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2001

Institute of Policy and Planning Sciences
discussion paper series ~ no. 931

URL http://hdl.handle.net/2241/648

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INSTITUTE OF POLICY AND PLANNING SCIENCES

Discussion Paper Series

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Optimal Two-Sided Test for the Location Parameter of the Uniform Distribution Based on Lagrange's Method

by

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June 2001

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Optimal two-sided test for the location parameter of the uniform distribution based on Lagrange's method.

Yoshiko Nogami

Abstract.

In this paper we deal with the uniform distribution with the density
\[ f(x|\theta) = \begin{cases} \frac{1}{(\theta_2 - \theta_1)}, & \text{for } \theta_1 \leq x < \theta_2 \\ 0, & \text{otherwise} \end{cases} \]

where \(-\infty < \theta < \infty\) and \(\theta_i (i=1, 2)\) are real values such that \(\theta_1 < \theta_2\). Based on a random sample \(X_1, \ldots, X_n\) from \(f(x|\theta)\) we construct the two-sided test for testing the hypotheses \(H_0: \theta = \theta_0\) versus \(H_1: \theta \neq \theta_0\) for a constant \(\theta_0\), inverting the shortest interval estimate for \(\theta_0\). We show that the power function of this test is minimized at \(\theta_0\) and exhibit its exact form. We also prove that this test has the greatest power among the tests symmetric about \(\theta_0\).
§1. Introduction

In this paper we deal with the uniform distribution over the interval \([\theta+e_1, \theta+e_2]\) with the density

\[
    f(x|\theta) = \begin{cases} 
        (e_2-e_1)^{-1}, & \text{for } e_1 \leq x - \theta < e_2 \\
        0, & \text{otherwise}
    \end{cases}
\]

(1.1) where \(-e < \theta < e\) and \(e_1, e_2 (i=1, 2)\) are real values such that \(e_1 < e_2\). Based on a random sample \(X_1, \ldots, X_n\) from \(f(x|\theta)\) we test the hypothesis \(H_0: \theta = \theta_0\) versus the alternative hypothesis \(H_1: \theta \neq \theta_0\) for a constant \(\theta_0\). Let \(X_{(i)}\) be the \(i\)-th smallest observation of \(X_1, \ldots, X_n\). We first estimate \(\theta\) by an unbiased estimate \(\bar{X} = (X_{(1)} + X_{(n)}) / 2\) with \(\bar{X} = e_1 + e_2\).

The author needs to remark on the estimator \(Y\). Although the author had been noticed in Y. Nogami(1992) (see p.1) that the unbiased estimator \(Y\) could be a good estimator, miscalculation in deriving the probability density function (p.d.f.) of \(Y\) had prevented her from reaching \(Y\). Later, in Y. Nogami(1995), the author corrected this and used \(Y\) to obtain the two-sided test based on the shortest interval estimate. This paper is the generalization of Sections 1 and 2 of Y. Nogami(1995) to the density (1.1) and introduce in Section 4 extra property which is the generalization of Y. Nogami(1997).

Let \(0 < \varepsilon < 1\). We call \((U_1, U_2)\) a \((1-\varepsilon)\) interval estimate for the parameter \(\theta\) if \(P[U_1 < \theta < U_2] = 1-\varepsilon\). We use Lagrange’s method to get the shortest \((1-\varepsilon)\) interval estimate for the parameter \(\theta\) based on \(Y\). Inverting this interval for \(\theta_0\) we obtain the acceptance region of the two-sided test in Section 2.

As the bibliography the problem of testing hypotheses \(H_0: \theta = 0\) versus \(H_1: \theta \neq 0\) is treated by A. Birnbaum(1954). He treats the case of \(\theta = -\theta_1 = 2^{-1}\) and states in his paper that there exists no uniformly most powerful unbiased test of \(H_0: \theta = 0\). However, he had not mentioned the estimator \(Y\). Also, the alternative hypothesis \(H_1: 0 < \theta < 2^{-1}(1-\varepsilon^{1/n})\) in his Example 2 is too close to the hypothesis \(H_0: \theta = 0\) as \(n \to \infty\) because \(\varepsilon^{1/n}\) becomes close to 1 when \(n\) is large. So, the author still believes goodness of our test introduced in Section 2. In fact, in Section 3 we see that the power function of this test is minimized at \(\theta_0\) with the minimum value \(c\) and exhibit its exact form. (see (3.1).) Furthermore, in Section 4 we state the theorem that this test has the greatest power among size-\(\varepsilon\) tests symmetric about \(\theta_0\) and prove it by usage of generalized Neyman-Pearson Lemma.

Hereafter, we let \(c = e_2 - e_1 (> 0)\). Let \(A\) be the defining property.
§2. The two-sided test for $\theta$.

Let $X_1, \ldots, X_n$ be a random sample of size $n$ taken from the density (1.1). We find the shortest $(1-\alpha)$ interval estimate for $\theta$ based on $Y$ defined by (1.2) using Lagrange's method and invert this interval estimate for $\theta_0$ to get the two-sided test for testing hypotheses $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$.

We first find the probability density function (p.d.f.) of $Y$. Applying the variable transformation $Y = (X_{(1)} + X_{(n)} - \theta_0)/2$ and $Z = X_{(1)}$ to the joint p.d.f. of $(X_{(1)}, X_{(n)})$ and taking the marginal p.d.f. we obtain the p.d.f. of $Y$ as follows:

\begin{equation}
 g_Y(y|\theta) = \begin{cases}
 n c^{-n} (c-2|y-\theta|)^{n-1}, & \text{for } -c/2 < y < c/2 \\
 0, & \text{otherwise.}
\end{cases}
\end{equation}

To get the shortest $(1-\alpha)$ interval estimate for $\theta$ we shall find real numbers $r_1$ and $r_2$ ($r_1 < r_2$) which minimize $r_2 - r_1$ subject to

\begin{equation}
 (2.1) \quad \int_{r_1}^{r_2} g_Y(y|\theta) \, dy = 1 - \alpha.
\end{equation}

Letting $\lambda$ be a real number we define

\begin{equation}
 (2.2) \quad P_{\theta} [r_1 < Y < r_2] = \int_{r_1}^{r_2} g_Y(y|\theta) \, dy = 1 - \alpha.
\end{equation}

\begin{align*}
 L & = L(r_1, r_2; \lambda) = r_2 - r_1 - \lambda \int_{r_1}^{r_2} g_Y(y|\theta) \, dy \quad = 1 + \lambda.
\end{align*}

By Lagrange's method $\partial L/\partial r_1 = 0 = \partial L/\partial r_2$, which leads to

\begin{equation}
 (2.3) \quad g_Y(\theta + r_1|\theta) = g_Y(\theta + r_2|\theta) = \lambda^{-1}, \quad \forall \theta.
\end{equation}

Since $\partial L/\partial \lambda = 0$ is equivalent to (2.2), (2.2) and (2.3) lead to $r_2 - r_1 = \lambda^{-1}$. Substituting these into (2.2), making a variable change $u = y - \theta$ and performing further calculations leads to

\begin{align*}
 r & = \text{the extreme left hand side of (2.2)} = 2 \int_{0}^{\infty} n c^{-n} (c-2u)^{n-1} \, du = 1 - (1 - (2r/c))^n.
\end{align*}

Solving $\alpha = (1 - (2r/c))^n$ we get

\begin{equation}
 (2.4) \quad r = (1-\alpha^{1/n})/2.
\end{equation}

Hence, the shortest $(1-\alpha)$ interval estimate for $\theta$ is
(2.5) \((Y-r, Y+r)\)
where \(r\) is given by (2.4).

Therefore, by inverting the shortest \((1-\gamma)\) interval estimate (2.5) for \(\theta_0\),
we get the acceptance region \((\theta_0-r, \theta_0+r)\). Namely, our test is to reject \(H_0\) if \(Y \in (-\infty, \theta_0-r) \cup (\theta_0+r, +\infty)\) and accept \(H_0\) if \(Y \in (\theta_0-r, \theta_0+r)\). Let \(\phi(y)\) be a randomized test
which chooses between two decisions, rejection or acceptance of \(H_0\) with probabili-
ties \(\phi(y)\) and \(1-\phi(y)\), respectively. Let \(\phi^*(y)\) be a randomized test defined by

\[
(2.6) \quad \phi^*(y) = \begin{cases} 
1, & \text{if } y \leq \theta_0 - r \text{ or } \theta_0 + r \\
0, & \text{if } \theta_0 - r < y < \theta_0 + r.
\end{cases}
\]

In the next section we show that the power function of our test is minimized
at \(\theta_0\) and exhibit its exact form.

§3. The power function.

Let \(Y_1 = Y - r\) and \(Y_2 = \theta_0 + r\) where \(r\) is given by (2.4). We define the power
function \(x(\theta)\) of our test as follows:

\[
x(\theta) = \frac{1}{\gamma} \int g_Y(y|\theta) \, dy - 1,
\]

where \(g_Y(y|\theta)\) is defined by (2.1). From (2.2) with \(r = -r_1(\theta)\) we have that \(x(\theta)\) = \(g\).

From (2.3) and \(d\theta/d\theta = g_Y(y_2|\theta) - g_Y(y_1|\theta)\) we obtain that \([d\theta/d\theta]_{\theta = \theta_0} = -1\).
Furthermore, from \(d\theta/d\theta = g_Y(y_2|\theta) - g_Y(y_1|\theta)\) and the definitions of \(g_Y(y_1|\theta)\)
\(i = 1, 2\) we can easily check that \(d\theta/d\theta < 0\) for \(\theta < \theta_0\) and \(d\theta/d\theta > 0\) for \(\theta < \theta_0\).
Since \(x(\theta) = x(-\theta) = 1\) and \(x(\theta_0) = 0\), \(x(\theta)\) is minimized at \(\theta = \theta_0\) with the minimum
value \(x(\theta_0) = \gamma\).

We can find the power function directly as follows:

\[
x(\theta) = \begin{cases} 
1, & \text{for } \theta < \theta_0 - r - c/2 \\
1 - (1 - 2c^{-1}(\theta_0 - r - \theta))^n/2, & \text{for } \theta_0 - r - c/2 < \theta < \theta_0 + r - c/2 \\
1 - (1 - 2c^{-1}(\theta_0 - r - \theta))^n/2 + (1 - 2c^{-1}(\theta_0 + r - \theta))^n/2, & \text{for } \theta_0 + r - c/2 < \theta < \theta_0 + r \\
1 - (1 - 2c^{-1}(\theta - \theta_0))^n/2, & \text{for } \theta_0 + r - c < \theta < \theta_0 + r + c/2 \\
1, & \text{for } \theta_0 + r + c < \theta.
\end{cases}
\]
Table. The Values of $s^{1/n}$.

<table>
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<th>$s$</th>
<th>.10</th>
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<td>n=4</td>
<td>.56</td>
<td>.47</td>
<td>.32</td>
<td>.18</td>
</tr>
<tr>
<td>5</td>
<td>.63</td>
<td>.55</td>
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<tr>
<td>100</td>
<td>.98</td>
<td>.97</td>
<td>.95</td>
<td>.93</td>
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Here, we remark that for $(s \geq 0.001; n \geq 10)$, $(s \geq 0.01; n \geq 7)$, $(s \geq 0.05; n \geq 5)$ and $(s \geq 0.10; n \geq 4)$, we have that $s^{1/n} \approx 1/2$. (See Table.) So, calculations led to (3.1) depend on $s$ and $n$ such that $s^{1/n} \approx 1/2$.

In the last section we show that our test has the greatest power among size-$s$ tests symmetric about $\theta_0$. To do so we use the generalized Neyman-Pearson lemma.


In this section we shall prove the following theorem:

**Theorem.** Let $X_1, \ldots, X_n$ be a random sample from the p.d.f. $f(x; \theta)$ given by (1.1). The test $*$ given by (2.6) for testing the hypothesis $H_0 : \theta = \theta_0$ versus the alternative hypothesis $H_1 : \theta \neq \theta_0$ has the greatest power among size-$s$ tests symmetric about $\theta_0$.

**Proof.** We first introduce a size-$s$ test $*$ symmetric about $\theta_0$. Using the generalized Neyman-Pearson Lemma we obtain the test $*$ given by (4.8) which has the greatest power. Hence, the test $*$ given by (2.6) will be the best symmetric two-sided test of size-$s$.

Let $X_{(1)}$ be the $i$-th smallest observation such that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. Let $Y$ be as defined by (1.2). The p.d.f. $g_Y(y; \theta)$ of $Y$ is given by (2.1). Let $*$ be a size-$s$ test symmetric about $\theta_0$ (namely, $*(Y) = *(2\theta_0 - y)$, $\forall y$). Then, it follows that

\[ (4.1) \quad E_\theta(*(Y)) = 0 \]

and

\[ (4.2) \quad E_\theta(*(Y - \theta_0)) = \theta_0 * \delta. \]

Above (4.2) holds because $E_\theta((Y - \theta_0)*) = 0$ and (4.1) holds.

Hence, by generalized Neyman-Pearson Lemma $*$ defined by (4.8) maximizes the
integral
\[ \phi(y) = \{ 1, \text{ if } (c-2|y-y'|)^{n-1}I_{(-c/2, c/2)}(y-y') \]
\[ \geq (k_2 + k_1 y)(c-2|y-y_0|)^{n-1}I_{(-c/2, c/2)}(y-y_0) \]
\( (4.3) \)
\[ \phi(y) = \{ 0, \text{ if } (c-2|y-y'|)^{n-1}I_{(-c/2, c/2)}(y-y') \]
\[ \leq (k_2 + k_1 y)(c-2|y-y_0|)^{n-1}I_{(-c/2, c/2)}(y-y_0) \]
\( (4.4) \)

We check existence of such \( k_1 \) and \( k_2 \) and show that the test \( \phi \) is of form \( (4.8) \). We first check existence of such \( k_1 \) and \( k_2 \) until the fifth line below \( (4.7) \).

When \( y' \leq y_0 - c/2 \) or \( y_0 + c/2 \leq y' \), we take \( k_1 = 0 \) and \( k_2 = 1 \). When \( y' \leq y_0 - c \) or \( y_0 + c \leq y' \), the inequality \( (4.3) \) or \( (4.4) \) trivially holds. When \( y_0 - c \leq y' \leq y_0 - c/2 \), the inequality \( (4.4) \) holds for \( (y_0 + y')/2 \leq y_0 + c/2 \) and the inequality \( (4.3) \) holds for \( y' - c/2 \leq y_0 + c \), the inequality \( (4.4) \) holds for \( y_0 - c/2 \leq y_0 + y' \leq y_0 + c/2 \) and the inequality \( (4.3) \) holds for \( (y_0 + y')/2 \leq y_0 + c/2 \).

We now consider the case of \( y_0 - c/2 < y' < y_0 + c/2 \). Let \( y_0 + y' < y_0 + c \). Then, for \( y_0 - c/2 < y < y_0 - c/2 \), the inequality \( (4.4) \) holds when \( k_1 = 0 \) and \( k_2 = 1 \). On the other hand, for \( y_0 + c/2 < y < y_0 + c/2 \), the inequality \( (4.3) \) always holds for any \( k_1 \) and \( k_2 \). Let \( y_0 - c/2 < y' < y_0 \). Then, for \( y_0 - c/2 < y < y_0 - c/2 \), the inequality \( (4.3) \) holds for any \( k_1 \) and \( k_2 \). On the other hand, for \( y_0 + c/2 < y < y_0 + c/2 \), the inequality \( (4.4) \) holds when \( k_1 = 0 \) and \( k_2 = 1 \). Henceforth, it is enough to consider the \( y' \)'s in \( (y_0 - c/2, y_0 + c/2) \) for \( y_0 + y' < y_0 + c/2 \) or in \( (y_0 - c/2, y_0 + c/2) \) for \( y_0 - c/2 < y' < y_0 \). We let

\[ h(y) = (c-2|y-y'|)/(c-2|y-y_0|) \]
\( (\delta 0) \)

and

\[ z(y) = (k_2 + k_1 y)^{n-1} \]
\( (\leq 0) \).

We also let

\( (4.5) \)

\[ y_0 = -k_2/k_1. \]
For $\theta_0 < \theta < \theta_0 + c/2$, take $y_0$ such that $\theta - c/2 < y_0 < \theta_0$. For $\theta_0 - c/2 < \theta < \theta_0$, take $y_0$ such that $\theta_0 < y_0 < \theta + c/2$. Let $p$ be a given number such that $0 < p < 1$. Take $y_0 = (1-p)\theta_0 + p(\theta - c/2)$ for $\theta_0 < \theta < \theta_0 + c/2$ and take $y_0 = (1-p)\theta_0 + p(\theta_0 + c/2)$ for $\theta_0 - c/2 < \theta < \theta_0$. Then, from (4.5), $k_3$ is taken as follows:

$$k_3 = \begin{cases} 
-k_1[(1-p)\theta_0 + p(\theta - c/2)], & \text{for } \theta_0 < \theta < \theta_0 + c/2 \\
-k_1[(1-p)\theta_0 + p(\theta_0 + c/2)], & \text{for } \theta_0 - c/2 < \theta < \theta_0.
\end{cases}$$

Substituting these values into $z(y)$ we have that

$$z(y) = \begin{cases} 
[k_1\{y - ((1-p)\theta_0 + p(\theta - 2^{-1}c))\}]^{n-1}, & \text{for } \theta_0 < \theta < \theta_0 + 2^{-1}c, \\
[k_1\{y - ((1-p)\theta_0 + p(\theta_0 + 2^{-1}c))\}]^{n-1}, & \text{for } \theta_0 - 2^{-1}c < \theta < \theta_0.
\end{cases}$$

which are drawn by stripe lines in the figure below. Since we must accept $H_0$ for $y = \theta_0$, we must have $h(\theta_0) < z(\theta_0)$. We take $k_1 = 2(pc)^{-1}$ for $\theta_0 < \theta < \theta_0 + 2^{-1}c$ and $k_1 = -2(pc)^{-1}$ for $\theta_0 - 2^{-1}c < \theta < \theta_0$. Then, $(z(\theta_0))^{n-1} = h(\theta_0)$. Hence, we have $h(\theta_0) < z(\theta_0)$ because $0 < 1 - 2 \theta_0 - \theta'/c < 1$. $k_3$ is obtained by substituting these values of $k_1$ into (4.6).

To show that the test $\phi$ is of form (4.8) we check the existence of two intersection points of $h(y)$ and $z(y)$. Since for $\theta_0 < \theta < \theta_0 + c/2$

$$h(y) = \begin{cases} 
1 - \{2(y - \theta_0)/(2y - c - 2\theta_0)\}, & \text{for } \theta - c/2 < y \leq \theta_0, \\
1 - \{2(c - (\theta - \theta_0))/(2y - c - 2\theta_0)\}, & \text{for } \theta < y \leq \theta_0, \\
1 - \{2(y - \theta_0)/(2y - c - 2\theta_0)\}, & \text{for } \theta' < y \leq \theta_0 + c/2,
\end{cases}$$

$h(y)$ is an increasing function for $\theta - c/2 < y \leq \theta_0 + c/2$. Since for $\theta_0 - c/2 < \theta < \theta_0$

$$h(y) = \begin{cases} 
1 + \{2(\theta_0 - \theta)/(2y - c - 2\theta_0)\}, & \text{for } \theta_0 - c/2 < y \leq \theta_0, \\
1 + \{2(c - \theta)/(2y - c - 2\theta_0)\}, & \text{for } \theta < y \leq \theta_0, \\
1 + \{2(\theta_0 - \theta)/(2y - c - 2\theta_0)\}, & \text{for } \theta < y \leq \theta_0 + c/2,
\end{cases}$$

$h(y)$ is a decreasing function for $\theta_0 - c/2 < y \leq \theta_0 + c/2$. On the other hand, when $\theta_0 < \theta < \theta_0 + c/2$ $dz(y)/dy > 0$ for all $y < y_0$ and when $\theta_0 - c/2 < \theta < \theta_0$ $dz(y)/dy < 0$ for all $y < y_0$. Since $0 = z(y_0) - h(y_0)(<1)$ and since for $\theta_0 < \theta < \theta_0 + c/2$ $z((\theta_0 + c/2) -) < \lim_{y \to (\theta_0 + c/2) -} h(y) = +\alpha$ and for $\theta_0 - c/2 < \theta < \theta_0$ $z((\theta_0 - c/2) +) < \lim_{y \to (\theta_0 - c/2) +} h(y) = +\alpha$. 


in view of the fact that \( h(\theta_0) \prec z(\theta_0) \) there must exist two intersection points of \( h(y) \) and \( z(y) \) for \( \theta_0 < \theta' < \theta_0 + c/2 \) and for \( \theta_0 - c/2 < \theta' < \theta_0 \), respectively. (See Figure.)

Let \( y_1 \) and \( y_2 \) be such \( y \)-coordinates of these intersection points with \( y_1 < y_2 \). Then, we finally have the optimal test of form

\[
\delta(y) = \begin{cases} 
1, & \text{if } y \leq y_1 \text{ or } y \geq y_2 \\
0, & \text{if } y_1 < y < y_2.
\end{cases}
\]  

(4.8)

Hence, the test provided by (2.6) is size-1 test with the greatest power among size-1 tests symmetric about \( \theta_0 \).

Figure. The graphs of \( h(y) \) and \( z(y) \)

\[(\theta_0 < \theta' < \theta_0 + c/2) \quad \quad (\theta_0 - c/2 < \theta' < \theta_0)\]  

\[\text{(g. e. d.)}\]
REFERENCES.


