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ON D-PARACOMPACT $p$- AND $\Sigma$-SPACES

By
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1. Introduction
All spaces are assumed to be $T_1$ topological spaces and all mappings to be continuous and onto. The letter $N$ always denotes all positive integers and $\tau_X$ the topology of a space $X$.

As well known as Dowker's Theorem, a $T_2$-space $X$ is paracompact if and only if for each open cover $\mathcal{U}$ of $X$ there exists a $\mathcal{U}$-mapping $f$ of $X$ onto a metric space $M$, where a mapping $f$ is called a $\mathcal{U}$-mapping if there exists an open cover $\mathcal{V}$ of $M$ such that $f^{-1}(\mathcal{V}) < \mathcal{U}$. Taking into account that developable spaces is one of the nicest generalizations of metric spaces, it is quite natural to substitute a metric space $M$ in the above with a developable space $D$ in order to get a generalization of both paracompact spaces and developable spaces.

DEFINITION 1.1 [12]. A space $X$ is called a $D$-paracompact if for each open cover $\mathcal{U}$ of $X$ there exists a $\mathcal{U}$-mapping of $X$ onto a developable spaces.

Pareek originally gave its inner characterization to $D$-paracompact spaces [12]. Besides many inner characterizations are given by Brandenburg [1], Chaber [6] and Mizokami [9]. As for the overview of $D$-paracompact spaces, refer to [2]. In this paper, we consider the mapping properties of $D$-paracompact spaces on the classes of $D$-paracompact $p$-spaces and $D$-paracompact $\Sigma$-spaces.

2. $D$-paracompact $p$-spaces
With respect to the mapping property of $D$-paracompact spaces, the following problem remains unsolved.

PROBLEM [1], [6]. Let $f : X \rightarrow Y$ be a perfect mapping of a $D$-paracompact space onto a space $Y$. Then is $Y$ $D$-paracompact?

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Let us note that $D$-paracompactness is preserved by neither of perfect preimages and closed images. The former is due to [6, Example 3.3] and the latter due to [9, Example 3]. But we have the following positive partial answers given by Chaber [6] and by Mizokami [9]: Let $\mathcal{C}$ be a class of spaces such that $\mathcal{C} \subset \{D$-paracompact spaces$\}$. Then $\mathcal{C}$ is closed under perfect images when $\mathcal{C}$ is either of the class of $D$-paracompact $p$-spaces [6] of $D$-paracompact $\sigma$-spaces [9]. According to his definition there [6], a space $X$ is a $D$-paracompact $p$-space if and only if for any open cover $\mathcal{U}$ of $X$ there exists a perfect $\mathcal{U}$-mapping of $X$ onto a Moore space, that is a regular developable space. Originally, $p$-spaces are defined for completely regular spaces by Arhangel'skii as follows: A completely regular space $X$ is a $p$-space if $X$ has a sequence $\{\mathcal{U}_n \mid n \in N\}$ of open covers of $X$ in $\beta X$ such that $\bigcap \{S(x, \mathcal{U}_n) \mid n \in N\} \subset X$ for each $x \in X$. A few inner characterizations are given by Burke [4], Burke and Stoltenberg [5] and Pareek [13]. But, as observed in Remark and the part preceding to Theorem 3.16 in [8, p. 442], since the Stone–Čech compactification $\beta X$ can be changed by any compactification of $X$, their discussions are applicable to regular spaces. In this sense, we consider here $p$-spaces, strict $p$-spaces, Pareek's $p$-spaces for regular spaces. Pareek gave the definition of $p$-spaces in his paper and showed the equivalence of (iv) and (v) below [12, Theorem 4.4]. But this was criticized to be based on a dubious lemma by Mack [1974, Math. Reviews 47 ( #1034)]. Here, we can show the equivalence by a different way.

**Theorem 2.1.** For a regular space $X$, the following are equivalent:

(i) $X$ is a $D$-paracompact w$\Delta$-space.

(ii) $X$ is a $D$-paracompact $p$-space in the sense of Burke [4]. (Refer to [8, Theorem 3.21]).

(iii) $X$ is a $D$-paracompact strict $p$-space in the sense of Burke and Stoltenberg [5]. (Refer to [8, Theorem 3.17]).

(iv) $X$ is a $D$-paracompact $p$-space in the sense of Pareek [12, Definition 4.6].

(v) For any open cover $\mathcal{U}$ of $X$, there exists a perfect $\mathcal{U}$-mapping of $X$ onto a Moore space.

(vi) $X$ is a $D$-paracompact space and has a perfect mapping of $X$ onto a Moore space.

**Proof.** Since $D$-paracompact spaces are submetacompact, the arguments of [8, Theorem 3.19 and 3.21] can apply to get the equivalence of (i), (ii) and (iii). If we again note the remark in [8, p. 442], the discussion of [13] holds true for regular spaces, so that we have the equivalence of (iv) and (iii). (iii) $\rightarrow$ (v): Let $\mathcal{U}$
be an open cover of $X$ and let $\{ \mathcal{G}_n : n \in \mathbb{N} \}$ be a strict $p$-sequence for $X$ satisfying the following:

1. $C_x = \bigcap \{ S(x, \mathcal{G}_n) : n \in \mathbb{N} \}$ is compact.
2. $\{ S(x, \mathcal{G}_n) : n \in \mathbb{N} \}$ is an open neighborhood base of $C_x$ in $X$.

Since $X$ is regular and $D$-paracompact, for some open cover $\tau_1$ of $X$ such that $\tau_1 < \mathcal{G}_1 \wedge \mathcal{U}$, there exists a $\tau_1$-mapping $f_1$ of $X$ onto a developable space $D_1$. Without loss of generality, we can assume that $D_1$ has a decreasing development $\{ \mathcal{A}_{1n} : n \in \mathbb{N} \}$ such that $f_1^{-1}(\mathcal{A}_{11}) < \tau_1$. By regularity of $X$, there exists an open cover $\tau_2$ of $X$ such that

$$\tau_2 < \mathcal{G}_2 \wedge f_1^{-1}(\mathcal{A}_{12}) \wedge \mathcal{U}.$$ 

Using $D$-paracompactness of $X$ again, there exists a $\tau_2$-mapping $f_2$ of $X$ onto a developable space $D_2$ which has a decreasing development $\{ \mathcal{A}_{2n} : n \in \mathbb{N} \}$ such that $f_2^{-1}(\mathcal{A}_{21}) < \tau_2$. Repeating this process, we can get sequences $\{ \tau_n : n \in \mathbb{N} \}$, $\{ \mathcal{A}_{in} : i \in \mathbb{N} \}$, $\{ f_n : n \in \mathbb{N} \}$ and $\{ D_n : n \in \mathbb{N} \}$ satisfying the following:

3. $D_n$ has a decreasing development $\{ \mathcal{A}_{nk} : k \in \mathbb{N} \}$ such that $f_n^{-1}(\mathcal{A}_{n1}) < \tau_n$.
4. For each $n$, $f_n$ is a $\tau_n$-mapping of $X$ onto $D_n$.
5. $\tau_n$ is an open cover of $X$ such that

$$\tau_n < \mathcal{G}_n \wedge \left( \bigwedge_{i=1}^{n-1} f_i^{-1}(\mathcal{A}_{in}) \right) \wedge \mathcal{U} \quad \text{for} \quad n \geq 2.$$ 

Let $f = \prod f_i : X \to \prod D_i$ be defined by $f(x) = (f_i(x))_i$, $x \in X$. Then it is easily seen from (4) and (5) that $f$ is a $\mathcal{U}$-mapping of $X$ onto a developable space $D = f(X) \subset \prod D_n$. We show that $f$ is a perfect mapping, and consequently $D$ is a Moore space. For each $p \in D$, by virtue of (3) and (5) we have

$$f^{-1}(p) \subset \bigcap_n S(x, \mathcal{G}_n),$$

where $x \in f^{-1}(p)$. So, because of (1), $f^{-1}(p)$ is compact. To see the closedness of $f$, it suffices to show that for each point $p = (p_i)_i \in D$ and each open subset $U$ of $X$ such that $f^{-1}(p) \subset U$, there exists a neighborhood $V$ of $p$ in $D$ such that $f^{-1}(V) \subset U$. Let

$$C_x = \bigcap_n S(x, \mathcal{G}_n), \quad x \in f^{-1}(p).$$

We can easily observe by virtue of (1) that $f(C_x \setminus U)$ is a compact subset of $D$ and $p \notin f(C_x \setminus U)$. Take a neighborhood $G$ of $p$ in $D$ such that
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\[ G = \left( \prod_{i=1}^{k} S(p_{n(i)}, A_{n(i)m(i)}) \times \prod \{ D_t : t \neq n(i) \} \right) \cap D \]

\[ G \cap f(C_x \setminus U) = \emptyset. \]

By virtue of (3), (4) and (5), we can find some \( n(0) \in N \) such that

\[ f_{n(0)}^{-1}(S(p_{n(0)}, A_{n(0)1})) \cap (C_x \setminus U) = \emptyset. \]

Set

\[ O = X \setminus (f_{n(0)}^{-1}(S(p_{n(0)}, A_{n(0)1})) \setminus U). \]

Then \( O \) is an open neighborhood of \( C_x \). By virtue of (2), there exists \( s \in N \) such that

\[ C_s \subset S(x, A_s) \subset O. \]

Using all of (3) through (7), we can find some \( t \in N \) such that

\[ V = \left( S(p_t, A_{ti}) \times \prod \{ D_n : n \neq t \} \right) \cap D \]

is an open neighborhood of \( p \) in \( D \) such that \( f^{-1}(V) \subset U \). Hence \( f \) is a perfect mapping. Since \( (vi) \rightarrow (i) \) is trivial, we have completed the proof. \( \square \)

Let us note that in most cases, \( D \)-paracompact \( p \)-spaces go parallel to paracompact \( p \)-spaces. For example, the following theorem on making the space Moore corresponds to the metrization theorem of paracompact \( p \)-spaces.

**THEOREM 2.2.** A regular \( D \)-paracompact \( p \)-space \( X \) is a Moore space if and only if \( X \) has a \( G_{\delta} \)-diagonal.

**PROOF.** Only if part is trivial. If part: Let \( \{ U_n : n \in N \} \) be a sequence of open covers of \( X \) such that \( \bigcap_n S(p, U_n) = \{ p \} \) for each point \( p \in X \). By the above theorem, for each \( n \) there exists a perfect \( U_n \)-mapping \( f_n \) of \( X \) onto a Moore space \( D_n \). Let \( f : X \rightarrow \prod_n D_n \) be defined by

\[ f(x) = (f_n(x)), \quad x \in X. \]

Then easily we can observe that \( f \) is a homeomorphism of \( X \) onto \( f(X) \subset \prod_n D_n \). Since Moore spaces have countably productive and hereditary properties, \( f(X) \) is a Moore space. This completes the proof. \( \square \)

Nagata characterized a paracompact \( p \)-space as a space which is embedded in the closed subspace of the product of a metrizable space and a compact space
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[11]. But this type of characterization does not work for $D$-paracompact $p$-spaces stated below:

**Theorem 2.3.** A regular $D$-paracompact $p$-space is embedded in a closed subspace of the product of a Moore space and a compact space. But the converse is not true.

**Proof.** The former is straightforward from [8, Lemma 3.13] and Theorem 2.1. For the latter, it suffices to consider the product space of a Moore space $S = N \cup \mathcal{A}$ and a compact space $Z = A(\mathbb{N}_1)$ for which $S \times Z$ is not $D$-paracompact [6, Example 3.3].

3. $D$-paracompact $\Sigma$-spaces

As stated above, $D$-paracompact $p$-spaces and $D$-paracompact $\sigma$-spaces are preserved by perfect mappings. Both are $\Sigma$-spaces in the sense of Nagami. So it is quite natural to ask whether $D$-paracompact $\Sigma$-spaces are preserved by perfect mappings. In this section, we give the positive answer to it. Here, we use the definition of $\Sigma$-spaces due to Michael, which is equivalent to the original one due to Nagami.

**Definition 3.1** [8, Definition 4.13]. A regular space $X$ is called a (strong) $\Sigma$-space if $X$ has a cover $\mathcal{G}$ by (resp. compact) countably compact subsets and has a $\sigma$-locally finite family $\mathcal{F}$ of closed subsets of $X$ such that for $C \in \mathcal{G}$ and $U \in \tau_X$, if $C \subset U$, then $C \subset F \subset U$ for some $F \in \mathcal{F}$.

Since $D$-paracompact space is subparacompact, a $D$-paracompact $\Sigma$-space is a strong $\Sigma$-space. We state the terminology used in the proof. We call $\mathcal{P}$ a pair-collection of a space $X$ if $\mathcal{P}$ is a collection of ordered pairs $P = (P_1, P_2)$ of subsets of $X$ such that $P_1 \subset P_2$ and $P_1$, $P_2$ are closed, open in $X$, respectively. We call $\mathcal{P}$ discrete, locally finite, $\sigma$-discrete or $\sigma$-locally finite in $X$ if the family $\{P : P \in \mathcal{P}\}$ is so in $X$, that is, each point $p$ of $X$ has a neighborhood in $X$ intersecting $P_1$ for at most one $P \in \mathcal{P}$, and so forth. Let $\mathcal{U}$ be a family of open subsets of $X$. Then we call that $\mathcal{P}$ is a pair-network for $\mathcal{U}$ in $X$ if for each point $p \in X$ and each $U \in \mathcal{U}$, if $p \in U$, then $p \in P_1 \subset P_2 \subset U$ for some $P = (P_1, P_2) \in \mathcal{P}$. As known already [7], a space $X$ is developable if and only if there exists a $\sigma$-discrete pair-network for the topology $\tau_X$ of $X$. We prepare two lemmas for the main theorem.
Lemma 3.2. Let $X$ be a subparacompact space and let $\mathcal{F}$ be a locally finite family of closed subsets of $X$ and $\{U(F): F \in \mathcal{F}\}$ its open expansion in $X$. Then there exists a $\sigma$-discrete pair-collection $\mathcal{P}$ of $X$ such that for each point $p \in X$ and each $F \in \mathcal{F}$, if $p \in F$, then $p \in P_1 \subset P_2 \subset U(F)$ for some $P = (P_1, P_2) \in \mathcal{P}$.

Proof. For each point $p \in X$, take an open neighborhood $V(p)$ of $p$ in $X$ such that

$$V(p) \subset X \setminus \bigcup\{F \in \mathcal{F} : p \notin F\}$$

and such that if $p \in \bigcup \mathcal{F}$, then

$$V(p) \subset \bigcap\{U(F) : p \in F \in \mathcal{F}\}.$$

By subparacompactness of $X$, there exists a $\sigma$-discrete closed refinement $\mathcal{H}$ of $\{V(p) : p \in X\}$. For each $H \in \mathcal{H}$ with $H \cap (\bigcup \mathcal{F}) \neq \emptyset$, choose an open subset $W(H)$ of $X$ such that

$$H \subset W(H) \subset \bigcap\{U(F) : F \cap H \neq \emptyset\}.$$

Then

$$\mathcal{P} = \{(H, W(H)) : H \in \mathcal{H} \text{ with } H \cap (\bigcup \mathcal{F}) \neq \emptyset\}$$

is the required pair-collection of $X$. □

For brevity, in the next lemma we call that a space $X$ satisfies the condition $(\ast)$ if for each discrete pair-collection $\{(F, U(F)) : F \in \mathcal{F}\}$ of $X$ there exists a pair $\langle \mathcal{V}, \mathcal{P} \rangle$ of a family $\mathcal{V}$ of subsets of $X$ and a $\sigma$-discrete pair-collection $\mathcal{P}$ of $X$ satisfying the following (1) and (2):

1. $\mathcal{V} = \{V(F) : F \in \mathcal{F}\}$ is an open expansion of $\mathcal{F}$ in $X$ such that $F \subset V(F) \subset U(F)$ for each $F \in \mathcal{F}$.

2. For each point $p \in X$ and each $F \in \mathcal{F}$ if $p \in V(F)$ then $p \in P_1 \subset P_2 \subset U(F)$ for some $P = (P_1, P_2) \in \mathcal{P}$.

(We call the pair $\langle \mathcal{V}, \mathcal{P} \rangle$ the $(\ast)$-pair for $\{(F, U(F)) : F \in \mathcal{F}\}$.)

Lemma 3.3. Let $X$ be a subparacompact space satisfying the condition $(\ast)$. Then $X$ is $\mathcal{D}$-paracompact.

Proof. By [1, Theorem 1, (iii)], it suffices to show that $X$ is $\mathcal{D}$-expandable, that is, for each discrete pair-collection $\{(F, U(F)) : F \in \mathcal{F}\}$ of $X$ with $F \cap U(F') = \emptyset$ if $F \neq F'$ and $F$, $F' \in \mathcal{F}$, there exists a “dissectable” family $\mathcal{V} =$
\{V(F): F \in \mathcal{F}\} of open subsets of X such that \( F \subseteq V(F) \subseteq U(F) \) for each \( F \in \mathcal{F} \). To show the existence of such \( \mathcal{V} \), by argument of the proof of [1, Theorem 1, (ii) \( \rightarrow \) (iii)], it suffices to find a \( \sigma \)-discrete pair-network \( \mathcal{P} \) for \( \mathcal{V} \) in X. Thus we will construct such \( \mathcal{V} \) and \( \mathcal{P} \) for a given discrete pair-collection \( \{(F, U(F)) : F \in \mathcal{F}\} \) of X. First, by (*) there exists a (*)-pair \( \langle \mathcal{V}_1, \mathcal{P}_1 \rangle \) for \( \{(F, U(F)) : F \in \mathcal{F}\} \) satisfying (1) and (2):

1. \( \mathcal{V}_1 = \{V_1(F) : F \in \mathcal{F}\} \) is an open expansion of \( \mathcal{F} \) such that \( F \subseteq V_1(F) \subseteq U(F) \) for each \( F \in \mathcal{F} \).

2. \( \mathcal{P}_1 \) is a \( \sigma \)-discrete pair-collection of X such that for each \( p \in X \) and each \( F \in \mathcal{F} \), if \( p \in V_1(F) \), then \( p \in P_1 \subseteq P_2 \subseteq U(F) \) for some \( P = (P_1, P_2) \in \mathcal{P}_1 \).

Write \( \mathcal{P}_1 = \bigcup \{\mathcal{P}_n : n \in \mathbb{N}\} \), where each \( \mathcal{P}_n = \{P_\alpha : \alpha \in A_{1n}\} \) is a discrete pair-collection of X. By (*), for each \( n \) there exists a (*)-pair

\[ \langle \{P_\alpha : \alpha \in A_{1n}\}, \mathcal{P}_{2n} \rangle \]

for \( \mathcal{P}_n \) satisfying the following (3) and (4):

3. \( P_{a1} \subseteq P_{a2} \subseteq P_{a2} \) for each \( \alpha \in A_{1n} \).

4. \( \mathcal{P}_{2n} \) is a \( \sigma \)-discrete pair-collection of X such that for each \( \alpha \in A_{1n} \) and each \( p \in X \), if \( p \in P_{a2} \), then \( p \in P_1 \subseteq P_2 \subseteq P_{a2} \) for some \( P = (P_1, P_2) \in \mathcal{P}_{2n} \).

For each \( F \in \mathcal{F} \) set

\[ V_2(F) = \bigcup \left\{ P_{a2} : \alpha \in \bigcup_n A_{1n}, p_{a1} \cap V_1(F) \neq \emptyset \text{ and } p_{a2} \subseteq U(F) \right\} \]

and set

\[ \mathcal{P}'_1 = \left\{ (P_{a1}, P_{a2}) : \alpha \in \bigcup_n A_{1n} \right\}. \]

Then \( \{V_2(F) : F \in \mathcal{F}\} \) is an open expansion of \( \mathcal{F} \) and \( \mathcal{P}'_1 \) is a \( \sigma \)-discrete pair-collection of X such that for each \( p \in X \) and each \( F \in \mathcal{F} \), if \( p \in V_1(F) \), then \( p \in P_1 \subseteq P_2 \subseteq V_2(F) \) for some \( P = (P_1, P_2) \in \mathcal{P}'_1 \). Write each \( \sigma \)-discrete pair-collection \( \mathcal{P}_{2n} \) as

\[ \mathcal{P}_{2n} = \bigcup \{\mathcal{P}_{2nm} : m \in \mathbb{N}\}, \]

where each \( \mathcal{P}_{2nm} = \{(P_{a1}, P_{a2}) : \alpha \in A_{2nm}\} \) is a discrete pair-collection of X. For each \( n, m \in \mathbb{N} \), by (*) there exists a (*)-pair

\[ \langle \{P_{a2} : \alpha \in A_{2nm}\}, \mathcal{P}_{3nm} \rangle \]

for \( \mathcal{P}_{2nm} \) satisfying the following (5) and (6):

5. \( P_{a1} \subseteq P_{a2} \subseteq P_{a2} \) for each \( \alpha \in A_{2nm} \).
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(6) \( \mathcal{P}_{3nm} \) is a \( \sigma \)-discrete pair-collection of \( X \) such that for each \( a \in A_{2nm} \) and each \( P \in X \), if \( p \in P'_{a,2} \), then \( p \in P_1 \subset P_2 \subset P_{a,2} \) for some \( P = (P_1, P_2) \in \mathcal{P}_{3nm} \).

Set
\[
V_3(F) = \bigcup \{ P'_{a,2} : a \in \bigcup \{ A_{2nm} : n, m \in N \}, P_{a,1} \cap V_2(F) \neq \emptyset \text{ and } P_{a,2} \subset U(F) \}
\]
for each \( F \in \mathcal{F} \) and set
\[
\mathcal{P}'_2 = \{(P_{a,1}, P'_{a,2}) : a \in \bigcup \{ A_{2nm} : n, m \in N \} \}.\]

Then \( \{ V_3(F) : F \in \mathcal{F} \} \) is an open expansion of \( \mathcal{F} \) satisfying the following (7) and (8):

(7) \( F \subset V_1(F) \subset V_2(F) \subset V_3(F) \subset U(F) \) for each \( F \in \mathcal{F} \).

(8) \( \mathcal{P}'_2 \) is a \( \sigma \)-discrete pair-collection of \( X \) such that for each \( p \in X \) and each \( F \in \mathcal{F} \), if \( p \in V_2(F) \), then \( p \in P_1 \subset P_2 \subset V_3(F) \) for some \( P = (P_1, P_2) \in \mathcal{P}'_2 \).

By repeating this process, we can construct a sequence \( \{ V_n(F) : F \in \mathcal{F} \} \) of open expansion of \( \mathcal{F} \) and a sequence \( \{ \mathcal{P}'_n : n \in N \} \) of \( \sigma \)-discrete pair-collections of \( X \) satisfying the following (9) and (10):

(9) \( F \subset V_1(F) \subset V_2(F) \subset \cdots \subset V_n(F) \subset V_{n+1}(F) \subset \cdots \subset U(F) \) for each \( F \in \mathcal{F} \).

(10) For each \( p \in X \) and \( F \in \mathcal{F} \), if \( p \in V_n(F) \), then \( p \in P_1 \subset P_2 \subset V_{n+1}(F) \) for some \( P = (P_1, P_2) \in \mathcal{P}'_n \).

Set
\[
V(F) = \bigcup \{ V_n(F) : n \in N \} \text{ for each } F \in \mathcal{F}
\]
and
\[
\mathcal{P}' = \bigcup \{ \mathcal{P}'_n : n \in N \}.
\]

Then each \( V(F) \) is an open subset of \( X \) such that \( F \subset V(F) \subset U(F) \) and obviously \( \mathcal{P}' \) is a \( \sigma \)-discrete pair-network for \( \{ V(F) : F \in \mathcal{F} \} \) in \( X \). This completes the proof.

For a closed mapping \( f : X \to Y \), we use the following notation: For each open subset \( U \) of \( X \), we write
\[
f^*(U) = Y \setminus f(X \setminus U),
\]
which is open in \( Y \).

**Theorem 3.4.** Let \( f \) be a perfect mapping of a space \( X \) onto a space \( Y \). If \( X \) is a \( D \)-paracompact \( \Sigma \)-space, then so is \( Y \).
Proof. By [10, Theorem 1.8], Y is a Σ-space. Since subparacompactness is preserved by perfect mappings, Y is subparacompact. Thus by Lemma 3.3, it suffices to show that Y satisfies the condition (*). Let \{F, U(F) : F \in \mathcal{F}\} be a discrete pair-collection of Y. We may assume that \(F \cap U(F') = \emptyset\) for \(F, F' \in \mathcal{F}\) with \(F \neq F'\). Since \(X\) is \(D\)-paracompact, there exists a \(\mathcal{U}_1\)-mapping \(g_1\) of \(X\) onto a developable space \(D_1\), where

\[
\mathcal{U}_1 = \{f^{-1}(U(F)) : F \in \mathcal{F}\} \cup \{X \setminus \bigcup f^{-1}(\mathcal{F})\}.
\]

Obviously there exists an open expansion \(\{V_1(F) : F \in \mathcal{F}\}\) of \(f^{-1}(\mathcal{F})\) in \(X\) such that for each \(F \in \mathcal{F}\)

\[
f^{-1}(F) \subset V_1(F) \subset f^{-1}(U(F))
\]

and \(V_1(F) = g_1^{-1}(O)\) with \(O\) open in \(D_1\). For each \(F \in \mathcal{F}\),

\[
V_1(F)^* = f^{-1}(f^*(V_1(F)))
\]

is an open subset of \(X\) such that

\[
f^{-1}(F) \subset V_1(F)^* \subset V_1(F) \subset f^{-1}(U(F)).
\]

Using the \(D\)-paracompactness of \(X\), there exists a \(\mathcal{U}_2\)-mapping \(g_2\) of \(X\) onto a developable space \(D_2\), where

\[
\mathcal{U}_2 = \{V_1(F)^* : F \in \mathcal{F}\} \cup \{X \setminus \bigcup f^{-1}(\mathcal{F})\}.
\]

Then there exists an open expansion \(\{V_2(F) : F \in \mathcal{F}\}\) of \(f^{-1}(\mathcal{F})\) in \(X\) such that for each \(F \in \mathcal{F}\)

\[
f^{-1}(F) \subset V_2(F) \subset V_1(F)^*
\]

and \(V_2(F) = g_2^{-1}(O)\) with \(O\) open in \(D_2\). Let \(g : X \to g(X) \subset D_1 \times D_2\) be a mapping defined by

\[
g(x) = (g_1(x), g_2(x)) \text{ for each } x \in X.
\]

Obviously both \(V_1(F)\) and \(V_2(F)\) are the inverse images of open subsets of \(X' = g(X)\) for each \(F \in \mathcal{F}\). Since \(X'\) is a developable space, there exists a \(σ\)-discrete pair-network \(\mathcal{P}'\) for the topology of \(X'\). Set

\[
\mathcal{P} = \{(g^{-1}(P_1), g^{-1}(P_2)) : P = (P_1, P_2) \in \mathcal{P}'\}.
\]

and write newly

\[
\mathcal{P} = \{(F_\alpha, V_\alpha) : \alpha \in \mathcal{A}_n \text{ and } n \in N\}.
\]
where for each \( n \), \( \{ F_a : a \in A'_n \} \) is a discrete family of closed subsets of \( X \). Obviously \( \mathcal{P} \) satisfies the following (1):

(1) \( \mathcal{P} \) is a pair-network for \( \{ V_1(F), V_2(F) : F \in \mathcal{F} \} \) in \( X \).

By the definition of a strong \( \Sigma \)-space, \( Y \) has a cover \( \mathcal{C} \) by compact subsets and has a \( \sigma \)-locally finite family \( \mathcal{H} = \{ H_\lambda : \lambda \in \Lambda \} \) of closed subsets of \( Y \) such that:

(2) For each \( O \in \tau_Y \) and each \( C \in \mathcal{C} \), if \( C \subseteq O \), then \( C \subseteq H_\lambda \subseteq O \) for some \( \lambda \in \Lambda \).

Without loss of generality, we can assume that \( \mathcal{H} \) is closed under any finite intersections. For each \( n \), let \( A_n = \bigcup \{ A'_i : i \leq n \} \). Then \( \{ F_a : a \in A_n \} \) is locally finite in \( X \) and \( A_n \subseteq A_{n+1} \). For each \( n \), let \( \Delta_n \) be the totality of finite subsets of \( A_n \) and for each \( (\delta, \lambda) \in \Delta_n \times \Lambda \), \( (\delta, \delta') \in \Delta_n \times \Delta_m \), \( n, m \in N \), set

\[
\begin{align*}
F(\delta) &= \bigcap \{ f(F_a) : a \in \delta \}, \\
f(\delta, \lambda) &= F(\delta) \cap H_\lambda, \\
W(\delta) &= f^*\left( \bigcup \{ V_a : a \in \delta \} \right), \\
W(\delta, \delta') &= W(\delta) \cup W(\delta').
\end{align*}
\]

For each \( n, m \in N \) let \( T(m, n) \) be the set of all combinations \( (\delta_1, \lambda, n) \in \Delta_m \times \Lambda \times \{ n \} \) such that

\[
A_n(\delta_1, \lambda) = \{ a \in A_n : f(F_a) \cap (F(\delta_1, \lambda) \setminus W(\delta_1)) \neq \emptyset \}
\]

is finite. \( (T(m, n) \) may be empty for some \( m, n \).) For each combination \( (\delta_1, \lambda, n) \in T(m, n) \), let

\[
\Delta(\delta_1, \lambda, n) = \{ \delta_2 \in \Delta_n : \delta_2 \subseteq A_n(\delta_1, \lambda) \text{ and } F(\delta_1, \lambda) \subset W(\delta_1) \cup W(\delta_2) \}.
\]

From the definition of \( T(m, n) \), \( \Delta(\delta_1, \lambda, n) \) is finite. For each \( \delta_2 \in \Delta(\delta_1, \lambda, n) \) with \( (\delta_1, \lambda, n) \in T(m, n) \), \( m, n \in N \), construct an order pair of subsets of \( Y \)

\[
P(\delta_1, \lambda, \delta_2) = (P_1(\delta_1, \lambda, \delta_2), P_2(\delta_1, \lambda, \delta_2))
\]

where

\[
P_1(\delta_1, \lambda, \delta_2) = F(\delta_1, \lambda)
\]

and

\[
P_2(\delta_1, \lambda, \delta_2) = W(\delta_1, \delta_2).
\]

Set

\[
\mathcal{P}(\delta_1, \lambda, n) = \{ P(\delta_1, \lambda, \delta_2) : \delta_2 \in \Delta(\delta_1, \lambda, n) \}
\]
and

\[ \mathcal{A} = \bigcup \{ \mathcal{P}(\delta_1, \lambda, n) : (\delta_1, \lambda, n) \in T(m, n) \text{ and } m, n \in N \}. \]

Then obviously \( \mathcal{A} \) is a \( \sigma \)-locally finite pair-collection of \( Y \). We establish the following claim:

**Claim:** For each \( p \in Y \) and each \( F \in \mathcal{F} \), if \( p \in f^*(V_2(F)) \), then \( p \in Q_1 \subseteq Q_2 \subseteq f^*(V_1(F)) \) for some \( Q = (Q_1, Q_2) \in \mathcal{A} \).

Suppose \( p \in f^*(V_2(F)) \). Then \( f^{-1}(p) \subseteq V_2(F) \). By the compactness of \( f^{-1}(p) \) and by (1), there exists \( n_0 \in N \) such that for each \( n \geq n_0 \) there exists \( \delta_n \in \Delta_n \) such that

\[
\begin{align*}
& f^{-1}(p) \cap F_a \neq \emptyset \text{ for each } \alpha \in \delta_n, \\
& f^{-1}(p) \subseteq \bigcap \{ V_a : \alpha \in \delta_n \} \subseteq V_2(F)
\end{align*}
\]

and \( \delta_n \subseteq \delta_{n+1} \), which imply

\[
p \in f(F(\delta_n)) \cap W(\delta_n), \quad W(\delta_n) \subseteq f^*(V_2(F)).
\]

Take \( C \in \mathcal{C} \) with \( p \in C \) and let \( \{ H_{\lambda(i)} : i \in N \} \) be a decreasing sequence of members of \( \mathcal{H} \) containing \( C \) satisfying the following (3):

(3) For each \( O \in \tau_Y \), if \( C \subseteq O \), then \( C \subseteq H_{\lambda(i)} \subseteq O \) for some \( i \).

In fact, such a sequence \( \{ H_{\lambda(i)} \} \) exists because of (2) and of the assumption on \( \mathcal{H} \). We show the following (4):

(4) For each \( t \in N \), there exists \( i_0 \in N \) such that

\[(\delta_{i_0}, \lambda(i_0), t) \in T(n_0, t).\]

To show (4), assume the contrary, i.e., for some \( s \in N \) \( A_s(\delta_{i_0}, \lambda(i)) \) is infinite for each \( i \). Then, since \( \{ f(F_a) : \alpha \in A_s \} \) is locally finite in \( Y \), we can choose a sequence \( \{ a_i : i \in N \} \subseteq A_s \) and a sequence \( \{ p_i : i \in N \} \) of points of \( Y \) such that

\[
p_i \in Y \setminus \{ p_1, \ldots, p_{i-1} \}
\]

and \( F_{a_i} \neq F_{a_j} \) whenever \( i \neq j \). By (3) \( \{ p_i : i \in N \} \) has a cluster point in \( Y \). But this is a contradiction, because \( p_i \in f(F_{a_i}) \) for each \( i \). This establishes (4). Since

\[
C \cap (f(\delta_{i_0}) \setminus W(\delta_{i_0}))
\]

is a compact subset and is contained in \( f^*(V_1(F)) \), there exists \( n_1 \geq n_0 \) and \( \delta_1 \in \Delta_{n_1} \) such that

\[
C \cap (f(\delta_{i_0}) \setminus W(\delta_{i_0})) \subseteq W(\delta_1) \subseteq f^*(V_1(F)).
\]
Using (4), there exists $i_1 \in \mathbb{N}$ such that $(\delta_{m_0}, \lambda(i_1), n_1) \in T(n_0, n_1)$. By (3), we can easily find $i_2 \geq i_1$ such that

$$F(\delta_{m_0}, \lambda(i_2)) \subset W(\delta_{m_0}, \delta_1).$$

Since $\{H_k(i_j)\}$ is decreasing, it is obvious that $(\delta_{m_0}, \lambda(i_2), n_1) \in T(n_0, n_1)$. Recalling the definition of $\mathcal{P}(\delta_{m_0}, \lambda(i_2), \delta_1)$, we have

$$p \in P_1(\delta_{m_0}, \lambda(i_2), \delta_1) \subset P_2(\delta_{m_0}, \lambda(i_2), \delta_1) \subset f^*(V_1(F))$$

and $P(\delta_{m_0}, \lambda(i_2), \delta_1) \in \mathcal{P}$. This establishes the validity of the claim. Using Lemma 3.3, we can conclude that $Y$ is $D$-paracompact. This completes the proof. \(\square\)

Finally, we give a positive result to the mapping property of $D$-paracompact spaces. To state it, we need the definition of $\beta$-spaces. $\Sigma$-spaces and Moore spaces are $\beta$-spaces [8, Theorem 7.8(i)].

**Definition 3.5** [8, Definition 7.7]. A space $X$ is called a $\beta$-space if there exists a $\beta$-function $g : \mathbb{N} \times X \to \tau_X$ such that

(i) $x \in g(n, x)$ for each $n \in \mathbb{N}$, $x \in X$.

(ii) If $x \in g(n, x_n)$ for each $n \in \mathbb{N}$, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point in $X$.

**Theorem 3.6.** Let $f : X \to Y$ be a perfect mapping. If $X$ is a $D$-paracompact $\beta$-space with a $G_\delta$-diagonal, then $Y$ is a $D$-paracompact $\beta$-space.

**Proof.** Since as easily checked $\beta$-spaces are preserved by perfect mappings, $Y$ has a $\beta$-function $g : \mathbb{N} \times Y \to \tau_Y$. To see that $Y$ satisfies the condition $(\ast)$ in Lemma 3.3, let $\{(F, U(F)) : F \in \mathcal{F}\}$ be a discrete pair-collection. Without loss of generality, we can assume that $U(F) \cap F' = \emptyset$ whenever $F \neq F'$. Since $X$ is subdevelpable [12, Proposition 5.1], in the sense of [3], there exists a one-to-one $\mathcal{U}$-mapping $h$ of $X$ onto a developable space $D$, where

$$\mathcal{U} = \{f^{-1}(U(F)) : F \in \mathcal{F}\} \cup \{X \setminus \bigcup f^{-1}(\mathcal{F})\}.$$ 

Then there exists a family $\mathcal{V} = \{V(F) : F \in \mathcal{F}\}$ of open subsets of $X$ and a $\sigma$-locally finite pair-network

$$\mathcal{P} = \{(F_{\alpha}, V_{\alpha}) : \alpha \in A_n, n \in \mathbb{N}\}$$

for $\mathcal{V} \cup h^{-1}(\tau_D)$ in $X$ satisfying the following:

(1) $f^{-1}(F) \subset V(F) \subset f^{-1}(U(F))$, $F \in \mathcal{F}$.

(2) For each $n$, $\{F_{\alpha} : \alpha \in A_n\}$ is locally finite in $X$ and $A_n \subset A_{n+1}$.
For each \( p \in X \) and \( F \in \mathcal{F} \), if \( p \in V(F) \), then there exists \( \alpha \in A_n, n \in \mathbb{N} \), such that \( p \in F_\alpha \subset V_\alpha \subset V(F) \).

Let \( \Delta_n \) be the totality of finite subsets of \( A_n \) and for each \( \delta \in \Delta_n, k \in \mathbb{N} \), let

\[
H(\delta, k) = \bigcap \{ f(F_\alpha) : \alpha \in \delta \} \setminus \bigcup \{ \theta(k, y) : y \in K(\delta) \},
\]

\[
K(\delta) = \bigcap \{ f(F_\alpha) : \alpha \in \delta \} \setminus f^*(\bigcup \{ V_\alpha : \alpha \in \delta \})
\]

and

\[
W(\delta, k) = f^*(\bigcup \{ V_\alpha : \alpha \in \delta \}).
\]

Then obviously \( H(\delta, k) \subset W(\delta, k) \) for each \( \delta \) and \( k \), and by virtue of (2), \( \{ H(\delta, k) : \delta \in \Delta_n \} \) is locally finite in \( Y \). Construct the pair-collection of \( Y \)

\[
\mathcal{Q} = \{ (H(\delta, k), W(\delta, k)) : \delta \in \Delta_n, k, n \in \mathbb{N} \}.
\]

Then we show that \( \mathcal{Q} \) is a \( \sigma \)-locally finite pair-network for \( \mathcal{W} = \{ W(F) : F \in \mathcal{F} \} \) in \( Y \), where \( W(F) = f^*(V(F)) \), \( F \in \mathcal{F} \). It is trivial that \( \mathcal{Q} \) is \( \sigma \)-locally finite in \( Y \).

To see that \( \mathcal{Q} \) is a pair-network for \( \mathcal{W} \) in \( Y \), let \( p \in W(F), F \in \mathcal{F} \). Then there exists a sequence \( \{ \delta_n : n \geq n_0 \} \) with \( \delta_n \in \Delta_n \) for each \( n \geq n_0 \), satisfying for each \( n \geq n_0 \)

\[
p \in W(\delta_n, k), \quad \delta_n \subset \delta_{n+1}
\]

and

\[
\delta_n = \{ \alpha \in A_n : F_\alpha \cap f^{-1}(p) \neq \emptyset \text{ and } V_\alpha \subset V(F) \}.
\]

In this case we have \( \bigcap \{ K(\delta_n) : n \geq n_0 \} = \emptyset \). For, if \( q \in \bigcap_n K(\delta_n) \), then \( q \in \bigcap f(F_\alpha) : \alpha \in \delta_n \) for each \( n \), which implies

\[
h(f^{-1}(p)) \cap h(f^{-1}(q)) \neq \emptyset,
\]

but this is a contradiction to \( f^{-1}(p) \cap f^{-1}(q) = \emptyset \). Assume \( p \notin H(\delta_n, n) \) for each \( n \). Then \( p \in g(n, p_n) \) for some point \( p_n \in K(\delta_n) \). Since \( g \) is a \( \beta \)-function, \( \{ p_n \} \) has a cluster point \( p_0 \), which must belong to \( \bigcap_n K(\delta_n) \). But this is a contradiction to the above. Hence we have

\[
p \in Q_1 \subset Q_2 \subset W(F)
\]

for some \( Q = (Q_1, Q_2) \in \mathcal{Q} \). This completes the proof.

**Remark.** (i) \( Y \) need not have a \( G_\delta \)-diagonal. In fact, there exists a perfect mapping of a disjoint topological sum of two Michael lines onto a space which has no \( G_\delta \)-diagonal [14].
(ii) This theorem is not a corollary to the result in [9] that if $X$ is a perfect image of a perfect $D$-paracompact space, then so is $X$ because there exists a compact subdevelopable space $X$ but not perfect.

References


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