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THE JACOBI IDENTITY BEYOND LIE ALGEBRAS

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Abstract
Frölicher and Nijenhuis recognized well in the middle of the previous century that the Lie bracket and its Jacobi identity could and should exist beyond Lie algebras. Nevertheless, the conceptual status of their discovery has been obscured by the genuinely algebraic techniques they exploited. The principal objective in this paper is to show that the double dualization functor in a Cartesian closed category as well as synthetic differential geometry provides an adequate framework, in which their discovery’s conceptual meaning appears lucid. The general Jacobi identity discovered by the author [13] will play a central role.

1. Introduction

Lie groups and their infinitesimal counterparts called Lie algebras were introduced by Norwegian mathematician Sophus Lie in the 19th century. Lie algebras are nonassociative algebras obeying the Jacobi identity instead. It was Frölicher and Nijenhuis (cf. [3] and [12]) in the middle of the preceding century that realized the far-reaching nature of the Lie bracket and its Jacobi identity (i.e., beyond Lie algebras) for the first time. They have shown that tangent-vector-valued differential forms enjoy a kind of Lie bracket, which abides by a sort of the Jacobi identity. Nevertheless, because of the genuinely algebraic techniques they used in

2000 Mathematics Subject Classification: 58A03.

Keywords and phrases: Jacobi identity, synthetic differential geometry, double dualization functor, general Jacobi identity, Schwartz distribution, tangent-vector-valued differential form.

Received June 3, 2009
order to establish their marvelous discovery, the ubiquitous nature of the Lie bracket and its Jacobi identity themselves has remained to be explored.

Now synthetic differential geometry, which is the avant-garde of differential geometry, liberalizes ourselves. In particular, the general Jacobi identity discovered by the author [13] more than a decade ago, which lies behind the Jacobi identity of vector fields on a microlinear space, will play a crucial role in this paper. For standard textbooks on synthetic differential geometry the reader is referred to [6] or [7].

The principal objective in this paper is to show that the double dualization functor in a Cartesian closed category as well as the general Jacobi identity established in synthetic differential geometry provides us with the desired framework. Our approach is completely combinatorial or geometric in sharp contrast to Frölicher and Nijenhuis’ genuinely algebraic approach. After some preliminaries, we present our discovery in the most abstract form in Section 3. This abstract Jacobi identity for the double dualization functor is then specialized in two distinct ways. In Section 4, we specialize the abstract Jacobi identity to tangent-vector-valued differential forms, while we do so for Schwartz distributions in Section 5.

2. Preliminaries

2.1. The double dualization functor

Let $\mathcal{E}$ be a Cartesian closed category. It is well known that Cartesian closed categories and typed $\lambda$-calculi are essentially equivalent, for which the reader is referred to, e.g., Chapter 4 of [1] or Chapter 6 of [2], so that we can speak about $\mathcal{E}$ in terms of typed $\lambda$-calculi. Given two objects $A, B$ in $\mathcal{E}$, we denote by $[A \to B]$ the exponential of $A$ over $B$, which is often written $B^A$. We now fix an object $M$ in $\mathcal{E}$, which gives rise to the double dualization functor assigning $[[A \to M] \to M]$ to each object $A$ in $\mathcal{E}$. Given $f \in [[A \to M] \to M]$ and $g \in [[B \to M] \to M]$, two kinds of convolution of $f$ and $g$, both of which belong to $[[A \times B \to M] \to M]$, are defined, as is familiar in the theory of distributions, to be

$$\lambda h \in [A \times B \to M] \cdot f(\lambda a \in A \cdot g(\lambda b \in B \cdot h(a, b))),$$

$$\lambda h \in [A \times B \to M] \cdot g(\lambda b \in B \cdot f(\lambda a \in A \cdot h(a, b))).$$

The former is denoted by $f \ast g$, while the latter is denoted by $f \ast g$. By identifying
$A \times B$ and $B \times A$ naturally, we can say that $f \ast g$ is no other than $g \ast f$. It should be obvious that

**Lemma 1.** Given $f \in [[A \to M] \to M]$, $g \in [[B \to M] \to M]$ and $h \in [[C \to M] \to M]$, we have

\[
(f \ast g) \ast h = f \ast (g \ast h),
\]

\[
(f \ast g) \ast h = f \ast (g \ast h).
\]

This lemma enables us to write, e.g., $f \ast g \ast h$ without parentheses in place of $(f \ast g) \ast h$ or $f \ast (g \ast h)$.

If $a$ is an element of $A$ (i.e., $a$ is a global section $1 \to A$), then $\lambda f \in [A \to M] \cdot f(a)$ is denoted by $\delta_a$ and, exploiting the terminology in the theory of distributions, is called the *Dirac distribution at $a$*. The following lemma should be obvious.

**Lemma 2.** If one of $f \in [[A \to M] \to M]$ and $g \in [[B \to M] \to M]$ is a Dirac distribution, then $f \ast g$ and $\tilde{g} \ast g$ coincide.

If both $A$ and $B$ are a terminal object $1$, then both $[[A \to M] \to M]$ and $[[B \to M] \to M]$ can naturally be identified with $[M \to M]$, so that the above two convolutions degenerate into the composition of mappings in such a way that

\[
f \ast g = f \circ g,
\]

\[
\tilde{g} \ast g = g \circ f.
\]

### 2.2. Synthetic differential geometry

We assume that the reader is familiar with Lavendhomme’s textbook [7] on synthetic differential geometry up to Chapter 4. From now on our discussion will be done within an adequate universe of synthetic differential geometry, as in Lavendhomme’s textbook [7]. We denote by $D$ the subset of $\mathbb{R}$ (the extended set of real numbers satisfying the generalized Kock-Lawvere axiom so that $\mathbb{R}$ is microlinear) consisting of elements $d$ of $\mathbb{R}$ with $d^2 = 0$. We shall let $M$ and $N$ with or without subscripts denote microlinear spaces in the sense of Definition 1 in Section 2.3 of [7].

Given $\gamma \in [D^p \to M]$, $\alpha \in \mathbb{R}$ and a natural number $i$ with $1 \leq i \leq p$, we
define \( \alpha \cdot \gamma \in [D^p \to M] \) to be
\[
(\alpha \cdot \gamma)(d_1, ..., d_{i-1}, d_i, d_{i+1}, ..., d_p) = \gamma(d_1, ..., d_{i-1}, \alpha d_i, d_{i+1}, ..., d_p)
\]
for any \((d_1, ..., d_{i-1}, d_i, d_{i+1}, ..., d_p) \in D^p\). We write \( \mathbb{S}_p \) for the permutation group of the first \( p \) natural numbers, namely, 1, ..., \( p \). Given \( \sigma \in \mathbb{S}_p \), we denote by \( \varepsilon_\sigma \) its signature. Given \( \gamma \in [D^p \to M] \) and \( \sigma \in \mathbb{S}_p \), we define \( \gamma^\sigma \in [D^p \to M] \) to be
\[
\gamma^\sigma(d_1, ..., d_p) = \gamma(d_{\sigma(1)}, ..., d_{\sigma(p)})
\]
for any \((d_1, ..., d_p) \in D^p\). Given \( \phi \in [[D^p \to M] \to M] \) and \( \alpha \in \mathbb{R} \), we define
\[
\alpha_i \cdot \phi \in [[D^p \to M] \to M] \quad (1 \leq i \leq p)
\]
to be
\[
(\alpha_i \cdot \phi)(\gamma) = \phi(\alpha \cdot \gamma)
\]
for any \( \gamma \in M^{D^p} \). Given \( \phi \in [[D^p \to M] \to M] \) and any \( \sigma \in \mathbb{S}_p \), we define \( \phi^\sigma \in [[D^p \to M] \to M] \) to be
\[
\phi^\sigma(\gamma) = \phi(\gamma^\sigma)
\]
for any \( \gamma \in [D^p \to M] \). Given \( \phi \in [[D^p \to M] \to M] \) and \( \sigma, \tau \in \mathbb{S}_p \), it is easy to see that
\[
\phi^{\sigma \tau}(\gamma) = \phi((\gamma^\sigma)^\tau) = \phi^\tau(\gamma^\sigma) = (\phi^\tau)^\sigma(\gamma)
\]
for any \( \gamma \in [D^p \to M] \), so that \( \phi^{\sigma \tau} = (\phi^\tau)^\sigma \).

2.3. Vector fields

In synthetic differential geometry, vector fields on \( M \) can be viewed in three distinct but equivalent ways, which is based upon the following familiar exponential laws:
\[
[M \to [D \to M]] = [M \times D \to M] = [D \to [M \to M]]
\]
The first viewpoint, which is based upon the first exponential form in the above and is highly orthodox in traditional differential geometry, is to regard a vector field on \( M \) as a section of the canonical projection \([D \to M] \to M\). The second viewpoint, which is based upon the middle exponential form in the above, is to look upon a vector field on \( M \) as an infinitesimal flow on \( M \). The third viewpoint, which is most radical and is based upon the last exponential form in the above, is to speak of a vector field on \( M \) as an infinitesimal transformation of \( M \). For the detailed exposition of these three viewpoints on vector fields and their equivalence, the reader is referred to Section 3.2 of [7].

2.4. The general Jacobi identity

The notion of strong difference \( \overset{\sim}{\cdot} \) was introduced by Kock and Lavendhomme [5] into synthetic differential geometry. The notion of strong difference \( \overset{\sim}{\cdot} \) can be relativized. Since \([D^3 \to M] = [D^2 \to [D \to M]]\), microcubes on \( M \) can be viewed as microsquares on \([D \to M]\). According to which \( D \) in the right-hand side of \( D^3 = D \times D \times D \) appears in the subformula \([D \to M]\) of \([D^2 \to [D \to M]]\), we get the three relativized strong differences \( \overset{\sim}{\cdot}^i \) \((i = 1, 2, 3)\), for which we have the following general Jacobi identity.

**Theorem 3.** Let \( \gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in [D^3 \to M]\). As long as the following three expressions are well defined, they sum up only to vanish:

\[
\left( \gamma_{123} \overset{\sim}{\cdot}^1 \gamma_{132} \right) = \left( \gamma_{231} \overset{\sim}{\cdot}^1 \gamma_{321} \right),
\]

\[
\left( \gamma_{231} \overset{\sim}{\cdot}^2 \gamma_{213} \right) = \left( \gamma_{312} \overset{\sim}{\cdot}^2 \gamma_{132} \right),
\]

\[
\left( \gamma_{312} \overset{\sim}{\cdot}^3 \gamma_{321} \right) = \left( \gamma_{123} \overset{\sim}{\cdot}^3 \gamma_{213} \right).
\]

The theorem was established by the author in [13] and has been reproved twice by himself in [14] and [15], where K. Osoekawa aided the author in computer algebra in the latter paper. The Jacobi identity of vector fields on \( M \) follows from the above theorem at once, as was noted in [13].
3. The Jacobi Identity for the Double Dualization Functor

Let \( n \) be a natural number. Let \( A \) be a space with \( \xi \in A^n \). Let \( \xi(0, \ldots, 0) = \delta_{a_0} \). Let \( B \) be another object in \( \mathcal{E} \) with \( b_0 \in B \). Let \( m \) be a natural number. Given an \( (A, a_0) \)-icon \( \xi_1 \) on \( M \) and a \( (B, b_0) \)-icon \( \xi_2 \) on \( M \), their compositions \( \xi_1 \circ \xi_2 \) and \( \xi_1 \circ \xi_2 \), both of which are \( (A \times B, (a_0, b_0)) \)-(\( m + n \))-icons, are defined to be

\[
(\xi_1 \circ \xi_2)(d_1, d_2) = \xi_1(d_1) \xi_2(d_2),
\]

\[
(\xi_1 \circ \xi_2)(d_1, d_2) = \xi_1(d_1) \circ \xi_2(d_2),
\]

for any \((d_1, d_2) \in D^n \times D^m = D^{m+n}\). In particular, if \( m = n = 1 \), then, by Lemma 2, we have \( \xi_1 \circ \xi_2 |_{D(2)} = \xi_1 \circ \xi_2 |_{D(2)} \), so that we can define their strong difference, called their \textit{Lie bracket} and denoted by \([\xi_1, \xi_2]\), to be

\[
[\xi_1, \xi_2] = \xi_1 \circ \xi_2 - \xi_2 \circ \xi_1.
\]

Let \( A \) and \( B \) be objects in \( \mathcal{E} \) with \( a_0 \in A \) and \( b_0 \in B \).

We will show that the bracket \([\cdot, \cdot]\) is antisymmetric.

**Theorem 4.** Let \( A \) and \( B \) be spaces with \( a_0 \in A \) and \( b_0 \in B \). Let \( \xi_1 \) be an \( (A, a_0) \)-icon on \( M \) and \( \xi_2 \) be a \( (B, b_0) \)-icon on \( M \). Then we have the following antisymmetry:

\[
[\xi_1, \xi_2] + [\xi_2, \xi_1] = 0.
\]

**Remark 5.** Since \( A \times B \) and \( B \times A \) can naturally be identified, not only \([o_1, o_2]\) but also \([o_2, o_1]\) is to be regarded as a mapping \( D \rightarrow [(A \times B) \rightarrow M] \rightarrow M \). The reader should be aware that the permutation sigma, which shall be omitted intentionally for notational simplicity in this section, should be inserted in the identification of \( A \times B \) and \( B \times A \) from a very strict viewpoint. This comment should be recalled in the next section.
Proof. This follows from Propositions 4 and 6 in Section 3.4 of Lavendhomme [7]. More specifically, we have

\[
\begin{align*}
[\xi_1, \xi_2] + [\xi_2, \xi_1] &= (\xi_1 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_2 - \xi_1 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_2) + (\xi_2 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_1 - \xi_2 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_1) \\
&= (\xi_1 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_2 - \xi_1 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_2) + (\xi_2 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_1 - \xi_2 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_2).
\end{align*}
\]

[By Proposition 6 in Section 3.4 of Lavendhomme [7],

since \(\xi_2 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_1\) can be identified with

\((d_1, d_2) \in D^2 \mapsto \xi_1(d_2) \ast \xi_2(d_1) \in [[A \times B, M], M],\) and

similarly for \(\xi_2 \ast \xi_1\)]

\[= 0\]

[By Proposition 4 in Section 3.4 of Lavendhomme [7]].

\[\square\]

Theorem 6. Let \(A, B\) and \(C\) be objects in \(\mathcal{E}\) with \(a_0 \in A, b_0 \in B\) and \(c_0 \in C\). Let \(\xi_1\) be an \((A, a_0)\)-1-icon on \(M\), \(\xi_2\) be a \((B, b_0)\)-1-icon on \(M\), and \(\xi_3\) be a \((C, c_0)\)-1-icon on \(M\). Then we have the following Jacobi identity:

\[
[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] = 0.
\]

Remark 7. As in Theorem 4, not only \([\xi_1, [\xi_2, \xi_3]]\) but also both \([\xi_2, [\xi_3, \xi_1]]\) and \([\xi_3, [\xi_1, \xi_2]]\) are to be regarded as mappings \(D \to [[A \times B \times C \to M] \to M]\).

In order to establish this theorem, we need the following simple lemma, which is a tiny generalization of Proposition 2.6 of [13].

Lemma 8. Let \(\xi\) be an \((A, a_0)\)-1-icon on \(M\), and \(\xi_1\) and \(\xi_2\) be \((B, b_0)\)-2-icons on \(M\) with \(\xi_1|_{D(2)} = \xi_2|_{D(2)}\). Then the following formulas are both meaningful and valid:

\[
\xi \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_1 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_2 = \xi \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \left(\xi_1 \ast \xi_2\right),
\]

\[
\xi \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_1 \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \xi_2 = \xi \underset{\raisebox{-3pt}{\scriptsize \text{\textsuperscript{\(\circ\)}}}}{\circ} \left(\xi_1 \ast \xi_2\right).
\]
\[ \xi \circledast \xi_1 \overset{\gamma}{\rightarrow} \xi \circledast \xi_2 = \xi \circledast \left( \xi_1 \overset{\gamma}{\rightarrow} \xi_2 \right), \]

\[ \xi_1 \circledast \xi \overset{\gamma}{\rightarrow} \xi_2 \circledast \xi = \left( \xi_1 \overset{\gamma}{\rightarrow} \xi_2 \right) \circledast \xi, \]

\[ \xi_1 \overset{\gamma}{\rightarrow} \xi \overset{\gamma}{\rightarrow} \xi_2 \circledast \xi = \left( \xi_1 \overset{\gamma}{\rightarrow} \xi_2 \right) \circledast \xi. \]

**Proof of Theorem 6.** Our present discussion is a tiny generalization of Proposition 2.7 in [13]. We define six \((A \times B \times C, (a_0, b_0, c_0))-3\)-icons on \(M\) as follows:

\[ \xi_{123} = \xi_1 \circledast \xi_2 \circledast \xi_3, \]

\[ \xi_{132} = \xi_1 \circledast (\xi_2 \circledast \xi_3), \]

\[ \xi_{213} = (\xi_1 \circledast \xi_2) \circledast \xi_3, \]

\[ \xi_{231} = \xi_1 \overset{\gamma}{\rightarrow} (\xi_2 \circledast \xi_3), \]

\[ \xi_{312} = (\xi_1 \circledast \xi_2) \overset{\gamma}{\rightarrow} \xi_3, \]

\[ \xi_{321} = \xi_1 \overset{\gamma}{\rightarrow} \xi_2 \overset{\gamma}{\rightarrow} \xi_3. \]

Then it is easy, by dint of Lemma 8, to see that

\[ \left[ \xi_1, \left[ \xi_2, \xi_3 \right] \right] = \left( \xi_{123} \overset{\gamma}{\rightarrow} \xi_{132} \right) - \left( \xi_{231} \overset{\gamma}{\rightarrow} \xi_{321} \right), \quad (1) \]

\[ \left[ \xi_2, \left[ \xi_3, \xi_1 \right] \right] = \left( \xi_{231} \overset{\gamma}{\rightarrow} \xi_{213} \right) - \left( \xi_{312} \overset{\gamma}{\rightarrow} \xi_{132} \right), \quad (2) \]

\[ \left[ \xi_3, \left[ \xi_1, \xi_2 \right] \right] = \left( \xi_{312} \overset{\gamma}{\rightarrow} \xi_{321} \right) - \left( \xi_{123} \overset{\gamma}{\rightarrow} \xi_{213} \right). \quad (3) \]

Therefore, the desired Jacobi identity follows directly from the general Jacobi identity.

**Remark 9.** In order to see that the right-hand side of (1) is meaningful, we have to check that all of

\[ \xi_{123} \overset{\gamma}{\rightarrow} \xi_{132}. \]
The Jacobi Identity Beyond Lie Algebras

\[ \xi_{231} \overset{T}{\rightarrow} T \xi_{321}. \]

\[ \left( \xi_{123} \overset{T}{\rightarrow} T \xi_{312} \right) \overset{\rightarrow}{\rightarrow} \left( \xi_{231} \overset{T}{\rightarrow} T \xi_{321} \right) \]

are meaningful. Since \( \xi_2 \otimes \xi_3 \overset{T}{\rightarrow} T \xi_2 \otimes \xi_3 \) is meaningful by Lemma 2, \( \xi_{123} \overset{T}{\rightarrow} T \xi_{132} \)

is also meaningful and we have

\[ \xi_{123} \overset{T}{\rightarrow} T \xi_{312} = \xi_1 \otimes \left( \xi_2 \otimes \xi_3 \overset{T}{\rightarrow} T \xi_2 \otimes \xi_3 \right), \]

by Lemma 8. Similarly \( \xi_{231} \overset{T}{\rightarrow} T \xi_{321} \) is meaningful and we have

\[ \xi_{231} \overset{T}{\rightarrow} T \xi_{321} = \xi_1 \otimes \left( \xi_2 \otimes \xi_3 \overset{T}{\rightarrow} T \xi_2 \otimes \xi_3 \right). \]

Therefore, \( \left( \xi_{123} \overset{T}{\rightarrow} T \xi_{312} \right) \overset{\rightarrow}{\rightarrow} \left( \xi_{231} \overset{T}{\rightarrow} T \xi_{321} \right) \)

is meaningful by Lemma 2. Similar considerations apply to (2) and (3).

4. The Jacobi Identity for Tangent-vector-valued Differential Forms

Our three distinct but equivalent viewpoints of tangent-vector-valued differential forms on \( M \) are based upon the following exponential laws:

\[
\begin{align*}
[[D^p \rightarrow M] \rightarrow [D \rightarrow M]] &\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow = \left[ [D^p \rightarrow M] \times D \rightarrow M \right] \\
&\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow = [D \rightarrow [[D^p \rightarrow M] \rightarrow M]].
\end{align*}
\]

If \( p = 0 \), then the above laws degenerate into the corresponding ones in Subsection 2.3.

The first viewpoint, which is highly orthodox, is to regard \([D \rightarrow M]-valued p\)-forms on \( M \) as mappings \( \omega : [D^p \rightarrow M] \rightarrow [D \rightarrow M] \) with \( \gamma(0, ..., 0) = \omega(\gamma)(0) \), for any \( \gamma \in [D^p \rightarrow M] \) and satisfying the \( p \)-homogeneity and the alternating property in the sense of Definition 1 in Section 4.1 of Lavendhomme [7]. By
dropping the alternating property, we get the weaker notion of a \([D \to M]\)-valued p-semiform on \(M\).

The second viewpoint goes as follows:

**Proposition 10.** \([D \to M]\)-valued p-forms on \(M\) can be identified with mappings \(\omega : [D^p \to M] \times D \to M\) pursuant to the following conditions:

1. \(\omega(\gamma, 0) = \gamma(0, ..., 0)\) for any \(\gamma \in [D^p \to M]\).

2. \(\omega(\gamma, ad) = \omega(\alpha \cdot \gamma, d)\) for any \(d \in D\), any \(\alpha \in \mathbb{R}\), any \(\gamma \in [D^p \to M]\) and any natural number \(i\) with \(1 \leq i \leq p\).

3. \(\omega(\gamma^\sigma, d) = \omega(\gamma, e_\sigma d)\) for any \(d \in D\), any \(\gamma \in [D^p \to M]\) and any \(\sigma \in \mathbb{S}_p\).

By dropping the third condition, we get entities corresponding to \([D \to M]\)-valued p-semiforms on \(M\).

The third viewpoint, which is most radical, goes as follows:

**Proposition 11.** \([D \to M]\)-valued p-forms on \(M\) can be identified with mappings \(\omega : D \to [[D^p \to M] \to M]\) satisfying the following conditions:

1. \(\omega(0) = \delta_{(0, ..., 0)}\).

2. \(\alpha \cdot \omega(d) = \omega(\alpha d)\) for any \(d \in D\), any \(\alpha \in \mathbb{R}\) and any natural number \(i\) with \(1 \leq i \leq p\).

3. \((\omega(d))^\sigma = \omega(e_\sigma d)\) for any \(d \in D\) and any \(\sigma \in \mathbb{S}_p\).

By dropping the third condition, we get entities corresponding to \([D \to M]\)-valued p-semiforms on \(M\).

**Remark 12.** By dropping the second and third conditions, we find the notion of \((D^p, (0, ..., 0))\)-1-icon on \(M\).

The following proposition is simple but very important.
Proposition 13. The addition for $[D \to M]$-valued $p$-semiforms on $M$ in the first sense (i.e., using the fiberwise addition of the vector bundle $[D \to M] \to M$) and that in the third sense (i.e., as the addition of tangent vectors to the microlinear space $[[D^p \to M] \to M]$ at $\delta_{(0,\ldots,0)}$) coincide.

Proof. This follows mainly from the exponential law
\[
[[D^p \to M] \to [D(2) \to M]] = [D(2) \to [[D^p \to M] \to M]].
\]
The details can safely be left to the reader. \qed

Unless stated to the contrary, we will use the terms $[D \to M]$-valued $p$-semiforms on $M$ and $[D \to M]$-valued $p$-forms on $M$ in the third sense.

The following lemma should be obvious.

Lemma 14. If $\omega_1$ is a $[D \to M]$-valued $p$-semiform on $M$ and $\omega_2$ is a $[D \to M]$-valued $q$-semiform on $M$, then we have
\[
\alpha_i ((\omega_1 \oplus \omega_2)(d_1, d_2)) = (\omega_1 \oplus \omega_2)(\alpha d_1, d_2),
\]
for any $(d_1, d_2) \in D^2$ and natural number $i$ with $1 \leq i \leq p$, while we have
\[
\alpha_i ((\omega_1 \oplus \omega_2)(d_1, d_2)) = (\omega_1 \oplus \omega_2)(d_1, \alpha d_2),
\]
for any $(d_1, d_2) \in D^2$ and any natural number $i$ with $p + 1 \leq i \leq p + q$.

Corollary 15. If $\omega_1$ is a $[D \to M]$-valued $p$-semiform on $M$ and $\omega_2$ is a $[D \to M]$-valued $q$-semiform on $M$, then $[\omega_1, \omega_2]$ is a $[D \to M]$-valued $(p + q)$-semiform on $M$.

Proof. It suffices to see that
\[
\alpha_i ([\omega_1, \omega_2](d)) = [\omega_1, \omega_2](\alpha d),
\]
for any \( d \in D \), any \( \alpha \in \mathbb{R} \) and any natural number \( i \) with \( 1 \leq i \leq p + q \), which follows easily from the above lemma and Proposition 5 in Section 3.4 of Lavendhomme [7]. □

Given a \([D \to M]\)-valued \( p \)-semiform \( \omega \) on \( M \) and \( \sigma \in \mathbb{S}_p \), we define a \([D \to M]\)-valued \( p \)-semiform \( \omega^\sigma \) to be

\[
\omega^\sigma(d) = \omega(d)^\sigma.
\]

Now we are ready to state that

**Theorem 16.** If \( \omega_1 \) is a \([D \to M]\)-valued \( p \)-semiform on \( M \) and \( \omega_2 \) is a \([D \to M]\)-valued \( q \)-semiform on \( M \), then we have

\[
[\omega_1, \omega_2] = -[\omega_2, \omega_1]^p,
\]

where \( \rho \) is the permutation mapping of the sequence \( 1, \ldots, q, q + 1, \ldots, p + q \) to the sequence \( q + 1, \ldots, p + q, 1, \ldots, q \).

**Proof.** This follows simply from Theorem 4. □

**Theorem 17.** If \( \omega_1 \) is a \([D \to M]\)-valued \( p \)-semiform on \( M \), \( \omega_2 \) is a \([D \to M]\)-valued \( q \)-semiform on \( M \) and \( \omega_3 \) is a \([D \to M]\)-valued \( r \)-semiform on \( M \), then the following Jacobi identity holds for the three \([D \to M]\)-valued \((p + q + r)\)-semiforms on \( M \):

\[
[\omega_1, [\omega_2, \omega_3]] + [\omega_2, [\omega_3, \omega_1]] + [\omega_3, [\omega_1, \omega_2]] = 0,
\]

where \( \rho_1 \) and \( \rho_2 \) are the following permutations:

\[
\rho_1 = \begin{pmatrix}
1 & \cdots & q & q + 1 & \cdots & q + r & q + r + 1 & \cdots & p + q + r \\
p + 1 & \cdots & p + q & p + q + 1 & \cdots & p + q + r & 1 & \cdots & p
\end{pmatrix},
\]

\[
\rho_2 = \begin{pmatrix}
1 & \cdots & r & r + 1 & \cdots & p + r & p + r + 1 & \cdots & p + q + r \\
p + q + 1 & \cdots & p + q + r & 1 & \cdots & p & p + 1 & \cdots & p + q
\end{pmatrix}.
\]

**Proof.** The desired Jacobi identity is a direct consequence of Theorem 6. □

Now we turn to forms. Given a \([D \to M]\)-valued \( p \)-semiform \( \omega \) on \( M \) and
σ ∈ S_p, we define a \([D \to M]\)-valued \(p\)-semiform \(\omega^\sigma\) on \(M\) to be

\[
\omega^\sigma(d) = \omega(d)^\sigma.
\]

Given a \([D \to M]\)-valued \(p\)-semiform \(\omega\) on \(M\), we define a \([D \to M]\)-valued \(p\)-semiform \(A\omega\) on \(M\) to be

\[
A\omega = \sum_{\sigma \in S_p} e_\sigma \omega^\sigma.
\]

We write \(A_{p,q}\omega\) for \((\lfloor p!q! \rfloor)A\omega\) in case that \(\omega\) is a \([D \to M]\)-valued \((p + q)\)-semiform on \(M\). We write \(A_{p,q,r}\omega\) for \((\lfloor p!q!r! \rfloor)A\omega\) in case that \(\omega\) is a \([D \to M]\)-valued \((p + q + r)\)-semiform on \(M\).

Given a \([D \to M]\)-valued \(p\)-form \(\omega_1\) on \(M\) and a \([D \to M]\)-valued \(q\)-form \(\omega_2\) on \(M\), we are going to define their Frölicher-Nijenhuis bracket \([\omega_1, \omega_2]\) to be

\[
[\omega_1, \omega_2] = A_{p,q}([\omega_1, \omega_2]),
\]

which is undoubtedly a \([D \to M]\)-valued \((p + q)\)-form on \(M\).

**Theorem 18.** If \(\omega_1\) is a \([D \to M]\)-valued \(p\)-form on \(M\) and \(\omega_2\) is a \([D \to M]\)-valued \(q\)-form on \(M\), then we have

\[
[\omega_1, \omega_2] = -(-1)^{pq}[\omega_2, \omega_1].
\]

**Proof.** We have

\[
[\omega_1, \omega_2] = A_{p,q}([\omega_1, \omega_2])
\]

\[
= -A_{p,q}(\omega_2, \omega_1)^p \quad \text{[By Theorem 16]}
\]

\[
= -\frac{1}{p!q!} \sum_{\tau \in S_{p+q}} e_\tau (\omega_2, \omega_1)^p
\]

\[
= -\frac{1}{p!q!} \sum_{\tau \in S_{p+q}} e_\tau [\omega_2, \omega_1]^p
\]
\[
= -\frac{1}{p!q!}\varepsilon_p \sum_{\tau \in \Sigma_{p+q}} \varepsilon_{\tau \rho} [\omega_2, \omega_1]^{\tau \rho}
= -\varepsilon_{\rho} [\omega_2, \omega_1].
\]

Since \( \varepsilon_{\rho} = (-1)^{pq} \), the desired conclusion follows. \(\square\)

**Lemma 19.** If \( \omega_1 \) is a \( [D \to M] \)-valued \( p \)-form on \( M \), \( \omega_2 \) is a \( [D \to M] \)-valued \( q \)-form on \( M \) and \( \omega_3 \) is a \( [D \to M] \)-valued \( r \)-form on \( M \), then we have

\[
A_{p, q+r}([\omega_1, A_{q, r}([\omega_2, \omega_3])]) = A_{p, q+r}([\omega_1, [\omega_2, \omega_3]]).
\]

**Proof.** By the same token as in the familiar associativity of wedge products in differential forms. \(\square\)

**Theorem 20.** If \( \omega_1 \) is a \( [D \to M] \)-valued \( p \)-form on \( M \), \( \omega_2 \) is a \( [D \to M] \)-valued \( q \)-form on \( M \) and \( \omega_3 \) is a \( [D \to M] \)-valued \( r \)-form on \( M \), then the following graded Jacobi identity holds for the three \( [D \to M] \)-valued \( (p + q + r) \)-forms on \( M \):

\[
[\omega_1, [\omega_2, \omega_3]] + (-1)^{p(q+r)}[\omega_2, [\omega_3, \omega_1]] + (-1)^{q(p+q)}[\omega_3, [\omega_1, \omega_2]] = 0.
\]

**Proof.** By the same token as in the proof of Theorem 18. This follows mainly from Theorem 17 with the help of Lemma 19 and the simple fact that \( \varepsilon_{\rho_1} = (-1)^{p(q+r)} \) and \( \varepsilon_{\rho_2} = (-1)^{q(p+q)} \). The details can safely be left to the reader. \(\square\)

5. The Jacobi Identity for Schwartz Distributions

By a *distribution with compact support on \( M \)* we mean a mapping \( u : [M \to \mathbb{R}] \to \mathbb{R} \) with the property that

\[
u(\alpha f) = \alpha u(f)
\]

for any \( f \in [M \to \mathbb{R}] \) and any \( \alpha \in \mathbb{R} \). Let us suppose that we are given \( x \in M \).

By a *Dirac \( x \)-flow on \( M \)*, we mean a \( (M, x) \)-1-icon \( \xi \) on \( \mathbb{R} \) with the property that \( \xi(d) \) is a distribution with compact support on \( M \) for any \( d \in D \).
Lemma 21. If \( u \) is a distribution with compact support on \( M \) and \( v \) is a distribution with compact support on \( N \), then \( u \ast v \) as well as \( \overset{\sim}{u} \ast v \) is a distribution with compact support on \( M \times N \).

**Proof.** We note that, given \( \alpha \in \mathbb{R} \) and \( h \in [M \times N \rightarrow \mathbb{R}] \), we have

\[
(u \ast v)(\alpha h) = u(\lambda x \in M \cdot v(\lambda y \in N \cdot \alpha h(x, y)))
\]

\[
= u(\lambda x \in M \cdot v(\alpha(\lambda y \in N \cdot h(x, y))))
\]

\[
= \alpha(u(\lambda x \in M \cdot v(\lambda y \in N \cdot h(x, y))))
\]

\[
= \alpha(u \ast v)(h)
\]

so that \( u \ast v \) is a distribution with compact support on \( M \times N \). Similarly for \( \overset{\sim}{u} \ast v \).

\[\square\]

**Proposition 22.** If \( \xi_1 \) is a Dirac \( x \)-flow on \( M \) and \( \xi_2 \) is a Dirac \( y \)-flow on \( N \), then \( [\xi_1, \xi_2] \) is a Dirac \((x, y)\)-flow on \( M \times N \).

**Proof.** It suffices to note that the space of distributions with compact support on \( M \) forms a microlinear space, from which the desired result follows from the above lemma.

\[\square\]

**Theorem 23.** If \( \xi_1 \) is a Dirac \( x_1 \)-flow on \( M_1 \), \( \xi_2 \) is a Dirac \( x_2 \)-flow on \( M_2 \) and \( \xi_3 \) is a Dirac \( x_3 \)-flow on \( M_3 \), then we have

\[
[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] = 0.
\]

**Proof.** This is a direct consequence of Theorem 6.

\[\square\]

**Acknowledgement**

I gladly acknowledge my indebtedness to Professor Anders Kock (Aarhus University), who kindly helped me pay due attention to the double dualization functor in a Cartesian closed category. His sincere and detailed advice has improved the previous paper [16] considerably.
References


