \( i \in I \) such that \( \langle \alpha_i^\vee, w \lambda \rangle < 0 \); note that \( r_i w \in W^J \) and \( \ell(r_i w) = \ell(w) - 1 \). Since \( \langle \alpha_i^\vee, e \lambda \rangle = \langle \alpha_i^\vee, \lambda \rangle \geq 0 \), it follows from Lemma 6.1.3 that there exists a directed path from \( e \) to \( r_i w \) of the form:
\[
e = v_0 \xrightarrow{\gamma_1} v_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_m} v_m = r_i v_{m+1} \xrightarrow{\gamma_{m+2}} \cdots \xrightarrow{\gamma_k} r_i v_k = r_i w \text{ in } \text{QB}(\lambda; a).
\]
Therefore, by our induction hypothesis, all the shortest directed paths from \( e \) to \( r_i w \) in \( \text{QB}^J \) are in \( \text{QB}(\lambda; a) \).

Now, let
\[
e = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_l} w_l = w
\]
be an arbitrary shortest directed path from \( e \) to \( w \) in \( \text{QB}^J \); note that \( l = \ell(w) \) by Lemma 6.1.2. By the same argument as above, there exists a directed path from \( e \) to \( r_i w \) of length \( l - 1 \) of the form
\[
e = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_n} w_n = r_i w_{n+1} \xrightarrow{\gamma_{n+2}} \cdots \xrightarrow{\gamma_l} r_i w_l = r_i w \text{ in } \text{QB}^J.
\]
Notice that the directed path (6.1.6) is shortest by Lemma 6.1.2 since \( \ell(r_i w) = \ell(w) - 1 \). Hence the directed path (6.1.6) is in \( \text{QB}(\lambda; a) \) by our induction hypothesis. Therefore, \( e = w_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_l} w_l = w \) are directed paths in \( \text{QB}(\lambda; a) \).

It remains to prove that \( a(\gamma_{n+1}, \lambda) \in Z \). Since \( w_{n+1} = [w_n r_{\gamma_{n+1}}] \), we have
\[
-a(\gamma_{n+1}, \lambda) w_n \gamma_{n+1} = a(w_{n+1} \lambda - w_n \lambda)
= a(w \lambda - e \lambda) - a(w \lambda - w_{n+1} \lambda) - a(w_n \lambda - e \lambda).
\]
Since there exists a directed path from \( e \) to \( w_l = w \) in \( \text{QB}(\lambda; a) \) by assumption, it follows from Lemma 6.1.4 that \( a(w \lambda - e \lambda) \in Q \). Similarly, since there exist directed paths from \( e \) to \( w_n \) and from \( w_{n+1} \) to \( w_l = w \) in \( \text{QB}(\lambda; a) \) as seen above, we have \( a(w_n \lambda - e \lambda) \in Q \) and \( a(w \lambda - w_{n+1} \lambda) \in Q \) by Lemma 6.1.4. Thus we get \(-a(\gamma_{n+1}, \lambda) w_n \gamma_{n+1} \in Q \). Since \( w_n \gamma_{n+1} \in \Delta \), it follows that \( a(\gamma_{n+1}, \lambda) \) is an integer. Thus we have proved Lemma 6.1.5. \( \square \)

**Proof of Proposition 6.1.1.** We first show that \( [W_{I(\lambda; a)}] \subset W^J_{e \rightarrow a}(\lambda; a) \). Let \( w \in [W_{I(\lambda; a)}] \subset W^J \), and let \( w = r_{i_1} \cdots r_{i_l} \) be a reduced expression of \( w \); note that \( i_1, \ldots, i_l \in I(\lambda; a) \). We see that 
\[
e \xrightarrow{\alpha_{i_l}} B \xrightarrow{r_{i_l}} a \xrightarrow{r_{i_l-1} a_{i_l}} B \xrightarrow{r_{i_l} r_{i_l-1} a_{i_l-2}} B \cdots \xrightarrow{r_{i_l} r_{i_l-1} \cdots r_2 a_{i_1}} B \xrightarrow{r_{i_l} \cdots r_{i_1}} B \xrightarrow{a_{i_1}} B = w \text{ in } \text{QB}^J.
\]

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Since \( i_1, \ldots, i_l \in I(\lambda; a) \), we deduce that this directed path is in \( QB(\lambda; a) \), and hence \( w \in W_{e\rightarrow}(\lambda; a) \).

We next show that \([W_{I(\lambda; a)}] \supset W_{e\rightarrow}(\lambda; a)\). Let \( w \in W_{e\rightarrow}(\lambda; a) \). We show by induction on \( l = \ell(w) \) that \( w \in [W_{I(\lambda; a)}] \). If \( l = \ell(w) = 0 \), then it is obvious that \( w = e \in [W_{I(\lambda; a)}] \). Assume that \( \ell(w) > 0 \). Recall from Lemma 6.1.2 that a shortest directed path from \( e \) to \( w \) in \( QB^J \) is of the form

\[
e = w_0 \stackrel{\gamma_1}{\rightarrow} w_1 \stackrel{\gamma_2}{\rightarrow} \cdots \stackrel{\gamma_{l-1}}{\rightarrow} w_{l-1} \stackrel{\gamma_l}{\rightarrow} w_l = w. \tag{6.1.7}
\]

Because \( w \in W_{e\rightarrow}(\lambda; a) \), the directed path (6.1.7) is contained in \( QB(\lambda; a) \) by Lemma 6.1.5. In particular, \( w_{l-1} \in W_{e\rightarrow}(\lambda; a) \), and hence \( w_{l-1} \in [W_{I(\lambda; a)}] \) by induction hypothesis.

Now, let \( w = r_{i_1}r_{i_2}\cdots r_{i_l} \) be a reduced expression of \( w \). Since \( w_{l-1} \stackrel{\gamma_l}{\rightarrow} w_l = w \), we have \( \ell(w) = \ell(w_{l-1}) + 1 \), and \( w > w_{l-1} \) with respect to the Bruhat order on \( W \). Thus, by the Subword Property (see [BB05, Theorem 2.2.2]), there exists \( 1 \leq p \leq l - 1 \) such that

\[
w_{l-1} = r_{i_{p+1}} \cdots r_{i_p}r_{i_{p-1}} \cdots r_{i_1}
\]

is a reduced expression of \( w_{l-1} \). Because \( w_{l-1} \in [W_{I(\lambda; a)}] \) as mentioned above, we have \( i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_l \in I(\lambda; a) \). Thus it remains to show that \( i_p \in I(\lambda; a) \), that is, \( a(\alpha_{i_p}', \lambda) \in \mathbb{Z} \). Since \( w_{l-1} = wr_{i_p} \), we deduce that \( \gamma_l = r_{i_p} \cdots r_{i_{p+1}} \alpha_{i_p} \), and hence \( \gamma_l \in \alpha_{i_p}' + \sum_{p+1 \leq q \leq l} Z \alpha_q \); notice that \( a(\alpha_q', \lambda) \in \mathbb{Z} \) for every \( p + 1 \leq q \leq l \) since \( i_q \in I(\lambda; a) \). Because the directed path (6.1.7) is contained in \( QB(\lambda; a) \), we have \( a(\gamma', \lambda) \in \mathbb{Z} \). Combining these, we see that \( a(\alpha_{i_p}', \lambda) \in \mathbb{Z} \). Thus we get \( i_p \in I(\lambda; a) \), as desired.

**Corollary 6.1.6.** Let \( w, v \in W_{e\rightarrow}(\lambda; a) \). If \( w \rightarrow_\alpha v \) in \( QB(\lambda; a) \), then \( \alpha \in \sum_{i \in I(\lambda; a)} Z_{\geq 0} \alpha_i \).

**Proof.** Since \( v = [wr_\alpha] \), there exists \( z \in W_J \) such that \( r_\alpha = w^{-1}vz \). By Proposition 6.1.1, \( w, v \in W_{I(\lambda; a)} \). Also, \( z \in W_J \subset \dot{W}_{I(\lambda; a)} \) since \( J \subset I(\lambda; a) \). Therefore, \( r_\alpha \in W_{I(\lambda; a)} \), and hence \( \alpha \in \sum_{i \in I(\lambda; a)} Z_{\geq 0} \alpha_i \). Thus we have proved the corollary.

### 6.2 \( J \)-adjusted coroots

Let \( p_{I \setminus J} : Q^\vee \rightarrow Q^\vee_{I \setminus J} \) be the projection from \( Q^\vee = \bigoplus_{i \in I} Z \alpha_i \) onto \( Q^\vee_{I \setminus J} = \bigoplus_{i \in I \setminus J} Z \alpha_i \).

**Proposition 6.2.1.** For each \( i \in I \setminus J \), there exists a unique positive root \( \tilde{\alpha}_i \in \Delta^+ \) satisfying the following conditions:

1. \( p_{I \setminus J}(\tilde{\alpha}_i^\vee) = \alpha_i^\vee \);
2. \( \ell(r_{\tilde{\alpha}_i}) = 2(\tilde{\alpha}_i^\vee, \rho) - 1 \);
3. \( \tilde{\alpha}_i^\vee \in Q^\vee_{I, \text{adj}} \);
4. \( z_{\tilde{\alpha}_i^\vee}r_{\tilde{\alpha}_i} \in W^J \), and \( z_{\tilde{\alpha}_i^\vee}r_{\tilde{\alpha}_i} \frac{\tilde{\alpha}_i}{Q} \rightarrow e \) in \( QB^J \).

For a subset \( K \subset I \), let \( \theta_K = \Delta^+_K \) be the highest root of \( \Delta^+_K \); we write \( \theta = \theta_I \in \Delta^+ \).

**Lemma 6.2.2.** For every \( \gamma = w^{-1} \theta \in W\theta \cap (\Delta \setminus \Delta_J) \), the element \( z\gamma^\vee \in Q^\vee \) is \( J \)-adjusted, where \( z \in W_J \) is defined by \( r_\theta w = [r_\theta w] z \).
Proof. Let us show that \( \langle z\gamma^\vee, \alpha \rangle \in \{-1, 0\} \) for all \( \alpha \in \Delta_j^+ \). Let \( \alpha \in \Delta_j^+ \). We have

\[
\langle z\gamma^\vee, \alpha \rangle = \langle zw^{-1}\theta^\vee, \alpha \rangle = -\langle zw^{-1}r_\theta\theta^\vee, \alpha \rangle = -\langle \theta^\vee, r_\theta w z^{-1}\alpha \rangle = -\langle \theta^\vee, [r_\theta w] \alpha \rangle.
\]

we need only to show that \( \langle \theta^\vee, [r_\theta w] \alpha \rangle \in \{0, 1\} \). Notice that \( [r_\theta w] \alpha \in \Delta^+ \) by (2.1.8). Let us show that \( [r_\theta w] \alpha \neq \theta \). Suppose that \( [r_\theta w] \alpha = \theta \). Then we have

\[
\gamma = w^{-1}\theta = -w^{-1}r_\theta \theta = -z^{-1}[r_\theta w]^{-1}\theta = -z^{-1}\alpha.
\]

Because \( z \in W_J \) and \( \alpha \in \Delta_J \), we see that \( \gamma = -z^{-1}\alpha \in \Delta_J \), which contradicts the assumption. Thus we get \( [r_\theta w] \alpha \neq \theta \). Hence, it follows from [Bou68, Proposition 25 (iv)] that \( \langle \theta^\vee, [r_\theta w] \alpha \rangle \in \{0, 1\} \), as desired. \( \Box \)

We know the following lemma from [BMO11, Lemma 7.2]: in a simply-laced root system, we understand that all roots are long.

Lemma 6.2.3. Let \( \alpha \in \Delta^+ \), and write it as \( \alpha = \sum_{i \in I} c_i \alpha_i \) with \( c_i \in \mathbb{Z}_{\geq 0} \), \( i \in I \). We have \( \ell(r_\alpha) = 2(\alpha^\vee, \rho) - 1 \) if and only if \( \alpha \) is a long root, or \( \alpha \) is a short root, satisfying the condition that \( c_i = 0 \) for all \( i \in I \) such that \( \alpha_i \) is long.

Proof of Proposition 6.2.1. We know from [Woo05, Lemma/Definition 1] (see also [LS10, Theorem 10.15]) that an element satisfying conditions (1) and (3) is unique.

Let \( i \in I \setminus J \). We first show that there exists \( \tilde{\alpha}_i^\vee \in \Delta^+ \) satisfying conditions (1), (2), and (3).

Case 1. If \( \alpha_i \) is a long root of \( \Delta \), then it follows that \( \alpha_i \in W_\theta \cap (\Delta \setminus \Delta_J) \). By Lemma 6.2.2, there exists \( z \in W_J \) such that \( z\alpha_i^\vee \) is \( J \)-adjusted. Set \( \tilde{\alpha}_i := z\alpha_i \in \Delta^+ \). Since \( z \in W_J \), we get \( p_{J,i}(\tilde{\alpha}_i^\vee) = p_{I,J}(z\alpha_i^\vee) = \alpha_i^\vee \). Because \( \tilde{\alpha}_i \) is a long root of \( \Delta \), it follows from Lemma 6.2.3 that \( \ell(r_{\tilde{\alpha}_i}) = 2(\tilde{\alpha}_i^\vee, \rho) - 1 \). Thus, \( \tilde{\alpha}_i = z\alpha_i \in \Delta^+ \) satisfies conditions (1), (2), and (3).

Case 2. Assume that \( \alpha_i \) is a short root. Let \( J \cup \{i\} = K \sqcup J_1 \sqcup \cdots \sqcup J_m \) be the decomposition of (the Dynkin subdiagram corresponding to) \( J \cup \{i\} \) into its connected components, with \( i \in K \); note that \( \alpha_i \in \Delta_K \).

Subcase 2-1. If \( \Delta_K \) contains a long root of \( \Delta \), then \( \Delta_K \) is of type \( B, C, F \), or \( G \). We see from [Bou68] that \( \sum_{k \in K} \alpha_k^\vee \) is a short coroot for \( \Delta_K \). Let \( \gamma = \sum_{k \in K} \alpha_k^\vee \); notice that \( \gamma \in W_\theta \cap (\Delta \setminus \Delta_J) \). Hence, by Lemma 6.2.2, there exists \( z \in W_J \) such that \( z\gamma \) is \( J \)-adjusted. Set \( \tilde{\alpha}_i := z\gamma \). Since \( z \in W_J \), we get \( p_{J,i}(\tilde{\alpha}_i^\vee) = p_{J,i}(z\gamma) = p_{J,i}(\gamma) = \alpha_i^\vee \). Because \( \tilde{\alpha}_i = z\gamma \) is a long root of \( \Delta \), it follows from Lemma 6.2.3 that \( \ell(r_{\tilde{\alpha}_i}) = 2(\tilde{\alpha}_i^\vee, \rho) - 1 \). Thus, \( \tilde{\alpha}_i = z\gamma \in \Delta^+ \) satisfies conditions (1), (2), and (3).

Subcase 2-2. Assume that \( \Delta_K \) does not contain a long root of \( \Delta \); in this case, \( \Delta_K \) is a simply-laced root system. Because \( \alpha_i \in W_\theta \cap (\Delta \setminus \Delta_{K \setminus \{i\}}) \), we see from Lemma 6.2.2 that there exists \( z \in W_{K \setminus \{i\}} \) such that \( z\alpha_i^\vee \) is \( K \setminus \{i\} \)-adjusted. We claim that \( z\alpha_i^\vee \) is \( J \)-adjusted. Remark that \( J = (K \setminus \{i\}) \sqcup J_1 \sqcup \cdots \sqcup J_m \), which implies that \( \Delta^+_J = \Delta^+_{K \setminus \{i\}} \sqcup \Delta^+_J \sqcup \cdots \sqcup \Delta^+_J \). Since \( z\alpha_i^\vee \) is \( K \setminus \{i\} \)-adjusted, we have \( \langle z\alpha_i^\vee, \gamma \rangle \in \{-1, 0\} \) for all \( \gamma \in \Delta^+_{K \setminus \{i\}} \). Because \( K \) is a connected component of \( K \sqcup \{i\} \), it follows that \( \langle z\alpha_i^\vee, \gamma \rangle = 0 \) for all \( \gamma \in \Delta^+_J \sqcup \cdots \sqcup \Delta^+_J \). Combining these, we get \( \langle z\alpha_i^\vee, \gamma \rangle \in \{-1, 0\} \) for all \( \gamma \in \Delta^+_J \), which implies that \( z\alpha_i^\vee \) is \( J \)-adjusted. Since \( z \in W_{K \setminus \{i\}} \), we get \( p_{J,i}(z\alpha_i^\vee) = \alpha_i^\vee \). Because every root in \( \Delta_K \) is a short root of \( \Delta \), we
deduce from Lemma 6.2.3 that \( \ell(r_{\alpha_i}) = 2(z_\alpha, \rho) - 1 \). Thus, \( \tilde{\alpha}_i := z_\alpha \in \Delta^+ \) satisfies conditions (1), (2), and (3).

Next, we show that \( \tilde{\alpha}_i \in \Delta^+ \) above satisfies condition (4). Since \( \tilde{\alpha}_i \) is \( J \)-adjusted by condition (3), we see from Lemma 4.3.6 that

\[
\text{Inv}(z_{\tilde{\alpha}_i}) = \{ \gamma \in \Delta^+_J \mid \langle \tilde{\alpha}_i, \gamma \rangle = -1 \}. \tag{6.2.1}
\]

Let \( \gamma \in \Delta^+_J \); note that \( \langle \tilde{\alpha}_i, \gamma \rangle \in \{-1, 0\} \). If \( \langle \tilde{\alpha}_i, \gamma \rangle = -1 \), then \( r_{\tilde{\alpha}_i}(\gamma) = \gamma + \tilde{\alpha}_i \in \Delta^+ \setminus \Delta^+_J \) by condition (1), and hence \( z_{\tilde{\alpha}_i}r_{\tilde{\alpha}_i}(\gamma) \in \Delta^+ \) since \( z_{\tilde{\alpha}_i} \in W_J \). If \( \langle \tilde{\alpha}_i, \gamma \rangle = 0 \), then \( r_{\tilde{\alpha}_i}(\gamma) = \gamma \notin \text{Inv}(z_{\tilde{\alpha}_i}) \) by (6.2.1), and hence \( z_{\tilde{\alpha}_i}r_{\tilde{\alpha}_i}(\gamma) \in \Delta^+ \). Thus, we get \( z_{\tilde{\alpha}_i}r_{\tilde{\alpha}_i}(\gamma) \in \Delta^+ \) for all \( \gamma \in \Delta^+_J \), and hence \( z_{\tilde{\alpha}_i}r_{\tilde{\alpha}_i} \in W_J \) by (2.1.8).

Now, we have \( r_{\tilde{\alpha}_i}^{-1}(\text{Inv}(z_{\tilde{\alpha}_i})) \subset \Delta^+ \) by (6.2.1). Therefore, by [Mac03, (2.2.4)],

\[
\ell(z_{\tilde{\alpha}_i}r_{\tilde{\alpha}_i}) = \ell(z_{\tilde{\alpha}_i}) + \ell(r_{\tilde{\alpha}_i}); \tag{6.2.2}
\]

note that \( \ell(z_{\tilde{\alpha}_i}) = -2(\tilde{\alpha}_i, \rho_J) \) by Lemma 4.3.6 and condition (3), and that \( \ell(r_{\tilde{\alpha}_i}) = 2(\tilde{\alpha}_i, \rho) - 1 \) by condition (2). Combining these, we have \( \ell(z_{\tilde{\alpha}_i}r_{\tilde{\alpha}_i}) = 2(\tilde{\alpha}_i, \rho - \rho_J) - 1 \). Thus,

\[
\ell(z_{\tilde{\alpha}_i}r_{\tilde{\alpha}_i}) + 1 - 2(\tilde{\alpha}_i, \rho - \rho_J) = 0 = \ell(e),
\]

which implies that \( z_{\tilde{\alpha}_i}r_{\tilde{\alpha}_i} \to e \) in \( QB(J) \). Thus we have proved Proposition 6.2.1. \( \square \)

**Remark 6.2.4.** The element \( \tilde{\alpha}_i \in \Delta^+, i \in I \setminus J \), is explicitly described as follows: let \( K \) be the connected component of \( J \cup \{i\} \) containing \( i \), and let \( k \in K \) be the (unique) most nearest node whose corresponding simple root \( \alpha_k \) is a long root of \( \Delta_K \). If \( \alpha_i \) is a long root of \( \Delta_K \), then \( k = i \). If \( \alpha_i \) is a short root of \( \Delta_K \), then \( k \neq i \) and \( K \) contains either of the following Dynkin subdiagrams:

(i) \begin{align*}
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\
& i = k_1 & k_2 & & k_n & k
\end{array}

(n \in \mathbb{Z}_{\geq 1})
\end{align*}

(ii) \begin{align*}
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\longleftrightarrow & i & \quad \bigstar \\
& & k
\end{array}
\end{align*}

If \( \alpha_i \) is a long root of \( \Delta_K \) (and hence \( k = i \)), then we have \( \tilde{\alpha}_i = \alpha_i \); if \( K \) contains (i), then we have \( \tilde{\alpha}_i = r_{k_1}r_{k_2}r_{k_3} \cdots r_{k_n} \alpha_k \); and if \( K \) contains (ii), then we have \( \tilde{\alpha}_i = r_i \alpha_k \).

### 6.3 Directed paths from \( e \) to \( e \) in \( QB(J) \)

For a directed path \( d : v = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_l} w_l = w \) in \( QB(J) \),

we define the weight \( \text{wt}(d) \) of \( d \) by

\[
\text{wt}(d) := \sum_{1 \leq u \leq l \text{ s.t. } w_{u-1} \gamma_u w_u} \gamma_u^\vee \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee. \tag{6.3.1}
\]
Proposition 6.3.1. Let $0 < a \leq 1$ be a rational number such that $I(\lambda; a) \supseteq J$. For each $i \in I(\lambda; a) \setminus J$, there exists a directed path $d(i)$ from $e$ to $e$ in $Q(B(\lambda; a))$ such that $\text{wt}(d(i)) = \lambda\bar{\alpha}_i^\vee$, where $\lambda\bar{\alpha}_i^\vee$ is the positive root given in Proposition 6.2.1.

Proof. Let $i \in I(\lambda; a) \setminus J$. We see from condition (1) in Proposition 6.2.1 that $r\lambda_i \in W_{I(\lambda; a)} \subset W_{I(\lambda; a)}$ since $i \in I(\lambda; a)$ and $J \subset I(\lambda; a)$. Also, $z\lambda_i^\vee \in W_{I(\lambda; a)}$. Hence, $z\lambda_i^\vee r\lambda_i \in W_{I(\lambda; a)}$. Because $z\lambda_i^\vee r\lambda_i = |z\lambda_i^\vee r\lambda_i| \in W^J$ by condition (4) in Proposition 6.2.1, we have $z\lambda_i^\vee r\lambda_i \in W_{I(\lambda; a)}$. Thus, for each $i \in I(\lambda; a)$, let $d(i)$ be a directed path from $e$ to $e$ in $Q(B(\lambda; a))$ such that $\text{wt}(d(i)) = \lambda\bar{\alpha}_i^\vee$.

Concatenating these, we obtain a directed path $d(i)$ of the form

$$e = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_l} w_l = z\lambda_i^\vee r\lambda_i$$

in $Q(B(\lambda; a))$. Also, by condition (4) in Proposition 6.2.1, we have $z\lambda_i^\vee r\lambda_i \xrightarrow{\bar{\alpha}_i} e$ in $Q(B(\lambda; a))$. We see that $\lambda\bar{\alpha}_i^\vee = \{\bar{\alpha}_i^\vee, \lambda\}$ by condition (1) in Proposition 6.2.1. Since $i \in I(\lambda; a)$, it follows immediately that $a(\bar{\alpha}_i^\vee) = a(\bar{\alpha}_i^\vee, \lambda) \in \mathbb{Z}$, and hence the edge $z\lambda_i^\vee r\lambda_i \xrightarrow{\bar{\alpha}_i} e$ is contained in $Q(B(\lambda; a))$. Thus, for each $i \in I(\lambda; a)$, let $d(i)$ be a directed path from $e$ to $e$ in $Q(B(\lambda; a))$ such that $\text{wt}(d(i)) = \lambda\bar{\alpha}_i^\vee$. Thus we have proved Proposition 6.3.1.

Corollary 6.3.2. For a rational number $0 < a \leq 1$,

$$\{p_{I \setminus J}(\text{wt}(d)) \mid d \text{ is a directed path from } e \text{ to } e \text{ in } Q(B(\lambda; a))\} = \sum_{i \in I(\lambda; a) \setminus J} \mathbb{Z}_{\geq 0}\lambda\bar{\alpha}_i^\vee.$$

Proof. Let $d : e = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_p} w_p = e$ be a directed path from $e$ to $e$ in $Q(B(\lambda; a))$. Then, $w_0, w_1, \ldots, w_p \in W_{e \rightarrow}$. Applying Corollary 6.1.6 to $w_{q-1} \xrightarrow{\gamma_q} w_q$ for each $1 \leq q \leq p$, we see that $\gamma_q \in \sum_{i \in I(\lambda; a)} \mathbb{Z}_{\geq 0}\lambda\bar{\alpha}_i^\vee$ for every $1 \leq q \leq p$. Therefore, $\text{wt}(d) = \sum_{i \in I(\lambda; a)} \mathbb{Z}_{\geq 0}\lambda\bar{\alpha}_i^\vee$, and hence $p_{I \setminus J}(\text{wt}(d)) = \sum_{i \in I(\lambda; a) \setminus J} \mathbb{Z}_{\geq 0}\lambda\bar{\alpha}_i^\vee$. Thus the inclusion $(\text{LHS}) \subset (\text{RHS})$ holds.

Let us show the opposite inclusion. Remark that $(\text{LHS})$ contains 0 since the weight of the “trivial” directed path $e = w_0 = e$ in $Q(B(\lambda; a))$ is equal to 0. Hence, if $I(\lambda; a) = J$, then the assertion is obvious. Assume that $I(\lambda; a) \supseteq J$. For each $i \in I(\lambda; a)$, let $d(i)$ be a directed path from $e$ to $e$ such that $\text{wt}(d(i)) = \lambda\bar{\alpha}_i^\vee$ (see Proposition 6.3.1); note that $p_{I \setminus J}(\lambda\bar{\alpha}_i^\vee) = \alpha_i$ by condition (1) in Proposition 6.2.1. Thus, for each $\xi \in (\text{RHS})$, we obtain a directed path $d$ from $e$ to $e$ in $Q(B(\lambda; a))$ such that $p_{I \setminus J}(\text{wt}(d)) = \xi$, by concatenating these $d(i)$’s, $i \in I(\lambda; a) \setminus J$. Therefore the opposite inclusion $(\text{LHS}) \supset (\text{RHS})$ holds. Thus we have proved Corollary 6.3.2.

6.4 Existence condition for directed paths in $S(B(\lambda; a))$

Lemma 6.4.1. Let $x, y \in (W^J)_{af}$, and write them as $x = w_1z_{\xi_1}t_{\xi_1}$, $y = w_2z_{\xi_2}t_{\xi_2}$ with some $w_1, w_2 \in W^J$ and $\xi_1, \xi_2 \in Q^J_{\text{adj}}$. Assume that there exists a directed path

$$x = w_1z_{\xi_1}t_{\xi_1} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_l} w_2z_{\xi_2}t_{\xi_2} = y$$

(6.4.1)
from $x$ to $y$ in $\text{SB}^J$. Let
\[
\mathbf{d} : w_1 \overset{\gamma_1}{\rightarrow} \cdots \overset{\gamma_l}{\rightarrow} w_2
\]
be the directed path in $\text{QB}^J$ obtained by applying Proposition 4.3.7 (1) to the directed path (6.4.1) repeatedly. Then, $p_{I \setminus J}(\xi_2 - \xi_1) = p_{I \setminus J}(\text{wt}(\mathbf{d}))$.

Proof. It suffices to show the assertion when $l = 1$. Assume that $x = w_1 z_{\xi_1} t_{\xi_1} \overset{\beta}{\rightarrow} r_\beta x = w_2 z_{\xi_2} t_{\xi_2}$ in $\text{SB}^J$ for $\beta \in \Delta_{af}^+$; by Lemma 4.3.5, $\beta = w_1 \alpha + \chi \delta \in \Delta_{af}^+$ for some $\alpha \in \Delta^+ \setminus \Delta_f^+$ and $\chi \in \{0, 1\}$. Note that $r_\beta x = w_1 r_\alpha z_{\xi_1} t_{\xi_1 + \chi \delta^{-1} \alpha^\vee}$, and hence $\xi_2 = \xi_1 + \chi \delta^{-1} \alpha^\vee$.

Now, the directed path in $\text{QB}^J$ obtained from $x \overset{\beta}{\rightarrow} r_\beta x$ is
\[
\mathbf{d} : w_1 \overset{\alpha}{\rightarrow} [w_1 r_\alpha] = w_2;
\]
ote that this edge is Bruhat (resp., quantum) if $\chi = 0$ (resp., $\chi = 1$). If $\chi = 0$, then $\xi_1 - \xi_2 = 0 = \text{wt}(\mathbf{d})$. If $\chi = 1$, then $\xi_2 - \xi_1 = z_{\xi_1} \delta^{-1} \alpha^\vee$ and $\text{wt}(\mathbf{d}) = \alpha^\vee$. Hence, $p_{I \setminus J}(\xi_2 - \xi_1) = p_{I \setminus J}(z_{\xi_1} \delta^{-1} \alpha^\vee) = p_{I \setminus J}(\alpha^\vee) = p_{I \setminus J}(\text{wt}(\mathbf{d}))$. This proves the lemma.

**Proposition 6.4.2.** Let $\xi, \zeta \in Q^\vee$, and let $0 < a \leq 1$ be a rational number. There exists a directed path from $\Pi^J(t_\xi)$ to $\Pi^J(t_\zeta)$ in $\text{SB}(\lambda; a)$ if and only if $p_{I \setminus J}(\xi - \zeta) \in \sum_{i \in I(\lambda, a) \setminus J} Z \geq 0 \alpha_i^\vee$.

Proof. Assume that there exists a directed path
\[
\Pi^J(t_\xi) \overset{\beta_1}{\rightarrow} \cdots \overset{\beta_l}{\rightarrow} \Pi^J(t_\zeta) \quad (6.4.2)
\]
with $\beta_1, \ldots, \beta_l \in \Delta_{af}^+$. Recall from Lemma 2.2.6 that $\Pi^J(t_\xi) = z_\xi t_{\xi_1 + \phi_J(\zeta)}$ and $\Pi^J(t_\zeta) = z_\xi t_{\xi_1 + \phi_J(\xi)}$. Hence, by applying Proposition 4.3.7 (1) to the directed path (6.4.2) repeatedly, we obtain a directed path $\mathbf{d}$ from $e$ to $e$ in $\text{QB}(\lambda; a)$ of the form
\[
\mathbf{d} : e \overset{\gamma_1}{\rightarrow} \cdots \overset{\gamma_l}{\rightarrow} e \quad (6.4.3)
\]
for some $\gamma_1, \ldots, \gamma_l \in \Delta^+ \setminus \Delta_f^+$. Then we have
\[
p_{I \setminus J}(\text{wt}(\mathbf{d})) = p_{I \setminus J}(\xi + \phi_J(\xi) - \zeta - \phi_J(\zeta)) \quad (\text{by Lemma 6.4.1})
\]
\[
= p_{I \setminus J}(\xi - \zeta) \quad (\text{since } \phi_J(\xi), \phi_J(\zeta) \in Q^\vee).
\]
Therefore, we get $p_{I \setminus J}(\xi - \zeta) = p_{I \setminus J}(\text{wt}(\mathbf{d})) \in \sum_{i \in I(\lambda, a) \setminus J} Z \geq 0 \alpha_i^\vee$ by Corollary 6.3.2.

Conversely, let us show that there exists a directed path from $\Pi^J(t_\xi)$ to $\Pi^J(t_\zeta)$ in $\text{SB}(\lambda; a)$ for every $\xi, \zeta \in Q^\vee$ such that $p_{I \setminus J}(\xi - \zeta) \in \sum_{i \in I(\lambda, a) \setminus J} Z \geq 0 \alpha_i^\vee$; it suffices to prove the assertion in the case that $p_{I \setminus J}(\xi - \zeta) = \alpha_i^\vee$ for some $i \in I(\lambda, a) \setminus J$. Let $\mathbf{d}(i)$ be a directed path
\[
e = w_0 \overset{\gamma_i}{\rightarrow} w_1 \overset{\gamma_i}{\rightarrow} \cdots \overset{\gamma_i}{\rightarrow} w_l = z_{\alpha_i^\vee} r_{\alpha_i} \overset{\alpha_i}{\rightarrow} Q \quad e
\]
from $e$ to $e$ in $\text{QB}(\lambda; a)$ of weight $\alpha_i^\vee$ obtained in (6.3.2); note that $w_l = r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l}$ by Remark 4.2.2 (1), and $|r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\alpha_i}| = |w_l r_{\alpha_i}| = e$. By applying Proposition 4.3.7 (2) to $\mathbf{d}(i)$, we obtain the following directed path
\[
\Pi^J(t_\xi) \overset{\beta_1}{\rightarrow} r_{\beta_1} \Pi^J(t_\xi) \overset{\beta_2}{\rightarrow} \cdots \overset{\beta_{l+1}}{\rightarrow} r_{\beta_{l+1}} \cdots r_{\beta_{l+1}} r_{\alpha_i} \Pi^J(t_\xi) \quad (6.4.4)
\]
with \( r_{\beta_1} \cdots r_{\beta_2}r_{\beta_1} = r_{\gamma_1}r_{\gamma_2} \cdots r_{\gamma_i}r_{\alpha_i}t_{\alpha_i}^\vee \); note that \( r_{\beta_1} \cdots r_{\beta_2}r_{\beta_1} \in (W)^J_{af} \) by Lemma 2.2.8. Therefore, we obtain
\[
\begin{align*}
    r_{\gamma_1}r_{\gamma_2} \cdots r_{\gamma_i}r_{\alpha_i}t_{\alpha_i}^\vee &= \Pi^J(r_{\gamma_1}r_{\gamma_2} \cdots r_{\gamma_i}r_{\alpha_i}t_{\alpha_i}^\vee) \\
    &= |r_{\gamma_1}r_{\gamma_2} \cdots r_{\gamma_i}r_{\alpha_i}|z_{\alpha_i}^\vee t_{\alpha_i}^\vee \\
    &= z_{\alpha_i}^\vee t_{\alpha_i}^\vee
\end{align*}
\]
(see \( r_{\gamma_1}r_{\gamma_2} \cdots r_{\gamma_i}r_{\alpha_i}t_{\alpha_i}^\vee \in (W)^J_{af} \))
by Lemma 2.2.6 (3) along with \( \alpha_i^\vee \in Q^J_{J,adj} \).

By substituting this equality into (6.4.4), we obtain a directed path from \( \Pi^J(t_\xi) \) to \( \Pi^J(t_\xi) \) in \( SB(\lambda; a) \), as desired. Thus we have proved Proposition 6.4.2.

\[\square\]

6.5 Connected components of \( B_+^\vee (\lambda) \)

Write \( \lambda \in P^+ \) as \( \lambda = \sum_{i \in I} m_i \alpha_i \) with \( m_i \in \mathbb{Z}_{\geq 0} \), \( i \in I \). Set
\[
    \text{Turn}(\lambda) := \{ k/m_i \mid i \in I \setminus J, 0 \leq k \leq m_i \}.
\]

**Lemma 6.5.1.** Let \( 0 < a < 1 \) be a rational number, and let \( \xi, \zeta \in Q^J_{J,adj} \) with \( \zeta \neq \xi \). If there exists a directed path from \( z_\zeta t_\zeta \) to \( z_\zeta t_\xi \) in \( SB(\lambda; a) \), then \( a \in \text{Turn}(\lambda) \).

**Proof.** Let \( z_\zeta t_\zeta \xrightarrow{\beta} r_\beta z_\zeta t_\zeta \) be the initial edge of a directed path from \( z_\zeta t_\zeta \) to \( z_\zeta t_\xi \) in \( SB(\lambda; a) \). By applying Proposition 4.3.7 (1) to this edge in \( SB(\lambda; a) \), we obtain an edge \( e \xrightarrow{\gamma} [r_\gamma] \) in \( QB(\lambda; a) \) for some \( \gamma \in \Delta^+ \setminus \Delta^+_J \). Note that \( e \xrightarrow{\gamma} [r_\gamma] \) is a Bruhat edge by Lemma 6.1.2 (see also Remark 4.2.2 (3)). Hence, \( [r_\gamma] = r_\gamma \) by Remark 4.2.2 (1), and \( 1 = \ell(r_\gamma) - \ell(e) = \ell(r_\gamma) \), which implies that \( \gamma = \alpha_i \) for some \( i \in I \). Then we have
\[
am_i = a(\alpha_i^\vee, \lambda) = a(\gamma^\vee, \lambda) \in \mathbb{Z}_{\geq 0},
\]
which implies that \( a \in \text{Turn}(\lambda) \). This proves the assertion. \[\square\]

The next proposition follows immediately from Proposition 6.4.2 and Lemma 6.5.1; for simplicity of notation, we set \( T_\xi := \Pi^J(t_\xi) = z_\zeta t_\xi \in (W)^J_{af} \) for \( \xi \in Q^J_{J,adj} \).

**Proposition 6.5.2.** Let \( \xi_1, \ldots, \xi_{s-1} \in Q^J_{J,adj} \). An element
\[
    \eta = (T_{\xi_1}, \ldots, T_{\xi_{s-1}}, e; a_0, a_1, \ldots, a_{s-1}, a_s)
\]  
(6.5.1)
is contained in \( B_+^\vee (\lambda) \) if and only if \( a_u \in \text{Turn}(\lambda) \) and \( p_{1,J}(\xi_u - \xi_{u+1}) \in \sum_{i \in I(1; a_u) \setminus J} \mathbb{Z}_{\geq 0} \alpha_i^\vee \setminus \{0\} \) for all \( 1 \leq u \leq s - 1 \), where we set \( \xi_s := 0 \).

**Proposition 6.5.3.** Each connected component of \( B_+^\vee (\lambda) \) contains a unique element of the form (6.5.1).
In order to prove this proposition, we need some notation and lemmas. Let \( N \in \mathbb{Z}_{>0} \). For simplicity of notation, we set

\[
[y_1, y_2, \ldots, y_N] := \eta_{y_1} \otimes \eta_{y_2} \otimes \cdots \otimes \eta_{y_N} \in \mathbb{F}^{N}(\lambda)^{\otimes N}
\]

for \( y_1, y_2, \ldots, y_N \in (W_J)^{af} \).

**Lemma 6.5.4.** Let \( N \in \mathbb{Z}_{>0} \) be as in Lemma 5.2.7. Let \( \eta \in \mathbb{F}^{N}(\lambda) \), and write \( \sigma_N(\eta) \) as

\[
\sigma_N(\eta) = [y_1, y_2, \ldots, y_N] \quad \text{with some} \quad y_1, y_2, \ldots, y_N \in (W_J)^{af}.
\]

Let \( X \) be a monomial in the root operators \( e_i \) and \( f_i \), \( i \in I_{af} \), and assume that \( X\eta \neq 0 \). Then,

\[
\sigma_N(X\eta) = [v_1y_1, v_2y_2, \ldots, v_Ny_N]
\]

with some \( v_1, v_2, \ldots, v_N \in W^{af} \) such that \( v_ny_n \in (W_J)^{af} \) for all \( 1 \leq n \leq N \).

**Proof.** It is essential to verify the assertion in the case where \( X = e_i \) or \( f_i \), \( i \in I_{af} \); the assertion for a general \( X \) follows immediately by an inductive argument. By the definition of the root operators (see also the proof of Theorem 3.1.6) and Lemma 5.2.7, we deduce that

\[
\sigma_N(X\eta) = [y_1, \ldots, y_k, r_1y_k, \ldots, r_ly_m, y_{m+1}, \ldots, y_N]
\]

for some \( 1 \leq k \leq m \leq N \), with \( r_iy_l \in (W_J)^{af} \) for all \( 1 \leq l \leq m \). Thus we have proved the lemma. \( \square \)

Let \( N \in \mathbb{Z}_{>0} \) be as in Lemma 5.2.7. Let \( \eta \in \mathbb{F}^{N}(\lambda) \) be of the form (6.5.1); by Lemma 5.2.7, \( \sigma_N(\eta) \) is of the form:

\[
\sigma_N(\eta) = [T_{\zeta_1}, T_{\zeta_2}, \ldots, T_{\zeta_N}]
\]

for some \( \zeta_1, \zeta_2, \ldots, \zeta_N \in Q_{J, \text{adj}}^{\prime} \). Let \( X \) be a monomial in the root operators \( e_i \) and \( f_i \), \( i \in I_{af} \). Assume that \( X\eta \neq 0 \), and write \( \sigma_N(X\eta) \) as (see Lemma 6.5.4):

\[
\sigma_N(X\eta) = [v_1T_{\zeta_1}, v_2T_{\zeta_2}, \ldots, v_NT_{\zeta_N}]
\]

with some \( v_1, v_2, \ldots, v_N \in W^{af} \) such that \( v_nT_{\zeta_n} = v_nz_{\zeta_n}t_{\zeta_n} \in (W_J)^{af} \), \( 1 \leq n \leq N \); note that \( v_n \in (W_J)^{af} \) for all \( 1 \leq n \leq N \) by Lemma 2.2.8.

**Lemma 6.5.5.** Keep the notation and setting above. Let \( \eta' \in \mathbb{F}^{N}(\lambda) \) be also of the form (6.5.1), and write \( \sigma_N(\eta') \) as

\[
\sigma_N(\eta') = [T_{\zeta'_1}, T_{\zeta'_2}, \ldots, T_{\zeta'_N}]
\]

for some \( \zeta'_1, \zeta'_2, \ldots, \zeta'_N \in Q_{J, \text{adj}}^{\prime} \). Then, \( X\eta' \neq 0 \), and

\[
\sigma_N(X\eta') = [v_1T_{\zeta'_1}, v_2T_{\zeta'_2}, \ldots, v_NT_{\zeta'_N}];
\]

note that \( v_nT_{\zeta'_n} \in (W_J)^{af} \) for all \( 1 \leq n \leq N \) by Lemma 2.2.8 since \( v_n \in (W_J)^{af} \).
Proof. Let \( X = g_p g_{p-1} \cdots g_{2g_1} \), where \( g_q \in \{ e_i, f_i \mid i \in I_{af} \} \) for each \( 1 \leq q \leq p \). We show the assertion by induction on \( p \). If \( p = 0 \), then the assertion is obvious since \( X = \text{id} \). Assume that \( p > 0 \). Set \( Y := g_{p-1} \cdots g_{2g_1} \). Since \( X \eta \neq 0 \), it follows that \( Y \eta \neq 0 \). Write \( \sigma_N(Y \eta) \) as

\[
\sigma_N(Y \eta) = [u_1 T_{\zeta_1}, u_2 T_{\zeta_2}, \ldots, u_N T_{\zeta_N}]
\]

with some \( u_1, u_2, \ldots, u_N \in W_{af} \) such that \( u_n T_{\zeta_n} = u_n z_{\zeta_n} t_{\zeta_n} \in (W^J)_{af} \) for all \( 1 \leq n \leq N \). By induction hypothesis, we have \( Y \eta' \neq 0 \), and

\[
\sigma_N(Y \eta') = [u_1 T_{\zeta_1'}, u_2 T_{\zeta_2'}, \ldots, u_N T_{\zeta_N'}].
\]

Now, assume that \( g_p \) is either \( e_i \) or \( f_i \) for some \( i \in I_{af} \). We see from the definition of the root operator \( e_i \) or \( f_i \) and Lemma 5.2.7 that

\[
\sigma_N(X \eta) = \sigma_N(g_p Y \eta) = [u_1 T_{\zeta_1}, \ldots, u_{k-1} T_{\zeta_{k-1}}, r_i u_k T_{\zeta_k}, \ldots, r_i u_m T_{\zeta_m}, u_{m+1} T_{\zeta_{m+1}}, \ldots, u_N T_{\zeta_N}]
\]

for some \( 0 \leq k \leq m \leq N \), with \( r_i u_l T_{\zeta_l} \in (W^J)_{af} \) for every \( k \leq l \leq m \); we should remark that \( k \) and \( m \) are determined by the function \( H^\eta(t) = \langle \alpha_i^\eta, \pi_{Y \eta}(t) \rangle \) (see the definition of \( t_0, t_1 \in [0, 1] \) in Definition 3.1.5 and Lemma 5.2.7). Since

\[
\langle \alpha_i^\eta, u_n T_{\zeta_n} \rangle = \langle \alpha_i^\eta, u_n \lambda \rangle = \langle \alpha_i^\eta, u_n T_{\zeta_n} \rangle \quad \text{for all} \quad 1 \leq n \leq N,
\]

we deduce that \( H^\eta(t) = H^\eta'(t) \) for all \( t \in [0, 1] \), which implies that \( t_0, t_1 \) for \( Y \eta' \) coincides with those for \( Y \eta \), respectively. Therefore, if follows from the definition of the root operator \( e_i \) or \( f_i \) and Lemma 5.2.7 that \( X \eta' = g_p Y \eta' \neq 0 \), and

\[
\sigma_N(X \eta') = \sigma_N(g_p Y \eta') = [u_1 T_{\zeta_1'}, \ldots, u_{k-1} T_{\zeta_{k-1}'}, r_i u_k T_{\zeta_k'}, \ldots, r_i u_m T_{\zeta_m'}, u_{m+1} T_{\zeta_{m+1}'}, \ldots, u_N T_{\zeta_N'}].
\]

Thus we have proved the lemma.

\[ \square \]

Proof of Proposition 6.5.2. First we show that each connected component of \( B^\infty_\lambda(\lambda) \) contains an element of the form (6.5.1). Let \( \eta \in B^\infty_\lambda(\lambda) \); recall that \( \pi_\eta \in B(\lambda) \). By [NS08, Theorem 3.1 (2)], there exists a monomial \( X \) in the root operators \( e_i \) and \( f_i \), \( i \in I_{af} \), such that \((X \pi_\eta)(t) \equiv t \lambda \mod \mathbb{R} \delta \) for all \( t \in [0, 1] \). Because the map \( B^\infty_\lambda(\lambda) \to B(\lambda), \eta \mapsto \pi_\eta \), is a strict morphism of crystals (see Remark 4.4.1 (1)), it follows immediately that \( X \eta \neq 0 \), and \( X \eta \in B^\infty_\lambda(\lambda) \) is of the form:

\[
X \eta = (T_{\xi_1'}, T_{\xi_2'}, \ldots, T_{\xi_{s-1}'}, T_{\xi_s'}, a_0, a_1, \ldots, a_{s-1}, a_s)
\]

for some \( \xi_1', \xi_2', \ldots, \xi_{s-1}' \in Q^{J, \text{adj}} \).

Now, observe that \( \eta_{\xi_1} = (T_{\xi_1}; 0, 1) \in B^\infty_0(\lambda) \) (see §5.1). Let \( Y \) be a monomial in the root operators \( e_i \) and \( f_i \), \( i \in I_{af} \), such that \( Y \eta_{\xi_1} = \eta_\epsilon \). Take \( N \in \mathbb{Z}_{>0} \) as in Lemma 5.2.7. Then,

\[
\sigma_N(\eta_{\xi_1}) = [T_{\xi}, T_{\xi}, \ldots, T_{\xi}] \quad (N \text{ times}),
\]

\[
\sigma_N(\eta_\epsilon) = [e, e, \ldots, e] \quad (N \text{ times}).
\]
Also, we see from Lemma 6.5.5 that
\[ \sigma_N(Y_{\eta_\xi}) = [v_1 T_{\xi_1}, v_2 T_{\xi_2}, \ldots, v_N T_{\xi_N}] \]
for some \( v_1, v_2, \ldots, v_N \in W_{af} \) with \( v_n T_{\xi} \in (W^J)_{af} \) for every \( 1 \leq n \leq N \). Because \( Y_{\eta_\xi} = \eta_e \), and hence \( \sigma_N(Y_{\eta_\xi}) = \sigma_N(\eta_e) \), it follows that \( v_n T_{\xi} = e \) for all \( 1 \leq n \leq N \). Thus, \( v_n = T_{\xi}^{-1} \) for all \( 1 \leq n \leq N \).

We see from Lemma 5.2.7 that \( \sigma_N(X\eta) \) is of the form:
\[ \sigma_N(X\eta) = [T_{\zeta_1}, T_{\zeta_2}, \ldots, T_{\zeta_N}] \]
for some \( \zeta_1, \zeta_2, \ldots, \zeta_N \in Q_{J, \text{adj}}^Y \); we should remark that \( \zeta_N = \xi \). Then we deduce from Lemma 6.5.5 that
\[
\begin{align*}
\sigma_N(Y X \eta) &= [v_1 T_{\zeta_1}, v_2 T_{\zeta_2}, \ldots, v_N T_{\zeta_N}] \\
&= [T_{\xi}^{-1} T_{\zeta_1}, T_{\xi}^{-1} T_{\zeta_2}, \ldots, T_{\xi}^{-1} T_{\zeta_N}]
\end{align*}
\]
with \( v_n T_{\zeta_n} = T_{\xi}^{-1} T_{\zeta_n} \in (W^J)_{af} \) for every \( 1 \leq n \leq N \). Since
\[
(W^J)_{af} \ni T_{\xi}^{-1} T_{\zeta_n} = t_{-\xi} z_{\zeta_{\xi}}^{-1} z_{\zeta_{\xi}} t_{\zeta_n} = z_{\xi} z_{\zeta_{\xi}} t_{\zeta_n}, \quad \text{with } \zeta_n := \zeta_n - z_{\zeta_{\xi}} z_{\xi},
\]
we see from (2.2.5) that \( \zeta_n \in Q_{J, \text{adj}}^Y \) and \( z_{\xi} z_{\zeta_{\xi}} = z_{\zeta_n} \). Thus we get
\[
\sigma_N(Y X \eta) = [T_{\zeta_1}, T_{\zeta_2}, \ldots, T_{\zeta_{N-1}}, e].
\]
Because the final factor above is equal to \( e \), we deduce from Lemma 5.2.7 that \( Y X \eta \) is of the form (6.5.1). Thus we have shown that each connected component of \( B_{\tilde{F}}(\lambda) \) contains an element of the form (6.5.1).

Next we prove the uniqueness. Let \( \eta, \eta' \in B_{\tilde{F}}(\lambda), \eta \neq \eta', \) be of the form (6.5.1), and suppose that \( X \eta = \eta' \) for some monomial \( X \) in the root operators \( e_i \) and \( f_i, i \in I_{af} \). As above, let \( N \in \mathbb{Z}_{>0} \) be as in Lemma 5.2.7. Then, \( \sigma_N(\eta) \) and \( \sigma_N(\eta') \) are of the form (note that \( e = T_0 \)):
\[
\begin{align*}
\sigma_N(\eta) &= [T_{\zeta_1}, T_{\zeta_2}, \ldots, T_{\zeta_{N-1}}, T_0], \\
\sigma_N(\eta') &= [T'_{\zeta_1}, T'_{\zeta_2}, \ldots, T'_{\zeta_{N-1}}, T_0]
\end{align*}
\]
for some \( \zeta_n, \zeta'_n \in Q_{J, \text{adj}}^Y, 1 \leq n \leq N - 1 \). Since \( \eta \neq \eta' \) and the map \( \sigma_N : B_{\tilde{F}}(\lambda) \to B_{\tilde{F}}(\lambda)^{\otimes N} \) is injective, there exists \( 1 \leq n \leq N - 1 \) such that \( \zeta_n \neq \zeta'_n \); set
\[
m := \max\{1 \leq n \leq N - 1 \mid \zeta_n \neq \zeta'_n\}.
\]
Then we see that \( \zeta_m - \zeta'_m \notin Q_{J}^Y \). Indeed, suppose that \( \zeta_m - \zeta'_m \in Q_{J}^Y \). Since \( t_{\zeta_m - \zeta'_m} \in (W_J)_{af} \),
\[
T_{\zeta_m} = \Pi(t_{\zeta_m}) = \Pi(t_{\zeta'_m} t_{\zeta_m - \zeta'_m}) = \Pi(t_{\zeta'_m}) = T_{\zeta'_m},
\]
which is a contradiction. Thus, \( p_{I,J}(\zeta_m - \zeta'_m) \neq 0 \); we may assume, without loss of generality, that
\[
p_{I,J}(\zeta'_m - \zeta_m) \notin \sum_{i \in I \setminus J} \mathbb{Z}_{\geq 0} a_i^Y.
\]
Now, by Lemma 6.5.4, \[
\sigma_N(X\eta) = [v_1T_{\zeta_1}, v_2T_{\zeta_2}, \ldots, v_{N-1}T_{\zeta_{N-1}}, v_NT_0]
\] for some \(v_1, v_2, \ldots, v_N \in W_{af}\) with \(v_nT_{\zeta_n} \in (W^J)_{af}\) for every \(1 \leq n \leq N-1\) and \(v_NT_0 \in (W^J)_{af}\); note that \(v_1, v_2, \ldots, v_N \in (W^J)_{af}\) by Lemma 2.2.8. Since \(X\eta = \eta\), it follows that \(v_nT_{\zeta_n} = T_{\zeta_n}\) for every \(1 \leq n \leq N-1\), and \(v_NT_0 = T_0\). Hence, we get \(v_n = T_{\zeta_n}^{-1}T_{\zeta_n}\) for every \(1 \leq n \leq N-1\), and \(v_N = e\); in particular, \(v_{m+1} = \cdots = v_{N-1} = v_N = e\) by the definition of \(m\). Also, as above, we see that \(v_n = T_{\zeta_n}^{-1}T_{\zeta_n}^{1} \in (W^J)_{af}\), \(1 \leq n \leq m\), is of the form: \(v_n = T_{\zeta_n}^{1}\) for some \(\zeta_n^1 \in Q^\vee_{J_{adj}}\) such that \(p_{I, J}(\zeta_n^1) = p_{I, J}(\zeta_n - \zeta_n)\). We deduce from Lemma 6.5.5 that \(X\eta_e \neq 0\), and
\[
\sigma_N(X\eta_e) = [v_1, v_2, \ldots, v_N] = [T_{\zeta_1}, T_{\zeta_2}, \ldots, T_{\zeta_m}, e, \ldots, e].
\]
Then, we see that \(X\eta_e \in \mathcal{B}_F(\lambda)\) is of the form:
\[
X\eta_e = (T_{\xi_1}, \ldots, T_{\xi_{-1}}, e; a_0, a_1, \ldots, a_{s-1}, a_s)
\]
for some \(\xi_1, \ldots, \xi_{-1} \in Q^\vee_{J_{adj}}\) with \(\xi_{-1} = \zeta_m^m\). However, since
\[
p_{I, J}(\xi_{-1}) = p_{I, J}(\zeta_m^m) = p_{I, J}(\zeta_m - \zeta_m) \notin \sum_{i \in I \setminus J} \mathbb{Z}_{\geq 0} \alpha_i^\vee,
\]
this contradicts Proposition 6.5.2. This completes the proof of Proposition 6.5.3. \(\square\)

**Proposition 6.5.6.** There exists a one-to-one correspondence between Par(\(\lambda\)) and the connected components of \(\mathcal{B}_F(\lambda)\).

**Proof.** Let \(\text{Conn}(\lambda)\) be the set of connected components of \(\mathcal{B}_F(\lambda)\). First we define a map \(\Theta:\)
\[
\text{Conn}(\lambda) \to \text{Par}(\lambda)\)
\]
as follows: let \(C \in \text{Conn}(\lambda)\). By Proposition 6.5.2, \(C\) contains a unique element of the form
\[
\eta = (T_{\xi_1}, T_{\xi_2}, \ldots, T_{\xi_{-1}}, e; a_0, a_1, \ldots, a_{s-1}, a_s)
\]
for some \(\xi_1, \xi_2, \ldots, \xi_{-1} \in Q^\vee_{J_{adj}}\); by convention, we set \(\xi_s := 0\). Write \(\text{Turn}(\lambda) = \{k/m_i \mid i \in I \setminus J, 0 \leq k \leq m_i\}\) as:
\[
\text{Turn}(\lambda) = \{0 = \tau_0 < \tau_1 < \cdots < \tau_{\kappa} = 1\};
\]
ote that \(i \in \text{Turn}(\lambda) \setminus J\) if and only if \(i \notin J\) and \(\tau_i = k/m_i\) for some \(0 \leq k \leq m_i\). Recall from Proposition 6.5.2 that \(a_0, a_1, \ldots, a_{s-1}, a_s \in \text{Turn}(\lambda)\). So, for each \(0 \leq u \leq s\), let \(0 \leq \kappa_u \leq \kappa\) be such that \(a_u = \tau_{\kappa_u}\). Then we define \(\zeta_l, 1 \leq l \leq \kappa\), by
\[
\zeta_l := \xi_u \text{ if } \kappa_{u-1} + 1 \leq l \leq \kappa_u;
\]
that is,
\[
\zeta_1, \ldots, \zeta_{\kappa_1}, \zeta_{\kappa_1+1}, \ldots, \zeta_{\kappa_2}, \ldots, \zeta_{\kappa_{s-2}+1}, \ldots, \zeta_{\kappa_{s-1}}, \zeta_{\kappa_{s-1}+1}, \ldots, \zeta_{\kappa}.
\]
Then we deduce from Proposition 6.5.2 that
\[
p_{I, J}(\zeta_l - \zeta_{l+1}) \in \sum_{i \in \text{Turn}(\lambda) \setminus J} \mathbb{Z}_{\geq 0} \alpha_i^\vee \text{ for } 1 \leq l \leq \kappa - 1.
\]
(6.5.2)
For each \(i \in I \setminus J\), let \(p^{(i)}_l\) be a coefficient of \(\alpha^\vee_i\) in \(\zeta_l\) for \(1 \leq l \leq \kappa\). We see from (6.5.2) that
\[
p^{(i)}_1 \geq p^{(i)}_2 \geq \cdots \geq p^{(i)}_{\kappa-1} \geq p^{(i)}_\kappa = 0,
\]
and \(p^{(i)}_l = p^{(i)}_{l+1}\) for \(1 \leq l \leq \kappa - 1\) such that \(i \notin I(\lambda; \tau_l) \setminus J\), that is, \(\tau_l \notin \{k/m_i \mid 0 \leq k \leq m_i\}\). So, for each \(1 \leq k \leq m_i\), set \(\rho_k^{(i)} := p^{(i)}_l\) if \(k/m_i = \tau_l\), and define a partition \(\rho^{(i)}\) of length less than \(m_i\) by
\[
\rho^{(i)} := (\rho^{(i)}_1 \geq \rho^{(i)}_2 \geq \cdots \rho^{(i)}_{m_i-1} \geq \rho^{(i)}_{m_i} = 0).
\]

Then we define \(\Theta(C) := (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda)\), where for every \(j \in J\), we set \(\rho^{(j)}\) to be the empty partition for every \(j \in J\).

Next we define a map \(\Xi : \text{Par}(\lambda) \to \text{Conn}(\lambda)\) as follows: let \(\rho = (\rho^{(i)}) \in \text{Par}(\lambda)\) with \(\rho^{(i)} = (\rho^{(i)}_1 \geq \rho^{(i)}_2 \geq \cdots \rho^{(i)}_{m_i-1} \geq 0)\) for \(i \in I \setminus J\). Define \(\zeta_l \in Q^\vee, 1 \leq l \leq \kappa\), by
\[
\zeta_\kappa = 0, \quad \zeta_l - \zeta_{l+1} = \sum_{i \in I(\lambda; \tau_l) \setminus J} \rho^{(i)}_{m_i \tau_l^\vee} \alpha^\vee_i \quad \text{for} \; 1 \leq l \leq \kappa - 1;
\]
note that for \(1 \leq l \leq \kappa - 1\), if \(i \notin I(\lambda; \tau_l) \setminus J\), then \(m_i \tau_l \in \mathbb{Z}\) with \(1 \leq m_i \tau_l \leq m_i - 1\). Set \(s := \#\{1 \leq l \leq \kappa - 1 \mid \zeta_l \neq \zeta_{l+1}\} + 1\), and assume that
\[
\{1 \leq l \leq \kappa - 1 \mid \zeta_l \neq \zeta_{l+1}\} = \{\kappa_1 < \kappa_2 < \cdots < \kappa_{s-1}\}.
\]
Namely,
\[
\zeta_1 = \cdots = \zeta_{\kappa_1} \neq \zeta_{\kappa_1+1} = \cdots = \zeta_{\kappa_2} \neq \cdots \neq \zeta_{\kappa_{s-1}+1} = \cdots = \zeta_\kappa = 0.
\]

Then we define (for the definition of \(\phi_J(\zeta_{\kappa_u})\), see Lemma 2.2.6)
\[
\xi_u := 0, \quad \xi_u := \zeta_{\kappa_u} + \phi_J(\zeta_{\kappa_u}) \quad \text{for} \; 1 \leq u \leq s - 1,
\]
and
\[
a_0 := 0; \quad a_u := \tau_{\kappa_u} \quad \text{for} \; 1 \leq u \leq s - 1; \quad a_s := 1.
\]

We deduce from Proposition 6.5.2 that
\[
\eta_\rho := (T_{\xi_1}, T_{\xi_2}, \ldots, T_{\xi_{s-1}}, e; a_0, a_1, \ldots, a_{s-1}, a_s) \in \mathbb{B}_\rho^\infty(\lambda). \tag{6.5.3}
\]

So, let us define \(\Xi(\rho)\) to be the connected component of \(\mathbb{B}_\rho^\infty(\lambda)\) containing this \(\eta_\rho\).

We deduce from the definitions that the maps \(\Theta\) and \(\Xi\) are mutually inverse. Thus we have proved the proposition. \(\square\)

### 6.6 Proof of Proposition 3.2.4

Let \(\rho \in \text{Par}(\lambda)\), and let \(\eta_\rho \in \mathbb{B}_\rho^\infty(\lambda)\) be defined as (6.5.3), which is a unique element of the form (6.5.1) contained in the connected component \(\mathbb{B}_\rho^\infty(\lambda) := \Xi(\rho)\). We will show that there exists a unique isomorphism \(\mathbb{B}_\rho^\infty(\lambda) \cong \{\rho\} \otimes \mathbb{B}_0^\infty(\lambda)\) of crystals that maps \(\eta_\rho\) to \(\rho \otimes \eta_e\). For this, it suffices to show that the following hold for monomials \(X, Y\) in the Kashiwara operators:

1. \(X \eta_\rho \neq 0\) in \(\mathbb{B}_\rho^\infty(\lambda)\) if and only if \(X(\rho \otimes \eta_e) \neq 0\) in \(\{\rho\} \otimes \mathbb{B}_0^\infty(\lambda)\),
Lemma 4.3.6 and
Assertion (1) follows immediately from Lemma 6.5.5 and the equality 
Proposition A.1.2.

A.1 Another definition of semi-infinite Bruhat order
\[ xJ \text{ note that } \]

Thus we have proved Proposition 3.2.4.

2.2.8 and 5.2.6, 

\[ u \text{ with some } \]

Let \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) be a multiple of \( N_\lambda \) (see Lemma 5.2.7), and write \( \sigma_N(\eta_\rho) \) as \( \sigma_N(\eta_\rho) = [T_{\xi_1}, T_{\xi_2}, \ldots, T_{\xi_N}] \) with some \( \xi_1, \xi_2, \ldots, \xi_N \in Q_{J, \text{adj}}^\prime \). Also, by Lemmas 2.2.8 and 5.2.6, \( \sigma_N(X_{\eta_\rho}) \) and \( \sigma_N(Y_{\eta_\rho}) \) are of the form 

\[ \sigma_N(X_{\eta_\rho}) = [u_1 T_{\xi_1}, u_2 T_{\xi_2}, \ldots, u_N T_{\xi_N}], \quad \sigma_N(Y_{\eta_\rho}) = [v_1 T_{\xi_1}, v_2 T_{\xi_2}, \ldots, v_N T_{\xi_N}] \]

with some \( u_1, u_2, \ldots, u_N \in (W^J)_f \) and \( v_1, v_2, \ldots, v_N \in (W^J)_f \), respectively. Then, by Lemma 6.5.5,

\[ \sigma_N(X_{\eta_\rho}) = [u_1, u_2, \ldots, u_N], \quad \sigma_N(Y_{\eta_\rho}) = [v_1, v_2, \ldots, v_N]. \]

Since \( X_{\eta_\rho} = Y_{\eta_\rho} \), we have \( u_p = v_p \) for every \( 1 \leq p \leq N \). Therefore, we obtain \( \sigma_N(X_{\eta_\rho}) = \sigma_N(Y_{\eta_\rho}) \), and hence \( X_{\eta_\rho} = Y_{\eta_\rho} \) by the injectivity of \( \sigma_N \). Thus we have proved the “only if” part of assertion (2). Thus we obtain an isomorphism \( \mathbb{B}^\infty_{\rho}(\lambda) \cong \{ \rho \} \otimes \mathbb{B}^\infty_0(\lambda) \) of crystals for each \( \rho \in \text{Par}(\lambda) \). Hence,

\[ \mathbb{B}^\infty(\lambda) = \bigsqcup_{\rho \in \text{Par}(\lambda)} \mathbb{B}^\infty_0(\lambda) \cong \bigsqcup_{\rho \in \text{Par}(\lambda)} \{ \rho \} \otimes \mathbb{B}^\infty_0(\lambda) = \text{Par}(\lambda) \otimes \mathbb{B}^\infty_0(\lambda). \]

Thus we have proved Proposition 3.2.4. \( \square \)

A Appendix

A.1 Another definition of semi-infinite Bruhat order

Let \( J \) be a subset of \( I \). For \( vt_\zeta \in W_{af} \) with \( v \in W \) and \( \zeta \in Q^J \), we define

\[ \ell_J^\infty(vt_\zeta) := \ell([v]) + 2(\zeta, \rho - \rho_J). \tag{A.1.1} \]

Lemma A.1.1. The equalities \( \ell_J^\infty(x) = \ell_J^\infty(\Pi^J(x)) = \ell_J^\infty(\Pi^J(x)) \) hold for all \( x \in W_{af} \).

Proof. Write \( \Pi^J(x) = w z \ell_\xi \) with \( w \in W^J \) and \( \zeta \in Q^J_{J, \text{adj}} \). The second equality follows from Lemma 4.3.6 and \( \ell(w z \ell_\xi) = \ell(w) + \ell(z) \).

In order to prove the first equality, we write \( x = x_1 x_2 \) with \( x_1 \in (W^J)_f \) and \( x_2 \in (W^J)_f \); note that \( \Pi^J(x) = x_1 \). By (2.2.5), \( x_1 = w_1 z_\xi t_\zeta \) for some \( w_1 \in W^J \) and \( \zeta \in Q^J_{J, \text{adj}} \), and by (2.2.3), \( x_2 = w_2 \ell_\zeta \) for some \( w_2 \in W^J \) and \( \zeta \in Q^J \). Since \( x = x_1 x_2 = w_1 z_\xi w_2 \ell_\zeta \), we have

\[ \ell_J^\infty(x) = \ell([w_1 z_\xi w_2]) + 2(w_2^{-1} \xi_1 + \xi_2, \rho - \rho_J) \]

(see (2.1.7))

\[ = \ell(w_1) + 2(w_2^{-1} \xi_1 + \xi_2, \rho - \rho_J) \]

(see (2.1.7))

Thus we have proved the lemma. \( \square \)

Proposition A.1.2. Let \( x, y \in (W^J)_f \) and \( \beta \in \Delta^+. \) We have \( x \xrightarrow{\beta} y \) in \( SB^J \) if and only if the following three conditions hold:

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(a) \( y = \Pi^J(r_\beta x) \);

(b) \( \ell_\beta^\infty(x) = \ell_\beta^\infty(x) + 1 \);

(c) Write \( x = wz\xi^\ell \xi \) with \( w \in W^J \) and \( \xi \in Q^J_{J-\text{adj}} \). Then, \( \beta = w\alpha + \chi\delta \) for some \( \alpha \in \Delta^+ \setminus \Delta_J^- \) and \( \chi \in \{0, 1\} \).

**Proof.** The “only if” part follows immediately from Lemmas 4.3.5 and A.1.1. We show the “if” part. By (c), we have \( r_\beta x = wr_\alpha z\xi^\ell \xi_z \xi^{-1}_z \alpha^y \). We deduce that

\[
1 = \ell_\beta^\infty(x) - \ell(x) = \ell(\lceil wr_\alpha \rceil) + 2(\xi + \chi \xi^{-1} \alpha^y, \rho - \rho_J) - \ell(w) - 2(\xi, \rho - \rho_J)
\]

which implies that \( w \xrightarrow{\alpha} [wr_\alpha] \) (resp., \( w \xrightarrow{\alpha} [wr_\alpha] \)) in \( QB^J \) if \( \chi = 0 \) (resp., \( \chi = 1 \)). Therefore, we obtain \( y = r_\beta x \in (W^J)_{af} \) and \( x \xrightarrow{\beta} y \) in \( SB^J \) by Proposition 4.3.7 (2). \( \square \)

A.2 Relation between the semi-infinite Bruhat order and the generic Bruhat order

In this subsection, we assume that \( J = \emptyset \); note that \( (W^J)_{af} = W_{af} \). Fix an arbitrary element \( \xi \in Q^J \) such that \( \langle \xi, \alpha_i \rangle > 0 \) for all \( i \in I \). We know from [Pet97] (see also [LNSSS13a, Theorem 5.2] together with Proposition 4.3.7) that for \( x, y \in (W^J)_{af} = W_{af} \), \( x \leq_{\mathfrak{F}} y \) if and only if there exists \( N \in \mathbb{Z}_{>0} \), depending on \( x, y, \) and \( \xi \), such that \( yt_n\xi^{-1} \leq xt_n\xi \) (or equivalently, \( t_n\xi^{-1} \leq t_n\xi \)) for all \( n \in \mathbb{Z}_{>N} \); here, recall that \( \leq \) is an (ordinary) Bruhat order on \( W_{af} \). On the other hand, in [Lus80, §1.5], Lusztig introduced a partial order \( \leq_L \) on \( W_{af} \), which we call Lusztig’s generic Bruhat order; we know from [Soe97, Claim 4.14 in the proof of Lemma 4.13] that \( x \leq_L y \) if and only if there exists \( N \in \mathbb{Z}_{>0} \), depending on \( x, y, \) and \( \xi \), such that \( t_n\xi x \leq t_n\xi y \) for all \( n \in \mathbb{Z}_{>N} \). Combining these facts, we obtain

**Lemma A.2.1.** Let \( x, y \in W_{af} \). We have \( x \leq_{\mathfrak{F}} y \) if and only if \( y^{-1} \leq_L x^{-1} \).

**References**


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