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ON CODIMENSION ONE ISOMETRIC IMMERSIONS BETWEEN INDEFINITE SPACE FORMS

By

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Introduction

This paper considers codimension one isometric immersions between manifolds which carry nondegenerate, though possibly indefinite, metrics and which have the same constant sectional curvature. Its major purpose is to study the completeness properties of the relative nullity foliation of such an immersion in the event that the source manifold is geodesically complete. In addition, those immersions in the case of zero curvature (and source manifold completeness) are classified.

In [G], [HN], and [N1], a similar program has been accomplished for flat Euclidean and Lorentz spaces. For Riemannian manifolds, the completeness properties of the relative nullity foliation have been studied extensively. See, e.g., [N2].

Section 1 of the present paper presents notation and necessary preliminary results. Symmetric tensors of a relevant type and associated nullity distributions, including the relative nullity foliation, are examined in Section 2. The completeness properties of the relative nullity foliations of the immersions under consideration are developed in the third section, Theorem (3.11) being the major result. This theorem is combined with techniques from [G] to classify the immersions between flat indefinite spaces in Section 4; the classification appears as Theorem (4.4).

1. Preliminaries

Consider an isometric immersion $f: M^n \to M^{n+1}(c)$ between manifolds carrying nondegenerate metrics, denoted unambiguously by $\langle \cdot, \cdot \rangle$, and where the target manifold has constant sectional curvature $c$. Since the metrics are nondegenerate, each point of $M$ has a neighborhood (in $M$) on which is defined a vector field, denoted by $\xi$, of unit normals (i.e., $|\langle \xi, \xi \rangle| = 1$).

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Let the corresponding Levi-Civita connections on the source and target manifolds be denoted by $\nabla$ and $\nabla'$ respectively. Then $\nabla$ and $\nabla'$ satisfy the following formulas. If $X$ and $Y$ are vector fields on (an open set in) $M$, then the Gauss formula

$$\nabla'_XY = \nabla_XY + h(X,Y)\xi$$

(1.1)

gives an orthogonal decomposition of $\nabla'_XY$ into components tangential and normal to $M$. In (1.1), $h$ is a symmetric bilinear form, the second fundamental form. A field $A$ of tangent space endomorphisms, called the second fundamental tensor, is defined by the following Weingarten formula:

$$\nabla'_X\xi = -f_*(AX).$$

(1.2)

$A$ and $h$ satisfy

$$h(X,Y)\cdot\langle\xi,\xi\rangle = \langle AX, Y \rangle$$

(1.3)

and

$$\langle AX, Y \rangle = \langle X, AY \rangle.$$  

(1.4)

If $R$ is the curvature tensor of the connection $\nabla$ on $M$, then the equation of Gauss relates $R$, $A$, and the curvature $c$ of the target manifold:

$$c(X\wedge Y) = R(X,Y) - \langle \xi, \xi \rangle (AX\wedge AY),$$

(1.5)

where the operation $\wedge$ is defined by

$$(X\wedge Y)Z = \langle Z, Y \rangle X - \langle Z, X \rangle Y.$$  

(1.6)

Finally, the second fundamental tensor satisfies the equation of Codazzi:

$$\nabla_X(AY) = \nabla_Y(AX) + A([X,Y]).$$

(1.7)

We recite some standard facts about nondegenerate metrics. If $V$ is a finite-dimensional vector space, with nondegenerate metric (inner product) $\langle , \rangle$, of which $W$ is a (non-empty) subspace, then

$$W' = \{ v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W \}$$

is a subspace of $V$ whose dimension complements that of $W$:

$$\dim W + \dim W' = \dim V.$$  

Moreover, $(W')' = W$. However, $V = W \oplus W'$ if and only if the metric induced on $W$ is nondegenerate.

The following lemma is from [GN]; see also, e.g., [W]. A subspace of an inner product space is (non) degenerate if the inner product induced on that subspace is (non) degenerate.
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(1.8) **Degeneracy Lemma.** Suppose $X_1$ and $X_2$ are linearly independent vectors. Then \( \text{Span} \{X_1, X_2\} \) is nondegenerate if and only if the 2x2 "degeneracy determinant"

\[
\det ||\langle X_i, X_j \rangle|| \quad i, j = 1, 2
\]

is nonzero.

3. **Nullity Distributions of Symmetric (1, 1)-Type Tensors of Rank One**

Let \( f: M^n \rightarrow M^{n+1} \) be an isometric immersion between manifolds with non-degenerate metrics and the same constant sectional curvature \( c \). Then the equation of Gauss (1.5) implies that the second fundamental tensor \( A \) has rank one when it is nontrivial. According to (1.4), it is symmetric with respect to the metric on \( M \). In view of these facts, we turn our attention to symmetric (1, 1)-type tensors with rank at most one on \( M \).

Let \( A \) denote such a tensor; if \( x \in M^n \), then define \( T_0(x) \) to be the kernel of \( A_x \). Then \( T_0(x) \) has dimension \( n \) or \( n-1 \). If \( W \) is the set of all \( x \in M^n \) such that \( T_0(x) \) has dimension \( n-1 \), then \( W \) is precisely the set of all \( x \in M^n \) such that \( A_x \) is nonzero. Therefore, \( W \) is open, and

\[
x \mapsto T_0(x)
\]

defines an \((n-1)\)-dimensional distribution on \( W \), called the **nullity distribution** of the tensor \( A \). In the case where \( A \) is the second fundamental tensor of an immersion \( f: M^n \rightarrow M^{n+1} \) as described above, \( T_0 \) is called the **relative nullity distribution** of \( f \), and \( W \) may be called the "umbilic-free" set (see [G], §4).

**Lemma (2.1).** If \( x \in W \), then the image of \( A_x \) is precisely the orthogonal complement of \( T_0(x) \) in \( T_x M \).

**Proof.** Both spaces are one-dimensional, and if \( X \in T_0(x) \), then \( \langle AZ, X \rangle = \langle Z, AX \rangle = 0 \). QED.

**Lemma (2.2).** For a symmetric (1, 1)-type tensor \( A \) with rank one, the following statements are equivalent.

(i) The kernel of \( A \) is degenerate.

(ii) The image of \( A \) is a light line.

(iii) \( A^2 \equiv 0 \).

**Proof.** (i) \( \Rightarrow \) (ii). If \( X \in T_0 \) satisfies \( \langle X, Y \rangle = 0 \) for all \( Y \in T_0 \), then \( X \in T_0^1 \), which is the image of \( A \). Thus, the image of \( A \) is \( \text{Span} \{X\} \), but \( \langle X, X \rangle = 0 \).
(ii) $\Rightarrow$ (iii). Denote the ambient (tangent) space by $R^n$. Write $R^n$ as an algebraic direct sum

$$R^n = T_x \oplus \text{Span } \{L\}.$$ 

If $X \in T_x$, then $\langle A^2 L, X \rangle = \langle AL, AL \rangle = 0$; but also $\langle A^2 L, L \rangle = \langle AL, AL \rangle = 0$. Thus, $A^2 L$ is orthogonal to $R^n$ and so $A^2 L = 0$, and $A^3 \equiv 0$.

(iii) $\Rightarrow$ (i). If $A^3 \equiv 0$, then the image of $A$ is contained in $T_x$. But the image of $A$ is orthogonal to $T_x$, and hence is the axis of degeneracy in $T_x$. QED.

(2.3) Corollary. If $G$ is the set of those $x$ in $W$ for which $T_0(x)$ is nondegenerate, then $G$ is open.

(2.4) Proposition. $T_x$ is a differentiable distribution on $W$.

Proof. Let $y \in W$. Choose $L \notin T_x(y)$ such that $\text{Span } \{L, AL\}$ is a nondegenerate plane. If $L$ is extended to a vector field near $y$, then the degeneracy determinant for $\{L, AL\}$ remains nonzero (perhaps in some smaller neighborhood of $y$). So, for points near $y$, $\text{Span } \{L, AL\}$ has a nondegenerate, hence algebraically complementary, orthogonal complement $E$, of dimension $n - 2$. By Lemma (2.1), $E$ is contained in $T_x$.

Now, $L \notin T_x$ so $\langle L, AL \rangle_x = \delta(x)$ is nonzero at points $x$ near $y$. Define $V \in \text{Span } \{L, AL\}$ by

$$V = AL - \langle AL, AL \rangle / \langle L, AL \rangle L.$$ 

That $\langle V, AL \rangle = 0$ implies that $V$ spans $T_x \cap \text{Span } \{L, AL\}$. It follows that

$$T_x = E \oplus \text{Span } \{V\}$$

near $y$, where the direct sum is an orthogonal sum as well. To see that $T_x$ is differentiable, it now suffices to see that the $(n - 2)$-dimensional distribution $x \mapsto E(x)$ is differentiable.

Give $E(y)$ an orthonormal basis $\{Y_j\}$ $(j = 1, \ldots, n - 2)$, and then extend $Y_j$ near $y$ to a vector field $Z_j$, for each $j$. Because the degeneracy determinant of $\{L, AL\}$ is nonzero near $y$, the solutions of the linear system

$$\begin{align*}
\langle L, Z_j \rangle &= c_1 \langle L, L \rangle + d_1 \langle AL, L \rangle \\
\langle AL, Z_j \rangle &= c_1 \langle AL, L \rangle + d_1 \langle AL, AL \rangle
\end{align*}$$

are smooth functions of inner products among $L, AL$, and $Z_j$. Hence,

$$Z_j' = Z_j - c_1 L - d_1 AL \quad j = 1, \ldots, n - 2$$

are smooth vector fields near $y$. Moreover, $(Z_j')_y = Y_j$ so the set $\{Z_j'\}$ $(j = 1, \ldots, n - 2)$
is linearly independent near $y$. Since each $Z'_j$ is orthogonal to $\text{Span}\{L, AL\}$, the set spans $E$. QED.

In the case when $A$ is a relative nullity distribution of such a codimension one isometric immersion as described at the beginning of this section, we can say more.

(2.5) **Proposition.** The relative nullity distribution of an isometric immersion $f : M^n(c) \rightarrow M^{n+1}(c)$ is integrable.

**Proof.** If $X, Y \in T_0$, then $A([X, Y]) = F_x(AY) - F_y(AX) = 0$, so $[X, Y] \in T_0$. QED.

Proposition (2.5) says that $T_0$, the relative nullity distribution of $f$, is a *foliation* of the umbilic-free set $W$. We shall refer to $T_0$ as the *relative nullity foliation*.

(2.6) **Proposition.** The image of $A$ is parallel in any $T_0$ direction.

**Proof.** If $X \in T_0$ and $L \not\in T_0$, then

$$F_x(AL) = F_x(AX) + A([X, L])$$

and so lies in the image of $A$. QED.

(2.7) **Proposition.** If $X, Y \in T_0$, then $F_x Y \in T_0$.

**Proof.** Choose $L \not\in T_0$. Then Proposition (2.6) and the fact that $A$ has rank one imply that

$$\langle F_x Y, AL \rangle = X \cdot \langle Y, AL \rangle - \langle Y, F_x(AL) \rangle$$

$$= k \cdot \langle Y, AL \rangle$$

for some constant $k$. By Lemma (2.1), $\langle Y, AL \rangle$ vanishes, and then also $F_x Y \in T_0$. QED.

A distribution (or foliation) $D$ which satisfies $F_x Y \in D$ whenever $X, Y \in D$ is said to be *totally geodesic*. (If $D$ is a foliation, which is so in the presence of zero torsion, then its integral submanifolds, called its *leaves*, are totally geodesic as submanifolds. For a discussion of totally geodesic submanifolds, see [KN] or [N2].)

We now have the following result for the relative nullity foliation $T_0$ of an isometric immersion $f : M^n(c) \rightarrow M^{n+1}(c)$. $W$ denotes the umbilic-free set.

(2.8) **Theorem.** $T_0$ is a totally geodesic foliation of $W$. 
3. Completeness Properties of the Relative Nullity Foliation

This section will study the relative nullity foliation of an isometric immersion \( f: M^n(c) \to M^{n+1}(c) \), where the source manifold \( M^n \) is (geodesically) complete; that is, any geodesic in \( M^n \) may be extended to all values of its affine parameter.

A totally geodesic foliation \( F \) has the following property (see, e.g., [KN]). Given a point \( x_0 \), if \( x_t \) is a geodesic whose tangent vector \( x_0 \) at \( x_0 \) lies in \( F(x_0) \), then \( x_t \) lies in the leaf of \( F \) through \( x_0 \) for all \( t \) in some neighborhood of 0. A totally geodesic foliation is called complete if every affinely-parametrized geodesic which is tangent to the foliation can be extended to all values of the parameter and still lie in a leaf of the foliation.

It has been seen (Theorem (2.8)) that the relative nullity foliation under consideration is totally geodesic. The purpose of this section is to show that (if \( M^n \) is complete) this foliation is complete.

\( T_0 \) will denote the foliation, and \( W \) will denote the "umbilic-free" set, on which the \((n-1)\)-dimensional \( T_0 \) lives. Let \( x_0 \in W \) and let \( X_0 \in T_0(x_0) \). Let \( x_t \) be an affinely-parametrized geodesic such that \( x_0 = X_0 \), and let \( x_t \) be extended to all values of \( t \) in the complete manifold \( M^n \). Suppose \( b > 0 \) satisfies that \( x_t \) lies in the leaf of \( T_0 \) through \( x_0 \) for \( 0 \leq t < b \); such a \( b \) exists by Theorem (2.8). The proof of the following lemma is essentially that of Lemma (5.9) of [G], which argument also appears in [N1].

(3.1) Lemma. If \( x_0 \in W \), then there exists a positive \( \varepsilon \) such that for \( 0 \leq t < b + \varepsilon \), \( x_t \) lies in the leaf of \( T_0 \) through \( x_0 \).

Hence, to show that \( T_0 \) is complete, it is imperative to show that \( x_0 \in W \). In what follows, \( y = x_{t_0} \) will always denote a fixed point of the geodesic \( x_t \) at which some pertinent differentiation or function evaluation will occur. The second fundamental tensor at a point \( x \) will be denoted \( A(x) \).

Let \( \Omega \) generate the image of \( A(x_0) \). Extend \( \Omega \) as a parallel vector field along all of \( x_t \) to a vector field \( \Omega_t \). By Proposition (2.6), \( \Omega_t \) generates the image of \( A(x_t) \) if \( t < b \). Choose \( L \notin T_0(x_0) \) such that \( \langle L, \Omega \rangle = -1 \). Extend \( L \) as a parallel vector field along \( x_t \) for all \( t \). For \( t < b, L \notin T_0(x_t) \). Also define a smooth function \( \rho: R \to R \) by

\[
\rho(t) = \langle AL, L \rangle_{x_t}.
\]

\( \rho \) is defined for all \( t \), but if \( t < b \), then

\[
AL_{x_t} = -\rho(t)\Omega_t.
\]

Near any point \( y \) on the geodesic, we may extend \( \varphi(t) \) to a \( T_0 \)-field \( X \) such
that \( X(x_t) = x_t \) for each point \( x_t \) of the geodesic near \( y \). \( X \) will be called a \( T_b \)-extension (of \( x(t_1) \)) near \( y \), and is constructed as follows. Choose a normal coordinate system \((x^1, \ldots, x^n)\) at \( y \) such that the geodesic \( x_t \) is described by \( x^1 = \cdots = x^n = 0 \). Choose a \( T_b \)-field on \( x^1 = 0 \) extending \( x_0 \), and consider the geodesics in those directions. Their tangent vectors form a \( T_b \)-extension near \( y \).

We will also need extensions of \( L \) and \( \Omega \) in certain directions transverse to the geodesic \( x_t \). Define a map \( h : \mathbb{R}^2 \to M^n \) by

\[
h(t, u) = \exp_x(u L_t).
\]

Since

\[
h_t(0, 0) \frac{\partial}{\partial t} = x_t, \quad h_u(0, 0) \frac{\partial}{\partial u} = L_u,
\]

for each \( t \), there is a neighborhood \( U \) of \((t, 0)\) such that \( h : U \to h(U) \) is an embedding. These \( U \)-neighborhoods form a neighborhood \( V \) of \((t, 0) : t \in \mathbb{R}\). The vector field \( h_t(0, 0) \frac{\partial}{\partial u} \) is an extension of \( L \) to \( h(V) \). By shrinking the neighborhood \( V \), if need be, and restricting \( t \) to \((0, b)\), we may assume that \( AL \neq 0 \); since \( L \notin T_0 \), \( \langle AL, L \rangle \neq 0 \). Now let

\[
\Omega = -\langle 1, \langle AL, L \rangle \rangle AL.
\]

This extends \( \Omega \) to \( h(V) \). Finally, note that if \( Z \) is a vector field near \( y = x(t_1) \), then \( (F_t Z)_y \) depends only on the behavior of \( Z \) along the curve \( u \to h(t, u) \). In particular, with the above extensions, \( (F_t \Omega)_x \) is a well-defined vector field along the geodesic \( x_t \).

Now we examine Codazzi's equation (1.7) near \( y \), using \( L \) and a \( T_b \)-extension \( X \) near \( y \). Using zero torsion, the Codazzi equation reduces to

\[
F_X(AL) = -A(F_X L),
\]

since \( L \) is parallel along \( x_t \), and \( X \in T_b \).

Next we consider the derivative of the function \( p \) of (3.2) and (3.3):

\[
\frac{dp}{dt} = X \cdot \langle AL, L \rangle
\]

\[
= \langle F_X(AL), L \rangle + \langle AL, F_X L \rangle
\]

\[
= -\langle A(F_X L), L \rangle
\]

\[
= -\langle F_X L, AL \rangle,
\]

so

\[
\frac{dp}{dt} = p(t) \langle F_X L, \Omega \rangle.
\]

Since \( p \) is well-defined and smooth for all \( t \), (3.4) implies that the function \( Q(t) = \)
Let us pause to study the behavior of \( Q(t) \) along the geodesic \( x_t \).

By the Gauss equation (1.5) of the immersion,

\[
R(X, L)X = c\langle X, L \rangle X - c\langle X, X \rangle L.
\]

Since there is no torsion,

\[
R(X, L)X = F_x F_L X - F_L F_x X - (F_{xL} - F_{Lx}) X.
\]

Equations (3.5) and (3.6) can be combined to give

\[
P_x F_L X = c\langle X, L \rangle X - c\langle X, X \rangle L + F_x F_L X + F_L F_x X - F_{Lx} X.
\]

Now,

\[
\frac{dQ}{dt} = X \cdot \langle F_L X, \Omega \rangle = \langle F_x F_L X, \Omega \rangle + \langle F_L F_x X, \Omega \rangle - \langle F_{Lx} X, \Omega \rangle,
\]

since \( \Omega \) is parallel along \( x_t \). From (3.7) the following equation obtains:

\[
\frac{dQ}{dt} = c\langle X, L \rangle \langle X, \Omega \rangle - c\langle X, X \rangle \langle L, \Omega \rangle + \langle F_x F_L X, \Omega \rangle - \langle F_{Lx} X, \Omega \rangle.
\]

(3.8)

**Lemma.** The function \( Q \) satisfies the following differential equation on the interval \( 0 \leq t < b \):

\[
\frac{dQ}{dt} = Q(t)^2 + c\langle x_t, \dot{x}_t \rangle.
\]

**Remark.** Note that \( c \) and \( \langle \dot{x}_t, x_t \rangle \) are constants.

**Proof.** Each term of the right-hand side of (3.8) is to be evaluated at \( y = x(t) \) where \( 0 \leq t < b \).

Since \( \Omega_y \) generates the image of \( A_y \), the first term vanishes. By design, \( \langle L, \Omega \rangle_t = -1 \); since \( X_t = x_t \), the second term equals \( c\langle \dot{x}_t, x_t \rangle \). At \( y \), \( F_x L \) is zero \( \langle L \) is parallel along \( x_t \); therefore, \( (F_{xL} X)_y \), and the fourth term, vanish.

That the third term vanishes can be seen as follows. At \( y \), near which a fixed \( T_x \)-extension has been established, consider the equation

\[
\langle F_L F_X X, \Omega \rangle_y = L_y \langle F_X X, \Omega \rangle - \langle (F_X X)_y, (F_L \Omega)_y \rangle = L_y \langle F_X X, \Omega \rangle.
\]
Near \( y, F_X \in T_y \). Since \( \Omega \) generates the image of \( A \) for points on the integral curves of \( L \) constructed previously, \( \langle F_X, \Omega \rangle \) vanishes on those curves. Therefore, \( L_y \langle F_X, \Omega \rangle \) is zero.

Finally, the fifth term depends on \( \langle F_L, X \rangle \), a vector well-defined at \( y \) since the \( T_y \)-extension \( X \) and the integral curve of \( L \) through \( y \) are fixed. Now, if \( Z = F_L X + \langle F_L, \Omega \rangle L \), then \( \langle Z, \Omega \rangle = 0 \), and hence \( Z \in T_0(y) \). As well, \( \langle F_L, X \rangle \) lies in \( T_0(y) \); so \( \langle F_L, X, \Omega \rangle \) is zero. This implies that

\[
\langle F_{L_1}, X, \Omega \rangle = -\langle F_{L_1}, \Omega \rangle = -\langle F_L, \Omega \rangle = -Q(t)^2.
\]

In summary, then, equation (3.8) reduces to

\[
\frac{dQ}{dt} \bigg|_{t=t_1} = \langle \bar{z}, \bar{x} \rangle - (-Q(t_1))^2
\]

independently of the \( T_y \)-extension near \( y \). QED.

Now, to see whether the relative nullity is complete, which (by Lemma (3.1)) is equivalent to whether \( x_0 \in W \), we consider the differential equations in (3.4) and (3.9), according to different cases for the constant \( c \langle \bar{z}, \bar{x} \rangle \).

First, suppose that \( c \langle \bar{z}, \bar{x} \rangle = 0 \). Then, from (3.9), \( Q \) satisfies the differential equation

\[
\frac{dQ}{dt} = Q^2.
\]

It follows that either \( Q \equiv 0 \) on \([0, b)\) or

\[
Q(t) = \frac{Q(0)}{1 - t \cdot Q(0)}.
\]

Then (3.4) implies that either \( p(t) \equiv p(0) \) on \([0, b)\) or

\[
p(t) = \frac{p(0)}{|1 - t \cdot Q(0)|}
\]

(see [G]). In either case \( \lim_{t \to 0} p(t) = p(b) \) is nonzero, and \( x_0 \in W \).

Next, suppose there is some \( r \) such that \( c \langle \bar{z}, \bar{x} \rangle = -r^2 \). The differential equation for \( Q \) given by (3.9) is

\[
\frac{dQ}{dt} = Q^2 - r^2,
\]

for which \( Q(t) \equiv r \) is a solution; in this event, \( p(t) = p(0)e^{rt} \), so \( p(b) \) is nonzero.
If $Q \neq r$, then it is easy to see that

$$Q(t) = r \coth (k-rt)$$

where $k$ is the inverse hyperbolic cotangent of $Q(0)/r$. $Q(t)$ is well-defined for $0 \leq t < b$, so $k/r \notin [0, b)$. Now, from (3.4) it follows that for $0 \leq t < b$,

$$p(t)=p(0) \exp \left( \int_0^t Q(s) ds \right)$$

$$=p(0) \exp \left[ \log \left( \frac{\sin h k}{\sin h(k-rt)} \right) \right].$$

(Note: since $k/r \notin [0, b)$, the quantities $\sin h k$ and $\sin h(k-rt)$ have the same sign.) So

$$p(t)=\frac{(p(0) \sin h k)}{\sin h(k-rt)}.$$

Since $k \neq 0$, $p(t) \neq 0$ ($0 \leq t < b$). Were $k$ equal to $rb$, $\lim_{t \to b} p(t)$ would be infinite. However, $p$ is well-defined and smooth for all $t$. Therefore, the limit exists and is nonzero, and $x_t \in W$.

In the cases where $c(x_t, x_t)$ is nonpositive, it has been shown that $x_t \in W$. Lemma (3.1) now implies that

$$\sup \{ t : x_t \in W \quad \text{for} \quad 0 \leq u < t \}$$

cannot be finite. Therefore, the (complete) geodesic $x_t$ lies in $W$.

Finally, consider the case where $c(x_t, x_t) = r^2$ for some $r$. Then (3.9) implies that

$$Q(t) = r \tan(rt + k)$$

where $k = \arctan(Q(0)/r)$. Note that $k \neq \pi/2$, since $Q(t)$ is well-defined on $[0, b)$. For the same reason, the quantity

$$b' = \frac{\pi - 2k}{2r}$$

does not lie in $[0, b)$. This fact and (3.4) give

$$p(t)=p(0) \exp \left( \int_0^t r \tan(rs + k) ds \right)$$

so

$$(3.10) \quad p(t) = (p(0) \cos k)/\cos(rt + k).$$

Now, $\cos k \neq 0$ since $Q(0)$ is not infinite. However, since $M^n$ is complete, $b'$ is an admissible parameter value, and so $p(b')$ must be well-defined (and smooth there).
But, by (3.10), \( \lim_{t \to -\infty} p(t) \) does not exist, engendering a contradiction. The only viable conclusion is that, if \( M^n \) is complete, the case where \( c\langle \xi, \xi \rangle \) is positive cannot occur.

The basic completeness properties of the relative nullity foliation, derived in this section, are summarized by the following theorem.

(3.11) Theorem. Let \( f: M^n \to M^{n+1} \) be an isometric immersion, where

(i) \( M^n \) and \( M^{n+1} \) have indefinite nondegenerate metrics;
(ii) \( M^n \) and \( M^{n+1} \) have the same constant curvature \( c \); and
(iii) \( M^n \) is (geodesically) complete.

Let \( W \) be the umbilic-free set of the immersion, and let \( x_t \) be an (affinely parametrized) geodesic passing through a point of \( W \). Then:

1. \( c\langle \xi, \xi \rangle \) is a nonpositive constant;
2. \( x_t \) lies in \( W \) for all \( t \).

If \( M^n \) is not complete, then it can be asserted that \( x_0 \in W \) for those values to which the geodesic can be extended from a particular value \( t_0 \) for which \( x_{t_0} \in W \). If \( c\langle \xi, \xi \rangle \) is positive, then the contradiction involving \( b' \) engendered during the development of Theorem (3.11) can at best imply that \( b' \) is not a value to which the geodesic can be extended.

4. Codimension One Isometric Immersions Between Indefinite Euclidean Spaces

Let \( R^s_n \) be the \( n \)-dimensional real vector space together with an indefinite metric (inner product) of signature \((s, n-s)\) given by

\[
\langle x, y \rangle = -\sum_{j=1}^{s} x_j y_j + \sum_{k=s+1}^{n} x_k y_k
\]

for \( x=(x_1, \ldots, x^n) \) and \( y=(y_1, \ldots, y^n) \). \( R^s_n \) will be called the indefinite \((n\)-dimensional\) Euclidean space \( \text{with signature} \ s \). If \( s=0 \), then \( R^0_n \) is just the ordinary Euclidean space \( E^n \). If \( s=1 \), then \( R^1_n \) is what is usually called the \( n \)-dimensional Lorentz space, \( L^n \). Note that there is a natural isomorphism between \( R^s_n \) and \( R^{s-1}_{n-1} \). Under this isomorphism, geometric properties of one correspond to geometric properties of the other. This "independence of sign convention" has long been exploited in the case of \( L^n \) versus \( R^{n-1}_s \).

The purpose of this section is to outline the classification of isometric immersions of \( R^s_n \) into \( R^{s+1}_{n+1} \). The case \( s=0 \) was done by Hartman and Nirenberg [HNN]; a proof also appears in [N1]. The classification in the case \( s=1 \) appears in [G]. All classifications are based upon the completeness properties of the relative
nullity foliation. The natural isomorphism between $R^n_1$ and $R^{n-1}_1$ may be applied to the classification of immersions $R^n_1 \to R^{n+1}_1$ to classify the immersions $R^{n-1}_1 \to R^n_{n-1}$ as well.

Let $M_0(x)$ denote the leaf through $x \in R^n_1$ of the relative nullity foliation of an immersion $R^n_1 \to R^n_{n+1}$.

(4.1) **Theorem.** The umbilic-free set $W$ is a union of parallel hyperplanes.

**Proof.** By (2) of Theorem (3.11), $M_0(x)$ is contained in the $(n-1)$-dimensional subspace tangent to it at $x$, for $x \in W$. Now, the hyperplanes $M_0(x)$ are the maximal connected integral submanifolds of the relative nullity foliation. Distinct hyperplanes therefore cannot intersect; but nonintersecting hyperplanes must be parallel. QED.

Since parallel hyperplanes inherit the same metric from $R^n_1$, the following theorem is immediate.

(4.2) **Theorem.** Either $G = \emptyset$ or $G = W$.

Now, if $x_0$ is a fixed origin in $W \subseteq R^n_1$, write $M_0 = M_0(x_0)$. Choose a vector $L$ at $x_0$ such that $L \perp T_0(x_0)$; if $T_0(x_0)$ is nondegenerate, choose $L$ to be orthogonal to $T_0$. Then

\[
M_0 \oplus \text{Span}(L) = R^n_1
\]

describes $R^n_1$ as a direct sum of vector spaces which is orthogonal if $M_0$ is nondegenerate. If $f$ is the immersion, then define $f_0 : M_0 \to R^n_{n+1}$ and $f_1 : \text{Span}(L) \to R^n_{n+1}$ by

\[
f_0(x) = f(x, 0), \quad \text{if } x \in M_0;
\]
\[
f_1(s) = f_1(sL) = f(x_0, s).
\]

Invoking the "Moore Lemma" in [G] gives

\[
f(x, s) = f_0(x) + f_1(s).
\]

Using the Gauss formula (1.1), it is easy to see that $f_0$ is an isometry of $M_0$ onto an $(n-1)$-plane in $R^n_{n+1}$. If $M_0$ is nondegenerate, and $L$ is chosen orthogonal to $M_0$, then $f_1$ maps \text{Span}(L) into the orthogonal complement of $f(M_0)$ in $R^n_{n+1}$. If $M_0$ is degenerate, then for some $(n-2)$-plane $E$ and nonzero vector $\Omega$,

\[
M_0 = E \oplus \text{Span}(\Omega)
\]

describes $M_0$ as a direct and orthogonal sum of a nondegenerate $(n-2)$-plane and an axis of degeneracy. Let the vector $L$ of (4.3) be chosen orthogonal to $E$. Since $R^n_1$ has a nondegenerate metric, $\langle L, \Omega \rangle \neq 0$. From the additional fact that $\langle \Omega, \Omega \rangle = 0$,
it follows that \( \text{Span}(L, \Omega) \) is an indefinite Euclidean plane with signature 1, i.e., an \( L^2 \), which is orthogonal to \( E \). The isometry \( f_0: M_0 \rightarrow f(M_0) \) induces an isometry of \( E \) onto an \((n-2)\)-plane \( f(E) \), and \( f(\text{Span}(L, \Omega)) \) lies in that \( L^3 \) which is the orthogonal complement of \( f(E) \) in \( R^{2+1}_3 \). The methods of [G] now apply virtually verbatim to establish the following theorem.

(4.4) **Theorem.** Up to a proper motion of \( R^{2+1}_3 \), an isometric immersion \( R^n_1 \rightarrow R^{2+1}_3 \) has one of the following forms.

(i) \( \text{id} \times c: R^n_1 \times L^2 \rightarrow R^n_1 \times L^2 \)
(ii) \( \text{id} \times c: R^n_1 \times E^1 \rightarrow R^n_1 \times E^2 \)
(iii) \( \text{id} \times g: R^n_1 \times L^2 \rightarrow R^n_1 \times L^2 \)

where: in (i), \( c: L^2 \rightarrow L^2 \) is a "unit-speed" time-like curve \( \langle dc/dt, dc/dt \rangle = -1 \) in \( L^2 \); in (ii), \( c: E^1 \rightarrow E^2 \) is a unit-speed Euclidean plane curve; in each case "\( \text{id} \)" is the appropriate identity map; and in (iii), \( g \) is an immersion of \( L^2 \) into \( L^3 \) with degenerate relative nullity (as classified in [G]). Moreover, class (iii) consists of precisely those immersions \( R^n_1 \rightarrow R^{2+1}_3 \) with degenerate relative nullities.

It should be noted that those immersions with nondegenerate relative nullities are cylinders over curves, in analogy with the Hartman-Nirenberg result, whereas the description (iii) is the best possible for those immersions with degenerate relative nullities.

**References**


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