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SCHWARTZ KERNEL THEOREM FOR THE FOURIER HYPERFUNCTIONS

By

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§ 0. Introduction

The purpose of this paper is to give a direct proof of the Schwartz kernel theorem for the Fourier hyperfunctions. The Schwartz kernel theorem for the Fourier hyperfunctions means that with every Fourier hyperfunction $K$ in $\mathcal{F}(\mathbb{R}^n \times \mathbb{R}^n)$ we can associate a linear map

$$\mathcal{K}: \mathcal{F}(\mathbb{R}^n) \rightarrow \mathcal{F}'(\mathbb{R}^n)$$

and vice versa, which is determined by

$$\langle \mathcal{K}(\phi), \psi \rangle = K(\phi \otimes \psi), \quad \phi \in \mathcal{F}(\mathbb{R}^n), \ \psi \in \mathcal{F}'(\mathbb{R}^n).$$

For the proof we apply the representation of the Fourier hyperfunctions as the initial values of the smooth solutions of the heat equation as in [3] which implies that if a $C^\infty$-solution $U(x, t)$ satisfies some growth condition then we can assign a unique compactly supported Fourier hyperfunction $u(x)$ to $U(x, t)$ (see Theorem 1.4). Also we make use of the following real characterizations of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions in [1, 3, 5]

$$\mathcal{F} = \left\{ \phi \in C^\infty \mid \sup_{a, z} \frac{|\partial^a \phi(x)| \exp k |x|}{h^{a_1} \alpha !} < \infty \text{ for some } h, k > 0 \right\}$$

$$= \left\{ \phi \in C^\infty \mid \sup |\phi(x)| \exp k |x| < \infty, \ \sup |\phi(\xi)| \exp h |\xi| < \infty \right\}$$

for some $h, k > 0$.

Also, we closely follow the direct proof of the Schwartz kernel theorem for the distributions as in Hörmander [2].

§ 1. Preliminaries

We denote by $x=(x_1, x_2) \in \mathbb{R}^n$ for $x_1 \in \mathbb{R}^n_1$ and $x_2 \in \mathbb{R}^n_2$, and use the multi-index notation $|\alpha| = a_1 + \cdots + a_n$, $\partial^\alpha = \partial^{a_1} \cdots \partial^{a_n}$ for $\alpha = (a_1, \cdots, a_n) \in \mathbb{N}_0^n$ where

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$N_0$ is the set of nonnegative integers.

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx, \quad \xi \in \mathbb{R}^n.$$  

We first give two equivalent definitions of the space $\mathcal{S}$ of test functions for the Fourier hyperfunctions in [1, 3, 5] as follows:

**Definition 1.1 ([3]).** An infinitely differentiable function $\varphi$ is in $\mathcal{S}(\mathbb{R}^n)$ if there are positive constants $h$ and $k$ such that

$$\varphi \in C^\infty_{h, k},$$

where

$$\varphi \in C^\infty_{h, k} = \left\{ \varphi \in C^\infty \mid \varphi \mid_{h, k} = \sup_{n, \alpha} \frac{\partial^n \varphi(x)}{h^{|\alpha|} \alpha!} \exp k |x| < \infty \right\}.$$

**Definition 1.2 ([1]).** The space $\mathcal{S}$ of test functions for the Fourier hyperfunctions consists of all $C^\infty$ functions such that for some $h, k > 0$

$$\sup_{x} |\varphi(x)| \exp k |x| < \infty,$$

$$\sup_{\xi} |\hat{\varphi}(\xi)| \exp h |\xi| < \infty.$$

We denote by $E_t(x)$ the $n$-dimensional heat kernel;

$$E_t(x) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

We now need the following Proposition 1.3 and Theorem 1.5 to prove the Main theorem in §2.

**Proposition 1.3 ([4]).** There are positive constants $C$ and $a$ such that

$$|\partial^a E_t(x)| \leq C |n|^{1/2} (t^{-(n+|a|)/2} \alpha!)^{1/2} \exp(-a |x|^2/4t),$$

where $a$ can be taken as close as desired to 1 and $0 < a < 1$.

From Proposition 1.3 we can easily obtain the following

**Corollary 1.4.** There exist positive constants $C, C' > 0$ such that for every $\varepsilon > 0$ and sufficiently small $t > 0$

$$|E_t(x)|_{C^2 \varepsilon^{-1/2}, \varepsilon} \leq C' \varepsilon^{-n/2} \exp(\varepsilon(1/|x|)).$$

**Proof.** By Proposition 1.3 we can easily see that there exist positive constants $C, C' > 0$ such that for every $\varepsilon > 0$

$$\sup_{x, \|a\|} \frac{|\partial^a E_t(x-y)| \exp|y|}{(C \varepsilon^{-1/2})^{|\alpha|/2} \alpha!} \leq C' \varepsilon^{-n/2} \exp(\varepsilon/2t) \exp(2\varepsilon^2t) \exp\varepsilon |x|.$$
In fact, we have

\[ |\partial_\xi E_t(x-y)| \leq C |a|^{(n+1)/2} \epsilon^{n/2} \exp(-a|x-y|^2/4t) \]

\[ \leq C |a|^{(n+1)/2} \exp(-a|x-y|^2/4t) \exp(2|n+|a||a|!\alpha^{1/2} \exp(-a|x-y|^2/4t) \exp(\sqrt{2}\epsilon^{-1/2})\alpha^{1/2} \exp(-a|x-y|^2/4t). \]

Thus, we obtain that for every \( \epsilon > 0 \) and small \( t > 0 \)

\[ |E_t(x-y)| < \epsilon \leq C \epsilon^{-n/2} \exp]\[ (1/t + |x|) \]

which completes the proof.

**Theorem 1.5 ([3]).** Let \( u \in \mathcal{D}' \) and \( T > 0 \). Then \( U(x, t) = u \tau(E(x-y, t)) \) is a \( C^\infty \) function in \( \mathbb{R}^n \times (0, T) \) and satisfies the following:

1. \( (\partial/\partial t - \Delta)U(x, t) = 0 \) in \( \mathbb{R}^n \times (0, T) \).
2. For every \( \epsilon > 0 \) there exists a constant \( C > 0 \) such that

\[ |U(x, t)| \leq C \epsilon \exp[\epsilon(1/t + |x|)] \text{ in } \mathbb{R}^n \times (0, T). \]

3. \( \lim_{t \to 0^+} U(x, t) = u \) in \( \mathcal{D}' \) i.e.,

\[ u(\varphi) = \lim_{t \to 0^+} \int U(x, t) \varphi(x) \, dx, \quad \varphi \in \mathcal{D}. \]

Conversely, every \( C^\infty \) function \( U(x, t) \) in \( \mathbb{R}^n \times (0, T) \) satisfying (i) and (ii) can be expressed in the form \( U(x, t) = u \tau(E(x-y, t)) \) with a unique element \( u \in \mathcal{D}' \).

**Theorem 1.6 ([3]).** If \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) then it follows that \( \varphi * E_t \) converges to \( \varphi \) in \( \mathcal{D}(\mathbb{R}^n) \) when \( t \to 0^+ \).

We shall prove the associativity for convolution in \( \mathcal{D}(\mathbb{R}^n) \).

**Theorem 1.7.** If \( u \in \mathcal{D}'(\mathbb{R}^n) \) and \( \varphi, \psi \in \mathcal{D}(\mathbb{R}^n) \) then

\[ (u * \varphi) * \psi = u * (\varphi * \psi). \]

The proof is an easy consequence of the following

**Theorem 1.8.** If \( \varphi \in \mathcal{D}_{h_1, h_2}(\mathbb{R}^n), \ \psi \in \mathcal{D}_{h_2, h_2}(\mathbb{R}^n) \), then the Riemann sum

\[ \sum_{j \in \mathbb{Z}^n} \varphi(x-js)s^\delta \psi(js) \]

converges to \( \varphi * \psi(x) \) in \( \mathcal{D}_{h_1, h_2} \) when \( s \to 0 \) for \( h > \max\{h_1, h_2, 2\sqrt{2}\} \), \( k < \min\{k_1, k_2\} \).

Before proving Theorem 1.8 we show the following refinement of Definition
1.2 which is the main theorem in [1].

**Lemma 1.9.** Let $h > 2 \sqrt{2}$ and $k > 0$. Then the following conditions are equivalent:

(i) $\varphi \in \mathcal{F}_{h,k}$

(ii) $$\sup_x |\varphi(x)| \exp k |x| < \infty \quad \sup_x |\partial^a \varphi(x)| \leq C(h/2 \sqrt{2})^a \alpha!.$$  

(iii) There exists an integer $a > 2 \sqrt{2}$ such that

\begin{align*}
(1.2) & \quad \sup_x |\varphi(x)| \exp k |x| < \infty , \\
(1.3) & \quad \sup_x |\hat{\varphi}(\xi)| \exp(2 \sqrt{2} |\xi| / ah) < \infty .
\end{align*}

**Proof.** It follows from Theorem 2.1 in [1] that (ii) is a sufficient condition for $\varphi \in \mathcal{F}_{h,k}$. So it suffices to prove the implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii)

(i) $\Rightarrow$ (iii): It suffices to show (1.3). We obtain from (i) that

\begin{align*}
|\xi^a \hat{\varphi}(\xi)| &= \left| \int e^{-iz \cdot \xi} \partial^a \varphi(x) dx \right| \\
&\leq C_1 h^n \alpha! \int \exp(-k |x|) dx \\
&\leq C_1 (ah/2 \sqrt{2})^n \alpha! (2 \sqrt{2} / a)^{n \alpha} \text{ for all } \alpha
\end{align*}

where $a > 2 \sqrt{2}$. Hence

$$\sum_1/a! (2 \sqrt{2} |\xi| / ah)^{n \alpha} |\hat{\varphi}(\xi)| \leq C_1 \sum_1 (2 \sqrt{2} / a)^{n \alpha} < \infty$$

Therefore we obtain (1.3).

(iii) $\Rightarrow$ (ii): By Hölder’s inequality we have

\begin{align*}
|\partial^a \varphi(x)|^{\frac{4a}{a}} &= \frac{1}{(2\pi)^{\frac{4a}{4a}}} \int \left| e^{ix \cdot \xi} \partial^a \varphi(\xi) d\xi \right|^{\frac{4a}{4a}} \\
&\leq \frac{1}{(2\pi)^{\frac{4a}{4a}}} \left( |\xi|^{\frac{4a}{4a}} |\hat{\varphi}(\xi)| \int d\xi \left( |\hat{\varphi}(\xi)|^{\frac{4a}{4a}} d\xi \right)^{\frac{4a}{4a} - 1} \right) \\
&\leq C \sup_{\xi} \exp(2 \sqrt{2} |\xi| / h) \\
&\leq C'(h/2 \sqrt{2})^{4a |\alpha|} (\alpha!)^{a}.
\end{align*}

Thus we obtain (ii).

**Lemma 1.10.** Let $k > 0$ and $j = (j_1, \ldots, j_n)$, $j_i \in \mathbb{N}_0$, and let $0 < s < A$ for some fixed $A$. Then
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\[ \sum_{j \in \mathbb{N}_0^n} s^n \exp(-k s |j|) < C \]

where \( C \) is independent of \( s \) and \( |j| \) is a Euclidean norm.

**Proof.** Note that \( \sqrt{n} |j| \leq \sum_j j_i \) for \( j \in \mathbb{N}_0^n \) and that the function \( x/(1 - \exp(-k x)) \) is strictly increasing for \( x > 0 \). If \( 0 < s < A \) then

\[ \sum_{j \in \mathbb{N}_0^n} s^n \exp(-k s |j|) \leq \sum_{j_1 \in \mathbb{N}_0} s \exp(-k s j_1 / \sqrt{n}) \times \cdots \times \sum_{j_n \in \mathbb{N}_0} s \exp(-k s j_n / \sqrt{n}) \]

\[ = \left( \frac{2s}{1 - \exp(-k s / \sqrt{n})} \right)^n \]

\[ < \left( \frac{2A}{1 - \exp(-k A / \sqrt{n})} \right)^n. \]

**Proof of Theorem 1.8.** Choose \( h, k > 0 \) such that \( h > \max\{h_1, h_2, 2 \sqrt{2}\} \) and \( k < \min\{k_1, k_2\} \). Let \( f_s(x) = \sum_j \varphi(x - js)s^n \phi(js) \), \( s > 0 \). By Lemma 1.9 we shall show that for any \( \varepsilon > 0 \) there exists a constant \( \delta > 0 \) such that if \( s < \delta \) then

(1.4) \[ \sup_x |f_s(x) - \varphi \ast \phi(x)| \exp k |x| < \varepsilon, \]

(1.5) \[ \sup_x |f_s(x) - \varphi \ast \phi(x)| \exp(2 \sqrt{2} |x| / ah) < \varepsilon \]

where \( a > 2 \sqrt{2} \). From now on we take \( a = 4 \sqrt{2} \). Choose \( k' \) such that \( k < k' < \min\{k_1, k_2\} \).

If \( s < A \) then \( f_s \in \mathcal{F}_{h, k'} \) by Lemma 1.10. In fact,

\[ |f_s|_{h, k'} \leq \sum_j |\partial^n \varphi(x - js)| s^n |\phi(js)| \exp k' |x - js| \exp k'| js | \]

\[ \leq C \sum_j s^n \exp(-(k_2 - k') |js|) \]

\[ \leq M_1 \]

where \( M_1 \) is independent of \( s < A \). Similarly we obtain \( \varphi \ast \phi \in \mathcal{F}_{h, k'} \). For any \( \varepsilon > 0 \) choose \( R = R_i > 0 \) such that

\[ \exp(-(k' - k) R) < \varepsilon, \quad \exp \left( -\frac{1}{2} \left( \frac{1}{h_1} - \frac{1}{h} \right) R \right) < \varepsilon. \]

Thus for all \( s < A \) we obtain

(1.6) \[ \sup_{|x| \geq R} |f_s(x) - \varphi \ast \phi(x)| \exp k |x| \leq \sup_{|x| \geq R} \left( |f_s(x)| + |\varphi \ast \phi(x)| \right) \exp k |x| \]

\[ \leq C \sup_{|x| \geq R} \exp(-k' |x|) \exp k |x| \]
Note that for any $s>0$ the function $f_s(x)$ is continuous on the compact set \{\(x\mid |x| \leq R\)$ and the sequence $\{f_s\}, 0<s<A$ is bounded and equicontinuous. In fact, for $|x| \leq R$ we have
\[
|f_s| \leq C' \sum_j \exp(-k_1 |x - js|) s^n \exp(-k_2 |js|)
\leq C'e^{k_1 R} \sum_j \exp(-(k_1 + k_2) |js|) s^n
\leq M_2
\]
where $M_2$ is independent of $s<A$. The last inequality is also obtained by Lemma 1.10. Also, for any $\varepsilon>0$ there exists $\delta_1>0$ such that if $|x_1 - x_2| < \delta_1$ then
\[
|f_s(x_1) - f_s(x_2)| = \sum_j |\psi(x_1 - js) - \psi(x_2 - js)| s^n |\phi(j s)|
\leq \sum_j |\nabla \psi(\xi)| |x_1 - x_2| s^n |\phi(j s)|
\leq M |x_1 - x_2|
\]
where the second inequality is obtained from (1.7). Thus, by Arzela-Ascoli's theorem we obtain that for $|x| \leq R$ the sequence $\{f_s\}$ converges uniformly to $\varphi*\psi(x)$, i.e., for any $\varepsilon>0$ there exists $\delta_2>0$ such that if $s<\delta_2$ then
\[
\sup_{|x| \leq R} |f_s(x) - \varphi*\psi(x)| \exp k |x| < \varepsilon
\]
If $\delta = \min \{A, \delta_1\}$ then (1.4) is obtained from (1.6) and (1.9). On the other hand, if $g_s(\xi) = \sum_j s^n \exp(-i(j s \cdot \xi)) \psi(j s)$ we obtain for some $B>0$ the sequence $\{g_s\}, 0<s < B$ is bounded and equicontinuous as (1.7) and (1.8). Thus for $|\xi| \leq R$ the sequence $\{g_s\}$ converges uniformly to $\psi(\xi)$, i.e., for any $\varepsilon>0$ there exists $\delta_3>0$ such that if $s<\delta_3$ then
\[
\sup_{|\xi| \leq R} |g_s(\xi) - \psi(\xi)| < \varepsilon.
\]
From the above fact we obtain (1.5). In fact, if $s<\delta = \min \{\delta_2, B\}$ then
\[
\sup_{\xi} |\hat{f}_s(\xi) - \varphi*\hat{\psi}(\xi)| \exp(|\xi|/2h)
= \sup_{\xi} \left| \sum_j \hat{\psi}(\xi) \exp(-i(j s \cdot \xi)) s^n \phi(j s) - \phi(\xi) \phi(\xi) \exp(|\xi|/2h) \right|
\leq \sup_{\xi} |\phi(\xi)| \exp(|\xi|/2h) |g_s(\xi)| - \phi(\xi)|
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\[ C \sup_{\xi \in \mathbb{R}} \exp \left( -\frac{1}{2} \left( \frac{1}{h_1} - \frac{1}{h} \right) |\xi| \right) |g_{x}(\xi) - \phi(\xi)| \\
+ C \sup_{\xi \in \mathbb{R}} \exp \left( -\frac{1}{2} \left( \frac{1}{h_1} - \frac{1}{h} \right) |\xi| \right) \left( |g_{x}(\xi)| + |\phi(\xi)| \right) \\
\leq C' \sup_{\xi \in \mathbb{R}} |g_{x}(\xi) - \phi(\xi)| + C \exp \left( -\frac{1}{2} \left( \frac{1}{h_1} - \frac{1}{h} \right) R \right) \\
\leq M_{h} \varepsilon ,
\]

which completes the proof.

**Theorem 1.11.** If \( u \in \mathcal{S}'(\mathbb{R}^{n}) \) then \( u*E_{t} \) converges to \( u \) in \( \mathcal{S}'(\mathbb{R}^{n}) \) as \( t \to 0^{+} \).

**Proof.** We note that \( u(\phi) = u*\phi(0) \) if \( \phi \in \mathcal{S}(\mathbb{R}^{n}) \) and \( \phi(x) = \phi(-x) \). This gives

\[ (u*E_{t})(\phi) = (u*E_{t})*\phi(0) = u*(E_{t})*\phi(0) = u(E_{t})*\phi. \]

By Theorem 1.6 \( E_{t} * \phi \) converges to \( \phi \) in \( \mathcal{S}(\mathbb{R}^{n}) \) as \( t \to 0^{+} \). So it follows that \( (u*E_{t})(\phi) \) converges to \( u(\phi) \) as claimed.

**§ 2. Main Theorem**

We are now in a position to state and prove the Schwartz kernel theorem for the space \( \mathcal{S}' \).

**Theorem 2.1.** If \( K \in \mathcal{S}'(\mathbb{R}^{n_{1}} \times \mathbb{R}^{n_{2}}) \) then a linear map \( \mathcal{K} \) determined by

\[ \langle \mathcal{K} \varphi, \phi \rangle = K(\varphi \otimes \phi), \quad \varphi \in \mathcal{S}(\mathbb{R}^{n_{1}}), \phi \in \mathcal{S}(\mathbb{R}^{n_{2}}) \]

is continuous in the sense that \( \mathcal{K} \varphi_{j} \) converges to \( 0 \) in \( \mathcal{S}'(\mathbb{R}^{n_{1}}) \) if \( \varphi_{j} \) converges to \( 0 \) in \( \mathcal{S}(\mathbb{R}^{n_{2}}) \). Conversely, for every such linear map \( \mathcal{K} \) there is one and only one Fourier hyperfunction \( K \) such that (2.1) is valid.

**Proof.** If \( K \in \mathcal{S}'(\mathbb{R}^{n_{1}} \times \mathbb{R}^{n_{2}}) \) then (2.1) defines a Fourier hyperfunction \( \mathcal{S} \varphi \), since the map \( \varphi \to K(\varphi \otimes \phi) \) is continuous. Also \( \mathcal{K} \) is continuous, since the map \( \varphi \to K(\varphi \otimes \phi) \) is continuous.

Let us now prove the converse. We first prove the uniqueness, i.e., if

\[ u(\phi \otimes \varphi) = 0 \quad \text{for} \quad \varphi \in \mathcal{S}(\mathbb{R}^{n_{1}}), \phi \in \mathcal{S}(\mathbb{R}^{n_{2}}), \]

then \( u = 0 \) in \( \mathcal{S}'(\mathbb{R}^{n_{1}} \times \mathbb{R}^{n_{2}}) \).

It follows from Theorem 1.11 that \( u*E_{t} \) converges to \( u \) in \( \mathcal{S}'(\mathbb{R}^{n}) \) as \( t \to 0^{+} \). However, \( u*E_{t} = 0 \), since \( E_{t}(x_{1} - y_{1}, x_{2} - y_{2}) \) is the product of a function of \( y_{1} \) and one of \( y_{2} \). Hence \( u = 0 \) in \( \mathcal{S}' \).
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We now prove the existence. Since \( \mathcal{K} \) is continuous, the bilinear form on 
\( \mathcal{F}_{h_1, k_1}(\mathbb{R}^{n_1}) \times \mathcal{F}_{h_2, k_2}(\mathbb{R}^{n_2}) \)
\[
\langle \phi, \varphi \rangle \mapsto \langle \mathcal{K} \varphi, \phi \rangle
\]
is separately continuous, therefore continuous, since \( \mathcal{F}_{h, k} \) is a Fréchet space
for all \( h, k > 0 \). Hence we obtain that there is a constant \( C(h_1, k_1, h_2, k_2) \) such that
\begin{equation}
|\langle \mathcal{K} \varphi, \phi \rangle| \leq C|h_1, k_1|\varphi|h_{h_2, k_2}.
\end{equation}

Set for \((x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\) and small \( t > 0 \)
\begin{equation}
K_t(x_1, x_2) = \langle \mathcal{K} E_{t, x}(x_2 - \cdot), E_{t, x}(x_1 - \cdot) \rangle
\end{equation}
where \( E_{t, x}(x) \) is the \( n \)-dimensional heat kernel.

We now show that \( K_t \) has a limit in \( \mathcal{F}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \) as \( t \to 0 \), and then show
that (2.1) is also satisfied by the limit. It follows from (2.2) and Corollary 1.4
that for every \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that
\[
|K_t(x_1, x_2)| \leq C_\varepsilon \exp(1/t + |x|).
\]
Since
\[
\frac{\partial E_t}{\partial t} = \Delta_x E_t, \quad t > 0
\]
we have
\[
\frac{\partial K_t}{\partial t} = \Delta_x K_t.
\]
It follows from Theorem 1.5 that there exists a limit \( K_0 \in \mathcal{F}' \) such that \( K_t \)
converges to \( K_0 \) in \( \mathcal{F}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \).

Let \( \varphi_j \in \mathcal{F}(\mathbb{R}^{n_j}), \ j = 1, 2 \) and form
\[
\langle K_t, \varphi_1 \otimes \varphi_2 \rangle = \int \int K_t(x_1, x_2) \varphi_1(x_1) \varphi_2(x_2) dx_1 dx_2.
\]
We have
\[
\int \int K_t(x_1, x_2) \varphi_1(x_1) \varphi_2(x_2) dx_1 dx_2
\]
\[
= \int \int \langle \mathcal{K} E_{t, x}(\cdot - x_2) \varphi_2(x_2), E_{t, x}(\cdot - x_1) \varphi_1(x_1) \rangle dx_1 dx_2.
\]
Approximating the above integral by the Riemann sum we obtain from Lemma
1.8 that
\[
\langle K_t, \varphi_1 \otimes \varphi_2 \rangle = \langle \mathcal{K} (\varphi_2 \ast E_{t, x}), \varphi_1 \ast E_{t, x} \rangle.
\]
Since \( \varphi_j \ast E_{t, x} \) converges to \( \varphi_j \) in \( \mathcal{F}(\mathbb{R}^{n_j}) \) as \( t \to 0 \), it follows from (2.2) that the
right hand side converges to \( \langle \mathcal{K} \varphi_j, \varphi_1 \rangle \) as \( t \to 0 \). Thus
\[
\langle K_0, \varphi_1 \otimes \varphi_2 \rangle = \langle \mathcal{K} \varphi_2, \varphi_1 \rangle
\]
which completes the proof.
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