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ON THE SUM OF A PRIME AND A SQUARE

By

Hiroshi Mikawa

1. Introduction.

In 1923 G.H. Hardy and J.E. Littlewood [3] conjectured that every large integer, not being a square, may be expressed as the sum of a prime and a square. Let \( \nu(n) \) be the number of representations of an integer \( n \) in this manner. They further stated the hypothetical asymptotic formula; As \( n \neq k^2 \to \infty \),

\[
\nu(n) \sim \Xi(n) \sqrt{\frac{n}{\log n}}
\]

with

\[
\Xi(n) = \prod_{p \leq n} \left(1 - \frac{n/p}{p-1}\right)
\]

where \( (\cdot) \) is the Legendre symbol.

Define \( \Xi(k^2) = 0 \). In 1968 R.J. Miech [5] proved that

\[
\sum_{n \leq x} \left| \nu(n) - \Xi(n) \sqrt{\frac{n}{\log n}} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) \right|^2 \ll \sqrt{x} \log x\]

for any \( A > 0 \), from which it follows that

\[
E(x) \ll x^{(\log x)^{-A}}
\]

where \( E(x) \) denotes the number of integers \( n \leq x \) with \( \nu(n) = 0 \). It seems difficult to sharpen the right hand side of (1). However (2) may be improved, see [1, 9, 12].

A.I. Vinogradov [12; p. 35] remarked that, for any \( \varepsilon > 0 \),

\[
E(x) \ll x^{2/3 + \varepsilon}
\]

under the extended Riemann hypothesis. First of all we shall show

PROPOSITION. Assume the extended Riemann hypothesis. Then

\[
E(x) \ll x^{3/4}(\log x)^{3/4}
\]
It is the main aim of this paper to prove the following unconditional results.

**Theorem 1.** Let $1/2 < \theta \leq 1$ and $A > 0$ be given. We have

$$
\sum_{x - \varepsilon \leq n \leq x} \nu(n) - \mathcal{S}(n) \sqrt{n} \left( 1 + O\left( \frac{\log \log n}{\log n} \right) \right) \ll x^{\theta+1} (\log x)^{-A}
$$

where the $O$-constant is absolute and the $\ll$-constant depends on $\theta$ and $A$ only.

**Theorem 2.** Let $7/24 < \theta \leq 1$ and $A > 0$ be given. We have

$$
E(x + x^{\theta}) - E(x) \ll x^{\theta} (\log x)^{-A}
$$

where the implied constant depends on $\theta$ and $A$ only.

Our assertion may be regarded as a refinement of Miech's work (1)(2), and must be compared with a conditional bound (3). Within the frame of Circle method, we appeal to the large sieve [7, 8] and R.C. Vaughan's method [11; Chap. 4] on Weyl sums.

I would like to thank Professor Uchiyama and Dr. Kawada for suggestion and encouragement.

### 2. Singular series.

In this section we collect the facts of $\mathcal{S}(n)$. For the proof, see [1, 5, 9, 12].

For integers $q$ and $n$, let $\rho(q, n)$ be the number of solutions of the congruence $x^2 \equiv n \pmod{q}$, and $\rho_1$ be the convolution inverse of $\rho$ with respect to $q$. Define, for $Q \geq 3$,

\begin{equation}
\mathcal{S}(n, Q) = \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \rho_1(q, n).
\end{equation}

Then, uniformly for $n$,

\begin{equation}
\mathcal{S}(n, Q) \ll \log Q.
\end{equation}

Let $\mathcal{D}$ be the set of fundamental discriminants. An integer $n$ may be uniquely written as $n = n_1 n_2^2$ with a square-free $n_1$. Put

$$
\delta(n) = \begin{cases} 
n_1 & \text{if } n_1 \equiv 1 \pmod{4} \\
4n_1 & \text{otherwise.}
\end{cases}
$$

Thus, if $n \neq k^2$ then $\delta(n) \in \mathcal{D}$. For $d \in \mathcal{D}$, the Kronecker symbol $(d/\cdot)$ is a primitive character to modulus $d$. Let $\mathcal{L} = \mathcal{L}(T), T \geq 3$, denote the set of $d \in \mathcal{D}$ for which $L(s, (d/\cdot))$ the Dirichlet $L$-function has no zero in the region:
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$\Re(s) \geq \frac{29}{30}$ and $|\Im(s)| \leq T$. Suppose $x \asymp T$. If $\delta(n) \in \mathcal{L}$ then there exists a constant $\eta > 0$ such that

$$\mathfrak{S}(n, Q) = \mathfrak{S}(n) + O(Q^{-\eta} \exp(\sqrt{\log x}))$$

uniformly for $n \leq x$. Moreover,

$$\# \{d : d \in \mathcal{O} \setminus \mathcal{L}, \ d \leq 4x\} \ll x^{1/4} (\log x)^{14}.$$ 

Finally, for $n \neq k^2$,

$$\mathfrak{S}(n) = \mathfrak{S}(n) L\left(1, \left(\frac{\delta(n)}{\varphi(n)}\right)\right) \ll \frac{n}{\varphi(n)}.$$

### 3. A conditional estimate.

In this section we illustrate our device with the proof of Proposition. We employ the Circle method [11].

Let $x$ be a large parameter. We divide the unit interval by the Farey dissections of order

$$Q = x^{1/2} (\log x)^3.$$ 

For $(a, q) = 1$, write

$$I_{q, a} := \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].$$

Put

$$M = \bigcup_{q \leq P} \bigcup_{0 < a < q} I_{q, a}, \quad P = x/100Q,$$

$$m = [Q^{-1}, 1 + Q^{-1}] \setminus M.$$ 

We define the exponential sum

$$W(\alpha) = \sum_{m \leq x} e(\alpha m^2)$$

where $e(t) = e^{it}$. By Weyl's inequality, we see that

$$|W(\alpha)|^2 \ll \left(\frac{x}{q} + x^{1/2} + q\right) \log qx,$$

for $|\alpha - (a/q)| \leq q^{-3}$ with $(a, q) = 1$. When $\alpha \in I_{q, a}$, $W(\alpha)$ is approximated by

$$V(\alpha) = q^{-1} g(a, q) v\left(\alpha - \frac{a}{q}\right)$$

where

$$g(a, q) = \sum_{m \leq x} e\left(\frac{a}{q} m^2\right) \quad \text{and} \quad v(\beta) = \sum_{m \leq x} \frac{e(\beta m)}{2 \sqrt{m}}.$$ 

Actually it follows from [11; Theorem 4.1] and [12; p. 38] that


For \( |\alpha - (a/q)| \leq (4q\sqrt{x})^{-1} \), in addition, we note that

\[
|g(a, q)|^3 \ll \log x,
\]

\[
|v(\beta)|^3 \ll \min(x, \|\beta\|^{-1})
\]

where \( \|t\| = \min_{n \in \mathbb{Z}} |t - n| \).

Put

\[
S(\alpha) = \sum_{n \in \mathbb{Z}} A(n)e(\alpha n)
\]

where \( A \) is the von Mangoldt function. It is expected that, for \( \alpha \in I_{q,a} \), \( S(\alpha) \) is nearly equal to

\[
T(\alpha) = \frac{\mu(q)}{\varphi(q)} t\left(\alpha - \frac{a}{q}\right)
\]

where

\[
t(\beta) = \sum_{n \in \mathbb{Z}} e(\beta n) \ll \min(x, \|\beta\|^{-1}).
\]

In order to show this, define

\[
J(q) = \sum_{\alpha \in I_{q,a}} |S(\alpha) - T(\alpha)|^2 d\alpha
\]

where \( * \) in \( \Sigma_{*} \) stands for \( (a, q) = 1 \). If \( \alpha \in I_{q,a} \),

\[
S(\alpha) - T(\alpha) = \frac{\mu(q)}{\varphi(q)} \sum_{\alpha \in I_{q,a}} X(n) A(n) e\left(\left(\alpha - \frac{a}{q}\right)n\right) + O((\log x)^3).
\]

Here \( # \) in \( \Sigma_{*} \) means that if \( X \) is principal then \( X(n) A(n) \) should be replaced by \( A(n) - 1 \). When \( q \leq P \), by \([2; \text{Lemma 1}]\), we have

\[
J(q) \ll \sum_{\chi(q)} |\chi(\bar{z})|^2 \left| \sum_{\alpha \in I_{q,a}} X(n) A(n) e(\beta n) \right|^2 d\beta + O((\log x)^4).
\]

On noting \( qQ \leq PQ = x/100 \), it is easy to show that the above integral is

\[
\ll qQ x(\log x)^4,
\]

under the extended Riemann hypothesis. Therefore, uniformly for \( q \leq P \),

\[
J(q) \ll Q^{-1} x(\log x)^4.
\]

Now, for \( n \leq x \),

\[
\sum_{l + m = n} A(n) = \sum_{q-1}^{l+Q-1} S(\alpha) W(\alpha) e(-n\alpha) d\alpha = \int_{q-1}^{l+Q-1} S(\alpha) W(\alpha) e(-n\alpha) d\alpha = \int_{m}^{l+Q-1} S(\alpha) W(\alpha) e(-n\alpha) d\alpha
\]
By Bessel's inequality and (3.1),

\[
\sum_{n \leq x} \left| \int_{m} S(\alpha)W(\alpha)e(-n\alpha) \, d\alpha \right|^2 \leq \sum_{n \leq x} |S(\alpha)W(\alpha)|^2 \, d\alpha \\
\leq \sup_{\alpha \in \mathbb{M}} |W(\alpha)|^2 \int_{m} |S(\gamma)|^2 \, d\gamma \\
\ll Qx(\log x)^2.
\]

On \( \alpha \in \mathbb{M} \) we first exchange \( S(\alpha) \) for \( T(\alpha) \). Thus, by (3.1) and (3.7),

\[
\sum_{n \leq x} \left| \int_{m} (S(\alpha)-T(\alpha))W(\alpha)e(-n\alpha) \, d\alpha \right|^2 \\
\leq \sum_{q \leq P} \sum_{\delta=1}^{\mathbb{Q}} \int_{I_q, \delta} \left| S(\alpha)-T(\alpha) \right|^2 |W(\alpha)|^2 \, d\alpha \\
\ll \sum_{q \leq P} \frac{x}{q} (\log x) f(q) \\
\ll x^2 Q^{-1}(\log x)^4.
\]

Next we replace \( W(\alpha) \) by \( V(\alpha) \). On using (3.2) and (3.5),

\[
\sum_{n \leq x} \left| \int_{M} T(\alpha)V(\alpha)e(-n\alpha) \, d\alpha \right|^2 \\
\ll \sup_{\alpha \in \mathbb{M}} |W(\alpha)-V(\alpha)|^2 \int_{M} |T(\gamma)|^2 \, d\gamma \\
\ll P(\log P)^3 \sum_{q \leq P} \sum_{\beta_1 \leq 1/q \leq \beta} \mu(q) \varphi(q) \xi(\beta) |t(\beta)|^2 \, d\beta \\
\ll Px(\log x)^3.
\]

Finally we extend the Farey arc \( I_{q, a} \) to \( I_{q, a} = [(a/q)-(1/2), (a/q)+(1/2)] \). The resulting remainder is then equal to

\[
\sum_{\delta=1}^{\mathbb{Q}} \int_{M} T(\alpha)V(\alpha)e(-n\alpha) \, d\alpha \\
= \sum_{q \leq P} \sum_{\delta=1}^{\mathbb{Q}} \int_{1/q < \beta_1 \leq 1/q} \mu(q) \varphi(q) \xi(\beta) t(\beta) q^{-1}g(a, q) v(\beta) t(\beta) q^{-1} g(a, q) \xi(\beta) \, d\beta \\
= \int_{1/q < \beta_1 \leq 1/q} t(\beta) v(\beta) t(\beta) q^{-1} g(a, q) \xi(\beta) \, d\beta.
\]

On using Cauchy's inequality and (3.4), we have

\[
\sum_{n \leq x} |r_n|^2 \leq \sum_{n \leq x} \left| \int_{1/q < \beta_1 \leq 1/q} t(\beta) q^{-1} g(a, q) \xi(\beta) \, d\beta \right|^2 \\
\ll (\log x) \int_{1/q < \beta_1 \leq 1/q} \left( \sum_{q \leq P} \mu(q) \varphi(q) g(a, q) \xi(\beta) \right)^2 \, d\beta.
\]
The large sieve inequality [7, 8] yields that
\[
\sum_{n \leq x} |r_n|^2 \ll (\log x) \int_{1/2 < |\beta| < 1/2} |t(\beta)|^2 \left( \sum_{q \leq P} \sum_{\chi(q) \neq 0} \left( \sum_{x+qP} |g(a, q)|^2 \right) d\beta \right).
\]
\[
\ll (\log x) \sum_{q \leq P} (x+qP) \frac{\mu^2(q)}{\phi(q)} \int_{1/2 < |\beta| < 1/2} |t(\beta)|^2 d\beta.
\]
\[
\ll (\log x) (x+P^2) Q \sum_{q \leq P} \frac{\mu^2(q)}{\phi(q)}.
\]
(3.13) \[\ll Q x (\log x)^{59.}
\]

Here we used the bounds (3.3) and (3.5).

It remains to calculate
\[
\sum_{q \leq P} \sum_{a=1}^{\varphi(q)} T(\alpha) V(\alpha) e(-n\alpha) d\alpha
\]
\[
= \sum_{q \leq P} \sum_{a=1}^{\varphi(q)} \mu(q) g(a, q) e\left( -\frac{a}{q} \right) \int_{-1/2}^{1/2} \sum_{m \geq x} \frac{e(\beta(m-n))}{2\sqrt{m}} d\beta.
\]

The above sum is \( \mathcal{S}(n, P) \) with the definition (2.1). The integral is equal to
\[
\sum_{l \leq 1} \frac{1}{2\sqrt{n-l}} = \sqrt{n} + O(1).
\]

Hence, by (2.2), (3.14) becomes
\[
\mathcal{S}(n, P) \sqrt{n} + O(\log x).
\]

On summing up the above argument (3.9)–(3.15), we obtain
\[
\sum_{n \leq x} | \sum_{p \leq \sqrt{n}} \log p - \mathcal{S}(n, P) \sqrt{n} |^2 \ll Q x (\log x)^{69} + x^2 Q^{-2} (\log x)^{89} + P x (\log x)^{59}
\]
(3.16) \[\ll x^{1/2} (\log x)^{69}.
\]

Now, the extended Riemann hypothesis implies that \( \mathcal{L} = 0 \) the sets introduced in section 2, and that \( \mathcal{S}(n) \gg (\log \log n)^{-2} \) for \( n \neq k^2 \) and \( n \gg 1 \). By (2.3) we then have that, for \( n \neq k^2 \) (\( \gg 1 \)),
\[
\mathcal{S}(n, P) \gg (\log \log n)^{-2}.
\]

Consequently (3.16) leads that
\[
x^{3/2} (\log x)^{69} \sum_{n \leq x} | \sum_{p \leq \sqrt{n}} \log p - \mathcal{S}(n, P) \sqrt{n} |^2
\]
\[
\gg \sum_{x^{3/2} \leq \sqrt{n} \leq x^{1/2}} \mathcal{S}(n, P) \big| \mathcal{S}(n, P) \big| n
\]
\[
\gg x (\log \log x)^{-4} \sum_{n \leq x^{1/2}} 1
\]
or
\[
E(x) \leq \sum_{\substack{n \leq x \atop \nu(n) = 0}} 1 + \sum_{\substack{n \leq x \atop \nu(n) \neq 0}} 1 \ll \sum_{r} \left(\frac{x}{2^r}\right)^{1/2} (\log x \log \log x)^t + x^{1/2} \ll x^{1/2} (\log x)^q,
\]
as required.

4. Proof of Theorems.

In this section we derive Theorems from the known results mentioned in section 2 and our main lemma below. Lemma will be verified in the next section.

**Lemma.** Let \( x \) be a large parameter. For given \( 1/2 < \Theta \leq 3/4 \) and \( 7/12 < \Xi \leq 1 \), put \( \Delta = y^\alpha \) and \( y = x^{\tau} \). Write
\[
\nu(n, y) = \# \{(p, m) : x-y < p \leq x, m^2 \leq y, p+m^2 = n\},
\]
and
\[
K(n, y) = \int_1^{n-(x-y)^{-\alpha}} \frac{dt}{2\sqrt{t} \log(n-t)}.
\]
Then, for any \( A > 0 \), we have
\[
\sum_{x-L \leq n \leq x} |\nu(n, y) - \Xi(n, \sqrt{x})K(n, y)|^2 \ll \Delta y (\log x)^{-A}
\]
where the implied constant depends on \( \Theta, \Xi \) and \( A \) only.

**Proof of Theorem 1.** Let \( L = L(x) \) in section 2. Choose \( \Xi = 1 \) in Lemma. Then, because of \( \nu(n, x) = \nu(n) \),
\[
\sum_{x-L \leq n \leq x} |\nu(n) - \Xi(n, \sqrt{x})K(n, x)|^2 \ll x^{\Theta+1}(\log x)^{-A}
\]
for any \( A > 0 \). We note that
\[
K(n, x) = \frac{\sqrt{n}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right)
\]
with an absolute \( O \)-constant. Combining the above with (2.3) and (2.5) we have
\[
S = \frac{\sum_{x-L \leq n \leq x} |\nu(n) - \Xi(n)K(n, x)|^2}{\sum_{d(n) \leq x} \sum_{n \leq k^2}}
\]
\[
\ll x \sum_{x-L \leq n \leq x} |\nu(n) - \Xi(n, \sqrt{x})K(n, x)|^2 + \sum_{x-L \leq n \leq x} |\Xi(n) - \Xi(n, \sqrt{x})|^2 \sum_{x-L \leq n \leq x} \frac{n}{(\log n)^2}
\]
\[
+ x \sum_{x-L \leq n \leq x} \left(\frac{\nu(n)^2 + (\Xi(n))^2}{\log n}\right) + \sum_{x-L \leq n \leq x} \nu(k^2)^2
\]
\[
\sum_{x \leq n < 2x} |\nu(n) - \Omega(n, \sqrt{x})| K(n, x)^{2} + x^{\theta+1} \sup_\delta |\Omega(n) - \Omega(n, \sqrt{x})|^{2} \\
+ x \left( 1 + \sup_{\delta(n) \in \mathcal{L}} \left( \frac{\Omega(n)}{\log n} \right) \right) \sum_{d \in \mathcal{L}} \sum_{n \leq x \leq 2x} \frac{1}{\delta(n) = d} \\
\ll x^{\theta+1}(\log x)^{-A} + x \left( 1 + \sup_{d \in \mathcal{L}} \left| L(1, \left( \frac{d}{\delta(n)} \right) \right|^{2} \right) \left( 1 + \sum_{d \in \mathcal{L}} \right) x^{\theta+1/2}.
\]

By (2.4) and Siegel's theorem \([10; \text{Kap. IV, §8}], S \) becomes
\[
\ll x^{\theta+1}(\log x)^{-A} + x^{\theta+1/2}(x) \left( 1 + x^{1/4}(\log x)^{14} \right)
\ll x^{\theta+1}(\log x)^{-A}.
\]

Hence we obtain Theorem 1 in case \(1/2 < \theta \leq 3/4\). If \(3/4 < \theta \leq 1\), Theorem 1 follows from the case of \(\theta = 2/3\), by splitting up the interval \([x - x^\theta, x]\) into the sum of smaller intervals of type \([u - u^{2/3}, u]\).

**Proof of Theorem 2.** Put \(\theta = \Theta \mathcal{L}\) in Lemma. Then, \(7/24 < \theta \leq 3/4\). It is sufficient to prove Theorem 2 for \(\theta\) in the above range only. Since \(\nu(n) = 0\) implies \(\nu(n, y) = 0\), Lemma yields that
\[
\sum_{x - x^\theta \leq n < 2x} |\nu(n, \sqrt{y})| K(n, y)^{2} \ll yx^\theta(\log x)^{-A+\delta}.
\]

Here, \(K(n, y)^{2} \ll y(\log x)^{-2}\). Thus,
\[
\sum_{x - x^\theta \leq n < 2x} 1 \ll x^\theta(\log x)^{-A+\delta}(\log x \log x)^{2}
\]
by (2.3) and (2.5) with \(\mathcal{L} = \mathcal{L}(x)\). Hence, by (2.4), we obtain
\[
E(x) - E(x - x^\theta) \ll \sum_{x - x^\theta \leq n < 2x} 1 + \sum_{x - x^\theta \leq n < 2x} \frac{1}{\delta(n) \in \mathcal{L}} \\
\ll x^\theta(\log x)^{-A} + \sum_{d \in \mathcal{L} \text{ or } d = 1} (x^{\theta+1/2} + 1) \\
\ll x^\theta(\log x)^{-A} + x^{1/4}(\log x)^{14} \\
\ll x^\theta(\log x)^{-A},
\]
as required.

**5. Proof of Lemma.**

Put
\[
S(\alpha) = \sum_{x - y \leq n < x} \sigma(\alpha n).
\]
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And we define the exponential sums $W(a)$ and $V(a)$ by the similar way in section 3, except for changing the parameter $x$ in section 3 by $y$. The Farey arcs are determined as follows:

$$M = \left\{ \bigcup_{q \in \mathbb{P}} \bigcup_{0 < a < q \ (a, q) = 1} I_{q, a} \right\}, \quad L = \left\{ \bigcup_{q \in \mathbb{P}} \bigcup_{0 < a < q \ (a, q) = 1} I_{q, a} \right\}, \quad P = (\log x)^4$$

Here $I_{q, a}$ and $I_{q, a}$ are similar to that in section 3. We then have

$$\nu(n, y) = \int_{q = 1}^{1 + Q^{-1}} S(\alpha)W(\alpha)e(-na)d\alpha$$

$$= \int_L SV - \int_{L \cap M} SV + \int_M SV + \int_{M \setminus m} S(W - V) + \int_M SV$$

$$= J_1 - J_3 + J_4 + J_5, \text{ say.}$$

First we evaluate $J_1$. An elementary calculation leads that

$$J_1(n) = \int_L S(\alpha)V(\alpha)e(-na)d\alpha$$

$$= \sum_{q \in \mathbb{P}} \sum_{\alpha = 1}^{q - 1} \int_{-1/2}^{1/2} S(\alpha q + \beta)q^{-1}g(a, q)v(\beta)e(-n(\alpha q + \beta))d\beta$$

$$= \sum_{q \in \mathbb{P}} \sum_{\alpha = 1}^{q - 1} q^{-1}g(a, q)e\left(-\frac{a}{q}n\right)\sum_{m \geq y} \frac{\phi((a/q)p)}{2\sqrt{m}} \int_{-1/2}^{1/2} e(\beta(p + m - n))d\beta$$

$$= \sum_{q \in \mathbb{P}} \sum_{\alpha = 1}^{q - 1} q^{-1}g(a, q)e\left(-\frac{a}{q}n\right)\sum_{m \geq y} \frac{\phi((a/q)p)}{2\sqrt{m}}$$

$$= \sum_{q \in \mathbb{P}} \sum_{\alpha = 1}^{q - 1} \sum_{m \geq y} \frac{\phi(p + m - n)}{2\sqrt{m}}$$

(5.2)

On using partial summation, the innermost sum is equal to

(5.3)

$$K(n, y) \varphi(d) + O\left(1 + \sup_{(n, d) = 1} \sup_{1 \leq t \leq y} \left| \sum_{p \leq y \leq n \ (p, d) = 1} \frac{1}{\varphi(p)} \log p - \frac{t}{\varphi(d)} \right| \right).$$

We now appeal to the well known result on primes in arithmetical progressions [10; Kap. IV. § 3]. It follows from [10; Kap. VII. Satz 6.2, Kap. IV. Satz 8.1] zero free region and [4, 6; Theorem 12.1] zero density estimates for the Dirichlet $L$-functions that, for given positive constants $\varepsilon$, $E$ and $F$,
\[ \sum_{X < p < X + 1} \log p = -\frac{Y}{\varphi(k)} + O(Y \log X^{-\varepsilon}) \]

uniformly for \((l, k) = 1, k \leq (\log X)^{\varepsilon}\) and \(X^{1/12+\varepsilon} \leq Y \leq X\). Hence the O-term in (5.3) is at most

\[ y^{1/2} (\log x)^{-3A-1}, \]

and contributes to (5.2)

\[ \ll y^{1/3} (\log x)^{-3A-1} \sum_{q \leq P} \tau(q) q \]

\[ \ll y^{1/3} (\log x)^{-3A-1} P(x \log P) \]

\[ \ll y^{1/2} P^{-1}. \]

On combining this with (5.2) and (5.3) we have

\[ J_1(n) = \mathcal{K}(n, y) \sum_{q \leq P} q^{-1} \sum_{m \mid q} \sum_{d \mid q} \mu(d) \frac{d}{\varphi(d)} + O(y^{1/3} P^{-1}), \]

Notice that the above sum is

\[ \sum_{q \leq P} \frac{\mu(q)}{\varphi(q)} \sum_{m \mid q} c_{\zeta}(m^2 - n) = \mathcal{E}(n, P) = \sum_{q \leq P} \frac{\mu(q)}{\varphi(q)} g(a, q) e\left(-\frac{a}{q} n\right). \]

We widen the range of \(q\) up to \(\sqrt{y}\). Let \(J_{11}(n)\) be the resulting cost. On employing the large sieve inequality [7, 8] and (3.3),

\[ \sum_{x - \Delta < n < x} |J_{11}(n)|^{2} \ll K(n, y)^{2} \sum_{P < q \leq y} \sum_{a \mid q} \frac{\mu(q)}{q \varphi(q)} g(a, n) e\left(-\frac{a}{q} n\right)^{2} \]

\[ \ll y (\log x)^{-2} \sum_{P < q \leq y} \sum_{a \mid q} \frac{\mu(q)}{q \varphi(q)} g(a, q) \left|\frac{\Delta + q \sqrt{y}}{P + \sqrt{y}}\right| \]

\[ \ll y (\log x)^{-2} \left(\frac{\Delta}{P} + \sqrt{y}\right) \sum_{q \leq y} \frac{\mu(q)}{q \varphi(q)} \]

(5.6)

\[ \ll y(\Delta P^{-1} + \sqrt{y}). \]

In conjunction with (5.4), (5.5) and (5.6) we obtain

\[ J_1(n) - \mathcal{E}(n, \sqrt{y}) \mathcal{K}(n, y) |^{3} \ll \Delta y P^{-1} + y^{3/2}. \]

We proceed to \(J_2\). On using Cauchy's inequality and (3.3),

\[ J_2(n) = \int_{L, M} S(\alpha) V(\alpha) e(-n \alpha) d \alpha \]

\[ = \sum_{q \leq P} \sum_{a = 1}^{q} q^{-1} g(a, q) e\left(-\frac{a}{q} n\right) \int_{1/2 < \beta < 1 + 1/2} S\left(\frac{a}{q} + \beta\right) v(\beta) e(-n \beta) d \beta \]

\[ \ll P \left( \sum_{q \leq P} \sum_{a = 1}^{q} q^{-1} \right)^{1/2} \left( \sum_{1/2 < \beta < 1 + 1/2} S\left(\frac{a}{q} + \beta\right) v(\beta) e(-n \beta) d \beta \right)^{1/2}. \]
By Bessel's inequality and (3.4), we have

\[ \sum_{x - \Delta < \alpha < x} |J_3(n)|^2 \ll \sum_{q \leq P} q^{-1} \sum_{\alpha = 1}^{\infty} \int_{1 < \beta \leq 1} \left| S\left( \frac{\alpha}{q} + \beta \right) \nu(\beta) \right|^2 \, d\beta \]

\[ \ll \sum_{q \leq P} q^{-1} \sum_{\alpha = 1}^{\infty} \int_{1 < \beta \leq 1} \left| S\left( \frac{\alpha}{q} + \beta \right) \right|^2 \, d\beta \]

(5.8)

\[ \ll P^{4} Q \gamma \log x. \]

Next we consider \( J_3 \). Changing the order of summation and integration, we use Cauchy's inequality and (3.4). Thus,

\[ J_3(n) = \int_{m} S(\alpha)V(\alpha)e(-n\alpha) \, d\alpha \]

\[ = \sum_{p \leq q \leq R} \sum_{a = 1}^{q} \int_{1 < \beta \leq 1/4} S\left( \frac{\alpha}{q} + \beta \right) q^{-1} g(a, q) \nu(\beta) e\left( -n\left( \frac{\alpha}{q} + \beta \right) \right) \, d\beta \]

\[ = \sum_{p \leq q \leq R} \sum_{a = 1}^{q} q^{-1} g(a, q) S\left( \frac{\alpha}{q} + \beta \right) e\left( -\frac{\alpha}{q} n \right) \, d\beta \]

or

\[ \sum_{x - \Delta < \alpha < x} |J_3(n)|^2 \ll \left( \log x \right) \sum_{1 < \beta, p \leq x} \sum_{p \leq q \leq R} \sum_{a = 1}^{q} q^{-1} g(a, q) S\left( \frac{\alpha}{q} + \beta \right) e\left( -\frac{\alpha}{q} n \right) \, d\beta. \]

The large sieve [7, 8] yields that

\[ \sum_{x - \Delta < \alpha < x} |J_3(n)|^2 \ll \left( \log x \right) \int_{1 < \beta, p \leq x} \sum_{p \leq q \leq R} \sum_{a = 1}^{q} \left| \left( \frac{\alpha}{q} + \beta \right) S\left( \frac{\alpha}{q} + \beta \right) \right|^2 \, d\beta \]

\[ \ll \left( \log x \right) \left( \frac{\Delta}{P} + R \right) \int_{m} |S(\alpha)|^2 \, d\alpha \]

(5.9)

\[ \ll (\Delta P^{-1} + R) \gamma. \]

We turn to \( J_4 \).

\[ \sum_{x - \Delta < \alpha < x} |J_4(n)|^2 = \sum_{x - \Delta < \alpha < x} \left| \int_{M \cup \infty} S(\alpha)(W(\alpha) - V(\alpha)) e(-n\alpha) \, d\alpha \right|^2 \]

\[ \ll \int_{M \cup \infty} |S(\alpha)|^2 |W(\alpha) - V(\alpha)|^2 \, d\alpha \]

\[ \ll R (\log x)^2 \int_{M \cup \infty} |S(\alpha)|^2 \, d\alpha \]

(5.10)

\[ \ll \gamma^{3/2}, \]

by Bessel's inequality and (3.2). Similarly, by (3.1),
\[ \sum_{x-d<\alpha \leq x} |J_\alpha(n)|^2 = \sum_{x-d<\alpha \leq x} \left| \int_n S(\alpha)W(\alpha)e(-n\alpha)d\alpha \right|^2 \]
\[ \leq \int_n |S(\alpha)W(\alpha)|^2 d\alpha \]
\[ \ll \left( \frac{y}{R} + Q \right)(\log x) \int_n |S(\alpha)|^2 d\alpha \]
\[ \ll y^{1/3}(\log x). \]

(5.11)

In conjunction with (5.7)-(5.11) and (5.1), we have that
\[ \sum_{x-d<\alpha \leq x} |\varepsilon(n, y)| \ll \Delta yP^{-1} + P^2 Q y(\log x)^{-1} + R y + y^{3/12}(\log x) \]
\[ \ll \Delta yP^{-1} + P^2 y^{3/12} \]
\[ \ll \Delta y(\log x)^{-A}, \]
as required.

This completes our proof.

References


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