A characterization of paracompactness of locally Lindelöf spaces

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Abstract. A space $X$ is said to have property $B$ if every infinite open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ such that every point $x \in X$ has a neighborhood $W$ with $|\{V \in \mathcal{V} : W \cap V \neq \emptyset\}| < |\mathcal{U}|$. It is proved that a locally Lindelöf space is paracompact iff it has property $B$.

All spaces are assumed to be regular $T_1$.

A well-known problem posed by Arhangel’skii and Tall is: Is every locally compact normal metacompact space paracompact? The problem is affirmative if we assume $V=L$ [10] or if the space is perfectly normal [1] or boundedly metacompact [5] or locally connected [6].

In connection with this problem, in this paper we give a characterization of paracompactness for locally Lindelöf spaces by using property $B$, and provide another partial answer to the problem.

Property $B$ was introduced originally by Zenor [12] as a generalization of paracompactness: a space $X$ is said to have property $B$, if for every monotone increasing open cover $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$ (that is, $U_\alpha \subseteq U_\beta$ if $\alpha < \beta$) of $X$, there exists a monotone increasing open cover $\mathcal{V} = \{V_\alpha : \alpha \in \kappa\}$ which is a shrinking of $\mathcal{U}$, i.e., $\bigcap V_\alpha \subseteq U_\alpha$ for $\alpha \in \kappa$.

It is proved in [11] that a space $X$ has property $B$ iff every open cover of $X$ of infinite cardinality $\kappa$ has an open refinement $\mathcal{V}$ such that every point $x \in X$ has a neighborhood $W$ with $|\{V \in \mathcal{V} : W \cap V \neq \emptyset\}| < \kappa$; we say such a refinement $\mathcal{V}$ is locally $\kappa$. It is known from Rudin [9] that normal spaces with property $B$ are not necessarily paracompact. However, Balogh and Rudin [3] recently proved that a monotonically normal space is paracompact iff it has property $B$. Using the idea in Balogh [2] we now prove the following theorem.

**Theorem 1.** A locally Lindelöf space is paracompact iff it has property $B$.

**Proof.** Let $X$ be a locally Lindelöf space with property $B$. Suppose $X$ is not paracompact. Then there exists a minimal cardinal $\kappa$ such that we have
some open cover \( \mathcal{U} \) of \( X \) of cardinality \( \kappa \) which has no locally finite open refinement. We will show \( \mathcal{U} \) has, however, a locally finite open refinement. Let \( \mathcal{U} = \{ U_\alpha : \alpha \in \kappa \} \). Since \( X \) is countably paracompact and locally Lindelöf we can assume that \( \kappa > \omega \) and each \( U_\alpha \) is Lindelöf. There are two cases to consider.

Case 1. \( \kappa \) is singular. Then \( \text{cf}(\kappa) = \tau < \kappa \). Let \( \{ \kappa_\mu : \mu \in \tau \} \) be an increasing cofinal subset of \( \kappa \) so that \( \{ \cup U_{\kappa_\mu} : \mu \in \tau \} \) is a monotone increasing open cover of \( X \), where \( U_\alpha = \{ U_\beta : \beta \in \alpha \} \) for every \( \alpha \in \kappa \). Since \( X \) has property \( \mathcal{B} \), there is a monotone increasing open cover \( \{ V_\mu : \mu \in \tau \} \) of \( X \) such that \( \bigcup V_\mu \subseteq \bigcup U_{\kappa_\mu} \) for every \( \mu \in \tau \). By the definition of \( \kappa \), there exists a locally finite open collection \( \mathcal{U}_\mu \) such that \( \mathcal{U}_\mu \) refines \( U_{\kappa_\mu} \) and \( \bigcup U_{\kappa_\mu} \subseteq \bigcup \mathcal{U}_\mu \). Let us consider the open cover \( \mathcal{U} = \bigcup \{ \mathcal{U}_\mu : \mu \in \tau \} \) of \( X \). Note that each member of \( \mathcal{U} \) has Lindelöf closure, it is easy to check that each member of \( \mathcal{U} \) meets at most \( \tau \) many other members of \( \mathcal{U} \). Using usual chaining argument, we may find some partition \( \{ \lambda_\alpha : \alpha \in A \} \) of \( \mathcal{U} \) such that \( (\cup \lambda_\alpha) \cap (\cup \lambda_{\alpha'}) = \emptyset \) if \( \alpha, \alpha' \in A \) with \( \alpha \neq \alpha' \), and \( | \lambda_\alpha | \leq \tau \) for every \( \alpha \in A \). By the definition of \( \kappa \), \( \lambda_\alpha \) has, since \( \cup \lambda_\alpha \) is clopen, a locally finite open refinement \( \mathcal{J}_\alpha \), so that \( \cup \{ \mathcal{J}_\alpha : \alpha \in A \} \) is the desired refinement of \( \mathcal{U} \).

Case 2. \( \kappa \) is regular. Using property \( \mathcal{B} \) find an open refinement \( \mathcal{U} \) of \( \mathcal{U} \) such that every point in \( X \) has a neighborhood \( V \) with

\[ | \{ G : G \in \mathcal{U}, G \cap V = \emptyset \} | < \kappa. \]

Clearly we may assume \( \mathcal{U} = \{ G_\alpha : \alpha \in \kappa \} \) with \( G_\alpha \subseteq U_\alpha \) for every \( \alpha \in \kappa \). Let us first show that

\[ S = \{ \alpha \in \kappa : \overline{G_\alpha} \cap G_\alpha^* = \emptyset \} \]

is a non-stationary subset in \( \kappa \), where \( G_\alpha^* = \cup \{ G_\beta : \beta \in \alpha \} \) for \( \alpha \in \kappa \).

Suppose the contrary that \( S \) is stationary. Then for every \( \alpha \in S \), pick a point \( x_\alpha \in \overline{G_\alpha} \setminus G_\alpha^* \) and let \( s(\alpha) = \sup \{ \mu \in \kappa : x_\alpha \in G_\mu \} \) which belongs to \( \kappa \), since \( \kappa \) is regular. Define a subset \( C \) of \( \kappa \) by

\[ C = \{ \alpha \in \kappa : \beta \in S \cap \alpha \text{ implies } s(\beta) < \alpha \}. \]

Let us check that \( C \) is a c. u. b. set in \( \kappa \). Indeed, if \( \alpha \in C \), then there is a \( \beta \in S \cap \alpha \) with \( s(\beta) \geq \alpha \), so that \( (\beta, \alpha] \) is a neighborhood of \( \alpha \) which misses \( C \). To see \( C \) is unbounded, let \( \alpha \in \kappa \) be given, since \( S \) is stationary, we may find an \( \alpha_t \in S \) such that \( \alpha < \alpha_t \). Proceeding by induction, find an \( \alpha_{n+1} \in S \) so that

\[ \alpha_{n+1} > \sup \{ s(\mu) : \mu \in S, \mu \leq \alpha_n \}. \]

Then we obtain an increasing sequence \( \{ \alpha_n : n \in \mathbb{N} \} \) such that \( \alpha < \sup \{ \alpha_n : n \in \mathbb{N} \} \in C \). This concludes that \( C \) is a c. u. b. set in \( \kappa \). Let \( S_1 = S \cap C \) and for every \( \alpha \in S_1 \) define \( m(\alpha) = \min \{ \mu \in \kappa : x_\alpha \in G_\mu \} \) so that \( \alpha \leq m(\alpha) \leq s(\alpha) \). It follows that
A characterization of paracompactness

$x_\neq G_{m(\beta)}$ and $x_\neq G_{m(\alpha)}$ whenever $\alpha, \beta \subseteq S$ with $\alpha \neq \beta$. This implies that the set $P = \{x_\neq : \alpha \subseteq S\}$ consists of distinct points of $X$, and $\{G_{m(\alpha)} : \alpha \subseteq S\}$ is an open expansion of $P$, i.e., $G_{m(\alpha)} \cap P = \{x_\neq\}$ for every $\alpha \subseteq S$. Now for every $\alpha \subseteq S$, since $x_\neq \subseteq G_{\beta} : \beta \subseteq \alpha]$, there is a $\beta(\alpha) \subseteq \alpha$ such that $G_{\beta} \cap G_{m(\alpha)} \neq \emptyset$. By Pressing Down Lemma, there are a $\beta \subseteq \kappa$ and a stationary set $S_\beta \subseteq S$ such that $\beta(\alpha) = \beta$ for all $\alpha \subseteq S_\beta$, consequently $G_{\beta} \cap G_{m(\alpha)} \neq \emptyset$ for all $\alpha \subseteq S_\beta$. This contradicts our assumption that $\overline{G}_{\beta}$ is Lindelöf.

Now take a c. u. b. set $C_\alpha$ in $\kappa$ such that $C_\alpha \cap S = \emptyset$ and thus $G^{*}_\beta$ is clopen for every $\alpha \subseteq C_\alpha$. Define $H_\alpha$ for $\alpha \subseteq C_\alpha$ by

$$H_\alpha = G^{*}_\beta \cap \{\mu \subseteq C_\alpha \cap \alpha\}$$

so that $X = \cup \{H_\alpha : \alpha \subseteq C_\alpha\}$. Furthermore for every $\alpha \subseteq C_\alpha$, we have

(1) either $H_\alpha = \emptyset$ or $H_\alpha = G^{*}_\alpha \cap G^{*}_\mu(\alpha)$ for some $\mu(\alpha) \subseteq C_\alpha \cap \alpha$. In fact, if $H_\alpha \neq \emptyset$, then there is an $x \subseteq H_\alpha$, and thus $x \subseteq C_\alpha \cap \alpha$. This shows $G^{*}_\alpha \cap C_\alpha = \emptyset$, because if there is some $\mu(\alpha) \subseteq G^{*}_\alpha \cap C_\alpha$, then $x \subseteq G^{*}_\alpha \cap G^{*}_\mu(\alpha)$ which is impossible. Define $\mu(\alpha) = \sup\{\mu \subseteq \gamma : \mu \subseteq C_\alpha\}$ which belongs to $C_\alpha$. Then for every $\mu \subseteq C_\alpha \cap \alpha$, since $\gamma(\alpha) \cap C_\alpha = \emptyset$, we must have $\mu(\alpha) \subseteq \gamma$. This implies $\mu(\alpha) \subseteq \mu(\alpha)$ from which it follows that $H_\alpha = G^{*}_\alpha \cap G^{*}_\mu(\alpha)$, i.e., (1) holds. By the definition of $\kappa$, we can find, for every $\alpha \subseteq C_\alpha$, a locally finite open cover of $\mathcal{A}_\alpha$ of $H_\alpha$ such that every member of $\mathcal{A}_\alpha$ is contained in some member of $\mathcal{U}$, so that $\cup \{\mathcal{A}_\alpha : \alpha \subseteq C_\alpha\}$ is, since $X$ is now the union of the disjoint clopen collection $\{H_\alpha : \alpha \subseteq C_\alpha\}$, a locally finite open refinement of $\mathcal{U}$. Thus the proof is complete.

In [9], by proving that the Navy's space has property $\mathcal{B}$, Rudin shows that normality plus property $\mathcal{B}$ does not imply paracompactness. But the Navy's space is metacompact [7], in connection with Arhangel'skii and Tall's problem, it is natural to ask if the Navy's space is locally compact. But our Theorem 1 even shows that

**Corollary 1.** The Navy's space is not locally Lindelöf.

Also from Theorem 1 the problem of Arhangel'skii and Tall can be stated as follows:

**Problem 1.** Does every locally compact normal metacompact space have property $\mathcal{B}$?

However note that normal metacompact spaces do not necessarily have property $\mathcal{B}$, see Example 4.9 (ii) in [4] or [8] for such a counterexample.
With a modification of proof of Theorem 1 we can prove Arhangel'skii's result mentioned above, even we have

**THEOREM 2.** Locally Lindelöf perfectly normal metacompact spaces are paracompact.

**Proof.** Since normal metacompact spaces are shrinking (thus countably paracompact), \( \kappa \) and a point-finite open cover \( \mathcal{G} = \{ G_\alpha : \alpha \in \kappa \} \) can be defined in the same way as Theorem 1. Clearly we need only consider the case of \( \kappa \) being regular, and it suffices to prove that

\[
S = \{ \alpha \in \kappa : \bigcup_{\beta \leq \alpha} G_\beta \setminus \bigcup_{\beta \leq \alpha} G_\beta = \emptyset \}
\]

is non-stationary.

Suppose indirectly that \( S \) is stationary. As in the proof of Theorem 1, define \( m(\alpha) \in \kappa \) for every \( \alpha \in S \). Without loss of generality, we may assume that there is a \( \beta \in \kappa \) such that

\[
G_{m(\alpha)} \setminus G_\beta \neq \emptyset
\]

for all \( \alpha \in S \).

For every \( n \in \omega \) let

\[
X_n = \{ x \in X : \text{ord}(x, \mathcal{G}) \leq n \}.
\]

Then \( X_n \) is closed in \( X \). Let

\[
S_n = \{ \alpha \in S : G_{m(\alpha)} \cap G_\beta \cap X_n \neq \emptyset \}
\]

so that \( S = \bigcup_{n \in \omega} S_n \) and thus there is a minimal \( n \in \omega \) with \( |S_n| = \kappa \).

Since

\[
G_\beta \cap X_n = G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1})) \cup (G_\beta \cap X_{n-1}),
\]

we can assume that

\[
G_{m(\alpha)} \cap G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1})) \neq \emptyset
\]

for all \( \alpha \in S_n \).

Now every point in \( G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1})) \) has a neighborhood which meets \( G_{m(\alpha)} \cap G_\beta \cap X_n \) for at most finitely many \( \alpha \in S_n \). Since \( X \) is perfect, the set \( G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1})) \) is Lindelöf, and hence

\[
G_{m(\alpha)} \cap G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1})) \neq \emptyset
\]

for at most countably many \( \alpha \in S_n \), a contradiction proving \( S \) is non-stationary. Thus the proof is complete.
A characterization of paracompactness

Note that normal submetacompact spaces are shrinking [11], but we do not know whether in Theorem 2 metacompactness can be replaced by submetacompactness, that is

**Problem 2.** Are locally Lindelöf perfectly normal and submetacompact spaces paracompact?

**References**


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