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Research article

A study of quadratic Diophantine fuzzy sets with structural properties and their application in face mask detection during COVID-19

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Abstract: During the COVID-19 pandemic, identifying face masks with artificial intelligence was a crucial challenge for decision support systems. To address this challenge, we propose a quadratic Diophantine fuzzy decision-making model to rank artificial intelligence techniques for detecting masks, aiming to prevent the global spread of the disease. Our paper introduces the innovative concept of quadratic Diophantine fuzzy sets (QDFSs), which are advanced tools for modeling the uncertainty inherent in a given phenomenon. We investigate the structural properties of QDFSs and demonstrate that they generalize various fuzzy sets. In addition, we introduce essential algebraic operations, set-theoretical operations, and aggregation operators. Finally, we present a numerical case study that applies our proposed algorithms to select a unique face mask detection method and evaluate the effectiveness of our techniques. Our findings demonstrate the viability of our mask identification methodology during the COVID-19 outbreak.

Keywords: quadratic Diophantine fuzzy sets; COVID-19; face mask detection; aggregation operators; decision-making; neural network; artificial intelligence
Mathematics Subject Classification: 03E72, 90B50, 94D05

1. Introduction

The first outbreak of coronavirus disease (COVID-19) was initially reported in Wuhan, China, in December 2019 and was later identified as a new type of coronavirus by the World Health Organization (WHO). The rapid spread of the virus worldwide prompted the WHO to declare COVID-19 a public health emergency. As a result, the use of face masks has become increasingly popular in society as a means of protecting against the virus. While some individuals choose not to wear masks to protect themselves from polluted air [15], many believe that using face masks can help prevent the spread of COVID-19.

Computer science-powered deep learning and machine learning techniques have been shown to be valuable tools in the fight against COVID-19 [1]. Machine learning can analyze massive amounts of data, act as an early warning system for potential epidemics, and classify vulnerable groups. Several nations have implemented laws mandating the wearing of facial masks in public places, reflecting the importance of face mask detection as a crucial problem for COVID-19 prevention [12]. Face mask detection is a challenging technique, particularly when dealing with masked face-extracting features versus traditional facial-extracting features.

Various face mask detection strategies are available in the literature [13, 21, 31], but selecting the most appropriate strategy for a particular circumstance can be challenging due to decision-making in an uncertain environment. A decision-making process, as shown in Figure 1, is often required.

Conventional mathematical methods may not always be the optimal choice for addressing such problems. As a result, the application of fuzzy set (FS) theory is necessary, and a basic understanding of its fundamental concepts is required. The concept of FS was first introduced by Zadeh in 1965 [37] as a means to address ambiguity in natural language. Since its inception, research on FS theory has been extensively conducted in the fields of decision-making and operational research.

However, in many real-world scenarios, the membership function alone may not be sufficient to capture the complexities of the situation. To address this limitation, Atanassov introduced the concept of an intuitionistic fuzzy set (IFS) in his works [7,8,20]. This extension of FSs allows objects to possess nonmembership degrees alongside their membership degrees as long as their sum does not exceed one. Atanassov further proposed IF-relations on IFSs [9], which have gained significant attention and are now utilized in a wide range of practical applications, including decision-making [11, 24] and optimization in intuitionistic fuzzy settings. The first lexicographic ordering of intuitionistic fuzzy values and their correlations was established by Feng et al. [16].

In certain real-world situations, the total of membership and nonmembership degrees of objects may exceed 1, posing a problem for traditional FS theory. To overcome this issue, Yager introduced Pythagorean fuzzy sets (PyFSs), where the sum of squares of membership and nonmembership degrees cannot exceed 1 [34, 35]. PyFSs have been extensively studied using various methodologies [26, 38], and Yager further extended the concept in [36] to introduce q-rung orthopair fuzzy sets (q-ROFSs), which have also been the subject of significant research [4, 22]. Atanassov's type 2 IFSs [6] are also well-known in the field. Cuong introduced picture fuzzy sets (PFSs) as a way to handle fuzzy information consisting of hesitance or ignorance [10]. Recent research on PFSs and various aggregation operators can be found in [5, 19, 27], while [2, 3, 23, 29] explored recent trends in fuzzy decision-making.

Fuzzy sets (FSs), interval-valued fuzzy sets (IFSs), Pythagorean fuzzy sets (PyFSs), and q-rung orthopair fuzzy sets (q-ROFSs) have been widely studied and applied in practice. However, these sets impose strict constraints on membership and nonmembership grades. To overcome these limitations, Riaz and Hashmi [30] proposed a new concept called the linear Diophantine fuzzy set (LDFS), which incorporates reference parameters for membership and nonmembership grades, thereby expanding the scope of these sets. Nevertheless, LDFS still contains acceptance- and rejection-type parameters, which may not fully capture fuzzy information related to denial, ignorance, or confusion. Therefore, the present study aims to introduce the notion of the quadratic Diophantine fuzzy set (QDFS), which includes a third parameter to address such scenarios. The goals of this paper are as follows:

- To propose the notion of QDFSs and their operational laws and explain their characteristics and comparison method.
- To compare QDFSs with existing FSs, such as IFSs, PyFSs, q-ROFSs, LDFSs, and PFSs, and demonstrate their superiority over them.
- To develop aggregation operators, such as the quadratic Diophantine fuzzy weighted averaging aggregation (QDFWAA) operator and the quadratic Diophantine fuzzy weighted geometric aggregation (QDFWGA) operator, and discuss their properties.
- To develop a new multiple attribute decision-making (MADM) method based on the proposed operators.
- To provide an illustrative example to demonstrate the flexibility and superiority of the proposed method.

This paper is organized as follows. Section 2 provides an introduction to FSs, IFSs, PyFSs, q-ROFSs, and LDFSs. In Section 3, we introduce the concept of QDFSs, compare them with existing FSs, and discuss algebraic and set-theoretical operations. Section 4 proposes aggregation operators, such as QDFWAA and QDFWGA, and examines their properties. Section 5 applies QDFSs to MADM using the proposed aggregation operators. Section 6 presents a comparative analysis of QDFSs with existing sets. Finally, in Section 7, we provide a comprehensive conclusion of the study.

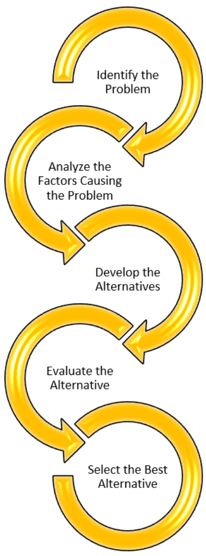


Figure 1. Steps involved in a decision-making problem.

2. Background

This section presents fundamental concepts related to FS theory. Various types of FSs are explored, each with a distinct approach to handling uncertainty and ambiguity. The discussion commences with an overview of FSs and their extension, IFSs. This extension introduces a second function to describe the degree of nonmembership, in addition to the degree of membership. Next, PyFSs are examined, which provide a more precise method for expressing uncertainty and ambiguity. After that, attention is shifted to q-ROFSs, which offer a wider range of possibilities for communicating ambiguous information. Finally, the definition of LDFSs is presented.

As noted earlier, Lotfi A. Zadeh independently developed FSs in 1965 as an extension of classical set theory. In classical set theory, the membership of elements in a set is evaluated in binary terms; an element is either a member of the set or not. In contrast, FS theory enables a gradual evaluation of an element's membership in a set by utilizing a membership function with a real unit range value between 0 and 1.

Definition 2.1. *([37]) An FS $\mathcal{A}_F$ on a universal set $\mathcal{X}$ is defined as*

$$\mathcal{A}_F = \{(x, f(x)) : x \in \mathcal{X}\}$$

where $f(x)$ is called membership function with $f(x) \in [0, 1]$.

IFSs represent an expansion of Lotfi Zadeh's FSs and were first introduced by Krassimir Atanassov, a Bulgarian mathematician, in 1983. While FSs rely on a membership function $f(x)$ to define the degree of membership of an element in a set, IFSs extend this concept by introducing a second function $g(x)$ that defines the degree of nonmembership of the element in the set. This additional degree of freedom enables the representation of uncertain and indeterminate information in a more effective and comprehensive manner.

Definition 2.2. *([8]) An IFS $\mathcal{A}_{IF}$ on a universal set $\mathcal{X}$ is defined as*

$$\mathcal{A}_{\mathcal{IF}} = \{(x, f(x), g(x)) : x \in \mathcal{X}\}$$

where $f(x)$ is called membership function and $g(x)$ is called nonmembership function with $f(x), g(x) \in [0, 1]$, satisfying

$$0 \leq f(x) + g(x) \leq 1.$$

Compared to IFSs, PyFSs, proposed by Yager, present a distinctive approach for expressing uncertainty and ambiguity with high precision and accuracy. This concept was explicitly designed to offer a structured framework for handling imprecision in real-life scenarios and to represent uncertainty and ambiguity mathematically.

Definition 2.3. *([34]) A PyFS $\mathcal{A}_{PYF}$ on a universal set $\mathcal{X}$ is defined as*

$$\mathcal{A}_{\mathcal{PYF}} = \{(x, f(x), g(x)) : x \in \mathcal{X}\}$$

where $f(x)$ is called membership function and $g(x)$ is called nonmembership function with $f(x), g(x) \in [0, 1]$, satisfying

$$0 \leq f^2(x) + g^2(x) \leq 1.$$

Yager introduced q-ROFs as a more powerful tool than IFSs and PyFSs for communicating ambiguous information. q-ROFs offer a broader range of possibilities due to the constraint that the total of the qth powers of the degrees of membership and nonmembership is no greater than one. This property ensures that q-ROFs are capable of conveying more complex and nuanced information than their FS counterparts.

Definition 2.4. *([36]) A q-ROFS $\mathcal{A}_{q-ROF}$ on a universal set $\mathcal{X}$ is defined as*

$$\mathcal{A}_{q-ROF} = \{(x, f(x), g(x)) : x \in \mathcal{X}\}$$

where $f(x)$ is called membership function and $g(x)$ is called nonmembership function with $f(x), g(x) \in [0, 1]$, satisfying

$$0 \leq f^q(x) + g^q(x) \leq 1, q \geq 1.$$

LDFSs extend the space of previously specified sets by incorporating reference parameters that correspond to membership and nonmembership grades. In MADM, LDFSs represent an ideal mathematical framework since decision-makers can flexibly choose the degrees.

Definition 2.5. *([30]) An LDFS $\mathcal{A}_{LDF}$ on a universal set $\mathcal{X}$ is defined as*

$$\mathcal{A}_{\mathcal{LDF}} = \{(x, < f(x), g(x) >, < \alpha, \beta >) : x \in \mathcal{X}\}$$

where $f(x)$ is called membership function and $g(x)$ is called nonmembership function and α, β are reference parameters with $f(x), g(x) \in [0, 1]$, satisfying

$$0 \leq \alpha f(x) + \beta g(x) \leq 1,$$

$$0 \leq \alpha + \beta \leq 1.$$

3. Quadratic Diophantine fuzzy set

In this section, we introduce the concept of QDFS, which is inspired by the idea of a general quadratic Diophantine equation in two variables. Specifically, for two variables x and y, a general second-order Diophantine equation takes the form

$$\alpha x^2 + \beta xy + \gamma y^2 = k.$$

Using this equation as motivation, we define QDFS as follows:

Definition 3.1. *Let X be a nonempty universal set. A QDFS $\mathcal{A}_{\mathcal{QDF}}$ on X is an object of the form*

$$\mathcal{A}_{\mathcal{QDF}} = (x, < f(x), g(x) >, < \alpha, \beta, \gamma >) : x \in X$$

where $f(x), g(x)$ are, respectively, the membership and nonmembership functions and α, β, γ are reference parameters such that $f(x), g(x), \alpha, \beta, \gamma \in [0, 1]$. The functions must satisfy the conditions

$$0 \leq \alpha f^2(x) + \beta f(x)g(x) + \gamma g^2(x) \leq 1 \text{ and } 0 \leq \alpha + \beta + \gamma \leq 1.$$

A quadratic Diophantine fuzzy number (QDFN) refers to the pair $(< f, g >, < \alpha, \beta, \gamma >)$.

The QDFS framework is highly relevant in situations where fuzzy information about acceptance (α), rejection (β), and hesitation or ignorance (γ) is needed. The pair $(< f, g >, < \alpha, \beta, \gamma >)$ is a QDFN, and the feasible space for the reference parameters α, β, and γ is illustrated in Figure 2 as a prism with the constraint $0 \leq \alpha + \beta + \gamma \leq 1$.

To provide further insight, Figure 3 shows the feasible regions of QDFNs for specific values of the reference parameters. For instance, Figure 3(a)–(c) illustrates scenarios where the reference parameters of acceptance, rejection, and hesitation are equal to 1, respectively. In contrast, Figure 3(d)–(f) corresponds to situations where any two reference parameters are equal to 0.5.

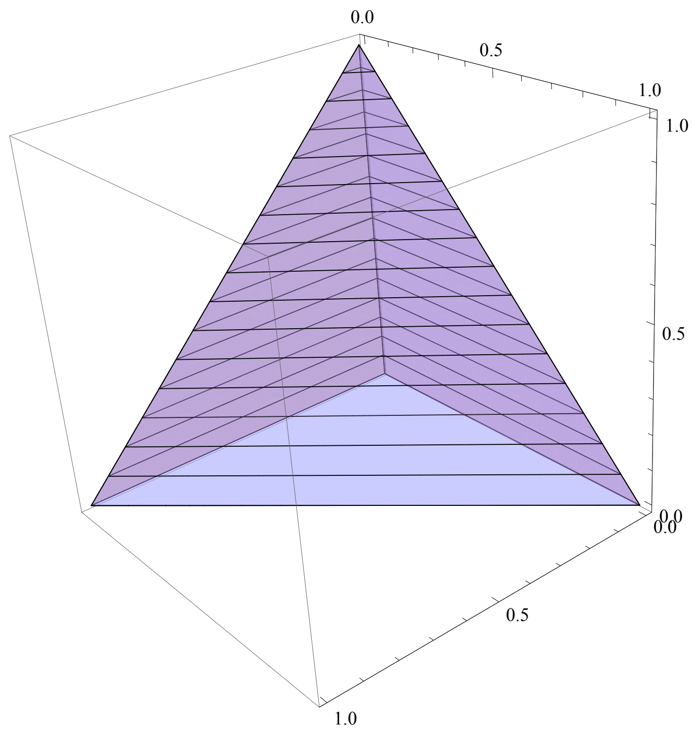
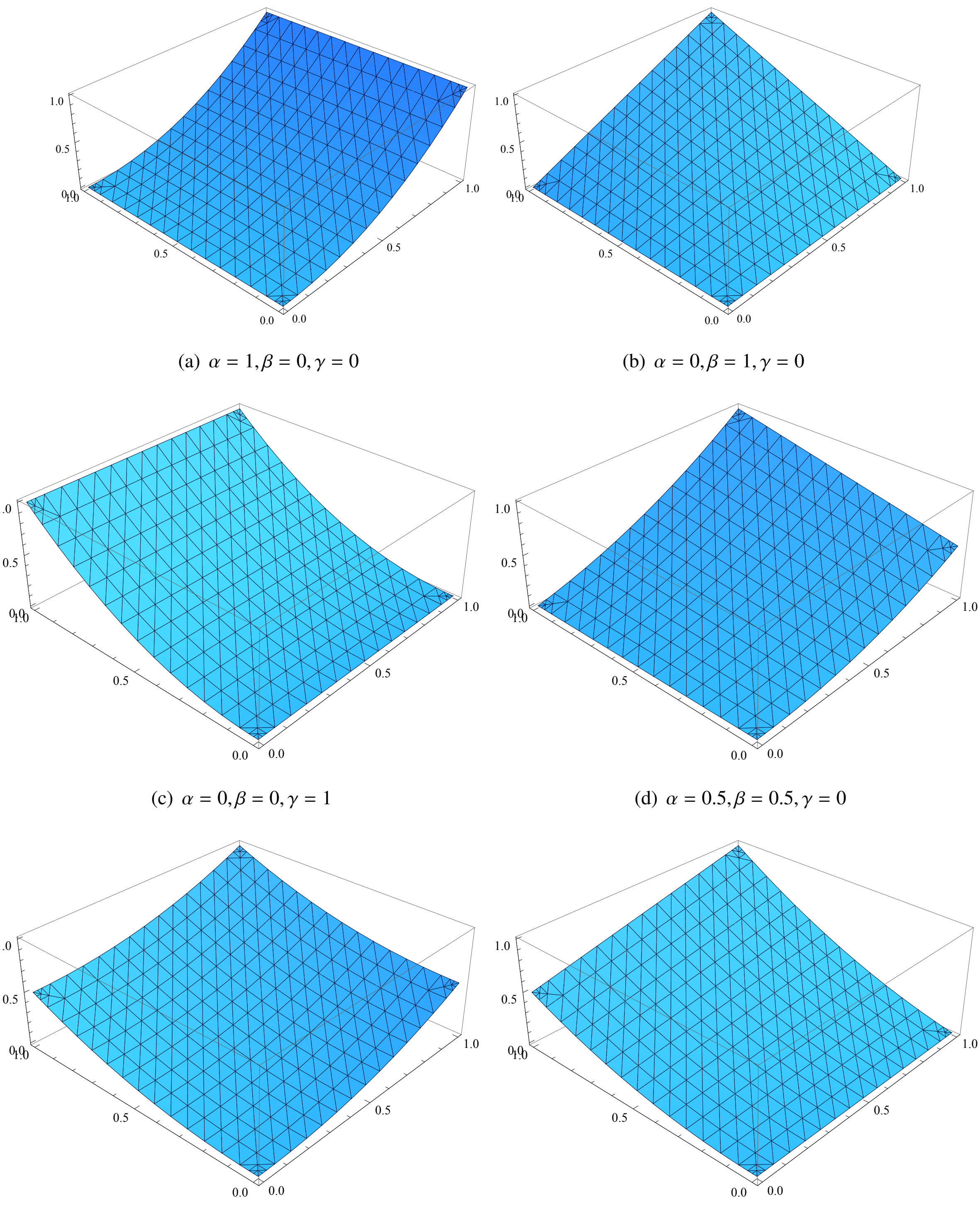


Figure 2. Graphical representation of space of reference parameters α, β, and γ.



(a) $\alpha = 1, \beta = 0, \gamma = 0$ (b) $\alpha = 0, \beta = 1, \gamma = 0$

(c) $\alpha = 0, \beta = 0, \gamma = 1$ (d) $\alpha = 0.5, \beta = 0.5, \gamma = 0$

(e) $\alpha = 0.5, \beta = 0, \gamma = 0.5$ (f) $\alpha = 0, \beta = 0.5, \gamma = 0.5$

Figure 3. Quadratic Diophantine fuzzy sets with different reference parameters.

Now, we demonstrate that the QDFS framework provides a larger feasible space for selecting membership and nonmembership values than IFSs, PyFSs, and q-ROFSs. To support this claim, we present the following theorem and then provide graphical visualizations of these sets (Figure 4).

Theorem 3.2. *The space of QDFNs is larger than the spaces of intuitionistic fuzzy numbers (IFNs), Pythagorean fuzzy numbers (PyFNs), and q-rung orthopair fuzzy numbers (q-ROFNs).*

Proof. Let $(< f, g >, < \alpha, \beta, \gamma >)$ be a QDFN, then the inequality

$$0 \leq \alpha f^2(x) + \beta f(x)g(x) + \gamma g^2(x) \leq 1 \text{ and } 0 \leq \alpha + \beta + \gamma \leq 1,$$

for $\beta = 0$ and arbitrary choice of α and γ holds for every IFN and PyFN. Hence, every IFN and PyFN is also a QDFN.

A QDFN with a given set of parameters may not necessarily be an IFN or PyFN.

For example, let $f(x) = 0.72$ and $g(x) = 0.8$, then

$$f(x) + g(x) = 1.52 > 1$$

and

$$f^2(x) + g^2(x) = 1.16 > 1$$

but $\alpha = 0.36, \beta = 0.27$, and $\gamma = 0.18$; we have

$$\alpha f^2(x) + \beta f(x)g(x) + \gamma g^2(x) = 0.46 < 1.$$

Similarly, it is easy to check that for a q-ROFS, whenever $f(x) \approx g(x) \to 1$, then $q \to \infty$.

For a special case $f(x) = g(x) = 1$, there does not exist any specific q; for this special case, no q-ROFN exists; however, for any choice of α, β, and γ such that $0 \leq \alpha + \beta + \gamma \leq 1$, we have

$$\alpha f^2(x) + \beta f(x)g(x) + \gamma g^2(x) = \alpha + \beta + \gamma \leq 1.$$

For illustration, let f(x)=g(x)=1 and $q \geq 1$, then

$$f(x)^q + g(x)^q = 2 > 1.$$

Hence, the pair $(< 1, 1 >)$ does not belong to any q-ROFS.

But, for $\alpha = 0.36, \beta = 0.27$, and $\gamma = 0.18$, we have

$$\alpha f^2(x) + \beta f(x)g(x) + \gamma g^2(x) = 0.36(1)^2 + 0.27(1)(1) + 0.18(1)^2 = 0.81 < 1$$

therefore, $(< 1, 1 >, < 0.36, 0.27, 0.18 >)$ is a QDFN.

So, it concludes that the space of QDFN consists of more points than the spaces of IFN and PyFN, providing more freedom to assign values to f and g. □

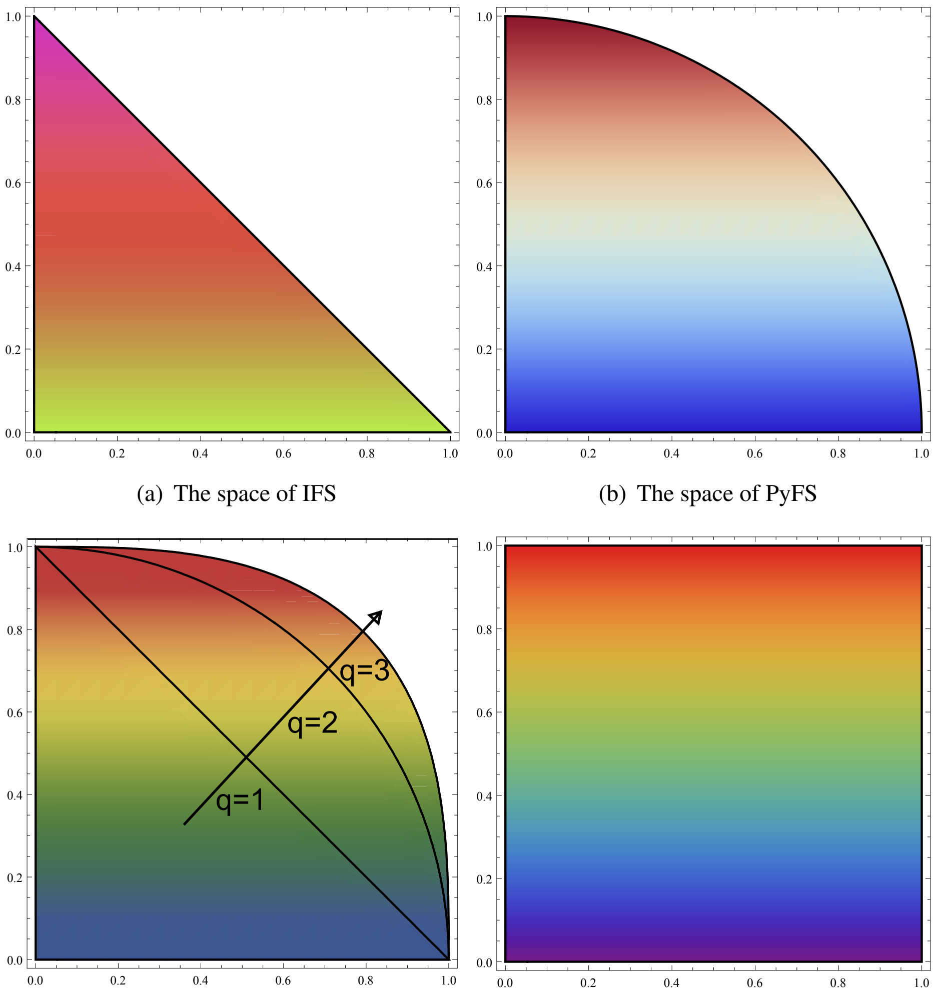
From the above arguments, it is evident that the QDFS is comparatively more feasible than other types of FSs.

3.1. Comparison with LDFS

Figure 5 shows that the feasible spaces of QDFS and LDFS are the same due to the reference parameters. This means that both QDFS and LDFS have reference parameters that enable the complete use of space.

However, QDFS is a more comprehensive tool than LDFS since it incorporates an additional parameter, γ, which caters to the hesitant type, allowing the phenomenon to be ignored. This feature is particularly beneficial in real-world scenarios where decision-makers may be indecisive or uncertain.

For example, in a real estate investment scenario, there may be three possibilities: investing in housing society X (acceptance type), investing in some other housing society (rejection type), or not investing in a housing society at all (ignorance type). As LDFS is unable to deal with such type of information, QDFS is the preferred tool for further investigation. In summary, QDFS offers a more comprehensive solution space than LDFS, making it a more useful tool for decision-making in situations where uncertain or imprecise information is present.



(a) The space of IFS

(b) The space of PyFS

(c) The space of q-ROFS

(d) The space of QDFS

Figure 4. Space comparison of QDFS with IFS, PyFS, and q-ROFS.

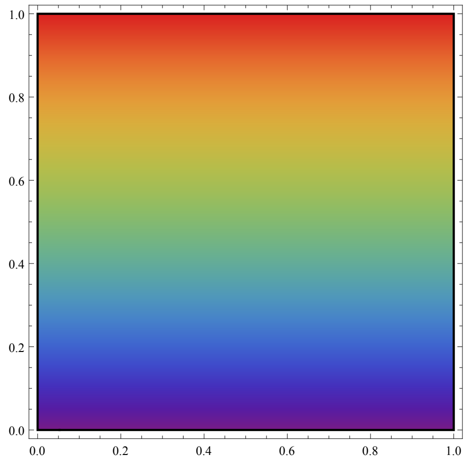


Figure 5. Space of LDFS and QDFS.

3.2. Comparison with picture fuzzy sets

In Subsection 3.1, it is shown that QDFS is advantageous over LDFS because of its ability to tackle hesitation-type scenarios. In the literature, there exists a type of FS named PFS proposed by Cuong [10], which also covers the hesitation-type fuzzy information. Cuong defined the PFS as follows:

Definition 3.3. *A PFS $\mathcal{A}_{PF}$ on a universal set $\mathcal{X}$ is defined as*

$$\mathcal{A}_{\mathcal{PF}} = \{(x, < f(x), h(x), g(x) >) : x \in \mathcal{X}\}$$

where $f(x)$ is called membership function, $h(x)$ is called neutral membership function, and $g(x)$ is called nonmembership function with $f(x), h(x), g(x) \in [0, 1]$, satisfying $0 \leq f(x) + h(x) + g(x) \leq 1$.

In Definition 3.3, the PFS imposes a limitation on membership, neutral, and nonmembership functions that their sum must not exceed 1; as a result, the feasible space of PFS becomes restricted. For example, if $f(x) = 0.43$, $h(x) = 0.52$, and $g(x) = 0.21$, then we have

$$f(x) + h(x) + g(x) = 0.43 + 0.52 + 0.21 = 1.16 > 1.$$

Hence, such f, g, and h do not represent the PFN. Now, in QDFS, for $\alpha = 0.43, \beta = 0.52$, and $\gamma = 0.21$ and the pair $F(x) = 0.62$ and $G(x) = 0.51$, we have

$$\alpha F^2(x) + \beta F(x)G(x) + \gamma G^2(x) = 0.38 < 1.$$

So, there exist numbers that are not PFNs but are QDFNs. By following the same arguments as in Theorem 3.2, it is easy to show that every PFN is in fact a QDFN.

From the above comparisons, it is evident that the proposed QDFS is a hybrid type of FS that combines the characteristics of LDFS and PFS. Figure 6 below effectively illustrates how QDFS extends and generalizes other fuzzy set extensions, making it a versatile option for accurate and flexible applications.

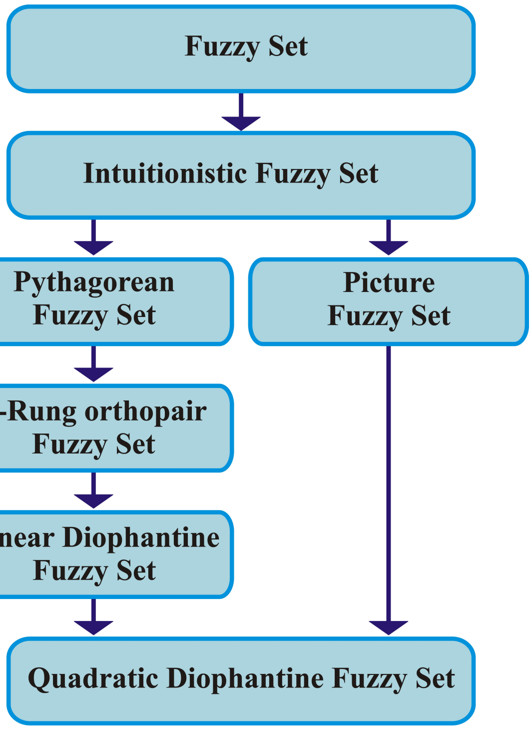


Figure 6. Hierarchical structure of quadratic Diophantine fuzzy set.

3.3. Algebraic operations on quadratic Diophantine fuzzy sets

Now, we proceed to define the algebraic operations on QDFSs in order to study their structural properties and establish aggregation operators. Specifically, we define the operations as follows:

Let $\mathcal{A}$ and $\mathcal{B}$ be two QDFSs and $\lambda > 0$. Then,

- $\mathcal{A} \subseteq \mathcal{B}$ iff $f_{\mathcal{A}} \leq f_{\mathcal{B}}, g_{\mathcal{A}} \geq g_{\mathcal{B}}, \alpha_{\mathcal{A}} \leq \alpha_{\mathcal{B}}, \beta_{\mathcal{A}} \leq \beta_{\mathcal{B}}, \gamma_{\mathcal{A}} \geq \gamma_{\mathcal{B}}$.
- $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.
- $\mathcal{A} \cup \mathcal{B} = \{(x, < max(f_{\mathcal{A}}, f_{\mathcal{B}}), min(g_{\mathcal{A}}, g_{\mathcal{B}}) >,$
  $< max(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}), min(\beta_{\mathcal{A}}, \beta_{\mathcal{B}}), min(\gamma_{\mathcal{A}}, \gamma_{\mathcal{B}}) >)\}$.
- $\mathcal{A} \cap \mathcal{B} = \{(x, < min(f_{\mathcal{A}}, f_{\mathcal{B}}), max(g_{\mathcal{A}}, g_{\mathcal{B}}) >,$
  $< min(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}), min(\beta_{\mathcal{A}}, \beta_{\mathcal{B}}), max(\gamma_{\mathcal{A}}, \gamma_{\mathcal{B}}) >)\}$.
- $\mathcal{A}^c = \{(x, < g_{\mathcal{A}}, f_{\mathcal{A}} >, < \gamma, \beta, \alpha >)\}$
- $\mathcal{A} \oplus \mathcal{B} = \{(x, < f_{\mathcal{A}} + f_{\mathcal{B}} - f_{\mathcal{A}} f_{\mathcal{B}}, g_{\mathcal{A}} g_{\mathcal{B}} >,$
  $< \alpha_{\mathcal{A}} + \alpha_{\mathcal{B}} - \alpha_{\mathcal{A}} \alpha_{\mathcal{B}}, \beta_{\mathcal{A}} \beta_{\mathcal{B}}, \gamma_{\mathcal{A}} \gamma_{\mathcal{B}} + \gamma_{\mathcal{A}} \beta_{\mathcal{B}} + \gamma_{\mathcal{B}} \beta_{\mathcal{A}} >)\}$
- $\mathcal{A} \otimes \mathcal{B} = \{(x, < f_{\mathcal{A}} f_{\mathcal{B}}, g_{\mathcal{A}} + g_{\mathcal{B}} - g_{\mathcal{A}} g_{\mathcal{B}} >,$
  $< \alpha_{\mathcal{A}} \alpha_{\mathcal{B}} + \alpha_{\mathcal{A}} \beta_{\mathcal{B}} + \alpha_{\mathcal{B}} \beta_{\mathcal{A}}, \beta_{\mathcal{A}} \beta_{\mathcal{B}}, \gamma_{\mathcal{A}} + \gamma_{\mathcal{B}} - \gamma_{\mathcal{A}} \gamma_{\mathcal{B}} >)\}$
- $\lambda \mathcal{A} = \{(x, < 1 - (1 - f_{\mathcal{A}})^{\lambda}, g_{\mathcal{A}}^{\lambda} >, < 1 - (1 - \alpha_{\mathcal{A}})^{\lambda}, \beta_{\mathcal{A}}^{\lambda}, (\gamma_{\mathcal{A}} + \beta_{\mathcal{A}})^{\lambda} - \beta_{\mathcal{A}}^{\lambda} >)\}$
- $\mathcal{A}^{\lambda} = \{(x, < f_{\mathcal{A}}^{\lambda}, 1 - (1 - g_{\mathcal{A}})^{\lambda} >, < (\alpha_{\mathcal{A}} + \beta_{\mathcal{A}})^{\lambda} - \beta_{\mathcal{A}}^{\lambda}, \beta_{\mathcal{A}}^{\lambda}, 1 - (1 - \gamma_{\mathcal{A}})^{\lambda} >)\}$.

The proposed algebraic operations can be illustrated with the following example.

Example 3.4. *Let* $\mathcal{A} = x, < 0.7, 0.15 >, < 0.1, 0.18, 0.36 >,$
$\mathcal{B} = x, < 0.44, 0.08 >, < 0.05, 0.49, 0.14 >$, *and* $\lambda = 2$, *then*
$\mathcal{A} \cup \mathcal{B} = \{(x, < 0.7, 0.08 >, < 0.1, 0.18, 0.14 >)\}$.

$\mathcal{A} \cap \mathcal{B} = \{(x, < 0.44, 0.15 >, < 0.05, 0.18, 0.36 >)\}$.
$\mathcal{A}^c = \{(x, < 0.15, 0.7 >, < 0.36, 0.18, 0.1 >)\}$
$\mathcal{A} \oplus \mathcal{B} = \{(x, < 0.83, 0.01 >, < 0.15, 0.09, 0.25 >)\}$
$\mathcal{A} \otimes \mathcal{B} = \{(x, < 0.31, 0.22 >, < 0.43, 0.09, 0.45 >)\}$
$2\mathcal{A} = \{(x, < 0.91, 0.02 >, < 0.19, 0.03, 0.03 >)\}$
$\mathcal{A}^\lambda = \{(x, < 0.49, 0.28 >, < (0.27, 0.03, 0.59 >)\}$.

3.4. Set operations on QDFS

The algebra of sets provides a framework for developing and describing the set-theoretic operations of union, intersection, and complementation, as well as the relationships between set equality and set inclusion. Moreover, this algebra furnishes systematic approaches for computing computations utilizing these operations and relations, as well as for evaluating expressions.

For any QDFS $\mathcal{A}, \mathcal{B}$, and C, the following set-theoretical laws hold:

a. if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq C$ then $\mathcal{A} \subseteq C$ (law of trichotomy)
b. $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$ (commutativity of intersection)
   $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$ (commutativity of union)
c. $(\mathcal{A} \cap \mathcal{B}) \cap C = \mathcal{A} \cap (\mathcal{B} \cap C)$ (associativity of intersection)
   $(\mathcal{A} \cup \mathcal{B}) \cup C = \mathcal{A} \cup (\mathcal{B} \cup C)$ (associativity of union)
d. $(\mathcal{A} \cap \mathcal{B}) \cup C = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap C)$ (distributive law of intersection over union)
   $(\mathcal{A} \cup \mathcal{B}) \cap C = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup C)$ (distributive law of union over intersection)
e. $(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$ (De Morgan's law)

3.5. Score and accuracy functions

In this section, we introduce the score and accuracy functions, which are essential for developing the ordering between QDFNs. The score function for a QDFN is defined as follows:

Definition 3.5. *Let $Q = (< f_Q, g_Q >, < \alpha_Q, \beta_Q, \gamma_Q >)$ be a QDFN, then the score function on Q can be defined by the mapping $S(Q) \to [-1, 1]$*

$$S(Q) = \frac{(f_Q - g_Q) + (\alpha + \beta - \gamma)}{2}.$$

Definition 3.6. *The accuracy function on Q can be defined by the mapping $A(Q) \to [0, 1]$*

$$A(Q) = \frac{(f_Q + g_Q) + (\alpha + \beta + \gamma)}{2}.$$

To compare two QDFNs Q_1 and Q_2, we employ score and accuracy function with the following criteria:

- If $S(Q_1) \leq S(Q_2)$, then $Q_1 \leq Q_2$.
- If $S(Q_1) \geq S(Q_2)$, then $Q_1 \geq Q_2$.
- If $S(Q_1) = S(Q_2)$, then
  - if $A(Q_1) \leq A(Q_2)$, then $Q_1 \leq Q_2$.
  - if $A(Q_1) \geq A(Q_2)$, then $Q_1 \geq Q_2$.
  - if $A(Q_1) = A(Q_2)$, then $Q_1 = Q_2$.

Example 3.7. *Let* $Q_1 = (<0.41, 0.17>, <0.1, 0.38, 0.17>)$ *and* $Q_2 = (<0.54, 0.23>, <0.21, 0.36, 0.33>)$ *be two QDFNs, then*

$$S(Q_1) = \frac{(0.41 - 0.17) + (0.1 + 0.38 - 0.17)}{2} = 0.275$$

and

$$S(Q_2) = \frac{(0.54 - 0.23) + (0.21 + 0.36 - 0.33)}{2} = 0.275$$

as $S(Q_1) = S(Q_2)$*, so check accuracy function, now*

$$A(Q_1) = \frac{(0.41 + 0.17) + (0.1 + 0.38 + 0.17)}{2} = 0.615$$

and

$$A(Q_2) = \frac{(0.54 + 0.23) + (0.21 + 0.36 + 0.33)}{2} = 0.835$$

since $A(Q_1) \leq A(Q_2)$ *hence* $Q_1 \leq Q_2$.

4. Aggregation operators

This section is dedicated to the introduction of two new aggregation operators, specifically designed for a collection Ω of QDFNs. We refer to these operators as QDFWAA and QDFWGA, respectively. We provide an exhaustive description of the properties of these operators.

4.1. Quadratic Diophantine fuzzy weighted averaging aggregation operator

Let $q_j = \left(\left\langle f_{q_j}, g_{q_j}\right\rangle, \left\langle \alpha_{q_j}, \beta_{q_j}, \gamma_{q_j}\right\rangle\right)$ $(j = 1, 2, ..., n)$ be a collection of CLDFNs. Then, the QDFWAA operator can be denoted and defined by

$$Q_{WA}(q_1, ..., q_n) = w_1 q_1 \oplus w_2 q_2 \oplus ... \oplus w_n q_n,$$

where w_j is the weight of q_j satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In particular, if $w = (1/n, 1/n, 1/n, ..., 1/n)^T$, then the Q_{WA} operator reduces to quadratic Diophantine fuzzy averaging aggregation ($QDFAA$) operator:

$$Q_A(q_1, ..., q_n) = \frac{1}{n}(q_1 \oplus q_2 \oplus ... \oplus q_n).$$

Theorem 4.1. *Let* $q_j = \left(\left\langle f_{q_j}, g_{q_j}\right\rangle, \left\langle \alpha_{q_j}, \beta_{q_j}, \gamma_{q_j}\right\rangle\right)$ $(j = 1, 2, ..., n)$ *be a collection of QDFNs,* w_j *is the weight of* q_j *satisfying* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$*. Then the fused result from* Q_{WA} *operator is also a QDFN and*

$$Q_{WA}(q_1, ..., q_n) = \left(\begin{array}{c} \left\langle \left(1 - \prod_{j=1}^{n}(1 - f_{q_j})^{w_j}\right), \prod_{j=1}^{n} g_{q_j}^{w_j} \right\rangle, \\ \left\langle 1 - \prod_{j=1}^{n}(1 - \alpha_{q_j})^{w_j}, \prod_{j=1}^{n} \beta_{q_j}^{w_j}, \prod_{j=1}^{n}(\gamma_{q_j} + \beta_{q_j})^{w_j} - \prod_{j=1}^{n} \beta_{q_j}^{w_j} \right\rangle \end{array} \right). \tag{4.1}$$

Proof. By employing algebraic operations presented in Subsection 3.3, one can easily verify the first result. Now we prove (4.1) by using mathematical induction on n.

When $n = 2$, we have

$$Q_{WA}(q_1, q_2) = w_1 q_1 \oplus w_2 q_2.$$

Since

$$w_1 q_1 = \left(\left\langle (1 - (1 - f_{q_1})^{w_1}), g_{q_1}^{w_1} \right\rangle, \left\langle 1 - (1 - \alpha_{q_1})^{w_1}, \beta_{q_1}^{w_1}, (\gamma_{q_1} + \beta_{q_1})^{w_1} - \beta_{q_1}^{w_1} \right\rangle\right),$$

$$w_2 q_2 = \left(\left\langle (1 - (1 - f_{q_2})^{w_2}), g_{q_2}^{w_2} \right\rangle, \left\langle 1 - (1 - \alpha_{q_2})^{w_2}, \beta_{q_2}^{w_2}, (\gamma_{q_2} + \beta_{q_2})^{w_2} - \beta_{q_2}^{w_2} \right\rangle\right).$$

Then

$$\begin{aligned} Q_{WA}(q_1, q_2) &= w_1 q_1 \oplus w_2 q_2 \\ &= \left(\left\langle 1 - (1 - f_{q_1})^{w_1}(1 - f_{q_2})^{w_2}, g_{q_1}^{w_1} g_{q_2}^{w_2} \right\rangle,\right. \\ &\left. \left\langle 1 - (1 - \alpha_{q_1})^{w_1}(1 - \alpha_{q_2})^{w_2}, \beta_{q_1}^{w_1}\beta_{q_2}^{w_2}, (\gamma_{q_1} + \beta_{q_1})^{w_1}(\gamma_{q_2} + \beta_{q_2})^{w_2} - \beta_{q_1}^{w_1}\beta_{q_2}^{w_2} \right\rangle\right), \end{aligned}$$

Suppose (4.1) holds for $n = k$, that is

$$\begin{aligned} Q_{WA}(q_1, q_2, ..., q_k) &= w_1 q_1 \oplus w_2 q_2 \oplus ... \oplus w_k q_k \\ &= \begin{pmatrix} \left\langle 1 - \prod_{j=1}^{k}(1 - f_{q_j})^{w_j}, \prod_{j=1}^{k} g_{q_j}^{w_j} \right\rangle, \\ \left\langle 1 - \prod_{j=1}^{k}(1 - \alpha_{q_j})^{w_j}, \prod_{j=1}^{k}\beta_{q_j}^{w_j}, \prod_{j=1}^{n}(\gamma_{q_j} + \beta_{q_j})^{w_j} - \prod_{j=1}^{k}\beta_{q_j}^{w_j} \right\rangle \end{pmatrix}. \end{aligned}$$

Then for $n = k + 1$, we have

$$\begin{aligned} &1 - \prod_{j=1}^{k}(1 - f_{q_j})^{w_j} + (1 - (1 - f_{q_k+1})^{w_{k+1}}) - \left(1 - \prod_{j=1}^{k}(1 - f_{q_j})^{w_j}\right)(1 - (1 - f_{q_k+1})^{w_{k+1}}) \\ &= 1 - \prod_{j=1}^{k+1}(1 - f_{q_j})^{w_j}. \end{aligned}$$

We have

$$\begin{aligned} Q_{WA}(q_1, ..., q_{k+1}) &= w_1 q_1 \oplus w_2 q_2 \oplus ... \oplus w_k q_k \oplus w_{k+1} q_{k+1} \\ &= (w_1 q_1 \oplus w_2 q_2 \oplus ... \oplus w_k q_k) \oplus w_{k+1} q_{k+1} \\ &= \left(\left\langle 1 - \prod_{j=1}^{k+1}(1 - f_{q_j})^{w_j}, \prod_{j=1}^{k+1} g_{q_j}^{w_j} \right\rangle,\right. \\ &\left. \left\langle 1 - \prod_{j=1}^{k+1}(1 - \alpha_{q_j})^{w_j}, \prod_{j=1}^{k+1}\beta_{q_j}^{w_j}, \prod_{j=1}^{k+1}(\gamma_{q_j} + \beta_{q_j})^{w_j} - \prod_{j=1}^{k+1}\beta_{q_j}^{w_j} \right\rangle\right). \end{aligned}$$

Proved. □

Theorem 4.2. *Let* $q_j = \left(\left\langle f_{q_j}, g_{q_j} \right\rangle, \left\langle \alpha_{q_j}, \beta_{q_j}, \gamma_{q_j} \right\rangle\right)$ *and* $q_{j^*} = \left(\left\langle f_{q_{j^*}}, g_{q_{j^*}} \right\rangle, \left\langle \alpha_{q_{j^*}}, \beta_{q_{j^*}}, \gamma_{q_{j^*}} \right\rangle\right)$ $(j = 1, 2, ..., n)$ *be two collections of QDFNs. Then the* Q_{WA} *operator satisfies the following properties.*

i) Idempotency: If all QDFNs q_j ($j = 1,2,...,n$) are equal, that is, $q_j = q$ for all j, then $Q_{WA}(q_1,...,q_n) = q$.

ii) Monotonicity: For q_j and q_{j^*} ($j = 1,2,...,n$), if $q_j \leq q_{j^*}$ for all j, then

$$Q_{WA}(q_1,...,q_n) \leq Q_{WA}(q_{1^*}, q_{2^*},...,q_{n^*}).$$

iii) Boundedness: Let q_j ($j = 1,2,...,n$) be a collection of QDFNs and $q^- = \min_j\{q_j\}, q^+ = \max_j\{q_j\}$, then

$$q^- \leq Q_{WA}(q_1,...,q_n) \leq q^+.$$

Proof. i) Idempotency: Let $q = (\left\langle f_q, g_q\right\rangle, \left\langle \alpha_q,\beta_q,\gamma_q\right\rangle)$. Then $q_j = q$ yield

$$\begin{aligned}
Q_{WA}(q_1,...,q_n) &= \begin{pmatrix} \left\langle 1-\prod_{j=1}^{n}(1-f_{q_j})^{w_j}, \prod_{j=1}^{n} g_{q_j}^{w_j}\right\rangle, \\ \left\langle 1-\prod_{j=1}^{n}(1-\alpha_{q_j})^{w_j}, \prod_{j=1}^{n}\beta_{q_j}^{w_j}, \prod_{j=1}^{n}(\gamma_{q_j}+\beta_{q_j})^{w_j} - \prod_{j=1}^{n}\beta_{q_j}^{w_j}\right\rangle \end{pmatrix} \\
&= \begin{pmatrix} \left\langle \left(1-\prod_{j=1}^{n}(1-f_q)^{w_j}\right), \prod_{j=1}^{n} g_q^{w_j}\right\rangle, \\ \left\langle 1-\prod_{j=1}^{n}(1-\alpha_q)^{w_j}, \prod_{j=1}^{n}\beta_q^{w_j}, \prod_{j=1}^{n}(\gamma_q+\beta_q)^{w_j} - \prod_{j=1}^{n}\beta_q^{w_j}\right\rangle \end{pmatrix} \\
&= \begin{pmatrix} \left\langle \left(1-(1-f_q)^{\sum_{j=1}^{n} w_j}\right), g_q^{\sum_{j=1}^{n} w_j}\right\rangle, \\ \left\langle 1-(1-\alpha_q)^{\sum_{j=1}^{n} w_j}, \beta_q^{\sum_{j=1}^{n} w_j}, (\gamma_q+\beta_q)^{\sum_{j=1}^{n} w_j} - \beta_q^{\sum_{j=1}^{n} w_j}\right\rangle \end{pmatrix} \\
&= \left(\left\langle\left(1-(1-f_q)\right), g_q\right\rangle, \left\langle 1-(1-\alpha_q), \beta_q, (\gamma_q+\beta_q)-\beta_q\right\rangle\right) \\
&= \left(\left\langle f_q, g_q\right\rangle, \left\langle \alpha_q,\beta_q,\gamma_q\right\rangle\right) = q.
\end{aligned}$$

Monotonicity: As $q_j \leq q_{j^*}$, so $f_{q_j} \leq f_{q_{j^*}}, \alpha_{q_j} \leq \alpha_{q_{j^*}}$ and $g_{q_j} \geq g_{q_{j^*}}, \beta_{q_j} \geq \beta_{q_{j^*}}, \gamma_{q_j} \geq \gamma_{q_{j^*}}$ for any j, we have

$$(1-f_{q_j})^{w_j} \geq (1-f_{q_{j^*}})^{w_j}, g_{L_j}^{w_j} \geq g_{q_{j^*}}^{w_j}.$$

Moreover

$$1-\prod_{j=1}^{n}(1-f_{q_j})^{w_j} \leq 1-\prod_{j=1}^{n}(1-f_{q_{j^*}})^{w_j},\ \prod_{j=1}^{n} g_{q_j}^{w_j} \geq \prod_{j=1}^{n} g_{q_{j^*}}^{w_j}.$$

Therefore

$$1-\prod_{j=1}^{n}(1-f_{q_j})^{w_j} - \prod_{j=1}^{n} g_{q_j}^{w_j} \leq 1-\prod_{j=1}^{n}(1-f_{q_{j^*}})^{w_j} - \prod_{j=1}^{n} g_{q_{j^*}}^{w_j}. \tag{4.2}$$

If $q = Q_{WA}(q_1, q_2,,, q_n)$ and $q^* = Q_{WA}(q_{1^*}, q_{2^*},,, q_{n^*})$. Then by (4.2) and Definition 3.5, we get $S(q) \leq S(q^*)$. If $S(q) = S(q^*)$, then it is easy to show that $A(q) = A(q^*)$.

Boundedness: Idempotency and monotonicity of Q_{WA} operator implies the following:

For $q_j \geq q^- = \min_j\{q_j\}$, we have

$$Q_{WA}(q_1, q_2,,, q_n) \geq Q_{WA}(q^-, q^-,,, q^-) = q^-.$$

For $q_j \leq q^+ = \max_j\{q_j\}$, we have

$$Q_{WA}(q_1, q_2, , , q_n) \leq Q_{WA}(q^+, q^+, , , q^+) = q^+.$$

Hence $q^- \leq Q_{WA}(q_1, q_2, , , q_n) \leq q^+$. □

4.2. Quadratic Diophantine fuzzy weighted geometric aggregation operator

Let $q_j = \left(\left\langle f_{q_j}, g_{q_j}\right\rangle, \left\langle \alpha_{q_j}, \beta_{q_j}, \gamma_{q_j}\right\rangle\right)$ $(j = 1, 2, ..., n)$ be a collection of QDFNs. Then, the QDFWGA operator can be denoted and defined by

$$Q_{WG}(q_1, ..., q_n) = q_1^{w_1} \otimes q_2^{w_2} \otimes ... \otimes q_n^{w_n},$$

where w_j is the weight of q_j satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In particular, if $w = (1/n, 1/n, 1/n, ..., 1/n)^T$, then the Q_{WG} operator reduces to quadratic Diophantine fuzzy geometric aggregation ($QDFGA$) operator:

$$QDFGA(q_1, ..., q_n) = (q_1 \otimes q_2 \otimes ... \otimes q_n)^{\frac{1}{n}}.$$

Theorem 4.3. *Let* $q_j = \left(\left\langle f_{q_j}, g_{q_j}\right\rangle, \left\langle \alpha_{q_j}, \beta_{q_j}, \gamma_{q_j}\right\rangle\right)$ $(j = 1, 2, ..., n)$ *be a collection of QDFNs,* w_j *is the weight of* q_j *satisfying* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. *Then, the fused result from* Q_{WG} *operator is also a QDFN and*

$$Q_{WG}(q_1, ..., q_n) = \left(\begin{array}{c} \left\langle \prod_{j=1}^{n} f_{q_j}^{w_j}, \left(1 - \prod_{j=1}^{n} (1 - g_{q_j})^{w_j}\right)\right\rangle, \\ \left\langle \prod_{j=1}^{n} (\alpha_{q_j} + \beta_{q_j})^{w_j} - \prod_{j=1}^{n} \beta_{q_j}^{w_j}, \prod_{j=1}^{n} \alpha_{q_j}^{w_j}, 1 - \prod_{j=1}^{n} (1 - \gamma_{q_j})^{w_j} \right\rangle \end{array} \right).$$

Proof. Straight forward. □

Theorem 4.4. *Let* $q_j = \left(\left\langle f_{q_j}, g_{q_j}\right\rangle, \left\langle \alpha_{q_j}, \beta_{q_j}, \gamma_{q_j}\right\rangle\right)$
and $q_k = \left(\left\langle f_{q_{j^*}}, g_{q_{j^*}}\right\rangle, \left\langle \alpha_{q_{j^*}}, \beta_{q_{j^*}}, \gamma_{q_{j^*}}\right\rangle\right)$ $(j = 1, 2, ..., n)$ *be two collections of QDFNs. Then, the* Q_{WG} *operator satisfies the following properties:*

i) Idempotency: If all QDFNs q_j $(j = 1, 2, ..., n)$ *are equal, that is,* $q_j = L$ *for all* j, *then*

$$Q_{WG}(q_1, ..., q_n) = L.$$

ii) Monotonicity: For q_j *and* q_{j^*} $(j = 1, 2, ..., n)$, *if* $q_j \leq q_{j^*}$ *for all* j, *then*

$$Q_{WG}(q_1, ..., q_n) \leq Q_{WG}(q_{1^*}, q_{2^*}, ..., q_{n^*}).$$

iii) Boundedness: Let q_j $(j = 1, 2, ..., n)$ *be a collection of QDFNs and* $q^- = \min_j\{q_j\}$, $q^+ = \max_j\{q_j\}$, *then*

$$q^- \leq Q_{WG}(q_1, ..., q_n) \leq q^+.$$

Proof. The proof is similar to the proof of Theorem 4.2. □

In this section, we present a new MADM technique that utilizes quadratic Diophantine fuzzy information and is based on the proposed aggregation operators. To address the challenges associated with complex decision-making problems, we consider a collection of alternatives and a collection of attributes with corresponding weight vectors to evaluate the effectiveness and reliability of the proposed approach.

Let $\mathcal{X} = \mathcal{X}1, \mathcal{X}2, ..., \mathcal{X}n$ denote the collection of attributes, $\mathcal{Y} = \mathcal{Y}1, \mathcal{Y}2, ..., \mathcal{Y}m$ denote the collection of alternatives, and $w = w_1, w_2, ..., w_n$ denote the associated attribute weights, where $\sum_{j=1}^{n} w_j = 1$. For each alternative $\mathcal{Y}i$ and attribute $\mathcal{X}j$, we use $q'_{ij} = \left(\left\langle f_{q_{ij}}, g_{q_{ij}} \right\rangle, \left\langle \alpha_{q_{ij}}, \beta_{q_{ij}}, \gamma_{q_{ij}} \right\rangle\right)$ to denote the QDFN that represents the characteristic of $\mathcal{Y}i$ with respect to $\mathcal{X}j$. By using a quadratic Diophantine fuzzy decision matrix (QDFDM), we can store the characteristics of all alternatives in relation to the attributes.

In this setting, we propose two aggregation operators, namely, QDFWAA and QDFWGA, to evaluate the overall performance of each alternative based on the given attributes and their corresponding weights. The properties of these operators are described in detail.

4.3. Algorithm for decision-making with Q_{WA} and Q_{WG} operators

The following are the steps involved in the algorithm used for decision-making using QDFS:

Step 1: Using the following transformation, normalize the QDFDM:

$$q_{ij} = \begin{cases} q'_{ij}, & \text{for benefit attribute } \mathcal{X}_j \\ \overline{q}'_{ij}, & \text{for cost attribute } \mathcal{X}_j \end{cases}.$$

Step 2: Make use of the Q_{WA} operator:

$$\overset{*}{q}_i = Q_{WA}(q_{i1}, q_{i2}, ..., q_{in}), \ (i = 1, 2, 3, ..., m)$$

or the Q_{WG} operator:

$$\overset{**}{q}_i = Q_{WG}(q_{i1}, q_{i2}, ..., q_{in}), \ (i = 1, 2, 3, ..., m).$$

Step 3: Calculate the scores $S(\overset{*}{q}_i)$ (or $S(\overset{**}{q}_i)$) of the overall attribute value $\overset{*}{q}_i$ (or $\overset{**}{q}_i$) of the alternative $\mathcal{Y}_i$ using the score function.

Step 4: Sort the alternatives according to their scores. If two scores are identical, the accuracy degree should be used to order the alternatives.

Step 5: Choose the alternative with the highest score (accuracy degree).

Figure 7 shows a flowchart for the aforementioned procedure for more explanation.

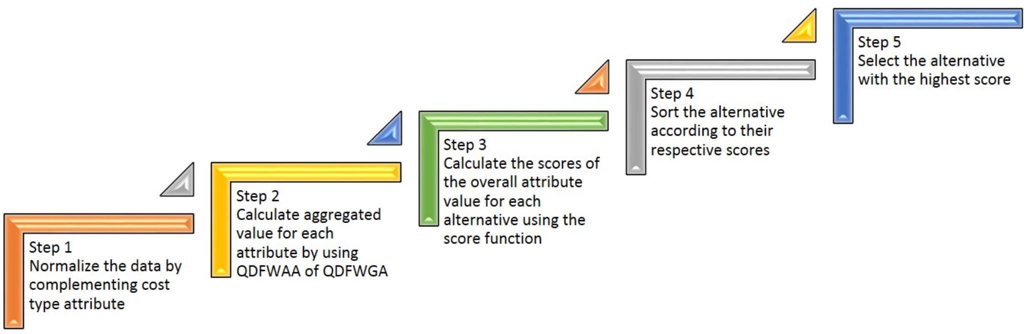


Figure 7. Graphical explanation of the proposed algorithm.

4.4. Application of QDFS in the selection of face mask detection algorithms

Since the discovery of the novel coronavirus, various countries' governments have implemented disciplinary measures to limit its spread. In the face of pandemics, the use of modern technology has become increasingly prevalent in enhancing the operations and performance of numerous organizations. One such technology is face mask detection, which aids in monitoring social distancing and, with artificial intelligence interpretation, identifying the use of face masks.

The WHO has issued several guidelines to mitigate the devastating consequences of COVID-19's spread on both society and the economy. The virus has spread to the point of becoming a global health hazard. These guidelines include the use of face masks, maintaining social distancing, and adopting a virtual workplace culture. Of all these guidelines, face mask detection is a crucial approach that could help determine the number of people wearing masks, maintain an appropriate social distance, and protect a significant population from the virus's severity.

The use of face mask technology often involves computer vision and deep learning. To utilize this technology, the user must subscribe to a face mask alert app on their smartphone or other electronic devices and connect their camera to the app. When someone nearby is not wearing a mask, the user must grant permission to receive warnings or notifications. In addition, users can report violations of mask-wearing rules to the administrator.

It is evident that the most appropriate algorithm selection for face mask identification is the traditional MADM problem. In this subsection, we present an example of algorithm selection for face mask detection with QDF information to demonstrate the approach provided in this work. An IT expert must choose the most suitable face mask detection algorithm from four preexisting algorithms, as follows:

$\mathcal{Y}_1$: **Convolutional Neural Network (CNN)** CNN is a form of neural network. CNN is effective in computer vision applications such as pattern classification due to its cheap computation complexity and ability to retrieve location data. CNN eliminates top-level features by combining convolutional sections with primary pictures. Advanced training algorithms are used in this technique to recognize numerous facial traits and detect whether or not a person is wearing a face mask. Face masks are recognized in real time by the technology, which helps limit the transmission rate. The system also has digital capabilities for receptionists and physicians.

$\mathcal{Y}_2$: **Region-Based Convolutional Neural Network (R-CNN)** The first CNN-focused two-stage object identification approach was R-CNN [25]. R-design CNNs are composed of three distinct blocks. Each source image uses a selective search strategy to produce class-independent regional notions during the first step. To acquire feature vectors of the needed length from each zone recommendation, the second block applies CNN with five convolutional layers and two fully connected layers. Following that, each region suggestion is fed into a distinct CNN to generate feature maps of a predetermined length. The last block uses a linear support vector machine for a specified group to categorize each area suggestion.

$\mathcal{Y}_3$: **Faster Region-Based Convolutional Neural Network (Faster-R-CNN)** Ren et al. [28] presented Faster-R-CNN as a substitute for the suggestion technique; nevertheless, they have since changed to a region proposal network (RPN) [17]. The RPN is just a fully convolutional network (FCN) that accepts any size image and produces a flurry of rectangular-shaped item recommendations. Every object concept has an object class score rate, which determines whether or not the proposal comprises an object.

$\mathcal{Y}_4$: **Mask Region-Based Convolutional Neural Network (Mask R-CNN)** Mask R-CNN, developed by Kaiming He [18], creates region proposals by utilizing the proposal networks of Faster-R-regional CNN. Instead of a pooling layer for regions of interest (RoI), the technique uses an RoI align layer on area suggestions to align recovered features with the input position of the object. The aligned RoIs are then sent through the Mask R-final CNN stage, which produces three outputs: a bounding box offset, a class label, and a binary object mask. To mask each RoI, a simple fully convolutional neural network is utilized.

The attributes considered to evaluate these alternatives are $\mathcal{X}_1$ =Accuracy, $\mathcal{X}_2$ = Simplicity, $\mathcal{X}_3$ =Effectiveness, and $\mathcal{X}_4$ =Scalability. The parameter α represents the expert's level of belief that an algorithm should have a specific attribute, the parameter β denotes the expert's level of confidence that an algorithm should not have a specific attribute, and the parameter γ represents the expert's degree of doubt of the specific attribute. Face mask detection algorithms are to be evaluated using the QDF information under the above four attributes. (Assume the attribute index's weighting vector is $(0.20, 0.16, 0.37, 0.27)$.)

Now, we make use of the algorithm proposed in Subsection 4.3.

Step 1: The expert information in the form of QDFS is shown in Table 1. As all attributes are of benefit type, there is no need to normalize QDFDM.

Table 1. Quadratic Diophantine fuzzy decision matrix.

	$\mathcal{X}_1$	$\mathcal{X}_2$
$\mathcal{Y}_1$	$(\langle 0.58, 0.69\rangle, \langle 0.14, 0.49, 0.21\rangle)$	$(\langle 0.47, 0.92\rangle, \langle 0.28, 0.28, 0.33\rangle)$
$\mathcal{Y}_2$	$(\langle 0.77, 0.30\rangle, \langle 0.15, 0.19, 0.54\rangle)$	$(\langle 0.80, 0.16\rangle, \langle 0.31, 0.15, 0.14\rangle)$
$\mathcal{Y}_3$	$(\langle 0.46, 0.67\rangle, \langle 0.26, 0.14, 0.40\rangle)$	$(\langle 0.73, 0.94\rangle, \langle 0.33, 0.22, 0.43\rangle)$
$\mathcal{Y}_4$	$(\langle 0.31, 0.73\rangle, \langle 0.42, 0.16, 0.41\rangle)$	$(\langle 0.75, 0.91\rangle, \langle 0.19, 0.44, 0.30\rangle)$
	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathcal{Y}_1$	$(\langle 0.62, 0.87\rangle, \langle 0.19, 0.12, 0.59\rangle)$	$(\langle 0.66, 0.31\rangle, \langle 0.60, 0.11, 0.26\rangle)$
$\mathcal{Y}_2$	$(\langle 0.62, 0.26\rangle, \langle 0.19, 0.18, 0.50\rangle)$	$(\langle 0.49, 0.60\rangle, \langle 0.23, 0.21, 0.37\rangle)$
$\mathcal{Y}_3$	$(\langle 0.48, 0.18\rangle, \langle 0.63, 0.18, 0.12\rangle)$	$(\langle 0.30, 0.11\rangle, \langle 0.66, 0.21, 0.10\rangle)$
$\mathcal{Y}_4$	$(\langle 0.90, 0.56\rangle, \langle 0.12, 0.15, 0.19\rangle)$	$(\langle 0.65, 0.58\rangle, \langle 0.64, 0.13, 0.11\rangle)$

Step 2: Now utilize the proposed aggregation operators Q_{WA} and Q_{WG} to calculate the aggregated QDF data shown in Table 2.

Table 2. The integrated assessment information by Q_{WA} and Q_{WG}.

	Q_{WA}	Q_{WG}
$\mathcal{Y}_1$	$(\langle 0.60, 0.63\rangle, \langle 0.34, 0.18, 0.40\rangle)$	$(\langle 0.60, 0.78\rangle, \langle 0.31, 0.26, 0.41\rangle)$
$\mathcal{Y}_2$	$(\langle 0.66, 0.31\rangle, \langle 0.21, 0.18, 0.39\rangle)$	$(\langle 0.63, 0.37\rangle, \langle 0.21, 0.21, 0.43\rangle)$
$\mathcal{Y}_3$	$(\langle 0.49, 0.27\rangle, \langle 0.54, 0.18, 0.20\rangle)$	$(\langle 0.45, 0.54\rangle, \langle 0.49, 0.48, 0.23\rangle)$
$\mathcal{Y}_4$	$(\langle 0.76, 0.64\rangle, \langle 0.37, 0.17, 0.21\rangle)$	$(\langle 0.65, 0.69\rangle, \langle 0.30, 0.26, 0.24\rangle)$

Step 3: The scores of attributes are computed in Table 3 and aggregate scores of alternatives are computed in Table 4.

Table 3. Quadratic Diophantine fuzzy score matrix.

	Score of $\mathcal{X}_1$	Score of $\mathcal{X}_2$	Score of $\mathcal{X}_3$	Score of $\mathcal{X}_4$
$\mathcal{Y}_1$	0.15	0.06	−0.23	0.23
$\mathcal{Y}_2$	0.19	0.23	−0.16	−0.28
$\mathcal{Y}_3$	0.11 (accuracy= 0.38)	0.21	−0.15	0.11 (accuracy= 0.54)
$\mathcal{Y}_4$	0.22	−0.03	−0.16	0.07

Table 4. The ranking distributions of alternatives.

	Q_{WA}		Q_{WG}	
	Score function	Ranking	Score function	Ranking
$\mathcal{Y}_1$	0.09	4th	0.14 (accuracy= 0.50)	3rd
$\mathcal{Y}_2$	0.13	2nd	0.12	4th
$\mathcal{Y}_3$	0.15	1st	0.17	1st
$\mathcal{Y}_4$	0.11	3rd	0.14 (accuracy= 0.60)	2nd

Step 4: The rankings of alternatives for Q_{WA} and Q_{WG} are given in Table 4.

Step 5: Since the alternative $\mathcal{Y}_3$ has the highest score for both Q_{WA} and Q_{WG}, the IT expert should select the Faster-R-CNN.

The problem considered above is solved by asymmetric weights based upon personal choice, and the same problem could also be solved for symmetric weights [33] based upon normal distribution; i.e., the attribute index's weighting vector is $(0.16, 0.35, 0.35, 0.16)$). The changes are highlighted in Tables 5 and 6.

Table 5. The integrated assessment information by Q_{WA} and Q_{WG} for symmetric weights.

	Q_{WA}	Q_{WG}
$\mathcal{Y}_1$	$(\langle 0.58, 0.72\rangle, \langle 0.30, 0.19, 0.41\rangle)$	$(\langle 0.56, 0.84\rangle, \langle 0.29, 0.24, 0.42\rangle)$
$\mathcal{Y}_2$	$(\langle 0.71, 0.25\rangle, \langle 0.24, 0.17, 0.33\rangle)$	$(\langle 0.67, 0.31\rangle, \langle 0.23, 0.22, 0.39\rangle)$
$\mathcal{Y}_3$	$(\langle 0.57, 0.35\rangle, \langle 0.51, 0.18, 0.24\rangle)$	$(\langle 0.50, 0.71\rangle, \langle 0.45, 0.44, 0.29\rangle)$
$\mathcal{Y}_4$	$(\langle 0.78, 0.69\rangle, \langle 0.31, 0.21, 0.24\rangle)$	$(\langle 0.67, 0.77\rangle, \langle 0.27, 0.22, 0.26\rangle)$

Table 6. The ranking distributions of alternatives for symmetric weights.

	Q_{WA}		Q_{WG}	
	Score function	Ranking	Score function	Ranking
$\mathcal{Y}_1$	−0.09	4th	−0.085	4th
$\mathcal{Y}_2$	0.27	2nd	0.21	1st
$\mathcal{Y}_3$	0.34	1st	0.20	2nd
$\mathcal{Y}_4$	0.19	3rd	0.07	3rd

5. Validation and robustness analysis

In this section, we present a sensitivity analysis to assess the impact of the aggregation operators and weights used during the decision-making process on the proposed method. As demonstrated in the previous section, the results indicate that the alternative $\mathcal{Y}_3$ is consistently ranked as the top choice for all aggregation operators, regardless of the weights employed. However, when *QWG* with normally distributed weights is utilized, $\mathcal{Y}_3$ is ranked as the second-best option. Based on this information, $\mathcal{Y}_3$ is designated as the optimal selection.

On the other hand, the alternative $\mathcal{Y}_1$ persistently occupies the fourth rank for all aggregation operators, except for *QWG* with user-defined weights, where it is positioned second to last. Therefore, it can be inferred that $\mathcal{Y}_1$ is not a viable choice for this decision-making process and should be excluded from consideration.

In addition, it is noteworthy that the rankings of the alternatives remain unaffected, even when the weights are modified for the operator Q_{WA}. The rankings of the alternatives for the various aggregation operators considered are presented in Table 7 for further analysis.

Table 7. The summarized ranking comparison of alternatives.

	Weights	Rankings
Q_{WA}	User defined	$\mathcal{Y}_3 \leq \mathcal{Y}_2 \leq \mathcal{Y}_4 \leq \mathcal{Y}_1$
Q_{WG}	User defined	$\mathcal{Y}_3 \leq \mathcal{Y}_4 \leq \mathcal{Y}_1 \leq \mathcal{Y}_2$
Q_{WA}	Normally distributed	$\mathcal{Y}_3 \leq \mathcal{Y}_2 \leq \mathcal{Y}_4 \leq \mathcal{Y}_1$
Q_{WG}	Normally distributed	$\mathcal{Y}_2 \leq \mathcal{Y}_3 \leq \mathcal{Y}_4 \leq \mathcal{Y}_1$

The proposed method is a versatile tool that has been designed to accommodate a wide range of input data. It is widely acknowledged that data can manifest in diverse shapes and sizes, and finding a model that can effectively handle such variations can be challenging. Nonetheless, the proposed model can handle uncertainties and covers an extensive spectrum of data types, including FSs, IFSs, PFSs, PyFSs, and q-ROFSs.

The proposed model possesses a unique feature of reference parameters, which play a critical role in expanding the membership and nonmembership grades' space. By altering the physical interpretation of these parameters, the model can be applied efficiently in different scenarios, imparting greater flexibility and adaptability to real-life situations.

In the context of MADM problems, we encounter varying types of criteria and input data, depending on the situation. For instance, investment opportunities may have different criteria and input data than recruitment or vendor selection. In such cases, the proposed QDFS model has proven to be highly effective. It is a straightforward and easy-to-understand method that can be applied seamlessly to different alternatives and attributes.

The model's simplicity and accessibility make it user-friendly, even for individuals who lack expertise in data analysis. With the proposed QDFS model, decision-makers can easily analyze various alternatives and make well-informed choices, thus saving time, resources, and costs by avoiding ineffective decisions.

Therefore, the proposed method is a powerful and adaptable tool that can handle different types of input data. Its unique feature of reference parameters and simplicity make it a highly effective tool for solving MADM problems. By employing the proposed QDFS model, organizations can make informed decisions, optimize their resources, and save time. Moreover, since the proposed algorithm yields nearly identical results under various weights and aggregation operators, it is considered valid and recommended for decision-making in uncertain situations. The validation of the algorithm is also illustrated in Figure 8.

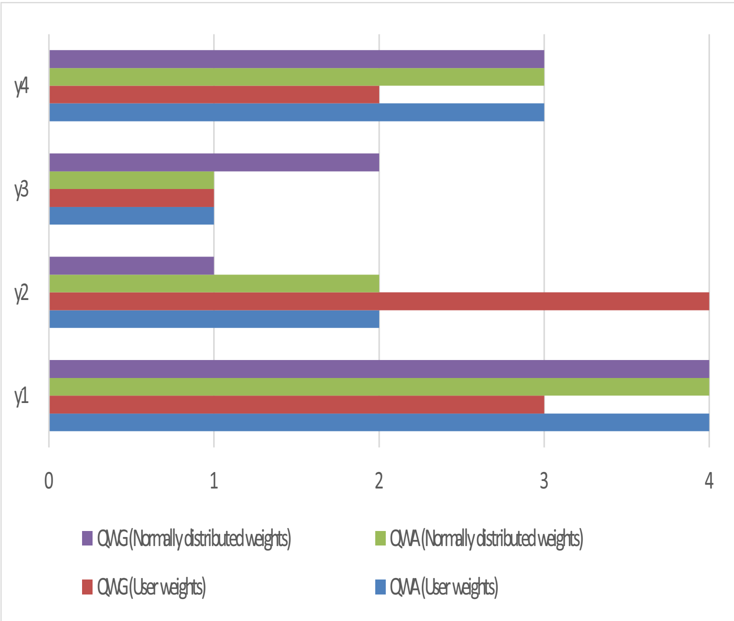


Figure 8. Comparison of alternatives for face mask detection algorithm selection problem.

6. Comparative analysis

In this subsection, we present a qualitative comparison of the proposed QDFS method with existing techniques. Table 8 provides a comparative analysis of the proposed QDFS model and several FS extensions.

Table 8. Comparison of QDFS with existing extensions of fuzzy sets.

Collection of sets	Acceptance Information	Rejection Information	Ignorance Information	Parameterization	Unrestricted Domain
FS	Yes	No	No	No	No
IFS	Yes	Yes	No	No	No
PyFS	Yes	Yes	No	No	No
q-ROFS	Yes	Yes	No	No	No
PFS	Yes	Yes	Yes	No	No
LDFS	Yes	Yes	No	Yes	Yes
QDFS	Yes	Yes	Yes	Yes	Yes

It is evident that the QDFS method proposed in this study provides a higher degree of flexibility and autonomy in handling fuzzy information.

7. Conclusions

The present article introduces QDFS, which is an effective tool for analyzing fuzzy information while taking into account the reference parameters of acceptance, ignorance, and rejection type. The complete framework of QDFS, including its algebraic operations, comparison functions, and aggregation operators, is described in detail. To further illustrate the versatility of the proposed approach, a numerical example is presented. In addition, detailed comparisons between the suggested method and current methods are also provided.

The proposed approach assumes the weights of attributes and experts, which are determined by decision specialists in advance. However, we observed that our approach only considers subjective factors for the weight of experts and does not consider the weight information generated from the decision matrix. Therefore, we recommend exploring the associative weight method to handle actual decision-making problems where the weight of attributes needs to be determined.

Future work will involve adapting the suggested approach to various environments and applying it to the disciplines of similarity measures, VIKOR and TOPSIS methods, and various other aggregation operators. This will enable us to further improve the QDFS method and expand its applicability to a wider range of decision-making scenarios. In addition, we plan to explore the use of QDFS in real-life problems such as site selection for wind power plants [32] and assessment of the innovative ability of universities [14]. By doing so, we hope to gain a deeper understanding of the principles of FSs and how they can be applied to practical decision-making problems.

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Conflict of interest

The authors declare that the publication of this article does not involve any conflicts of interest.

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