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著者	Takada Hideyuki, Sumita Ushio, Takahashi Kazuki
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Hideyuki Takada, Ushio Sumita and Kazuki Takahashi

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UNIVERSITY OF TSUKUBA
Tsukuba, Ibaraki 305-8573
JAPAN

Dynamic Analysis of Loss Distributions for Pricing Collateralized Debt Obligations

Hideyuki Takada[†], Ushio Sumita[‡] and Kazuki Takahashi[‡]
[†]Mizuho-DL Financial Technology Co., Ltd, Chiyoda-ku 100-0004, Japan
[‡]Graduate School of Systems and Information Engineering, University of Tsukuba,
Tsukuba-City 305-8573, Japan

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Abstract For valuation of CDO(Collateralized Debt Obligation) and Tranched Index, in practice, a standard approach is to employ the Gaussian copula model of Li [7]. However, this model is limited in that its framework is completely static, failing to capture the dynamic evolution of CDO tranches. In general, the portfolio credit derivatives are subject to two kinds of risks. One is a default event risk where any underlying firm involved in CDO fails to fulfill its obligation. The other is a credit spread risk due to the change of the default intensity over time. In dealing with either type of risks, it is absolutely necessary to develop a dynamic model incorporating the stochastic behavior of the macro economic condition and its influences on default intensity. For this purpose, a dynamic stochastic model is developed based on a Markov Modulated Poisson Process (MMPP). By exploiting the stochastic structure of the MMPP, efficient computational procedures are established for evaluating time dependent loss distributions and prices of CDO tranches. Some numerical results are presented, demonstrating the excellent consistency between the premium values obtained from the proposed theoretical model and the corresponding real market data.

Keywords: Loss Dynamics, CDO Tranches, Markov Modulated Default Intensity, Keilson's Uniformization Procedure, Laplace Transform, Convolution.

1. Introduction

A financial product called "Collateralized Debt Obligation (CDO)" is a structured product that securitizes a reference portfolio of default risky instruments such as loans or bonds. In case of a synthetic CDO, the reference portfolio consists of "Credit Default Swaps (single-name CDS's)". A credit default swap offers protection against default of a certain underlying entity over specified time horizon. Recently, portfolio credit derivatives such as Tranched Index are traded with growing liquidity.

For valuation of (synthetic) CDO and Tranched Index, in practice, a standard approach is to employ the Gaussian copula model of Li [7]. However, this model is limited in that its framework is completely static, failing to capture the dynamic evolution of CDO tranches. In general, the portfolio credit derivatives are subject to two kinds of risks. One is a default event risk where any underlying firm involved in CDO fails to fulfill its obligation. The other is a credit spread risk due to the change of the default intensity over time. In dealing with either type of risks, it is absolutely necessary to develop a dynamic model incorporating the stochastic behavior of the macro economic condition and its influences on default intensity.

In order to capture the dynamics of portfolio loss distribution, it is typically necessary to introduce a two layer process. The first layer describes the macro economic condition, which would affect portfolio loss distributions through the second layer. Recent papers along this line of research include Arndorf and Halperin [1], Bielecki, Vidozzi and Vidozzi [2], Frey and Backhaus [3], Kock, Kraft and Steffensen [5], Schonbucher [8] and Sidenius, Piterberg

and Andersen [9], where the default intensity function is assumed to satisfy a stochastic differential equation (or assumed to be driven by a certain Markov chain) characterized by the state of the external economic condition as well as the history of defaults up to the current time. While these pioneering models possess some structural elegance, they would not necessarily facilitate the computation of CDO tranches in continuous time, often resorting to Monte Calro simulation for this purpose. An alternative approach suggested by Lando [6] employs a Markov Modulated Poisson Process (MMPP) for describing the stochastic behavior of portfolio losses, thereby enabling one to use the computational procedure for general Markov Chains for evaluating loss distributions. The purpose of this paper is to recapture the MMPP model of Lando [6] within the context of CDO. In particular, by exploiting the stochastic structure of the MMPP, computational procedures are developed for evaluating loss distributions and pricing CDO tranches, which are much more efficient than those based on the general Markov chain approach.

The structure of this paper is as follows. A general framework for understanding CDO Tranches is introduced in Section 2, for which a mathematical model is developed in Section 3 based on an MMPP. Section 4 is devoted to analysis of the MMPP model deriving the Laplace transforms of the probabilities of just n defaults by time t explicitly. In Section 5, computational algorithms are developed for evaluating CDO Tranches based on the theoretical results of Section 4. Finally, in Section 6, some numerical results are presented, demonstrating the excellent consistency between the premium values obtained from the proposed theoretical model and the corresponding real market data. Some concluding remarks are given in Section 7.

Throughout the paper, vectors are denoted by the underbar, e.g. $\underline{v}$, $\underline{w}$, and matrices by the double-underbar, e.g. $\underline{\underline{a}}$, $\underline{\underline{b}}$. A vector having all components equal to one is denoted by $\underline{1}$.

2. CDO Tranches

In the CDO scheme, given a reference portfolio, the associated credit risk is divided into tranches of increasing seniority, where a tranche is defined by a pair of an attachment point and a detachment point of the cumulative aggregate loss of the reference portfolio. Here, the attachment point K_a means that the protection buyer (the CDO issuer) is fully responsible for the portfolio loss up to K_a. In principle, the protection seller (the tranche investor) compensates the portfolio loss beyond K_a up to K_d for the protection buyer, where K_d is the detachment point. Predetermined premiums are paid to the protection seller by the protection buyer according to a predetermined schedule up to the maturity year in such a way that no-arbitrage condition of the credit derivatives market is satisfied. The relationship between the protection seller and the protection buyer is depicted in Figure 2.1. Further procedural details of the CDO scheme will be discussed subsequently.

Multiple classes of securities are created by tranching the aggregate loss differently. In Japanese market, for example, the iTraxx Japan index is defined to be the average price of the most liquid 80 single name CDS's, and the standard tranches are available based on this index, consisting of $[0\%, 3\%]$-Tranche, $[3\%, 6\%]$-Tranche, $[6\%, 9\%]$-Tranche, $[9\%, 12\%]$-Tranche, and $[12\%, 22\%]$-Tranche, where the first figure denotes K_a and the second figure K_d. These tranches are traded mainly with 5year maturity. It is worth noting that the

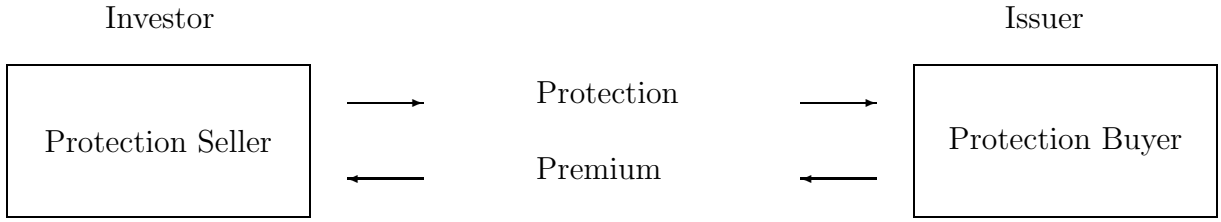

Figure 2.1: Cash flow of CDO Tranche

market convention of Tranched iTraxx Japan for [0%,3%]-Tranche is different from other standard tranches in that the former has a fixed premium rate of 300bp (basis points where one basis point corresponds to 0.01%) throughout the contract period and the necessary adjustment for the no-arbitrage condition is paid at the time of the contract agreement, while the latters do not require any initial payment and the premium rate itself is chosen so as to satisfy the no-arbitrage condition.

We now turn our attention to the procedural details of Tranched Index contracts. The principal entity of interest is a reference portfolio consisting of M single-name CDS's. Since the focus of this paper is to investigate Tranched Index, it is assumed that all single-name CDS's are equally weighted. As time progresses, some of them would default. Let $M(t)$ be the number of active corporations at time t. As we will see, the stochastic process $M(t)$ is modeled as a generalized pure death process with default intensities governed by a continuous time Markov chain representing the macro economic condition. Let $l(t)$ be the cumulative aggregate loss up to time t. Because of the equal weight assumption above, one can assume, without loss of generality, that $l(t)$ is proportional to $M - M(t)$. Using M as a scaling factor, $l(t)$ can then be scaled so that $0 \leq l(t) \leq 1$. More specifically, with R being the recovery rate of each of individual default losses, one can write

$$l(t) = \frac{M - M(t)}{M} \cdot (1 - R) = \Big(1 - \frac{M(t)}{M}\Big)(1 - R). \tag{2.1}$$

The protection seller taking the credit exposure to the tranche with an attachment point K_a and a detachment point K_d will bear full losses occurring in the portfolio in excess of K_a but up to K_d. Such a tranche is denoted by $[K_a, K_d]$-Tranche. Let $L_{[K_a,K_d]}(t)$ be the cumulative loss at time t of $[K_a, K_d]$-Tranche defined by

$$L_{[K_a,K_d]}(t) = \Big(l(t) - K_a\Big)^+ - \Big(l(t) - K_d\Big)^+ \tag{2.2}$$

where $(x)^+ = \max(0, x)$. Since $l(t)$ is scaled so as to satisfy $0 \leq l(t) \leq 1$, we postulate $0 \leq K_a \leq K_d \leq 1$. It should be noted that

$$L_{[K_a,K_d]}(t) = \begin{cases} 0 & \text{if } l(t) \leq K_a \\ l(t) - K_a & \text{if } K_a \leq l(t) \leq K_d \\ K_d - K_a & \text{if } K_d \leq l(t) \end{cases} . \tag{2.3}$$

The procedural details of a Tranched Index contract can now be described as follows. At predetermined time epochs $\tau_k,\ k = 0, 1, \cdots, K$, with $\tau_0 = 0$, the protection seller pays

to the protection buyer by the amount of $PAY_{sell\to buy}(\tau_k)$ specified by the tranched loss increment, i.e.

$$PAY_{sell\to buy}(\tau_k) = L_{[K_a,K_d]}(\tau_k) - L_{[K_a,K_d]}(\tau_{k-1}) \ , \quad k = 1, 2, \cdots, K \ . \tag{2.4}$$

It should be noted that once the cumulative payment from the protection seller to the protection buyer reaches $K_d - K_a$, $PAY_{sell\to buy}$ remains 0 from that point on.

In exchange, a payment would be made from the protection buyer to the protection seller at each time epoch τ_k $(k = 1, 2, \cdots, K)$. This payment amount, denoted by $PAY_{buy\to sell}$, is based on the unit premium $c_{[K_a,K_d]}$ agreed upon at time $\tau_0 = 0$, which is applied to the hedge interval, i.e. $(K_d - K_a)$ minus the cumulative payment made by the protection seller to the protection buyer up to time τ_k. More formally, one sees from (2.4) that

$$\begin{aligned} PAY_{buy\to sell}(c_{[K_a,K_d]}, \tau_k) &= c_{[K_a,K_d]}\Big((K_d - K_a) - \sum_{j=1}^{k} PAY_{sell\to buy}(\tau_j)\Big) \\ &= c_{[K_a,K_d]}\Big((K_d - K_a) - L_{[K_a,K_d]}(\tau_k)\Big). \end{aligned} \tag{2.5}$$

The financial interactions between the protection seller and the protection buyer are illustrated in Figure 2.2.

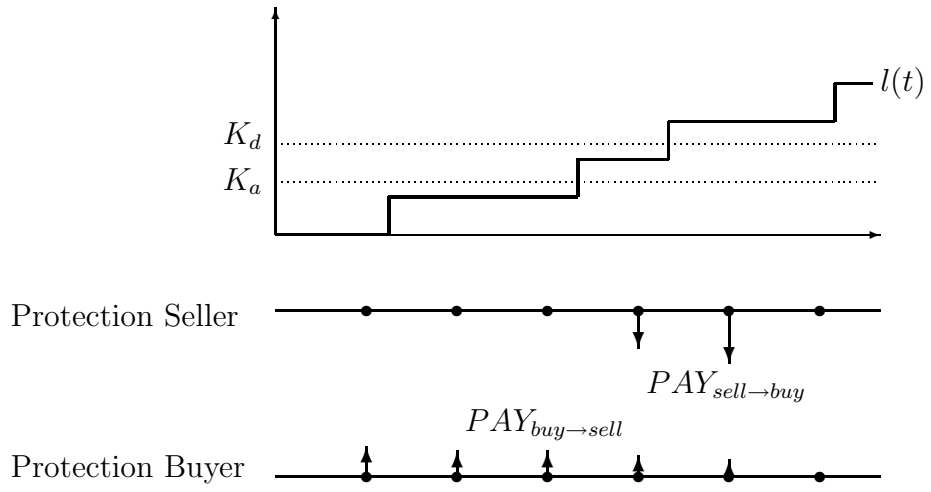

Figure 2.2: Protectionleg and Premiumleg

In formulating Tranched Index contracts, the key question is how to determine the unit premium $c^*_{[K_a,K_d]}$ at equilibrium which assures no-arbitrage in the credit derivatives market. We assume that there exists a risk-neutral martingale measure $\mathbb{P}$ under which all price processes discounted with the interest rate r are martingales. Furthermore, it is assumed that r is deterministic. In this paper, all expectations are taken with respect to this measure. Then the unit premium $c^*_{[K_a,K_d]}$ should satisfy

$$\sum_{k=1}^{K} e^{-r\tau_k}\mathbb{E}_{\mathbb{P}}\Big[PAY_{sell\to buy}(\tau_k)\Big] = \sum_{k=1}^{K} e^{-r\tau_k}\mathbb{E}_{\mathbb{P}}\Big[PAY_{buy\to sell}(c^*_{[K_a,K_d]}, \tau_k)\Big] \ . \tag{2.6}$$

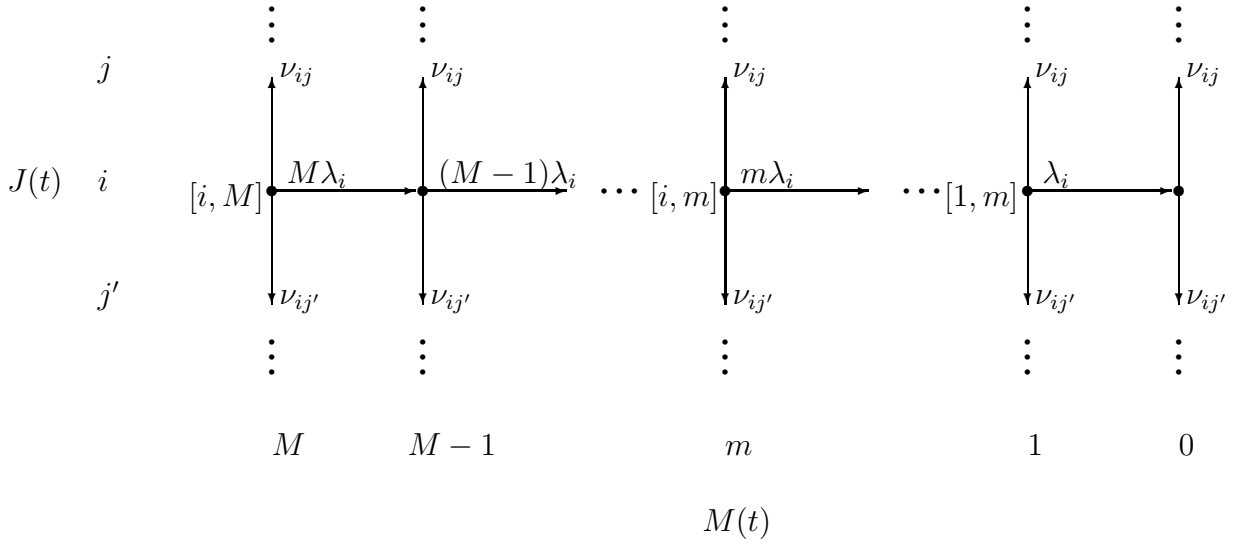

Figure 3.1: Transition Structure of $[J(t), M(t)]$

The left hand side of Equation (2.6) is called the protectionleg while the right hand side is called the premiumleg. From (2.4) and (2.5), Equation (2.6) can be solved for $c^*_{[K_a,K_d]}$ as

$$c^*_{[K_a,K_d]} = \frac{\sum_{k=1}^{K} e^{-r\tau_k}\mathbb{E}_{\mathbb{P}}\Big[L_{[K_a,K_d]}(\tau_k) - L_{[K_a,K_d]}(\tau_{k-1})\Big]}{\sum_{k=1}^{K} e^{-r\tau_k}\mathbb{E}_{\mathbb{P}}\Big[(K_d - K_a) - L_{[K_a,K_d]}(\tau_k)\Big]} \,. \tag{2.7}$$

The purpose of this paper is to develop efficient computational algorithms for evaluating $c^*_{[K_a,K_d]}$ based on (2.7). The validity of the model and the efficiency of the numerical procedures proposed in this paper are then tested by calibrating to real market data.

3. MMPP Formulation of CDO Scheme

We consider a finite Markov chain $J(t)$ in continuous time defined on $\mathcal{J} = \{0, 1, \cdots, J\}$ governed by $\underline{\underline{\mathcal{V}}} = [\nu_{ij}]$, describing the state of the macro-economic condition at time t. It is assumed that $J(t)$ is irreducible and therefore ergodic. There are M corporations under consideration constituting M single-name CDS's. All M corporations are active (non-default) at time $t = 0$. Given $J(t) = i$, each active corporation has a constant default intensity of λ_i. Once a corporation falls into default, it remains inactive from that point on. Apart from the fact that the default intensity governed by the macro-economic condition are common among all of the active corporations, individual corporations behave independently. Let $M(t)$ be the number of active corporations at time t with $M(0) = M$. The domain of $M(t)$ is denoted by $\mathcal{M} = \{0, 1, 2, \cdots, M\}$. It should be noted that the bivariate process $[J(t), M(t)]$ on $\mathcal{J}\times\mathcal{M}$ is an MMPP expressed as a row-continuous Markov chain governed by $[\underline{\underline{\mathcal{V}}}, \underline{\underline{\Lambda}}_D]$, where $\underline{\underline{\Lambda}}_D = [\delta_{ij}\lambda_i]$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Clearly, $[J(t), M(t)]$ is non-increasing with respect to $M(t)$, and $\{[i, m] : i \in \mathcal{J}, 1 \le m \le M\}$ is the transient set while $\{[i, 0] : i \in \mathcal{J}\}$ is the recurrent set as depicted in Figure 3.1.

In the next section, we analyze the bivariate Markov process $[J(t), M(t)]$, deriving the marginal distribution of $M(t)$ explicitly in a closed form. This result enables one to evaluate

the time dependent distribution of the cumulative aggregate loss $l(t)$ based on (2.1) and that of the cumulative loss $L_{[K_a,K_d]}(t)$ of $[K_a, K_d]$-Tranche from (2.2). This in turn provides a computational vehicle for assessing $PAY_{sell\to buy}$ from (2.4) and $PAY_{buy\to sell}$ from (2.5). Finally, the premium $c^*_{[K_a,K_d]}$ of interest can be computed based on (2.7).

4. Analysis of $[J(t), M(t)]$

As can be seen in Figure 3.1, the bivariate Markov process $[J(t), M(t)]$ is non-increasing in $M(t)$ and is lattice continuous. By exploiting this stochastic structure, given $M(t) = m$, we consider the absorbing bivariate Markov process $[J(t+\tau), M^*_m(t+\tau)]$ on $G_m \cup B_m$, where $G_m = \{[0,m]\cdots[J,m]\}$ is the transient set and $B_m = \{[0,m-1]\cdots[J,m-1]\}$ is the absorbing set, governed by transition rates as depicted in Figure 4.1. (We note that the rows and the columns are interchanged in Figure 4.1 so as to save the space.) Because of the underlying Markov property, we only consider the case of $t = 0$ without loss of generality. Of particular interest to our analysis is the first passage time of $[J(\tau), M^*_m(\tau)]$ from $[i,m] \in G_m$ to $[j, m-1] \in B_m$.

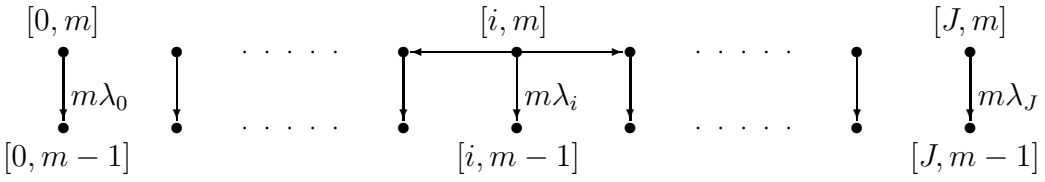

Figure 4.1: Transition rate from G_m to B_m

Formally, let $\underline{\underline{\mathcal{V}}}^*(m)$ be the transition rate matrix governing $[J(\tau), M^*_m(\tau)]$ where

$$\underline{\underline{\mathcal{V}}}^*(m) \stackrel{\text{def}}{=} \begin{array}{c} \\ G_m \\ B_m \end{array} \begin{array}{c} \begin{array}{cc} G_m & B_m \end{array} \\ \begin{bmatrix} \underline{\underline{\mathcal{V}}} & m\underline{\underline{\Lambda}}_D \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \end{array} . \tag{4.1}$$

We also introduce

$$\underline{\underline{\mathcal{V}}}^*_D(m) \stackrel{\text{def}}{=} \begin{bmatrix} \underline{\underline{\mathcal{V}}}_D + m\underline{\underline{\Lambda}}_D & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \;\; ; \;\; \underline{\underline{\mathcal{V}}}_D = [\delta_{ij}\nu_i] \;\; ; \;\; \nu_i = \sum_{j\in\mathcal{J}} \nu_{ij} \; , \tag{4.2}$$

so that the infinitesimal generator $\underline{\underline{\mathcal{Q}}}(m)$ of $[J(\tau), M^*_m(\tau)]$ is given by

$$\underline{\underline{\mathcal{Q}}}(m) = -\underline{\underline{\mathcal{V}}}^*_D(m) + \underline{\underline{\mathcal{V}}}^*(m) \; . \tag{4.3}$$

Let $\underline{\underline{P}}^*(m,\tau)$ be the transition probability matrix of $[J(\tau), M^*_m(\tau)]$. From the Kolmogorov forward equation, one has $\frac{d}{d\tau}\underline{\underline{P}}^*(m,\tau) = \underline{\underline{P}}^*(m,\tau)\underline{\underline{\mathcal{Q}}}(m)$. In the Laplase transform domain, i.e. $\underline{\underline{\pi}}^*(m,s) \stackrel{\text{def}}{=} \int_0^\infty e^{-s\tau}\underline{\underline{P}}^*(m,\tau)d\tau$, this leads to $s\underline{\underline{\pi}}^*(m,s) - \underline{\underline{I}} = \underline{\underline{\pi}}^*(m,s)\underline{\underline{\mathcal{Q}}}(m)$ so that

$$\underline{\underline{\pi}}^*(m,s) = \left[s\underline{\underline{I}} - \underline{\underline{\mathcal{Q}}}(m)\right]^{-1} \; . \tag{4.4}$$

One then sees that

$$\underline{\underline{P}}^*(m,\tau) = e^{\underline{\underline{\mathcal{Q}}}(m)\tau} = \sum_{k=0}^{\infty} \frac{1}{k!} \underline{\underline{\mathcal{Q}}}(m)^k \tau^k \ . \tag{4.5}$$

In order to facilitate our analysis further, we introduce the uniformization procedure of Keilson [4]. Let

$$\nu \geq \max\{\nu_i + m\lambda_i\} \ , \tag{4.6}$$

and define

$$\underline{\underline{a}}^*_{\nu}(m) \stackrel{\text{def}}{=} \underline{\underline{I}} - \frac{1}{\nu}\underline{\underline{\mathcal{V}}}^*_D(m) + \frac{1}{\nu}\underline{\underline{\mathcal{V}}}^*(m) \ . \tag{4.7}$$

From (4.3) and (4.7), one sees that $\underline{\underline{\mathcal{Q}}}(m) = -\nu[\underline{\underline{I}} - \underline{\underline{a}}^*_{\nu}(m)]$. Substituting this into (4.5) then yields

$$\underline{\underline{P}}^*(m,\tau) = e^{-\nu\tau[\underline{\underline{I}} - \underline{\underline{a}}^*_{\nu}(m)]} = \sum_{k=0}^{\infty} e^{-\nu\tau} \frac{(\nu\tau)^k}{k!} \underline{\underline{a}}^*_{\nu}(m)^k \ . \tag{4.8}$$

The uniformization procedure makes the dwell time of $[J(\tau), M^*_m(\tau)]$ at any state uniform in that all of such dwell times share the same exponential distribution with mean ν^{-1}, and the next state to be visited is governed by the stochastic matrix $\underline{\underline{a}}^*_{\nu}(m)$. This fact can be stated more formally in the following manner. Let $[\hat{J}(k), \hat{M}^*_m(k)]$ be the discrete time Markov chain governed by $\underline{\underline{a}}^*_{\nu}(m)$, and define $K_\nu(\tau)$ to be a Poisson process with parameter ν. One then has $[J(\tau), M^*_m(\tau)] = [\hat{J}(K_\nu(\tau)), \hat{M}^*_m(K_\nu(\tau))]$. The probabilistic meaning of Equation (4.8) is now clear.

From (4.4) and (4.7), one has $\underline{\underline{\pi}}^*(m,s) = \Big[(s+\nu)\underline{\underline{I}} - \nu\underline{\underline{a}}^*_{\nu}(m)\Big]^{-1}$ and hence

$$\underline{\underline{\pi}}^*(m,s) = \frac{1}{s+\nu}\Big[\underline{\underline{I}} - \frac{\nu}{s+\nu}\underline{\underline{a}}^*_{\nu}(m)\Big]^{-1} \ . \tag{4.9}$$

For the later use, we note that

$$\underline{\underline{\pi}}^*_{GG}(m,s) = \frac{1}{\nu}\sum_{k=0}^{\infty}\Big(\frac{\nu}{s+\nu}\Big)^{k+1} \underline{\underline{a}}^*_{\nu:GG}(m)^k \ , \tag{4.10}$$

and

$$\underline{\underline{\pi}}^*_{GB}(m,s) = \frac{1}{\nu}\sum_{k=0}^{\infty}\Big(\frac{\nu}{s+\nu}\Big)^{k+1} \underline{\underline{b}}^*_{\nu:GB}(m,k) \ , \tag{4.11}$$

where

$$\underline{\underline{b}}^*_{\nu:GB}(m,0) = \underline{\underline{0}} \ , \tag{4.12}$$

and

$$\underline{\underline{b}}^*_{\nu:GB}(m,k) = \sum_{j=0}^{k-1} \underline{\underline{a}}^*_{\nu:GG}(m)^j \underline{\underline{a}}^*_{\nu:GB}(m) \ . \tag{4.13}$$

In the real domain, these equations correspond to

$$\underline{\underline{P}}^*_{GG}(m,\tau) = \sum_{k=0}^{\infty} e^{-\nu\tau}\frac{(\nu\tau)^k}{k!}\underline{\underline{a}}^*_{\nu:GG}(m)^k \ , \tag{4.14}$$

$$\underline{\underline{P}}^*_{GB}(m,\tau) = \sum_{k=0}^{\infty} e^{-\nu\tau}\frac{(\nu\tau)^k}{k!}\underline{\underline{b}}^*_{\nu:GB}(m,k) \ . \tag{4.15}$$

The probabilistic entity of importance, which would play a key role as the computational vehicle for analysis of CDO Tranche contracts, is the joint probability of the first passage time of $[J(\tau), M^*_m(\tau)]$ from G_m to B_m starting from $[J(0), M^*_m(0)] = [i, m]$ and the absorbing state reached upon absorption. More specifically, let $T_{[i,m]B_m}$ be the first passage time of $[J(\tau), M^*_m(\tau)]$ from $[i, m] \in G_m$ to B_m and define

$$\begin{aligned} \underline{\underline{S}}(m,\tau) &= [S_{ij}(m,\tau)] \ ; \\ S_{ij}(m,\tau) &\stackrel{\text{def}}{=} \mathbb{P}\Big[T_{[i,m]B_m} \le \tau, J(\tau) = j | M^*_m(0) = m, J(0) = i\Big] \ . \end{aligned} \tag{4.16}$$

The corresponding Laplace transform matrix is defined by

$$\underline{\underline{\sigma}}(m,s) \stackrel{\text{def}}{=} [\sigma_{ij}(m,s)] \ \ ; \ \ \sigma_{ij}(m,s) = \int_0^{\infty} e^{-s\tau} dS_{ij}(m,\tau) \ . \tag{4.17}$$

The next theorem then holds true.

Theorem 4.1 *Let $\underline{\underline{S}}(m,\tau)$ and $\underline{\underline{\sigma}}(m,s)$ be as given in (4.16) and (4.17) respectively and define $\underline{\underline{s}}(m,\tau) = \frac{d}{d\tau}\underline{\underline{S}}(m,\tau)$. One then has:*

$$a) \quad \underline{\underline{\sigma}}(m,s) = \sum_{k=0}^{\infty}\Big(\frac{\nu}{s+\nu}\Big)^{k+1}\underline{\underline{a}}^*_{\nu:GG}(m)^k\underline{\underline{a}}^*_{\nu:GB}(m) \ ,$$

$$b) \quad \underline{\underline{s}}(m,\tau) = \sum_{k=0}^{\infty} e^{-\nu\tau}\frac{(\nu\tau)^k}{k!}\nu\underline{\underline{a}}^*_{\nu:GG}(m)^k\underline{\underline{a}}^*_{\nu:GB}(m) \ .$$

Proof

From the probabilistic interpretation of the uniformization procedure, one sees that

$$\sigma_{ij}(m,s) = \frac{\nu}{s+\nu}\Big[\delta_{ij}a^*_{\nu:[i,m][i,m-1]}(m) + \sum_{r\in\mathcal{J}} a^*_{\nu:[i,m][r,m]}(m)\sigma_{rj}(m,s)\Big] \ , \tag{4.18}$$

which can be explained in the following manner. Starting from $[i, m] = [J(0), M^*_m(0)]$, the dwell time at the state can be considered to have the common exponential distribution with the Laplace transform $\nu/(s+\nu)$ due to uniformization. The absorbing set B_m can be reached directly from $[i, m]$ only through $[i, m-1]$. When $i = j$, this direct transition occurs with probability $a^*_{\nu:[i,m][i,m-1]}(m)$. Otherwise, the next state to be visited would be $[r, m]$ with probability $a^*_{\nu:[i,m][r,m]}(m)$. In this case, the state $[j, m-1]$ should be reached anew from $[r, m]$, and the Laplace transform $\sigma_{rj}(m,s)$ should be multiplied.

In matrix notation, Equation (4.18) can be rewritten as

$$\underline{\underline{\sigma}}(m,s) = \frac{\nu}{s+\nu}\Big[\underline{\underline{a}}^*_{\nu:GB}(m) + \underline{\underline{a}}^*_{\nu:GG}(m)\underline{\underline{\sigma}}(m,s)\Big] \ , \tag{4.19}$$

which can be solved for $\underline{\underline{\sigma}}(m,s)$ as

$$\underline{\underline{\sigma}}(m,s) = \frac{\nu}{s+\nu}\Big[\underline{\underline{I}} - \frac{\nu}{s+\nu}\underline{\underline{a}}^*_{\nu:GG}(m)\Big]^{-1}\underline{\underline{a}}^*_{\nu:GB}(m)\ . \tag{4.20}$$

Part a) of Theorem 4.1 then follows by expanding (4.20) into a geometric series. The explicit inversion of part a) into the real domain yields part b), completing the proof.

□

We note that $\underline{\underline{s}}(m,\tau)$ can be expressed as a mixture of $\underline{\underline{a}}^*_{\nu:GG}(m)^k\underline{\underline{a}}^*_{\nu:GB}(m)$ with weights given as gamma functions. In principle, these gamma functions can be readily computed recursively given a vector of different values of τ. Since the mode of the gamma function of order k shifts to the right as $k \to \infty$, a little precaution is required for achieving numerical accuracy, which we will discuss later.

We are now in a position to derive the Laplace transform of the joint distribution of $[J(t), M(t)]$ explicitly. This expression facilitates the necessary computations substantially, thereby improving the brute force approach to apply the uniformization procedure to the entire state space $\mathcal{J}\times\mathcal{M}$. Formally, for $i,j\in\mathcal{J}$ and $m\in\mathcal{M}$, let $a_{ij}(m,t|M)$ be defined by

$$a_{ij}(m,t|M) = \mathbb{P}\{J(t)=j, M(t)=m|J(0)=i, M(0)=M\}\ . \tag{4.21}$$

The Laplase transform with respect to t is denoted by

$$\alpha_{ij}(m,s|M) = \int_0^\infty e^{-st}a_{ij}(m,t|M)dt\ . \tag{4.22}$$

The corresponding matrix function and matrix transform are written as

$$\underline{\underline{a}}(m,t|M) = \Big[a_{ij}(m,t|M)\Big]\ ;\ \underline{\underline{\alpha}}(m,s|M) = \Big[\alpha_{i,j}(m,s|M)\Big]\ . \tag{4.23}$$

One then has the following theorem.

Theorem 4.2 *Let $\underline{\underline{\alpha}}(m,s)$, $\underline{\underline{\sigma}}(m,s)$, $\underline{\underline{\pi}}^*_{GG}(m,s)$ and $\underline{\underline{\pi}}^*_{GB}(m,s)$ be as given in (4.23), (4.17), (4.10) and (4.11) respectively. Then the following statements hold.*

$$\begin{aligned}
&a) \quad \underline{\underline{\alpha}}(M,s|M) = \underline{\underline{\pi}}^*_{GG}(M,s),\\
&b) \quad \underline{\underline{\alpha}}(m,s|M) = \underline{\underline{\sigma}}(M,s)\underline{\underline{\sigma}}(M-1,s)\cdots\underline{\underline{\sigma}}(m+1,s)\underline{\underline{\pi}}^*_{GG}(m,s),\\
&\qquad\qquad (m = M-1, M-2, \cdots, 2, 1)\\
&c) \quad \underline{\underline{\alpha}}(0,s|M) = \underline{\underline{\sigma}}(M,s)\underline{\underline{\sigma}}(M-1,s)\cdots\underline{\underline{\sigma}}(2,s)\underline{\underline{\pi}}^*_{GB}(1,s).
\end{aligned}$$

Proof

Given $M(0)=M$, one has $M(t)=M$ if and only if no corporation has defaulted by time t. This means that the bivariate process $[J(t), M^*_M(t)]$ remains in G_M at time t starting from a state in G_M at time 0, proving part a). If $M(t)=m$ for $m=M-1, M-2, \cdots, 1$, the process $M(\cdot)$ has to reach m for the first time at some time τ with $0\le\tau\le t$. In the remaining period $[\tau,t]$, the bivariate process $[J(t), M^*_m(t)]$ has to remain in G_m. Statement b) then holds accordingly. For part c), when the process $M(\cdot)$ reaches 1 at some time τ with $0\le\tau\le t$, the bivariate process $[J(t), M^*_1(t)]$ has to reach B_0 in the remaining period $[\tau,t]$, completing the proof. □

Remark 4.3 *It should be noted that the sum of* $\underline{\underline{a}}(m,t|V)$ *over* $m \in \mathcal{M}$ *yields a stochastic matrix because of (4.21) and (4.23). More specifically, one has for any* $V \in \mathcal{M}$

$$\sum_{m=0}^{V} \underline{\underline{a}}(m,t|V)\underline{1} = \underline{1} \ . \tag{4.24}$$

In the subsequent sections, we develop the computational algorithms for evaluating the prices of CDO Tranches. Some numerical results will also be reported, demonstrating the consistency of the proposed model with real data.

5. Computational Algorithms

In order to evaluate the prices of CDO Tranches $c^*_{[K_a,K_d]}$, it is sufficient from (2.7) to develop computational procedures for assessing $\mathbb{E}_{\mathbb{P}}\big[L_{[K_a,K_d]}(t)|J(0)=i\big]$. From (2.2), the basic entity to constitute $L_{[K_a,K_d]}(t)$ is $(l(t)-K)^+$. It can be seen from (2.1) that this entity is positive if and only if

$$M(t) \le \Big(\frac{1-R-K}{1-R}\Big)M \ . \tag{5.1}$$

Accordingly, let $\mathcal{M}$ be decomposed into $\mathcal{M}_{\rm I}, \mathcal{M}_{\rm II}$ and $\mathcal{M}_{\rm III}$ defined by

$$\mathcal{M}_{\rm I} \stackrel{\rm def}{=} \Big\{m : 0 \le m \le \frac{1-R-K_d}{1-R}M\Big\}, \tag{5.2}$$

$$\mathcal{M}_{\rm II} \stackrel{\rm def}{=} \Big\{m : \frac{1-R-K_d}{1-R}M < m \le \frac{1-R-K_a}{1-R}M\Big\}, \tag{5.3}$$

and

$$\mathcal{M}_{\rm III} \stackrel{\rm def}{=} \Big\{m : \frac{1-R-K_a}{1-R}M < m \le M\Big\}. \tag{5.4}$$

We note that

$$L_{[K_a,K_d]}(t) = \begin{cases} K_d - K_a & \text{if } M(t) \in \mathcal{M}_{\rm I} \\ \Big(1-\frac{M(t)}{M}\Big)(1-R) - K_a & \text{if } M(t) \in \mathcal{M}_{\rm II} \\ 0 & \text{if } M(t) \in \mathcal{M}_{\rm III} \end{cases} . \tag{5.5}$$

It then follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\Big[L_{[K_a,K_d]}(t) \ \Big| \ J(0)=i\Big] &= (K_d-K_a)\sum_{m\in\mathcal{M}_{\rm I}}\sum_{j\in\mathcal{J}} a_{ij}(m,t|M) \\ &+ \sum_{m\in\mathcal{M}_{\rm II}}\sum_{j\in\mathcal{J}}\Big(\Big(1-\frac{m}{M}\Big)(1-R)-K_a\Big)a_{ij}(m,t|M) \ . \end{aligned} \tag{5.6}$$

Hence, the evaluation of the prices of CDO Tranches $c^*_{[K_a,K_d]}$ is nailed down to computation of $\underline{\underline{a}}(m,t|M)$.

From Theorem 4.2, the matrix Laplace transforms $\underline{\underline{\alpha}}(m,s|M)$ of $\underline{\underline{a}}(m,t|M)$ can be expressed in terms of $\underline{\underline{\sigma}}(m,s)$ in (4.20), $\underline{\underline{\pi}}^*_{GG}(m,s)$ in (4.10), and $\underline{\underline{\pi}}^*_{GB}(m,s)$ in (4.11). It should

be noted that these three types of Laplace transform matrices can be expressed as a linear combination of constant matrices with functional coefficients of the form

$$\gamma_k(s) = \int_0^\infty e^{-st} g_k(t) dt = \frac{\nu^k}{(s+\nu)^{k+1}} \; ; \; g_k(t) = e^{-\nu t} \frac{(\nu t)^k}{k!} \; . \tag{5.7}$$

The fact that the matrices in the series expression are independent of s and the functional parts are concentrated in the coefficients $\gamma_k(s)$ facilitates the matrix convolutions among the three types of the matrix functions greatly, with help from a nice property of $\gamma_k(s)$ given by

$$\gamma_m(s)\gamma_n(s) = \frac{1}{\nu}\gamma_{m+n+1}(s) \; . \tag{5.8}$$

In other words, the class of matrix Laplace transforms of a linear combination of constant matrices with functional coefficients $\gamma_k(s)$ is closed under matrix multiplications. Accordingly, the class of the corresponding matrix functions in real domain is closed under matrix convolutions.

Based on the observations discussed above, we define

$$\mathcal{A}_{\mathbf{LAPL}} \stackrel{\text{def}}{=} \left\{ \underline{\underline{\xi}}_{(n,\underline{\underline{X}}(n))_{n=0}^\infty}(s) : \underline{\underline{\xi}}_{(n,\underline{\underline{X}}(n))_{n=0}^\infty}(s) = \sum_{n=0}^\infty \gamma_n(s) \underline{\underline{X}}(n) \right\} , \tag{5.9}$$

and

$$\mathcal{A}_{\mathbf{REAL}} \stackrel{\text{def}}{=} \left\{ \underline{\underline{\mathcal{X}}}_{(n,\underline{\underline{X}}(n))_{n=0}^\infty}(t) : \underline{\underline{\mathcal{X}}}_{(n,\underline{\underline{X}}(n))_{n=0}^\infty}(t) = \sum_{n=0}^\infty g_n(t) \underline{\underline{X}}(n), \right\} . \tag{5.10}$$

We now introduce the following two mappings involving $\mathcal{A}_{\mathbf{LAPL}}$ and $\mathcal{A}_{\mathbf{REAL}}$.

$$\mathbf{INV} : \mathcal{A}_{\mathbf{LAPL}} \longrightarrow \mathcal{A}_{\mathbf{REAL}} \tag{5.11}$$

with

$$\underline{\underline{\mathcal{X}}}_{(n,\underline{\underline{X}}(n))_{n=0}^\infty}(t) = \mathbf{INV}\left[\underline{\underline{\xi}}_{(n,\underline{\underline{X}}(n))_{n=0}^\infty}(s) \right] \tag{5.12}$$

and

$$\mathbf{MULT} : \mathcal{A}_{\mathbf{LAPL}} \times \mathcal{A}_{\mathbf{LAPL}} \longrightarrow \mathcal{A}_{\mathbf{LAPL}} \tag{5.13}$$

with

$$\underline{\underline{\xi}}_{(\ell,\underline{\underline{Z}}(\ell))_{\ell=0}^\infty}(s) = \mathbf{MULT}\left[\underline{\underline{\xi}}_{(m,\underline{\underline{X}}(m))_{m=0}^\infty} \; , \; \underline{\underline{\xi}}_{(n,\underline{\underline{Y}}(n))_{n=0}^\infty} \right] . \tag{5.14}$$

Here **INV** is the matrix inversion operator specifying the matrix function in real domain given a matrix Laplace transform. **MULT** is the matrix multiplication operator implying

$$\underline{\underline{\xi}}_{(\ell,\underline{\underline{Z}}(\ell))_{\ell=0}^\infty}(s) = \underline{\underline{\xi}}_{(m,\underline{\underline{X}}(m))_{m=0}^\infty}(s) \times \underline{\underline{\xi}}_{(n,\underline{\underline{Y}}(n))_{n=0}^\infty}(s) \; . \tag{5.15}$$

From (5.8) and (5.15), it follows that

$$\begin{aligned}\underline{\underline{\xi}}_{(m,\underline{\underline{X}}(m))_{m=0}^{\infty}}(s) \times \underline{\underline{\xi}}_{(n,\underline{\underline{Y}}(n))_{n=0}^{\infty}}(s) &= \sum_{m=0}^{\infty}\gamma_m(s)\underline{\underline{X}}(m)\sum_{n=0}^{\infty}\gamma_n(s)\underline{\underline{Y}}(n) \\ &= \frac{1}{\nu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\gamma_{m+n+1}(s)\underline{\underline{X}}(m)\underline{\underline{Y}}(n) \\ &= \frac{1}{\nu}\sum_{\ell=1}^{\infty}\gamma_\ell(s)\sum_{m+n+1=\ell:m,n\geq 0}\underline{\underline{X}}(m)\underline{\underline{Y}}(n)\ .\end{aligned}$$

Consequently, one sees that

$$\underline{\underline{Z}}(\ell) = \frac{1}{\nu}\sum_{m+n+1=\ell:m,n\geq 0}\underline{\underline{X}}(m)\underline{\underline{Y}}(n)\ ,\ \ell = 1,2,\cdots\ . \tag{5.16}$$

We are now in a position to summarize computational procedures for evaluating $\underline{\underline{a}}(m,t|M)$.

Algorithm 5.1

Input:

$\triangleright\ \mathcal{M} = \{0,1,\cdots,M\}$

$\triangleright\ \mathcal{J} = \{0,1,\cdots,J\}$

$\triangleright\ \underline{\underline{\mathcal{V}}} = \left[\nu_{ij}\right]_{i,j\in\mathcal{J}}$

$\triangleright\ \underline{\lambda} = \left[\lambda_0,\lambda_1,\cdots,\lambda_J\right]^{\top}$

Output:

$\triangleleft\ \underline{\underline{a}}(m,t|M)\ ,\ m\in\mathcal{M}$

Procedure:

1] Set $\underline{\underline{\sigma}}(m,s) = \underline{\underline{\xi}}_{(k,\nu\underline{\underline{a}}^*_{\underline{\underline{\nu}}:GG}(m)^k\underline{\underline{a}}^*_{\underline{\underline{\nu}}:GB}(m))_{k=0}^{\infty}}(s)$, for $m = M, M-1,\cdots,2$.

2] Set $\underline{\underline{\pi}}^*_{GG}(m,s) = \underline{\underline{\xi}}_{(k,\underline{\underline{a}}^*_{\underline{\underline{\nu}}:GG}(m)^k)_{k=0}^{\infty}}(s)$, for $m = M, M-1,\cdots,2,1$.

3] Set $\underline{\underline{\pi}}^*_{GB}(m,s) = \underline{\underline{\xi}}_{(k,\underline{\underline{b}}^*_{\underline{\underline{\nu}}:GB}(m,k))_{k=0}^{\infty}}(s)$, for $m = 1$.

4] Set $\underline{\underline{\alpha}}(M,s|M) = \underline{\underline{\pi}}^*_{GG}(M,s)$ and $\underline{\underline{\beta}}(M,s|M) = \underline{\underline{I}}$.

5] $m \leftarrow M-1$

6] **LOOP** : $\underline{\underline{\beta}}(m,s|M) = \mathbf{MULT}\Big[\underline{\underline{\beta}}(m+1,s|M), \underline{\underline{\sigma}}(m+1,s)\Big]$.

7] Compute $\underline{\underline{\alpha}}(m,s|M) = \mathbf{MULT}\Big[\underline{\underline{\beta}}(m,s), \underline{\underline{\pi}}^*_{GG}(m,s)\Big]$.

8] Compute $\underline{\underline{a}}(m,t|M) = \mathbf{INV}\Big[\underline{\underline{\alpha}}(m,s|M)\Big]$.

9] $\rightarrow (1 \geq m \leftarrow m-1)/\mathbf{LOOP}$

10] Compute $\underline{\underline{\alpha}}(0,s|M) = \mathbf{MULT}\Big[\underline{\underline{\beta}}(1,s|M), \underline{\underline{\pi}}^*_{GB}(1,s)\Big]$.

11] Compute $\underline{\underline{a}}(0,t|M) = \mathbf{INV}\Big[\underline{\underline{\alpha}}(0,s|M)\Big]$.

12] END.

Algorithm 5.1 can be interpreted in the following manner. From (4.1), (4.2), (4.7) and (4.9), the matrices $\underline{\underline{\pi}}^*(m,s)$ can be readily obtained. Based on Theorem 4.1 a) together with (4.10) and (4.11), the matrices $\underline{\underline{\sigma}}(m,s)$, $\underline{\underline{\pi}}^*_{GG}(m,s)$ and $\underline{\underline{\pi}}^*_{GB}(m,s)$ can be expressed as linear combinations of constant matrices with functional coefficients of $\gamma_k(s)$ in (5.7), corresponding to 1] through 3] in Algorithm 5.1. Using **MULT** and **INV** given in (5.14) and (5.12) respectively, the matrices $\underline{\underline{a}}(m,t|M)$ can then be evaluated based on Theorem 4.2, explaining 4] through 11]. A few remarks are worth noting.

Remark 5.2 *a) Given $t > 0$, the sequence $(g_k(t))_{k=0}^{\infty}$ is unimodal with the peak shifting to the right as t becomes larger. In order to secure a truncation accuracy, say $\varepsilon > 0$, it is necessary to pick up terms which are grater than ε centered at the peak.*
b) Keeping the point in a) above in mind, the vector values $[g_{k+1}(t_1), g_{k+1}(t_2), \cdots, g_{k+1}(t_N)]$ can be generated recursively from $[g_k(t_1), g_k(t_2), \cdots, g_k(t_N)]$.

6. Numerical Results

In this section, we demonstrate the speed and accuracy of Algorithm 5.1 through numerical experiments. In addition, the model proposed in this paper for evaluating the prices of CDO Tranches is calibrated to real market data by fitting the underlying parameter values in a certain manner. As we will see, the calibration result proves to be excellent.

In order to design numerical experiments, we assume that $J(t)$ describing the macro economic condition is defined on $\mathcal{J} = \{0, 1, \cdots, 2V\}$ and can be expressed as an Ehrenfest process characterized by

$$\nu_{ij} = \begin{cases} v \cdot \frac{i}{2} & \text{if } \ j = i-1 \\ v \cdot \left(V - \frac{i}{2}\right) & \text{if } \ j = i+1 \\ 0 & \text{else} \end{cases} \quad . \tag{6.1}$$

It is known, see e.g. Sumita, Gotoh and Jin [10], that this Ehrenfest process converges in law to an Ornstein-Uhlenbeck process with appropriate scaling and shifting. The OU process has a tendency to return to the mean, should it depart from it in either direction. This property is desirable for describing the macro economic condition. It should be noted that the scaling parameter v enables one to adjust the speed of state transitions of the macro economic condition.

For the default intensity λ_j when $J(t) = j$, the following functional structure is assumed:

$$\lambda_j = \alpha e^{-\beta(j-V)} + \gamma e^{-\delta(j-V)} \ , \tag{6.2}$$

where $\alpha, \beta, \gamma, \delta > 0$. We note that $J(\cdot) = V$ corresponds to a normal economic condition which improves as $J(\cdot)$ increases and deteriorates as $J(\cdot)$ decreases. Accordingly, λ_j is defined as a decreasing function of j with $\alpha + \gamma$ being the default intensity in the normal economic condition.

In actually conducting numerical experiments, we set $v = 0.1$, $V = 3$ and $(\alpha, \beta, \gamma, \delta) = (0.0002, 2, 0.0015, 0.08)$. The corresponding default intensity function in λ_j takes values as given in Table 6.1, demonstrating the desired monotonicity property. Figures 6.1 through 6.6 show the term structure of the probabilities(z-axis) of just n defaults conditional to $J(\cdot)$, where the probabilities are expressed in %. As can be seen in Figure 6.1, the probability of

Table 6.1: Values of Default Intensity Vector $\underline{\lambda}$

λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0.08259	0.01268	0.00310	0.00170	0.00141	0.00128	0.00118

no default as a function of $J(\cdot)$ rises rather sharply around the normal economic condition at the beginning, and decreases monotonically as the term becomes longer. The peak point remains around the normal economic condition for a while but moves along the direction of the term length as if a child crawls into a blanket and continues to crawl with shift to the right under the blanket.

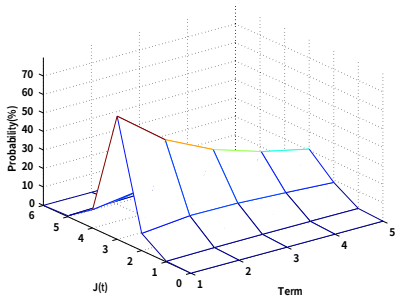

Figure 6.1: $a_{V,j}(M, t|M)$ for $j \in \mathcal{J}$ with $M = 80$

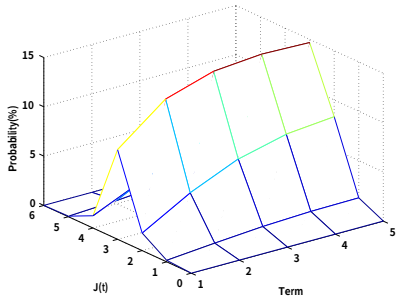

Figure 6.2: $a_{V,j}(M-1, t|M)$ for $j \in \mathcal{J}$ with $M = 80$

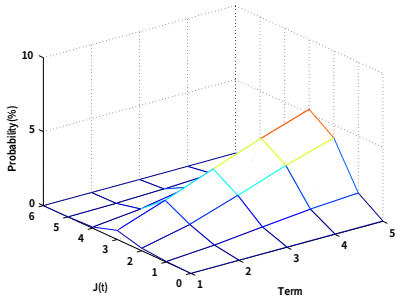

Figure 6.3: $a_{V,j}(M-2, t|M)$ for $j \in \mathcal{J}$ with $M = 80$

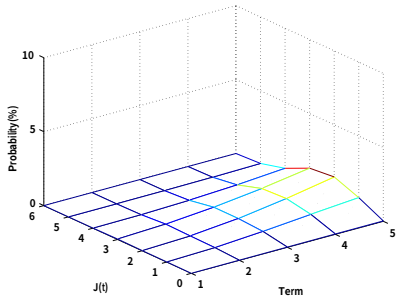

Figure 6.4: $a_{V,j}(M-3, t|M)$ for $j \in \mathcal{J}$ with $M = 80$

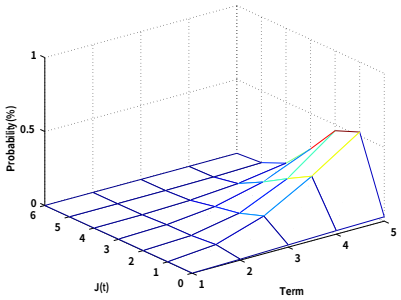

Figure 6.5: $a_{V,j}(M-4, t|M)$ for $j \in \mathcal{J}$ with $M = 80$

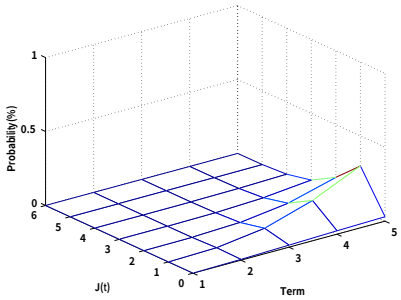

Figure 6.6: $a_{V,j}(M-5, t|M)$ for $j \in \mathcal{J}$ with $M = 80$

We next turn our attention to validate the proposed model by calibrating to the Tranched iTraxx Japan Series7(5Y), employing the market mid-quotes on 2007/5/25 obtained from Bloomberg. The state space size for the macro economic condition is set to be 7 with $V = 3$ in (6.1). For estimating the underlying parameter values $(\alpha, \beta, \gamma, \delta, v)$ so as to reflect the real market data, the objective is to minimize the squared sum of the differences of the theoretical Tranche premium values obtained from the model and the market quotes. More formally, let $c_i(\alpha, \beta, \gamma, \delta, v)$ be the theoretical premium value derived from the proposed model and let $c_i^{\mathbf{Market}}$ be the corresponding real market value, with i running over the four market quote sets as given in the first column of Table 6.3. The objective function can then be written as

$$f(\alpha, \beta, \gamma, \delta, v) = \sum_{i=1}^{4} \left\{ c_i(\alpha, \beta, \gamma, \delta, v) - c_i^{\mathbf{Market}} \right\}^2 , \tag{6.3}$$

which should be minimized. The optimal parameter values are estimated as

$$\hat{\alpha} = 0.00024922254304,\ \hat{\beta} = 1.98146548757474,\ \hat{\gamma} = 0.00157242963008,$$
$$\hat{\delta} = 0.07923944636996,\ \hat{v} = 0.09683163376248\ .$$

The corresponding values of λ_j can be obtained from (6.2) as given in Table 6.2. We note that λ_j is decreasing in j as it should be. The calibration results are summarized in Table 6.3, demonstrating the excellent consistency between the theoretical proposed model and the real data.

Table 6.2: Values of Default Intensity Vector $\underline{\lambda}$ for the optimal parameters

λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0.09800	0.01495	0.00351	0.00182	0.00149	0.00135	0.00124

Table 6.3: Proposed Model v.s. Real Market Data

iTraxx Japan S7	Model	Market Mid-Quote
0-3Tranche	300+9.449%	300+9.450%
3-6Tranche	17.749bp	17.750bp
6-9Tranche	4.999bp	5.000bp
9-12Tranche	2.628bp	2.625bp

A computer with CPU Core 2 Duo processor 2.40 GHz and 1.99 GB RAM is employed for all the computations presented in this paper, using MATLAB as a programming language. The required CPU times were around 72 seconds to compute CDO Tranches with $V = 3$.

7. Concluding Remarks

In this paper, computational procedures for evaluating loss distributions and pricing CDO tranches are developed by recapturing the MMPP model of Lando [6] within the context of CDO. In particular, by exploiting the stochastic structure of the MMPP, computational procedures are developed for evaluating loss distributions and pricing CDO tranches, which are much more efficient than those based on the general Markov chain approach. Some numerical results are presented, demonstrating the excellent consistency between the premium values obtained from the proposed theoretical model and the corresponding real market data.

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