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GRADED COALGEBRAS
AND MORITA-TAKEUCHI CONTEXTS

By
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0. Introduction

Viewing a $G$-graded $k$-coalgebra over the field $k$ as a right $kG$-comodule coalgebra it is possible to use a Hopf algebraic approach to the study of coalgebras graded by an arbitrary group that was started in [NT].

Let $C = \bigoplus_{g \in G} C_g$ be a $G$-graded coalgebra. The graded $C$-comodules may be viewed as comodules over the smash product $C \rtimes kG$, the general definition of which was given in [M]. Coalgebras graded by an arbitrary group have been considered in [FM] in order to introduce the notion of $G$-graded Hopf algebras. On the other hand, M. Takeuchi introduced in [T] the sets of pre-equivalence data connecting categories of comodules over two coalgebras (we call such a set a Morita-Takeuchi context). The main result of this note is a coalgebra version of a result established by M. Cohen, S. Montgomery in [CM] for group-graded rings: for a graded coalgebra $C$ the coalgebras $C$, and $C \rtimes kG$ are connected by a Morita-Takeuchi context in which one of the structure maps is injective. Most of the results in this note are consequences of the foregoing. As a first application we find that a coalgebra $C$ is strongly graded if and only if the other structure map of the context is also injective. The final section provides analogues of the Cohen-Montgomery duality theorems: if $C$ is a coalgebra graded by the finite group $G$ of order $n$, then $G$ acts on the smash coproduct as a group of automorphisms of coalgebras and $(C \rtimes kG) \rtimes kG^*$ is coalgebra isomorphic to the comatrix coalgebra $M^c(n, C)$. If $G$ is a finite group of order $n$, acting on the coalgebra $D$ as a group of coalgebra automorphisms, then the smash coproduct $D \rtimes kG^*$ is strongly graded by $G$ and moreover: $(D \rtimes kG^*) \rtimes kG \cong M^c(n, D)$. The second duality theorem is again a direct consequence of the Morita-Takeuchi context mentioned above.

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1. Graded Coalgebras and the Smash Coproduct

Throughout this paper $k$ is a field. We use Sweedler’s “sigma” notation [S] and further notation and conventions in [T], [D]. Let $G$ be a group with identity element 1. Recall that a $k$-coalgebra $(C, \Delta, \varepsilon)$ is graded by $G$ if $C$ is a direct sum of $k$-subspaces, $C = \bigoplus_{g \in G} C_g$, such that $\Delta(C_g) \subseteq \sum_{x+y=g} C_x \otimes C_y$, for all $g \in G$, and $\varepsilon(C_g) = 0$ for $g \neq 1$. A right $C$-comodule $M$ with structure map $\rho : M \to M \otimes C$ is a graded $C$-comodule if $M = \bigoplus_{g \in G} M_g$ as $k$-subspaces, such that $\rho(M_g) \subseteq \sum_{x+y=g} M_x \otimes C_y$ for all $g \in G$. For graded right $C$-comodules $M$ and $N$ a graded comodule morphism is a $C$-comodule morphism $f : M \to N$ such that $f(M_g) \subseteq N_g$ for all $g \in G$. The category of graded right $C$-comodules, denoted by $\text{grc}$, is a Grothendieck category, cf. [NT]. The main purpose of this section is to develop a Hopf algebraic approach to the graded theory. First we recall, see [S] or [A], some definitions.

1.1. Definition. Let $H$ be a bialgebra over the field $k$, $A$ a $k$-algebra and $(C, \Delta_c, \varepsilon_c)$ a $k$-coalgebra. Then:

i. $A$ is said to be a (right) $H$-module algebra if $A$ is a right $H$-module such that $(ab) \cdot h = \sum (a \cdot h_1)(b \cdot h_2)$ and $1_A \cdot h = \varepsilon(h)1_A$ for any $h \in H$, and $a, b \in A$.

ii. $C$ is a right $H$-comodule coalgebra if $C$ is an $H$-comodule by $c \to \sum c(0) \otimes c(1)$ such that we have:

\[ \sum c_1(0) \otimes c_2(0) \otimes c_1(1) c_2(1) = \sum c(0)_1 \otimes c(0)_2 \otimes c(1) , \]

\[ \sum \varepsilon_c(c(0)) c(1) = \varepsilon_c(c)1_H \quad \text{for all } c \in C \]

iii. $C$ is a (left) $H$-module coalgebra if $C$ is a left $H$-module such that:

\[ \Delta_c(h \cdot c) = \sum h_1 c_1 \otimes h_2 c_2, \varepsilon_c(h \cdot c) = \varepsilon_H(h) \varepsilon_c(c) \quad \text{for } c \in C, h \in H. \]

In the sequel we shall not refer to “right” or “left” as in the above definitions, the choice of “sides” shall remain fixed throughout.

For any group $G$ the group algebra $kG$ has a bialgebra structure defined by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$. The next result establishes the connection between $G$-graded coalgebras and $kG$-comodule coalgebras.

1.2. Proposition. A coalgebra $C$ graded by $G$ many in a natural way be viewed as a $kG$-comodule coalgebra; conversely every $kG$-comodule coalgebra is a $G$-graded coalgebra.

Proof. For a $G$-graded $C$ the map $\rho : C \to C \otimes kG$, $c \to c \otimes \sigma$ for all $\sigma \in G$, ...
Graded coalgebras and Morita-Takeuchi contexts

$c\in C_{\sigma}$ defines a $kG$-comodule coalgebra structure on $C$. Conversely, if $C$ is a $kG$-comodule coalgebra then any $c\in C$ has a unique presentation $\rho(c) = \sum_{g\in G} c_g \otimes g$. Put $C_g = \{c_g : c\in C, g\in G\}$. From $(I\otimes \varepsilon)(\rho(c)) = c\otimes 1$ we derive that $c = \sum_{g\in G} c_g$ and $C = \sum_{g\in G} C_g$. For $c\in C$, $g\in G$ we have that $c\in C_g$ if and only if $\rho(c) = c_g\otimes g$. We say that the group $G$ acts on the coalgebra $D$ whenever there is a group morphism $\varphi : G\to \text{Aut}(D)$, the latter denoting the set of all coalgebra automorphisms of $D$ with group structure defined as follows: if $f, g\in \text{Aut}(D)$, $f \cdot g = f \cdot g$.

1.3. Proposition. If $G$ acts on the coalgebra $D$ then $D$ has the structure of a $kG$-module coalgebra; conversely any $kG$-module coalgebra has a natural $G$-action.

Proof. Suppose that $\varphi : G\to \text{Aut}(D)$ determines that $G$ acts on $D$ then the map $kG\otimes D\to D, g\otimes d\mapsto \varphi(g)(d)$ defines a $kG$-module structure on $D$ as desired. Conversely, if $D$ is a $kG$-module coalgebra then we may define a $G$-action on $D$ by $\varphi : G\to \text{Aut}(D), \varphi(g)(d) = g\cdot d$ for $g\in G, d\in D$.

1.4. Remark. Let, for a finite group $G$, $kG^*$ be the dual bialgebra for the finite dimensional bialgebra $kG$. If the finite group $G$ acts on the coalgebra $D$ then $D$ is also a $kG^*$-comodule coalgebra. If $\{p_g, g\in G\}$ is the dual basis of $\{g, g\in G\}$ then $\{p_g, g\in G\}$ is a system of orthogonal idempotents of $kG^*$. The coalgebra structure of $kG^*$ is given in the usual way by: $\Delta(p_g) = \sum_{x,y=g} p_x \otimes p_y, \varepsilon(p_g) = \delta_{g,1}$.

The right comodule structure of $D$ is given by $\rho : D\to D \otimes kG^*, \rho(d) = \sum_{g\in G} (g\cdot d) \otimes p_g$.

In the sequel, the smash coproduct plays a central part. For a bialgebra $H$ and an $H$-module coalgebra $C$ the smash-coproduct $C \rtimes H$ is defined as the $k$-space $C \otimes H$ with $\Delta : C \rtimes H \to (C \rtimes H) \otimes (C \rtimes H)$ given by $\Delta(c \rtimes h) = \sum (c_1 \rtimes c_2) \otimes (c_1^{(0)} \rtimes h_1)$, and $\varepsilon : C \rtimes H \to k$ given by $\varepsilon(c \rtimes h) = \varepsilon(c)\varepsilon_H(h)$.
1.5. Proposition. \( C \times H \) with \( \Delta \) and \( \varepsilon \) as above is a coalgebra.

Proof. This is just the right hand version of Theorem 2.11 of [M], a proof is given in Proposition 2.3 of [FM]. \( \square \)

The smash coproduct is useful in general but has particular interest in some special cases frequently considered:

i. Graded smash coproduct

If the coalgebra \( C \) is graded by \( G \) then the coalgebra structure of \( C \times kG \) is given by: 
\[
\Delta(c \times g) = \sum (c_i \times \deg c_i g) \otimes (c_i \otimes g),
\]
for any homogeneous \( c \in C \) and \( g \in G \) (where we assumed, as we will always do in the sequel, that we have used the homogeneous decomposition \( \sum c_i \otimes c_i \)), whereas for all \( c \in C, g \in G \) we have that \( \varepsilon(c \times g) = \varepsilon_c(c) \).

ii. If the finite group \( G \) acts on the coalgebra \( D \), i.e. \( D \) is a \( kG \)-comodule coalgebra, then the coalgebra structure of \( D \times kG^* \) is given by:
\[
\Delta(d \times p_g) = \sum_{u \in G} (d \times p_g) \otimes (v \cdot d \times p_u),
\]
and
\[
\varepsilon(d \times p_g) = \varepsilon_g(d) \delta_{k,1}, \quad \text{for all } d \in D, g \in G.
\]

Note that the graded smash coproduct appears in a natural way when one studies graded comodules. Recall that a \( k \)-Abelian category is \( k \)-equivalent to a category of comodules \( \mathcal{M} \) over some coalgebra \( C \) if and only it is of finite type (Theorem 5.1 of [T]). The coalgebra giving the category as a category of comodules may, in general, be a somewhat mystical object. However for a \( G \)-graded coalgebra \( C \) the \( k \)-Abelian category of graded comodules, say \( \text{gr}^G \), is of finite type and it is therefore, equivalent to a category of comodules over the coalgebra given in the following.

Theorem 1.6. If \( C \) is a coalgebra graded by \( G \) then the categories \( \text{gr}^G \) and \( \mathcal{M}^{C \times kG} \) are isomorphic.

Proof. Take \( M \in \text{gr}^G \) with \( \rho : M \rightarrow M \otimes C, \rho(m) = \sum m_0 \otimes m_1 \). We make \( M \) into a right \( C \times kG \)-comodule by defining \( \rho' : M \rightarrow M \otimes (C \times kG), m \mapsto \sum m_0 \otimes (m_1 \otimes (\deg m)^{-1}) \) for homogeneous \( m \in M \). A morphism \( f : M \rightarrow N \) of \( G \)-graded \( C \)-comodules is also a morphism of \( C \times kG \)-comodules and we have defined a functor \( T : \text{gr}^G \rightarrow \mathcal{M}^{C \times kG} \).

Conversely, starting from an \( M \in \mathcal{M}^{C \times kG} \) we obtain on \( M \) a right \( C \)-comodule
structure and a right $kG$-comodule structure because the linear maps $\alpha: C \times kG \rightarrow C$, $c \times g \mapsto c$, and $\beta: C \times kG \rightarrow kG$, $c \times g \mapsto c(c)g^{-1}$ for $c \in C$, $g \in G$, are coalgebra morphisms. As in the proof of Proposition 1.2 it follows that $M = \bigoplus_{g \in G} M_g$ and a straightforward verification learns that $M$ becomes a graded $C$-comodule. Now, for $M, N \in \mathcal{M}^{C \times kG}$ and a morphism of $C \times kG$-comodules $f: M \rightarrow N$ it follows that $f$ is also a morphism of $G$-graded $C$-comodules when $M$ and $N$ are viewed as such. This defines the functors $S: \mathcal{M}^{C \times kG} \rightarrow \text{gr}C$ and it is easily seen that $T$ and $S$ are isomorphisms of categories and inverse to each other.

1.7. Remarks. 1. If the coalgebra $C$ is graded by a finite group $G$, then the dual algebra $C^*$ is graded by $G$ with $C^*_x = \{ f \in C^*, f(C_z) = 0 \text{ for all } x \neq g \}$. Hence $C^*$ is a $kG^*$-module algebra and we may construct the smash product $C^* \# kG^*$ with multiplication given by $(c^* \# h^*)(d^* \# g^*) = \sum (c^*(d^* \cdot h^*)) \# g^* h^*$, for all $c^*, d^* \in C^*$ and $h^*, g^* \in kG^*$. It is easy to see that the algebra $C^* \# kG^*$ is algebra-isomorphic to the dual algebra of $C \times kG$.

2. If $G$ acts on the coalgebra $D$ via $\varphi: G \rightarrow \text{Aut}(D)$, then the group morphism $\bar{\varphi}: G \rightarrow \text{Aut}(D^*)$ given by $\bar{\varphi}(g)(d^*) = d^* \varphi(g)$ for $g \in G$, $d^* \in D^*$, defines an action of $G$ on the algebra $D^*$. Note that $\text{Aut}(D^*)$ is a group with respect to $\sigma \cdot \tau = \tau \cdot \sigma$ for $\sigma, \tau \in \text{Aut}(D^*)$. Thus $D$ is a $kG$-module coalgebra and $D^*$ is a $kG$-module algebra. If $G$ is finite then $D$ is a $kG^*$-comodule coalgebra and the dual algebra of the smash coproduct $D \times kG^*$ is isomorphic to the skew group ring $D^* \# kG$.

2. The Morita-Takeuchi Context Associated to a Graded Coalgebra

The Morita-theorems for categories of comodules have been proved by M. Takeuchi in [T]; we call a set of pre-equivalence data as in [T] a Morita-Takeuchi context.

2.1. Definition. A Morita-Takeuchi context $(C, D, cP_D, dQ_C, f, g)$ consists of coalgebras $C$ and $D$, bicomodules $cP_D, dQ_C$ and bicolinear maps $f: C \rightarrow P \Box dQ$, $g: D \rightarrow Q \Box cP$ making the following diagrams commute:

\[
\begin{align*}
P & \xrightarrow{\cong} P \Box dQ \\
\downarrow & \cong \\
C \Box cP & \xrightarrow{f \Box 1} P \Box dQ \Box cP
\end{align*}
\]

\[
\begin{align*}
Q & \xrightarrow{\cong} Q \Box cC \\
\downarrow & \cong \\
D \Box dQ & \xrightarrow{g \Box 1} Q \Box cP \Box dQ
\end{align*}
\]

The context is called strict if $f$ and $g$ are injective, hence isomorphisms. In
this case the categories $\mathcal{M}^C$ and $\mathcal{M}^D$ of comodules over $C$, resp. $D$, are equivalent categories.

The following remark extends a corresponding one for Morita contexts given in [CRW].

2.2. Proposition. Let $(C, D, cP_d, cQ_c, f, g)$ be a Morita-Takeuchi context such that $f$ is injective. Then $\mathcal{M}^C$ is equivalent to a quotient category of $\mathcal{M}^D$.

Proof. Theorem 2.5 of [T] yields that $f$ is an isomorphism and the exact functor $S = \Box_{DQ}^c P : \mathcal{M}^C \to \mathcal{M}^D$, has a right adjoint $T = \Box_P^c D : \mathcal{M}^C \to \mathcal{M}^D$ such that the natural transformation $f^{-1} : ST \to Id$ is an isomorphism. By a result of P. Gabriel (cf. [G] or Proposition 15.18 of [F]) we have: ker $S = \{X \in \mathcal{M}^D, X \Box_P c Q = 0\}$ is a localizing subcategory of $\mathcal{M}^D$ and $S$ induces an equivalence from the quotient category $\mathcal{M}^D/\ker S$ to $\mathcal{M}^C$. ☐

2.3. Corollary. Let $(C, D, cP_d, cQ_c, f, g)$ be a Morita-Takeuchi context such that $f$ is injective then $g$ is injective (i.e. the context is strict) if and only if $\Box_P c Q$ is faithfully coflat.

Proof. By Proposition 2.2 the injectivity of $g$ is equivalent to $S$ being an equivalence, again equivalent to Ker $S = \{0\}$ or $\Box_P c Q$ being faithfully coflat. ☐

Before establishing the main result of this section let us point out that there is a natural way to associate a graded coalgebra to a given Morita-Takeuchi context. Indeed, if we have a Morita-Takeuchi context $(C, D, cP_d, cQ_c, f, g)$ let $x \to \Sigma x_{-1} \otimes x_0$, resp. $x \to \Sigma x_{(1)} \otimes x_{(2)}$, be the left, resp. right, comodule structure of $P$, resp. $Q$. The image of $u \in C$ (resp. $D$) under $f$ (resp. $g$) in $P \Box_P c Q$ (resp. $Q \Box_P c D$) will be denoted by $\Sigma f(u) \otimes f(u)_1$, (resp. $\Sigma g(u) \otimes g(u)_2$).

Put $\Gamma = \begin{pmatrix} C & P \cr Q & D \end{pmatrix} = \{(c, d) \in C, d \in D, p \in P, q \in Q\}.$

We make $\Gamma$ into a coalgebra by defining $\Delta : \Gamma \to \Gamma \otimes \Gamma$ as follows:

$$\Delta_{c, 0} = \Sigma (c_1, 0, 0) \otimes (c_2, 0, 0) + \Sigma (0, f(c)_1, 0) \otimes (0, f(c)_2, 0)$$
$$\Delta_{0, d} = \Sigma (0, d_1, 0) \otimes (0, d_2, 0) + \Sigma (0, g(d)_1, 0) \otimes (0, g(d)_2, 0)$$
$$\Delta_{0, p} = \Sigma (0, p_{-1}, 0) \otimes (0, p_0, 0) + \Sigma (0, p_0, 0) \otimes (0, p_{(1)}, 0)$$
$$\Delta_{q, 0} = \Sigma (0, q_{-1}, 0) \otimes (0, q_0, 0) + \Sigma (0, q_0, 0) \otimes (0, q_{(1)}, 0)$$
for \( c \in C, d \in D, p \in P, q \in Q \), and extended linearly, \( \varepsilon: \Gamma \to k \) given by \( \varepsilon(c, p) = \varepsilon(c) + s_p(d) \). Moreover \( \Gamma \) is \( \mathbb{Z} \)-graded by putting \( \Gamma_0 = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 0 & P \\ 0 & Q \end{pmatrix} \) and \( \Gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) for \( k = -1, 0, 1 \).

Let \( C = \bigoplus_{x \in G} C_x \) be a coalgebra, graded by \( G \). Recall from [NT] that \( C_1 \) is a coalgebra with comultiplication \( \Delta_1: C_1 \to C_1 \otimes C_1 \) given by \( \Delta_1(c) = \sum \pi(c_1) \otimes \pi(c_2) = \sum \sum c_1 \otimes c_2 \) for all \( c \in C_1 \), where \( \pi: C \to C_1 \) is the natural projection. The co-unit of \( C_1 \) is \( \varepsilon_c \) restricted to \( C_1 \). Since \( \pi \) is a coalgebra map, \( C \) becomes a left \( C_1 \)-comodule via the structure map \( \rho^1_c: C \to C_1 \otimes C \), \( c \mapsto \sum \pi(c_1) \otimes c_2 \) (\( c \) homogeneous) and it becomes a right \( C_1 \)-comodule via \( \rho^1_c: C \to C \otimes C_1 \), \( c \mapsto \sum c_1 \otimes \pi(c_2) \) (\( c \) homogeneous). Now \( C \) is a graded right \( C \)-comodule, so by Theorem 1.6 \( C \) is a right \( C \rtimes kG \)-comodule via the map

\[
\rho^2_c: C \to C \otimes (C \rtimes kG), \quad c \mapsto \sum c_1 \otimes (c_2 \otimes (\deg c_1)^{-1})
\]

for \( c \) homogeneous. For any homogeneous \( c \in C \), we have \( (I \otimes \rho^2_c)\rho^1_c(c) = (\rho^1_c \otimes I)\rho^2_c(c) = \sum \pi(c_1) \otimes c_2 \otimes (c_3 \otimes (\deg c_1)^{-1}) \); thus \( C \) becomes a left \( C_1 \), right \( C \rtimes kG \)-bicodule. In a similar way \( C \) becomes a left \( C \rtimes kG \), right \( C_1 \)-bicodule where the left \( C \rtimes kG \)-comodule-structure of \( C \) is given by \( \rho^1(c) = \sum (c_1 \otimes \deg c_2) \otimes c_3 \), for any homogeneous \( c \in C \).

Define \( f: C_1 \to C \square_{C \rtimes kG} C, c \mapsto \sum c_1 \otimes c_2 = \Delta_c(c) \). Observe that for any \( c \in C_1 \) we obtain:

\[
\sum \rho^2_{c_1}(c_2) \otimes c_3 = \sum c_1 \otimes c_2 \otimes (\deg c_1)^{-1} \otimes c_3 = \sum c_1 \otimes c_2 \otimes \deg c_2 \otimes c_3 = \sum c_1 \otimes c_2 \otimes c_3
\]

so the definition of \( f \) above is satisfactory. Moreover, \( f \) is a morphism of left and right \( C_1 \)-comodules as is easily verified. Note also that \( f \) is injective because it is the restriction of the comultiplication of \( C \) to \( C_1 \).

Next define \( g: C \times kG \to C \square_{C_1} C, c \times x \to \sum c_1 \otimes \pi_{x,-1}(c_2) \) for \( x \in G \) and homogeneous \( c \in C \), where \( \pi_x \) denotes the projection from \( C \) to \( C_x \). In order to have that \( g \) is well-defined it is necessary that: \( \sum (c_1_1 \otimes \pi_x(c_1_2) \otimes \pi_{x,-1}(c_2)) = \sum c_1 \otimes \pi_x(c_1_1) \otimes \pi_{x,-1}(c_2) \). However the left hand side is obtained from \( \sum c_1 \otimes c_2 \otimes c_3 \) by collecting the terms with \( \deg c_2 = 1 \) and \( \deg c_3 = x^{-1} \); on the other hand the right hand sum is an expression of the same thing. Moreover \( g \) is a morphism of right (and left) \( C \times kG \)-comodules; this follows from:

\[
\sum \deg c_2 = x^{-1}(c_1 \otimes c_2)
\]
Let \( C = \bigoplus_{g \in G} C_g \) be a graded coalgebra, then the following assertions are equivalent:

1. \( C \) is strongly \( G \)-graded
2. The context given in Theorem 2.4 is strict
3. \( C \) is faithfully coflat as a left \( C \times kG \)-comodule.
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Proof. 2. $\Rightarrow$ 1. Take $u, v \in G$ and $c \in C_{uG}$ such that we have: $g(c \times v^{-1}) = \sum \pi_u(c_1) \otimes \pi_u(c_2) = 0$. Then $g(c \times v^{-1}) = \sum c_1 \otimes \pi_u(c_2) = 0$, hence $c \times v^{-1} = 0$ and $c = 0$.

1. $\Rightarrow$ 2. Let $\alpha = \sum c_i \times x_i \in C \times kG$ with $c_i$ homogeneous of degree $\sigma_i$. Suppose that for $i \neq j$ we have $(\sigma_i, x_i) \neq (\sigma_j, x_j)$. If $g(\alpha) = 0$ then $\sum \pi_{\sigma_i x_i}((c_i)_1) \otimes \pi_{\sigma_j x_j}((c_j)_1) = 0$, therefore $\sum_{i, j} \pi_{\sigma_i x_i}((c_i)_1) \otimes \pi_{\sigma_j x_j}((c_j)_1) = 0$. On the other hand: $\pi_{\sigma_i x_i}((c_i)_1) \otimes \pi_{\sigma_j x_j}((c_j)_1) \in C_{\sigma_i x_i} \otimes C_{\sigma_j x_j}$. Since $C \otimes C = \bigoplus_{u, v \in G} C_u \otimes C_v$ we obtain for fixed $i$, the relation: $\sum \pi_{\sigma_i x_i}((c_i)_1) \otimes \pi_{\sigma_j x_j}((c_j)_1) = 0$. The latter yields $\pi_{\sigma_i x_i}((c_i)_1) = 0$ and therefore $c_i = 0$ for every choice of $i$, i.e. $\alpha = 0$ follows.

2. $\Leftarrow$ 3. Follows from Corollary 2.3. □

As a further application we reobtain Theorem 5.3 of [NT] which is a coalgebra version of a well-known result of E. Dade.

2.7. COROLLARY. The graded coalgebra $C$ is strongly graded if and only if the induced functor $- \square c_i C : \mathcal{M} \rightarrow \text{gr} C$ is an equivalence of categories.

2.8. REMARK. The functor $(-)_c : \text{gr} C \rightarrow \mathcal{M}, M \rightarrow M_c$, is naturally isomorphic to the functor $- \square c_i G$ since they are both left adjoints of the induced functor $- \square c_i C$ (see [NT] Proposition 4.1, [T] Remark 2.4). Therefore the localizing category implicit in Corollary 2.5 is just $\text{Ker}(-)_c = \text{Ker}(- \square c_{\text{M}G} C)$.

As a final application of these techniques let us include a short proof of Corollary 6.4 in [NT].

2.9. COROLLARY. If $C$ is a strongly graded coalgebra for the group $G$ then $G$ is a finite group.

Proof. If $G$ is infinite we could select a non-zero homogeneous $c \in C$ and $x \in G$ such that $x \neq \deg(c_n^{-1})$ for all $c_n$. Then $g(c \times x) = 0$, but that would contradict injectivity of $g$. □

3. Duality.

For a quasi-finite right $C$-comodule $M$, the so-called coalgebra of “co-endomorphisms” of $M$ has been defined in [T., 1.17] and it is denoted by $e_c(M)$. Unfortunately this coalgebra is not easy to use because of the rather complex comultiplication, so it will be useful to give a nicer description of $e_c(M)$ in some particular situation, e.g. in case $M$ is a finitely cogenerated free-comodule (that is, $M \cong X \otimes C$, for some finite dimensional $k$-vectorspace $X$, with the obvious
comodule structure).

Let \( C \) be a coalgebra, \( X \) an \( n \)-dimensional \( k \)-space with basis \( \{ x_1, \ldots, x_n \} \). Consider the \( n \times n \) comatrix coalgebra \( M^e(n, k) \) which is a \( k \)-space with basis \( \{ x_{ij}, 1 \leq i, j \leq n \} \) and \( \Delta, \varepsilon \) given as follows: \( \Delta(x_{ij}) = \sum_p x_{ip} \otimes x_{pj}, \varepsilon(x_{ij}) = \delta_{ij} \).

The \( n \times n \) comatrix coalgebras over \( C \), denoted by \( M^e(n, C) \) is defined to be the tensor product of coalgebra \( C \otimes M^e(n, k) \). We endow \( C \otimes X \) with a left \( C \)-and a right \( M^e(n, C) \)-bicomodule structure as follows. The left \( C \)-comodule structure is given by the map: \( \rho^L: C \otimes X \rightarrow C \otimes C \otimes X, c \otimes x \mapsto \sum c_i \otimes x_{ip} \otimes c_s \otimes x_p \).

The \( M^e(n, C) \)-comodule structure is given by the map: \( \rho^R: C \otimes X \rightarrow C \otimes X \otimes M^e(n, C), c \otimes x_{ij} \mapsto \sum_p c_i \otimes x_{ip} \otimes c_s \otimes x_p \).

In a similar way \( C \otimes X \) is a left \( M^e(n, C) \)-right \( C \)-bicomodule via the structure maps:

\[
\rho^L: C \otimes X \rightarrow C \otimes C \otimes X, c \otimes x \mapsto \sum c_i \otimes x_{ip} \otimes c_s \otimes x_p \\
\rho^R: C \otimes X \rightarrow M^e(n, C) \otimes C \otimes X, c \otimes x_{ij} \mapsto \sum_p c_i \otimes x_{ip} \otimes c_s \otimes x_p.
\]

Define \( f: C \rightarrow (C \otimes X) \longrightarrow M^e(n, C) \otimes C \otimes X, c \mapsto \sum_{c_{ij}} (c_i \otimes x_{ij}) \otimes (c_s \otimes x_p), \) which is obviously injective and \( C \)-bicolinear. Define \( g: M^e(n, C) \rightarrow (C \otimes X) \longrightarrow M^e(n, C) \otimes C \otimes X, c \otimes x_{ij} \mapsto \sum (c_i \otimes x_{ij}) \otimes (c_s \otimes x_p) \) which is also injective and \( M^e(n, C) \)-bicolinear. One easily verifies the following relations:

\[
(I \square f) \rho^L(c \otimes x_i) = (g \square I) \rho^R(c \otimes x_i) = \sum_p c_i \otimes x_{ip} \otimes c_s \otimes x_p \\
(f \square I) \rho^R(c \otimes x_i) = (I \square g) \rho^L(c \otimes x_i) = \sum_p c_i \otimes x_{ip} \otimes c_s \otimes x_p \otimes c_i \otimes x_i
\]

According to results of \([T]\) we immediately obtain:

3.1. **Proposition.** \((C, M^e(n, C), C \otimes X, C \otimes X, f, g)\) is a strict Morita-Takeuchi context. In particular we have coalgebra isomorphisms:

\[
e_C(C \otimes X) \cong M^e(n, C) \cong e_C(C \otimes X)
\]

3.2. **Theorem.** Let \( G \) be a finite group acting on the coalgebra \( D \), then \( D \rtimes k G^* \) is a strongly graded coalgebra and there exist coalgebra isomorphisms:

\[
(D \rtimes k G^* ) \rtimes k G \cong e_{D \rtimes k G^* } \cong (D \rtimes k G^* ) \rtimes M^e(n, D)
\]

where \( n = |G| \).

**Proof.** The map \( \rho: D \otimes k G^* , d \rightarrow \sum_g (g \cdot d) \otimes p_g \), makes \( D \) into a \( k G^* \)-comodule. The comultiplication of \( D \rtimes k G^* \) is given by \( \Delta(d \rtimes p_x) = \sum_{y \in G} (d \rtimes p_y) \otimes (\nu_d \rtimes p_w) \). This establishes that \( D \rtimes k G^* \) is a graded coalgebra of type \( G \) with grading given by \( (D \rtimes k G^*)_g = D \rtimes p_{g-1} \). The canonical morphism \( D \rtimes p_1 \rightarrow \)
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\[(D \times p_{e-1}) \otimes (D \times p_e), \quad d \times p_e \mapsto \sum (d_1 \times p_{e-1}) \otimes (\sigma^{-1} d_2 \times p_{e})\], is clearly injective. Thus\(D \times kG^*\) is a strongly graded coalgebra, and \((D \times kG^*)_1 = D \times p_1 = D\). Applying the Morita-Takeuchi context (constructed in Section 2) to \(D \times kG^*\), we have a strict context and so it provides us with coalgebra isomorphisms:

\[\left((D \times kG^*)_1 \otimes kG \cong (D \times kG^*)_1 \otimes (D \times kG^*)_1\right) \cong (D \times kG^*)_1\].

The left \((D \times p_1)\)-structure of \(D \times kG^*\) is given by \(d \times p_2 \mapsto \sum (d_1 \times p_1) \otimes (d_2 \times p_2)\), and this yields exactly the left \(D\)-comodule structure of \(D \otimes X\) where \(X = kG^*\) is a \(k\)-space of dimension \(n\). Proposition 3.1 yields the second isomorphism. \(\square\)

A similar result holds for graded coalgebras (or coactions).

3.3. Theorem. Let \(C\) be a coalgebra graded by the finite group \(G\). Then \(G\) acts on the coalgebra \(C \times kG\) and there are coalgebra isomorphisms:

\[(C \times kG) \times kG^* \cong (C \times kG) \cong \mathcal{M}(n, C)\]

Proof. An action of \(G\) on the coalgebra \(C \times kG\) is given by \(h \cdot (c \times g) = c \times gh^{-1}\), \(g, h \in G\) and \(c \in C\). Thus \(C \times kG\) becomes a \(kG^*\)-comodule coalgebra via the map:

\[c \times g \mapsto \sum y \cdot (c \times g) \otimes p_y = \sum (c \times g y^{-1}) \otimes p_y\]

The comultiplication of \((C \times kG) \times kG^*\) is given by

\[\Delta((c \times x) \times p_g) = \sum_{u,\bar{u} \in G} ((c_1 \times \deg x \cdot c_2) \times p_\bar{u}) \otimes ((c_3 \times x v^{-1}) \times p_u)\]

for any \(x, g \in G\) and homogeneous \(c \in C\). Now let \(\{e_x, y, x, y \in G\}\) be a basis for \(\mathcal{M}(n, k)\). Define a map \(F: (C \times kG) \times kG^* \to \mathcal{M}(n, C)\), \((c \times x) \times p_g \mapsto c \otimes e_{x, \beta}\) where \(\alpha = \deg c \cdot x, \beta = x g^{-1}\) for \(x, g \in G\) and homogeneous \(c \in C\). Let us check that \(F\) is a coalgebra morphism. Indeed,

\[\Delta(F((c \times x) \times p_g)) = \Delta(c \otimes e_{x, \beta}) = \sum (c_1 \otimes e_{x, x} \otimes c_2 \otimes e_{x, \beta})\]

and also

\[\Delta(F((c \times x) \times p_g)) = \sum (c \otimes e_{x, \deg c_2 x v^{-1}} \otimes (c_3 \otimes e_{\deg c_2 x v^{-1}, x v^{-1}, u v^{-1}, u v^{-1}}))\]

Since \(\{\deg c_2 x v^{-1}, v \in G\} = G\), both sums are equal. Now, consider \((c \times x) \times p_g \in (C \times kG) \times kG^*\) for \(x, g \in G\) and \(c\) homogeneous. Write \(\varepsilon\) for the co-unit of \((C \times kG) \times kG^*\) and \(\varepsilon'\) for the co-unit of \(\mathcal{M}(n, C)\). Then we have:
Therefore \( F \) is a coalgebra map as claimed. Now define \( H: \mathcal{M}^e(n, C) \to (C \times kG) \times kG^{*} \) by putting \( H(c(g)cu, v) = (c \times (\deg c)^{-1}u) \times p_{v^{-1}(\deg c)-1}u \), for \( u, v \in G \) and homogeneous \( c \in C \). Again \( H \) is a coalgebra morphism because:

\[
\Delta(H(c\otimes u, v)) = \sum_{z = v^{-1}(\deg c)-1}u ((c_1 \times \deg c)(\deg c)^{-1}u) \times p_z \otimes ((c_2 \times (\deg c)^{-1}u \times p_z)
\]

For fixed \( c_1 \) and \( u \) we have that \( \{h^{-1}(\deg c_1)^{-1}u, h \in G\} = G \) and if we write \( t = h^{-1}(\deg c_1)^{-1}u, z = v^{-1}(\deg c_3)^{-1}h \), then the above sums are clearly equal as desired. The fact that \( H \) preserves the co-unit too is obvious. Finally it is clear that \( F \cdot H \) and \( H \cdot F \) are the identities so that we do arrive at a coalgebra isomorphism. The isomorphism involving \( e_{c_1}(C \times kG) \) is obvious because of Proposition 3.1 (the left \( C \)-comodule structure of \( C \times kG \) is given by \( c \times g \mapsto \sum c_1 \otimes (c_2 \times g) \)).

3.4. COROLLARY. There exists a strict Morita-Tekeuchi context connecting \( C \) and \((C \times kG) \times kG^{*}\).

PROOF. \( C \times kG \) is a left \( C \)-comodule that is a quasi-finite injective cogenerator (in view of Proposition 3.1 and [T]). Moreover \( C \times kG \) is a right \((C \times kG) \times kG^{*}\)-comodule via \( c \times g \mapsto \sum (c_1 \times \deg c_2 g u) \otimes (c_3 \times g u) \times p_{u^{-1}} \), for \( g \in G \) and homogeneous \( c \in C \). Hence \( C \times kG \) is a \((C \times kG) \times kG^{*}\)-bicomodule. The assertion now follows from [T, Theorem 3.5 iv].

3.5. REMARKS. The Morita-Tekeuchi context of the above corollary may be given in detail. This may have an independent interest because it provides another proof of Theorem 3.3 and provides a hint for establishing a more general duality result we do not dwell upon here. The second bicomodule is also \( C \times kG \) with right \( C \)-comodule structure given by the map: \( c \times g \mapsto \sum (c_1 \times \deg c_2 g) \otimes c_3 \) (for homogeneous \( c \)) and left \((C \times kG) \times kG^{*}\)-comodule struc-
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The structure given by:

\[ c \times g \mapsto \sum_h (c_1 \times \deg c_2 g) \otimes p_h \otimes (c_3 \times g h) \]

for homogeneous \( c \), we have \( f : C \to (C \times kG)^{\text{op}} \), \( g : (C \times kG)^{\text{op}} \to C \times kG \), \( f(c) = \sum_h (c_1 \times \deg c_2 h) \otimes (c_3 \times h_2) \) for homogeneous \( c \in C \), \( g((c \times g) \times p_h) = \sum h (c_1 \times \deg c_2 g) \otimes (c_3 \times g h) \), for homogeneous \( c \in C \). It is also easily seen that \( f \) and \( g \) are injective maps.

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Added in proof. A general duality result for crossed coproducts was proved by S. Dăscălescu, S. Raianu, Y. Zhang in "Finite Hopf-Galois coextensions, crossed coproducts and duality", to appear in J. Algebra.