On excluded minors and biased graph representations of frame matroids

by

Daryl Funk

M.Sc., University of Victoria, 2009
B.Sc., Simon Fraser University, 1992

Dissertation Submitted in Partial Fulfillment
of the Requirements for the Degree of

Doctor of Philosophy

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Spring 2015

APPROVAL

Name: Daryl Funk

Degree: Doctor of Philosophy (Mathematics)

Title of Thesis: *On excluded minors and biased graph representations of frame matroids*

Examining Committee: Dr. Jonathan Jedwab, Chair
Professor, Department of Mathematics

Dr. Matthew DeVos
Senior Supervisor
Associate Professor, Department of Mathematics

Dr. Luis Goddyn
Co-Supervisor
Professor, Department of Mathematics

Dr. Bojan Mohar
Internal Examiner
Professor, Department of Mathematics

Dr. Daniel Slilaty
External Examiner
Professor, Department of Mathematics and Statistics
Wright State University

Date Defended: 8 January 2015

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ABSTRACT

A *biased graph* is a graph in which every cycle has been given a *bias*, either *balanced* or *unbalanced*. Biased graphs provide representations for an important class of matroids, the *frame* matroids. As with graphs, we may take *minors* of biased graphs and of matroids, and a family of biased graphs or matroids is *minor-closed* if it contains every minor of every member of the family. For any such class, we may ask for the set of those objects that are minimal with respect to minors subject to not belonging to the class — *i.e.*, we may ask for the set of *excluded minors* for the class. A frame matroid need not be uniquely represented by a biased graph. This creates complications for the study of excluded minors. Hence this thesis has two main intertwining lines of investigation: (1) excluded minors for classes of frame matroids, and (2) biased graph representations of frame matroids.

Trying to determine the biased graphs representing a given frame matroid leads to the necessity of determining the biased graphs representing a given graphic matroid. We do this in Chapter 3. Determining all possible biased graph representations of non-graphic frame matroids is more difficult. In Chapter 5 we determine all biased graphs representations of frame matroids having a biased graph representation of a certain form, subject to an additional connectivity condition.

Perhaps the canonical examples of biased graphs are *group-labelled* graphs. Not all biased graphs are group-labellable. In Chapter 2 we give two characterisations of those biased graphs that are group labellable, one topological in nature and the other in terms of the existence of a sequence of closed walks in the graph. In contrast to graphs, which are well-quasi-ordered by the minor relation, this characterisation enables us to construct infinite antichains of biased graphs, even with each member on a fixed number of vertices. These constructions are then used to exhibit infinite antichains of frame matroids, each of whose members are of a fixed rank.

In Chapter 4, we begin an investigation of excluded minors for the class of frame matroids by seeking to determine those excluded minors that are not 3-connected. We come close, determining a set $\mathcal{E}$ of 18 particular excluded minors and drastically narrowing the search for any remaining such excluded minors.

Keywords: Frame matroid; biased graph; excluded minors; representations; group-labelling; gain graph; well-quasi-ordering; lift matroid; graphic matroid

I am asked sometimes what a matroid is. I often revert to our sacred writings and recall the encounter of Alice with the grinning Cheshire cat. At one stage the cat vanishes away, beginning with the tip of its tail and ending with the grin, which persists long after the remainder of the cat.

“I expect you saw a lot of loose grins wandering around," said Humpty Dumpty.

“Yes, indeed," said Alice. “But with some of them you could see they belonged to cats. I kept trying to imagine what the cats behind them were like."

“What an Auslandish thing to do," said Humpty Dumpty.

“Oh it’s very interesting," said Alice. “I look at the grin and I see the eyes and whiskers and the ears and the warm furry body and the long sinuous tail."

“You put it very graphically," said Humpty Dumpty.

“But I can’t do it with all the grins," said Alice. “Some of them have the most uncatly shapes. Whatever can be behind them?"

“That’s what makes it interesting," said Humpty Dumpty. “You have to classify the Uncats.

> It now will be right to describe
> Each particular batch
> Distinguishing those that are Fanos
> and bite
> From K. Kuratowskis that scratch."

“I’ve heard something like that before," said Alice crossly. “The creatures here all recite far too much poetry." And she stalked angrily away.

- Tutte,

apparently apocryphal; found at http://userhome.brooklyn.cuny.edu/skingan/matroids/toast.html

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Overview

The objects of study in this thesis are *biased graphs*. These are graphs in which every cycle has been given a *bias*, either *balanced* or *unbalanced*. Biased graphs provide representations for an important class of matroids, the *frame* matroids. As with graphs, we may take *minors* of biased graphs, and of frame matroids. Some natural families of biased graphs and their corresponding classes of matroids are shown in the Venn diagram of Figure 1 (page 2). Each of these classes — of biased graphs and of matroids — is closed under taking minors, and each properly contains natural minor-closed families. Accordingly, for any such class, we may ask for the set of those objects that are minimal with respect to minors subject to not belonging to the class — *i.e.*, we may ask for the set of *excluded minors* for the class.

There may be many biased graphs representing a given frame matroid. As with investigations into other minor-closed classes of matroids, this creates complications for the study of excluded minors. Hence this thesis has two main intertwining lines of investigation: (1) excluded minors for classes of frame matroids, and (2) biased graph representations of frame matroids.

Trying to determine the biased graphs representing a given frame matroid leads to the necessity of determining the biased graphs representing a given graphic matroid. In Chapter 3, we exhibit five families of biased graphs, each defined as those biased graphs having a particular specific structure, whose frame matroids are graphic. The main result of Chapter 3 is that together with graphs these provide all biased graph representations of graphic matroids:

Theorem 3.1. *Let M be a connected graphic matroid, and suppose Ω is a biased graph representing M. Then Ω is a member of one of six explicit families of biased graphs.*

Determining all possible biased graph representations of non-graphic frame matroids is more difficult. In Chapter 5 we determine all biased graphs representations of frame matroids having a biased graph representation of a certain form. This completes one case of six required in order to answer the question of representability of frame matroids by biased graphs, subject to an additional connectivity condition.

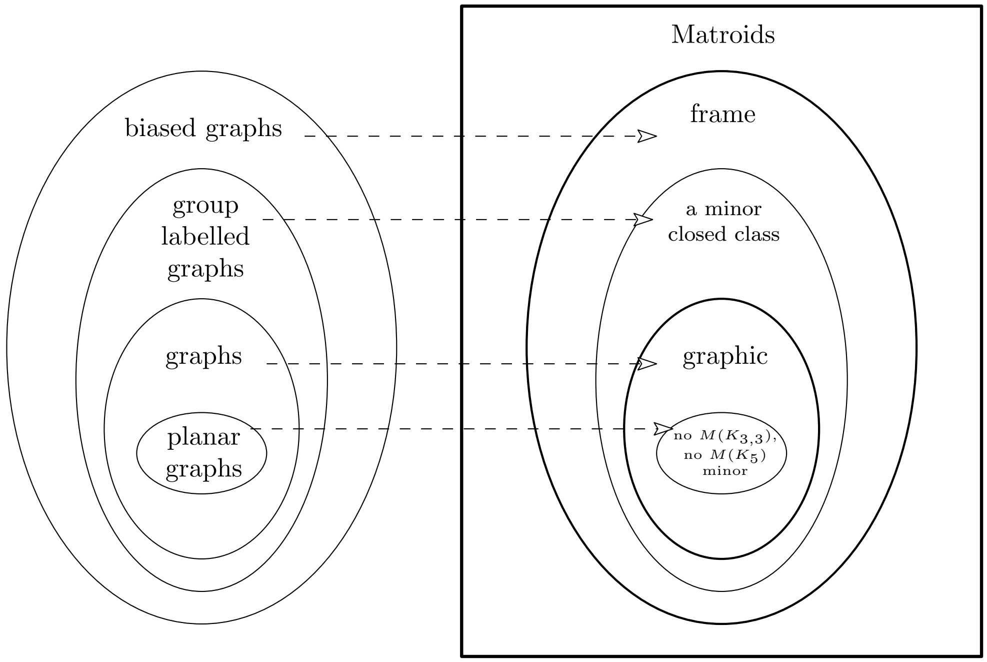


Figure 1: Some minor-closed classes of biased graphs and of matroids.

One of the minor-closed families of biased graphs appearing in Figure 1 is *group-labelled* graphs (also called *gain* graphs). Indeed, for any group Γ, there is a minor-closed class of biased graphs, the Γ-*labelled* graphs. Naturally, we would like to know which biased graphs are group-labellable. Since it is often convenient to describe the biases of cycles in a graph using a group-labelling, there is also a practical motivation for learning the answer to this question. In Chapter 2 we give two characterisations of those biased graphs that are group-labellable, one topological in nature and the other in terms of the existence of a sequence of closed walks in the graph.

Theorem 2.1. *Given a biased graph Ω, construct a 2-cell complex K by adding a disc with boundary C for each balanced cycle C. Then the following are equivalent:*

1. *Ω is group-labellable*
2. *The balanced cycles of Ω are precisely those contractible in K*
3. *No unbalanced cycle can be moved to a balanced cycle via a sequence of balanced reroutings of closed walks.*

Using Theorem 2.1, we find the behaviour of biased graphs to be in stark contrast to the behaviour of graphs. We exhibit natural minor-closed classes of biased graphs having infinite sets of excluded minors, and exhibit infinite antichains all whose members are on a fixed number of vertices.

Theorem 2.4. *Let Γ be an infinite group.*

(a) *For every $t \geq 3$ there exists an infinite antichain of Γ-labelled graphs on t vertices, each of whose underlying simple graph is K_t.*

(b) *For every $t \geq 3$, $t \neq 4$, there exists an infinite antichain of Γ-labelled graphs on t vertices, each of whose underlying simple graph is a cycle of length t.*

These results have parallels in the setting of frame matroids. Given a group Γ, the family $\mathcal{F}_\Gamma$ of frame matroids representable by a Γ-labellable biased graph is a natural minor-closed family to consider. Theorem 2.4 has the following corollary.

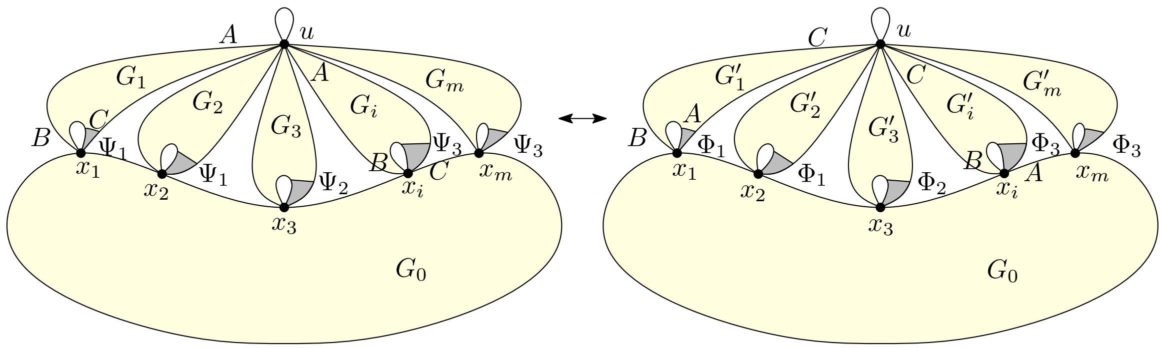
Corollary 2.31. *For every infinite group Γ and every $t \geq 3$, there exist infinite antichains of rank t matroids in $\mathcal{F}_\Gamma$.*

Little is known about excluded minors for the class of frame matroids. Zaslavski has exhibited several in [40]. Biased graphs share many properties with graphs — perhaps, like graphic matroids, frame matroids may be characterised by a finite list of excluded minors. On the other hand, we show in Chapter 2 that unlike graphs, both the classes of biased graphs and frame matroids contain infinite antichains. In Chapter 4, we begin by seeking to determine those excluded minors for the class of frame matroids that are not 3-connected. We come close, determining a set $\mathcal{E}$ of 18 particular excluded minors for the class and drastically narrowing the search for any remaining such excluded minors. The main result of Chapter 4 is:

Theorem 4.1. *Let M be an excluded minor for the class of frame matroids, and suppose M is not 3-connected. Then either $M \in \mathcal{E}$ or M is the 2-sum of $U_{2,4}$ and a 3-connected non-binary frame matroid.*

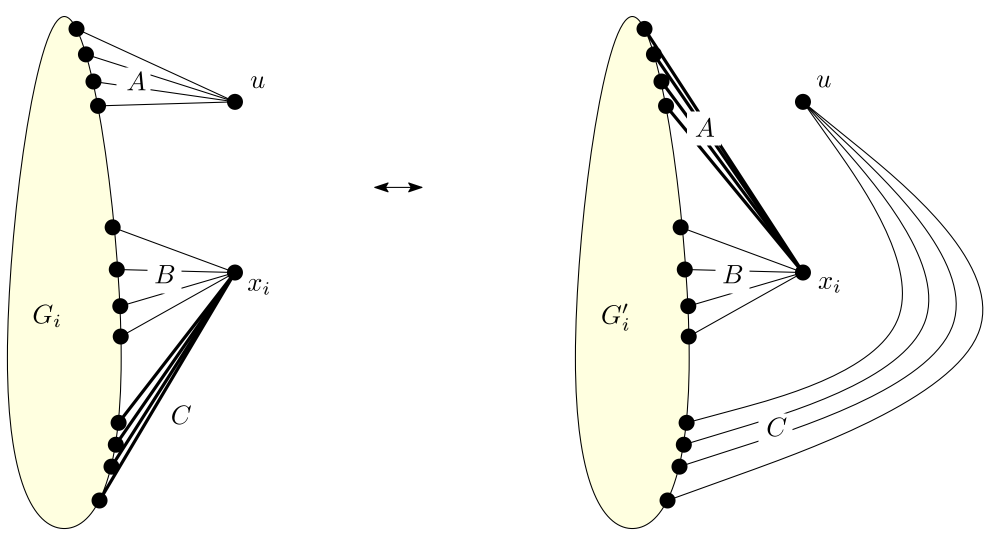
In the course of proving Theorem 4.1, we discover an operation that may be performed on a biased graph to produce a second biased graph with frame matroid isomorphic to the frame matroid of the first. This operation may be thought of as analogous to performing a Whitney twist on a graph to produce a second graph with cycle matroid isomorphic to the cycle matroid of the first. We call our operation a *twisted flip*. A precise definition is given near the end of Section 1.2.4. Informally, a twisted flip is performed on a biased graph of the form shown in Figure 2(a), in which a cycle is balanced if it has even intersection with each member of a collection of distinguished subsets of edges $\{\Psi_1, \ldots, \Psi_k\}$. The twisted flip operation produces a biased graph of the form shown in Figure 2(b). The operation as it is applied to each of the biased subgraphs G_i with $i \in \{1, \ldots, m\}$ is shown in Figure 3: each such biased subgraph meets the rest of the graph in just two vertices, u and x_i; edges incident to u become incident to x_i and are placed in a distinguished set Φ_j; edges incident to x_i contained in a distinguished set Ψ_j become incident to u. A cycle in the resulting biased graph is balanced if its intersection with each new distinguished set Φ_j ($j \in \{1, \ldots, k\}$) is even.

Theorem 4.2. *If Ω' is obtained from Ω by a twisted flip, then their frame matroids are isomorphic.*



(a) Ω (b) Ω'

Figure 2: A twisted flip.



(a) (b)

Figure 3: A twisted flip's effect on a single biased subgraph G_i. Edges contained in a distinguished set are bold. In G_i (a) edges marked C are in some Ψ_j, and in G'_i (b) edges marked A are then in Φ_j.

Chapter 1

Introduction

1.1 Matroids

A matroid is an abstract object underlying a notion of dependence, analogous to the way a group underlies a notion of symmetry and a topology underlies a notion of continuity. Their study was initiated in 1935 by Hassler Whitney [33]. Familiar notions of dependence abstracted by various classes of matroids include linear dependence in a vector space, algebraic dependence of elements of a field over a subfield, and geometric dependence of points in a geometry. There are many ways to axiomatically define a matroid. From our perspective, the following definition in terms of minimal dependent sets, is perhaps the most natural.

A *matroid* is a pair $(E, \mathcal{C})$ consisting of a finite *ground set* E and a collection $\mathcal{C}$ of subsets of E, its *circuits*, satisfying

(C1) $\emptyset \notin \mathcal{C}$;

(C2) If $C, C' \in \mathcal{C}$ and $C \subseteq C'$, then $C = C'$;

(C3) If C and C' are distinct members of $\mathcal{C}$ and $e \in C \cap C'$, then there exists $C'' \in \mathcal{C}$ such that $C'' \subseteq (C \cup C') \setminus \{e\}$.

Subsets of E containing a circuit are the *dependent* sets of a matroid. Axiom (C1) is a non-triviality condition, and (C2) just says that circuits are minimally dependent. Axiom (C3) is called the *circuit elimination axiom*. (Axiom (C3) is often called the "weak circuit elimination axiom", to distinguish it from the *strong circuit elimination axiom*, which may be deduced from it. The strong version of the circuit elimination axiom states: If $C, C' \in \mathcal{C}$, $e \in C \cap C'$, and $f \in C \setminus C'$, then there exists $C'' \in \mathcal{C}$ such that $f \in C'' \subseteq (C \cup C') \setminus \{e\}$.) That (C3) captures a combinatorial essence of "dependence" may perhaps be seen by considering some examples of matroids.

A canonical example of a matroid is that of a collection of vectors in a vector space. Let $\{v_1, \ldots, v_m\}$ be a set of n-dimensional vectors over a field $\mathbb{F}$. Place $v_1, \ldots, v_m$ as columns in an $n \times m$ matrix A, and let E be the set of column indices of A. The *vector matroid* (or *column matroid*) of A, denoted $M[A]$, is the pair $(E, \mathcal{C})$ where $\mathcal{C}$ is the collection of subsets of E that index minimally linearly dependent sets of columns of A. That axioms (C1), (C2), (C3) above hold for such a collection $\mathcal{C}$ follows immediately from what it means for a collection of vectors to be minimaly linearly dependent. If M is the vector matroid of a matrix A over the field $\mathbb{F}$, then M is $\mathbb{F}$-*representable*, *representable over* $\mathbb{F}$, or *linear over* $\mathbb{F}$; the matrix A *represents* M. A matroid that is linear over some field is said to be *linear* (such matroids are also often said to be simply *representable*, but we would like to avoid confusion when speaking about other types of representations of matroids, and use the word "representable" in a broader sense in this thesis). We denote the finite field of order q by $GF(q)$.

As this example suggests, much of the terminology of matroid theory comes from linear algebra. A set $X \subseteq E$ not containing a circuit is said to be *independent*. The *rank* of a set $X \subseteq E$, denoted $\mathrm{rank}(X)$ or $r(X)$, is the size of the largest independent set contained in X; the rank of a matroid $M{=}(E, \mathcal{C})$ is $\mathrm{rank}(E)$. A subset $B \subseteq E$ with $|B| = \mathrm{rank}(E)$ is a *basis*. The *closure* of a set $X \subseteq E$, denoted $\mathrm{cl}(X)$, is the set $\{e \in E : \mathrm{rank}(X \cup \{e\}) = \mathrm{rank}(X)\}$. Elements in $\mathrm{cl}(X)$ are said to be *spanned* by X.

The matroid axioms capture the essential combinatorial properties of linear dependence of a set of vectors in a vector space. However, matroids are much more general than vector spaces, and many matroids are not linear over any field. Indeed, it has been conjectured that among all matroids on n elements, the proportion of n element linear matroids tends to zero as $n \to \infty$ [18].

Another fundamental example of a matroid is that arising from a graph, in the following manner. Let $G{=}(V, E)$ be a graph. A *cycle* in G is a connected subgraph of G all of whose vertices have degree two. The *cycle matroid* of G, denoted $M(G)$, is the matroid $(E, \mathcal{C})$ on E in which $\mathcal{C}$ is the collection of edge sets of cycles in G. It is straightforward to verify that (C1), (C2), and (C3) are satisfied by $\mathcal{C}$. Independent sets of $M(G)$ are precisely edge sets of forests, and for a connected graph a basis consists of the edges of a spanning tree. If M is the cycle matroid of some graph G, then M is *graphic*, and we say G *represents* M.

As graphs provided another motivating example for matroids, much terminology in matroid theory is inherited from graph theory. Use of the word "circuit" for "minimal dependent set" comes from graph theory. If $\{e, f\}$ is a circuit of size two, then each of e and f spans the other, and e and f are said to be in *parallel*, and a circuit of size one is called a *loop*. A matroid is *simple* if it has no loops and no pair of parallel elements.

Graphic matroids are linear over every field. Given a graph $G{=}(V, E)$, with $V{=}\{v_1, \ldots, v_n\}$ and $E{=}\{e_1, \ldots, e_m\}$, let A be the $n \times m$ matrix whose rows are indexed by V and columns by E, in which for each edge $e{=}v_iv_j$ with $i < j$, the column indexed by e has 1 in the row indexed by v_i, -1 in the row indexed by v_j, and a zero in every other coordinate, and if $i = j$ then column e is identically zero. Matrix A is the *signed incidence* matrix of G. It is straightforward to verify that $M[A]$ is a linear representation of $M(G)$ over any field.

Given a rank r matroid $M{=}M[A]$ linear over a field $\mathbb{F}$, we may just as well consider each column vector as a point in a projective geometry, over $\mathbb{F}$ of dimension $r-1$. This view is useful, as projective geometries play a role for matroids linear over a field analogous to that of complete graphs in graph theory: every simple rank r matroid linear over a field is contained in the projective geometry of dimension $r-1$ over the field (constructed in one of the standard ways from the vector space) ([23], Theorem 6.1.3). Accordingly, matroid theory also takes much terminology from geometry. A set $X \subseteq E$ in a matroid M on ground set E with $\mathrm{cl}(X) = X$ is a *flat* of M. *Points* are flats of rank 1, *lines* are flats of rank 2, flats of rank 3 are *planes*, and a *hyperplane* is a flat of rank one less than that of M. This perspective allows us to illustrate matroids of small rank, representing points, lines, and planes in a geometric diagram.

Example 1. The rank 2 *uniform* matroid on four elements, $U_{2,4}$, is the matroid on four elements with circuits just those subsets of size three. It may be represented geometrically as the 4-point line of Figure 1.1. The matroid $U_{2,4}$ is ternary — that is, linear over $GF(3)$. The matrix in Figure 1.1 represents $U_{2,4}$ over $GF(3)$.

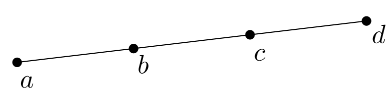
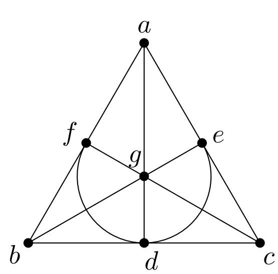


Figure 1.1: The 4-point line $U_{2,4}$

Example 2. The smallest projective plane, the *Fano matroid*, denoted F_7, is the matroid of rank 3 illustrated in Figure 1.2. The Fano matroid is linear over any field of characterstic 2, represented by the matrix accompanying F_7 in Figure 1.2 when viewed over any such field.



$$\begin{array}{c} \begin{matrix} a & b & c & d & e & f & g \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

F_7

Figure 1.2: The Fano matroid F_7.

1.1.1 Matroid minors

There is a natural notion of a minor of a matroid, which directly generalises graph minors. To describe it, we must first introduce matroid duality. For every matroid $M=(E,\mathcal{C})$, there is a unique matroid $M^*=(E,\mathcal{C}^*)$, the *dual* of M, defined as the matroid whose bases are the complements of the bases of M. The prefix "co-" is used to denote a collection of elements in the dual. For example, the bases of the dual M^* are *cobases* of M and the circuits of the dual are *cocircuits* of M. The set of cocircuits $\mathcal{C}^*$ consists of those subsets of E that are complements of hyperplanes of M; this follows almost immediately from the definitions and matroid axioms.

Let X and Y be sets of elements in a matroid $M=(E,\mathcal{C})$. The matroid $M\setminus X$ obtained by *deleting* X is the matroid on ground set $E\setminus X$ with circuits $\{C \subseteq E\setminus X : C \in \mathcal{C}\}$. The matroid M/Y obtained by *contracting* Y is defined as the dual of the matroid obtained by deleting Y from the dual of M; that is, $M/Y = (M^* \setminus Y)^*$. One may easily check that in the case e is an edge in a plane graph G, $G/e = (G^* \setminus e)^*$, where G^* denotes a plane dual of G. Moreover, in this case, it is not hard to see that $M(G)/e = M(G/e)$. Geometrically, contraction is the operation of projecting from Y onto a maximum rank flat contained in the complement of Y. This may perhaps be seen by considering the rank of a set $A \subseteq E \setminus Y$ in M/Y, which is given by $\mathrm{rank}_{M/Y}(A) = r_M(A\cup Y) - r_M(Y)$, where $r_{M/Y}$ and r_M respectively denote the rank of a set in M/Y and M, or considering that the circuits of M/Y are the minimal non-empty members of $\{C \setminus Y : C \in \mathcal{C}\}$ ([23], Propositions 3.1.6, 3.1.11).

A *minor* of a matroid N is any matroid M obtained by a sequence of the operations of deleting or contracting elements of N. A class of matroids $\mathcal{M}$ is *closed under minors*, or *minor-closed*, if every minor of a matroid in $\mathcal{M}$ is also in $\mathcal{M}$. The classes of graphic matroids and matroids linear over a field $\mathbb{F}$ are both closed under minors. For any minor-closed class, there is a set of *excluded minors* for the class — that is, those matroids not in the class all of whose proper minors are in the class. An excluded minor theorem is a theorem characterising a minor-closed class of matroids or graphs by exhibiting a list of excluded minors. The best known such theorem is Wagner's version (1937) of Kuratowski's theorem:

Theorem 1.1. *The excluded minors for the class of planar graphs are K_5 and $K_{3,3}$.*

In 1958, Tutte proved that the four-point line, $U_{2,4}$, is the single excluded minor for the class of binary matroids (*i.e.*, matroids linear over $GF(2)$) [23]. Tutte also gave the following excluded minor characterisation for the class of graphic matroids:

Theorem 1.2 (Tutte, 1959 [23]). *The set of excluded minors for the class of graphic matroids is $\{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})\}$.*

In 1970 two bold conjectures were made, one concerning graph minors and the other matroid minors. Wagner conjectured that in any infinite set of graphs, there is a pair one of which is a minor of the other; or in other words, when ordered by the minor relation there is no infinite antichain of graphs. This is equivalent to the statement that every minor-closed class of graphs has only a finite number of excluded minors. Wagner's conjecture was settled in 2001, when it was proved by Robertson and Seymour, in the twentieth paper of their Graph Minors Project (which required twenty-three papers totalling more than 700 pages, published between 1983 and 2010).

Theorem 1.3 (Robertson and Seymour [26]). *Every minor-closed class of graphs has a finite set of excluded minors.*

The analogous statement for matroids is false. In fact, infinite antichains of matroids are not difficult to find.

Example 3 (Lazarson, 1958 [15]). For each prime $p > 2$, the matroid S_p linear over the field of order p represented by

$$\left[\begin{array}{c|ccccc} & 0 & 1 & 1 & \cdots & 1 \\ & 1 & 0 & 1 & \cdots & 1 \\ I_{p+1} & 1 & 1 & 0 & \cdots & 1 \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 1 & 1 & 1 & \cdots & 0 \end{array}\right]$$

is an excluded minor for representability over the reals (where I_{p+1} indicates the identity matrix of size $p+1$).

The following more recent result shows that the difference in behaviour between classes of matroids linear over an infinite field and those linear over finite fields is large indeed.

Theorem 1.4 (Mayhew, Newman, & Whittle [19]). *Let $\mathbb{K}$ be an infinite field, and N be a matroid representable over $\mathbb{K}$. Then there exists an excluded minor for the class of $\mathbb{K}$-representable matroids that is not representable over any field and has N as a minor.*

We shall exhibit many other infinite antichains of matroids in Chapter 2. Nevertheless, in 1970 Rota (knowing it to be true for the two smallest fields) made the following sweeping conjecture.

Conjecture 1.5 (Rota). *For each finite field $\mathbb{F}$, there are only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.*

Progress toward proving Rota's conjecture has been a main focus of matroid theory since. While it appears to have been known at the time Rota made his conjecture that there are only four excluded minors for the class of matroids linear over the field of order three, it was not until 1979 that proofs were published, independently by Bixby and Seymour [23]. Further significant progress toward Rota's conjecture was not made until 2000, when

Geelen, Gerards, Kapoor [5] showed that seven particular matroids complete the list of excluded minors for the class of matroids linear over $GF(4)$ (for which they were awarded a 2003 Fulkerson Prize). This was an early piece in the Matroid Minors project of Geelen, Gerards, and Whittle, aimed at extending Robertson and Seymour's Graph Minors project to matroids linear over a finite field. The group have recently announced that they have proven Rota's conjecture, and that they anticipate writing and publishing the results to take "a few years" [7].

Certain subclasses of frame matroids play an important role here. A key piece in the Matroid Minors Project is the following analogue of Theorem 1.3.

Theorem 1.6 (Geelen, Gerards, and Whittle [7])**.** *For each finite field $\mathbb{F}$, every minor-closed class of $\mathbb{F}$-representable matroids has a finite set of $\mathbb{F}$-representable excluded minors.*

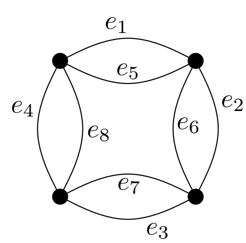
Geelen, Gerards, and Whittle report that their strategy for proving Theorem 1.6 parallels the proof of Theorem 1.3. Let us briefly consider this strategy. Suppose $\{H_1, H_2, \ldots\}$ is an infinite antichain of graphs. Then none of $H_2, H_3, \ldots$ has H_1 as a minor. Robertson and Seymour's Graph Minors Structure Theorem [25] gives a structural description of the graphs not containing H_1 as a minor. The theorem says that all graphs not containing H_1 as a minor may be constructed in a specified way from graphs that embed in a surface into which H_1 does not embed. Thus for any minor-closed class of graphs, graphs embedding into surfaces of low genus provide the fundamental classes of graphs from which all graphs in the class may be constructed, using the Graph Minors Structure Theorem. Similarly, if $\mathbb{F}$ is a finite field and $\{N_1, N_2, \ldots\}$ is an infinite antichain of $\mathbb{F}$-representable matroids, then none of $N_2, N_3, \ldots$ contain N_1 as a minor, and this imposes structure on the remaining matroids in the antichain. Geelen, Gerhards, and Whittle have proved an analogue of the Graph Minors Structure Theorem, a Matroid Minors Structure Theorem [4]. Analogous to the Graph Minors Structure Theorem, this describes, for a fixed finite field $\mathbb{F}$, how to construct, in a specified way, the members of a minor-closed class of $\mathbb{F}$-representable matroids from matroids contained in three fundamental minor-closed classes of $\mathbb{F}$-representable matroids. These fundamental classes are: matroids linear over subfields of $\mathbb{F}$, frame matroids over $\mathbb{F}$, and duals of frame matroids over $\mathbb{F}$. Minor-closed classes of frame matroids thus turn out to be of fundamental importance in matroid structure theory.

Having set the context into which this thesis should be read, we now precisely define the central objects of this thesis: biased graphs, group-labellings of graphs, frame matroids, minors of these objects, and other key notions we will require in the chapters that follow. We generally follow Oxley's notation [23]; in matters of graph theory, we follow Diestel [2].

1.2 Biased graphs and frame matroids

A *biased graph* Ω consists of a pair $(G, \mathcal{B})$, where G is a graph and $\mathcal{B}$ is a collection of cycles of G, called *balanced*, such that no theta subgraph contains exactly two balanced cycles; a *theta* graph consists of a pair of distinct vertices, called the *branch* vertices of the theta, and three internally disjoint paths between them. We say such a collection $\mathcal{B}$ satisfies the *theta property*.

Example 4. The pair $(2C_4, \mathcal{B})$, where $2C_4$ is the graph with edges named as in Figure 1.3, with set of balanced cycles $\mathcal{B} = \{e_1e_2e_3e_4, e_1e_2e_7e_8, e_3e_4e_5e_6\}$ is a biased graph. Since every balanced cycle is Hamiltonian and no two differ in just one edge, no two balanced cycles appear in any theta subgraph of $2C_4$. Hence no theta subgraph contains two balanced cycles, and the theta property is satisfied.



$$\mathcal{B} = \{e_1e_2e_3e_4, e_1e_2e_7e_8, e_3e_4e_5e_6\}$$

Figure 1.3: A biased graph, $(2C_4, \mathcal{B})$

Cycles not in $\mathcal{B}$ are called *unbalanced*; the membership or non-membership of a cycle in $\mathcal{B}$ is its *bias*. If the subgraph $G[X]$ of G induced by a set $X \subseteq E(G)$ contains no unbalanced cycle, it is *balanced*; otherwise it is *unbalanced*. If $G[X]$ contains no balanced cycle, it is *contrabalanced*. We denote by $V(X)$ the set of vertices incident with an edge in X, and by $b(X)$ the number of balanced components of $G[X]$. We write $\Omega = (G, \mathcal{B})$ and say G is the *underlying graph* of Ω. Throughout, graphs are finite, and may have loops and parallel edges. When it is important to distinguish an edge which is not a loop from one that is, we refer to an edge having distinct endpoints as a *link*. We denote the set of links incident with a vertex v in a biased graph by $\delta(v)$.

One natural example of a biased graph comes from a graph embedded in a surface. Given a graph G, and an embedding of G on a surface Σ, set $\mathcal{B} = \{C : C \text{ is a cycle contractible in } \Sigma\}$. The theta property is satisfied, for if two cycles of a theta are contractible then the third cycle is also contractible (Figure 1.4).

1.2.1 Group-labelled graphs

Perhaps the canonical source of examples of biased graphs are *group-labelled* graphs. These are obtained by orienting and labelling the edges of a graph using the elements

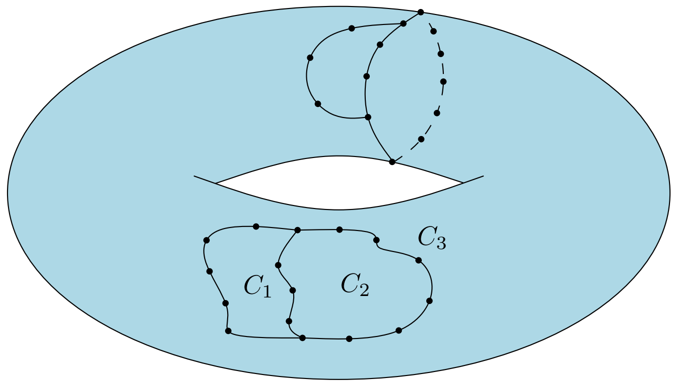


Figure 1.4: Graphs embedded on a surface give rise to biased graphs.

of a group, and letting these define the set of balanced cycles of the graph, as follows. Formally, a *group-labelling* of a graph G is an orientation of its edges together with a function $\gamma : E(G) \to \Gamma$, for some group Γ. We write Γ multiplicatively, with identity element 1. We say G has been Γ-*labelled*, or simply *labelled*, by γ. We presume such a labelling comes equipped with an orientation of $E(G)$. Now extend γ to the walks in G: if W is a walk in G with edge sequence $e_1, e_2, \ldots, e_n$, define $\gamma(W) = \prod_{i=1}^{n} \gamma(e_i)^{\epsilon_i}$, where

$$\epsilon_i = \begin{cases} 1 & \text{if } e_i \text{ is traversed forward in } W, \\ -1 & \text{if } e_i \text{ is traversed backward in } W. \end{cases}$$

Observe that if W is a simple closed walk traversing a cycle, then the walk W^{-1} obtained by traversing W in reverse has $\gamma(W^{-1}) = \gamma(W)^{-1}$. Moreover, if W and W' are two simple closed walks traversing a cycle in the same direction, then W and W' have the same cyclic sequence of edges (*i.e.*, if W has edge sequence $e_1, \ldots, e_n$ then W' has edge sequence $e_i, e_{i+1}, \ldots, e_n, e_1, \ldots, e_{i-1}$ for some $i \in \{1, \ldots, n\}$), and so $\gamma(W)$ and $\gamma(W')$ are conjugate. Hence for any two simple closed walks W and W' traversing a cycle C, $\gamma(W) = 1$ if and only if $\gamma(W') = 1$. Therefore for a graph G group-labelled by a function γ, we may unambiguously define $\mathcal{B}_\gamma$ to be the set of cycles C in G for which there is a simple closed walk traversing C with $\gamma(W) = 1$. Such a set of cycles $\mathcal{B}_\gamma$ always satisfies that theta property: if $u, v \in V(G)$ and P, Q, R are three internally disjoint u-v paths, with $\gamma(P \cup Q) = \gamma(R \cup Q) = 1$, then $\gamma(P) = \gamma(Q)^{-1} = \gamma(R)$, so $\gamma(P \cup R) = 1$. Therefore group-labelled graphs are biased graphs.

A biased graph labellable by the group of order two is called a *signed* graph. We use multiplicative notation, denoting the cyclic group of order n by $\mathbf{C}_n = \{1, g, g^2, \ldots, g^{n-1}\}$, and labelling a signed graph with a function $\sigma : E(G) \to \mathbf{C}_2 = \{+1, -1\}$. In a $\mathbf{C}_2$-labelled graph, since $(-1)^{-1} = -1$, the orientations of edges is irrelevant. Indeed, a labelling of the edges of a graph using $\mathbf{C}_2$ may be defined without orienting edges, as follows. Choose a labelling

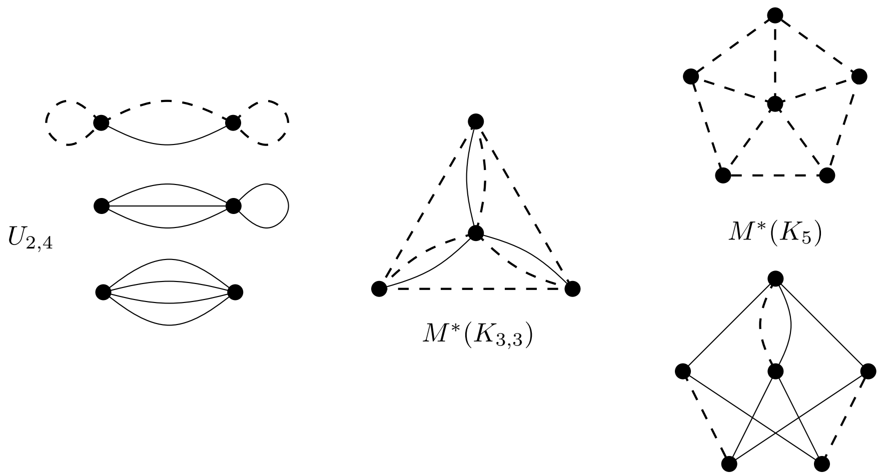


Figure 1.5: The biased graphs representing excluded minors for the class of graphic matroids. Those with dashed edges are signed graphs; the dashed edges are labelled -1. The other two biased graphs, representing $U_{2,4}$, have all cycles unbalanced.

$\sigma : E(G) \to \mathbf{C}_2$, and set $\Sigma = \{e \in E(G) : \sigma(e) = -1\}$. Now define the set of balanced cycles $\mathcal{B}_\Sigma$ to consist of those cycles C of G for which $|C \cap \Sigma|$ is even. In this case, we call Σ a *signature* for the graph. Choosing a $\mathbf{C}_2$-labelling of, or equivalently, a signature for a graph is referred to as *signing* the graph. In figures of signed graphs we indicate edges in the signature by dashed or bold edges, or with a shaded area around a vertex to indicate that all edges incident with the vertex in that area are in the signature.

Example 5. Three of the excluded minors for the class of graphic matroids (given in Theorem 1.2) are represented by signed graphs. These are shown in Figure 1.5. There are two signed graphs representing $M^*(K_5)$, and three biased graphs representing $U_{2,4}$, one of which is a signed graph. The other two biased graphs representing $U_{2,4}$ have no balanced cycles, and may be labelled by any group having enough elements that do not pairwise multiply to the identity. Neither F_7 nor its dual has any biased graph representation — in fact, both F_7 and F_7^* are excluded minors for the class of frame matroids [40].

In figures illustrating more general Γ-labelled graphs, we likewise leave edges labelled by the identity unmarked (since the orientation of such an edge is irrelevant), and either explicitly indicate the orientations and labels of other edges, or label with, say $\alpha \in \Gamma$, an area around a vertex v to indicate that all edges incident to v in that area are labelled by α. Such a label α is always assumed to be different than the group identity element, and all such edges are assumed to be oriented out from v, unless explicitly stated otherwise.

Given a graph labelled by $\gamma : E(G) \to \Gamma$, there are in general many different Γ-labellings of G having precisely the same set of balanced cycles as given by $\mathcal{B}_\gamma$. One way to move to a different labelling having the same set of balanced cycles is by *relabelling*. For a graph G labelled by $\gamma : E(G) \to \Gamma$, the operation of *relabelling* is one of the following operations.

(1) Reverse the orientation of an edge and replace its label $\gamma(e)$ with $\gamma(e)^{-1}$. (2) Choose a vertex v and an element $\alpha \in \Gamma$, and define a new labelling $\gamma_{v,\alpha} : E(G) \to \Gamma$ by

$$\gamma_{v,\alpha}(e) = \begin{cases} \gamma(e) & \text{if } e \notin \delta(v) \\ \alpha\gamma(e) & \text{if } e \in \delta(v) \text{ is oriented out from } v \\ \gamma(e)\alpha^{-1} & \text{if } e \in \delta(v) \text{ is oriented into } v. \end{cases}$$

Alternatively, a sequence of the second type of relabellings may equivalently be accomplished via the following more general relabelling operation: Choose a function $\eta : V(G) \to \Gamma$. Let $\gamma_\eta : E(G) \to \Gamma$ be the labelling defined by $\gamma_\eta(e) = \eta(u)^{-1}\gamma(e)\eta(v)$ if e has endpoints u, v and is oriented from u to v. Relabelling at a single vertex v is then the relabelling obtained by choosing $\eta(v) = \alpha$ for some $\alpha \in \Gamma$ and $\eta(u) = 1$ for all $u \in V(G) \setminus \{v\}$. Note that for a signed graph, relabelling at a vertex consists of switching the signs on each link incident to a vertex.

Evidently, relabelling does not change the set of cycles in $\mathcal{B}_\gamma$ | *i.e.* for any function $\eta : V(G) \to \Gamma$, $\mathcal{B}_\gamma = \mathcal{B}_{\gamma_\eta}$. Since the set of balanced cycles of a Γ-labelled graph is invariant under relabelling, relabelling defines an equivalence relation on the set of all Γ-labellings of G. Since we are interested mainly in the set of balanced cycles of a group-labelled graph, as opposed to particular labellings, we generally consider equivalence classes of Γ-labelled graphs. We think of a particular Γ-labelling of a graph G as a representative of its equivalence class under relabelling, and consider all Γ-labellings of G in the same relabelling class as giving rise to the same biased graph.

Given a biased graph $(G, \mathcal{B})$, if there is a group-labelling $\gamma : E(G) \to \Gamma$ for some group Γ with $\mathcal{B}_\gamma = \mathcal{B}$, we say the group-labelling given by γ *realises* $\mathcal{B}$, and say in this case that $(G, \mathcal{B})$ is Γ-*labellable*. If $(G, \mathcal{B})$ is Γ-labellable for some group Γ, we say $(G, \mathcal{B})$ is *group-labellable*. Not all biased graphs are group-labellable.

Observation 1.7. The biased graph $(2C_4, \mathcal{B})$ of Example 4 is not Γ-labellable by any group Γ.

Proof. Suppose to the contrary that, for some group Γ, there is a labelling $\gamma : E(2C_4) \to \Gamma$ with $\mathcal{B}_\gamma = \mathcal{B}$. By relabelling, we may assume all edges are oriented clockwise as drawn in Figure 1.3, and that e_1, e_2, e_3, and e_4 are all labelled 1. That $e_3e_4e_5e_6 \in \mathcal{B}$ implies $\gamma(e_5) = \gamma(e_6)^{-1}$, and that $e_1e_2e_7e_8 \in \mathcal{B}$ implies $\gamma(e_7) = \gamma(e_8)^{-1}$. But this implies that $\gamma(e_5e_6e_7e_8) = \gamma(e_5)\gamma(e_6)\gamma(e_7)\gamma(e_8) = 1$, a contradiction, since $e_5e_6e_7e_8 \notin \mathcal{B}$. $\square$

The theory of biased graphs was developed by Zaslavsky in a series of foundational papers [34, 35, 37, 38, 39, 40, 41]. As is apparent from the references, much of the foundational material surveyed here originated with Zaslavsky.

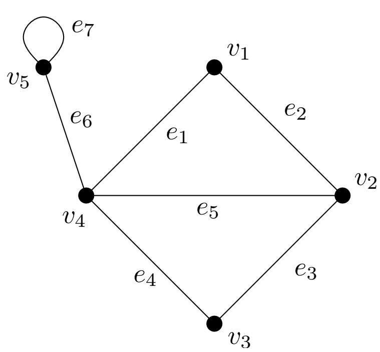
1.2.2 Biased graphs represent frame matroids

Depending on the view one would like to take, there are interesting classes of matroids that arise naturally from biased graphs, and biased graphs arise naturally from the study of these classes of matroids. We describe the relationship most relevant for us next (another will be briefly described in Section 2.1.1). In doing so, we often consider a matroid M on ground set E and a graph $G = (V, E)$ with edge set E. For the sake of readability, in the following and throughout this thesis, when the meaning is clear by context, we often make no distinction between a subset $X \subseteq E$, the set of edges in X in G, and the subgraph $G[X]$ of G induced by X. We often use the word *is* to mean, "is isomorphic to". Nevertheless, if any of these distinctions are particularly important or unclear in context, we will be explicit.

An *extension* of a matroid M is a matroid N containing a set of elements T such that $N \setminus T = M$.

Definition. A matroid is *frame* if it may be extended so that it contains a basis B such that every element is spanned by two elements of B.

It seems natural to call the distinguished basis B of the extension a *frame*, and so we do. Frame matroids are a natural generalisation of graphic matroids. The cycle matroid of a graph $G = (V, E)$ is defined as a matroid on E. Upon meeting $M(G)$ for the first time, a graph theorist may feel that the role of the vertices is somehow neglected. If so, he or she may be comforted by the following construction of $M(G)$ as a frame matroid. The cycle matroid $M(G)$ of G is naturally extended by adding V as a basis, and declaring each non-loop edge to be minimally spanned by its endpoints. The resulting matroid N can be represented over any field by adding $|V|$ columns to the signed incidence matrix A of G, in the form of a $|V| \times |V|$ identity matrix, with each vertex represented by one of the additional columns. Then N is represented by the matrix $[I_{|V|} \mid A]$, over any field. An example is shown in Figure 1.6.



$$\begin{array}{cc} & \begin{array}{cccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Figure 1.6: Extending $M(G)$ by V provides a frame for E.

Thus graphs are the prototypical frame matroid. Conversely, naturally associated with an arbitrary frame matroid there is a graph. Assigning biases to its cycles according to dependencies in the matroid, we obtain a biased graph which completely describes the matroid (whence these have also been called *bias* matroids). We now describe how to construct such biased graphs.

Let M be a frame matroid on ground set E, with frame B. By adding elements in parallel if necessary, we may assume $B \cap E = \emptyset$. Hence $M = N \setminus B$ where B is a basis for N and every element $e \in E$ is spanned by at most two elements in B. Let G be the graph with vertex set B and edge set E, in which e is a loop with endpoint f if e is in parallel with $f \in B$, and otherwise e is an edge with endpoints $f, f' \in B$ if $e \in \mathrm{cl}\{f, f'\}$. By (C3), each element $e \in E$ so defines a unique edge in G. Setting $\mathcal{B} = \{C : C$ is a cycle for which $E(C)$ is a circuit of $M\}$ yields a biased graph $(G, \mathcal{B})$. The theta property is easily seen to hold: If C, C' are two balanced cycles in a theta subgraph, say sharing non-trivial path P, then since each of C, C' are circuits, for any edge $e \in P$ there is a circuit contained in $(C \cup C') \setminus \{e\}$, by (C3). This implies that the third cycle of the theta subgraph $C \cup C'$ is also balanced. We say such a biased graph $(G, \mathcal{B})$ *represents* the frame matroid M, and write $M \cong F(G, \mathcal{B})$. Theorem 1.8 tell us what the circuits of M look like in $(G, \mathcal{B})$.

Theorem 1.8 (Zaslavski [40]). *Let M be frame matroid on E represented by biased graph $(G, \mathcal{B})$. A set $C \subseteq E$ is a circuit of M if and only if C induces one of following in $(G, \mathcal{B})$:*

1. *a balanced cycle,*
2. *two edge-disjoint unbalanced cycles intersecting in just one vertex,*
3. *two vertex-disjoint unbalanced cycles along with a path connecting them, or*
4. *a contrabalanced theta.*

A subgraph as in (2) or (3) is a *pair of handcuffs*, *tight* or *loose*, respectively.

Proof of Theorem 1.8. The edge set of a forest is independent, since any leaf edge is not spanned by the elements remaining after removing its leaf. The edge set of any subgraph having $k + 1$ edges on k vertices is dependent, since these edges are spanned by a set of size k (namely, the k elements of B represented by those k vertices). Hence if $C \subseteq E$ is a circuit on k vertices, C has no leaf edge and has either k or $k + 1$ edges. Moreover, C must be connected in G: otherwise, since each component induced by C in G has rank in N equal to the number of its vertices, we find C is not connected in N, a contradiction (details regarding connectivity of matroids may by found in Section 1.3.5). It follows that if $|C| = k$, then C is a cycle, and if $|C| = k + 1$, then C is either a pair of handcuffs or a contrabalanced theta. $\square$

Observe that for a biased graph $(G, \mathcal{B})$, if $\mathcal{B}$ contains all cycles in G, then $F(G, \mathcal{B})$ is the cycle matroid $M(G)$ of G. We therefore view a graph as a biased graph with all cycles balanced. At the other extreme, when no cycles are balanced, $F(G, \emptyset)$ is the bicircular matroid of G, introduced by Simões-Pereira [28] and further investigated by Matthews [17], Wagner [32], and others (for instance, [16, 21]). DeVos, Goddyn, Mayhew, and Royle [1] have shown that an excluded minor for the class of bicircular matroids has fewer than 16 elements, and thus that the set of excluded minors for the class is finite.

The classes of frame and linear matroids are different

We note here that while the class of frame matroids and that of matroids linear over a field certainly have large intersection, neither class is contained in the other. The Fano matroid F_7, for example, is linear over any field of characteristic 2 (as we observed in Example 2), but not frame (as we observed in Example 5). There are also frame matroids not linear over any field.

For example, the matroid Q_8 is obtained from the real affine cube by relaxing a circuit-hyperplane — that is, declaring the elements of a hyperplane that forms a circuit to be instead a basis. A geometric representation of Q_8 is shown in Figure 1.7. The 4-element

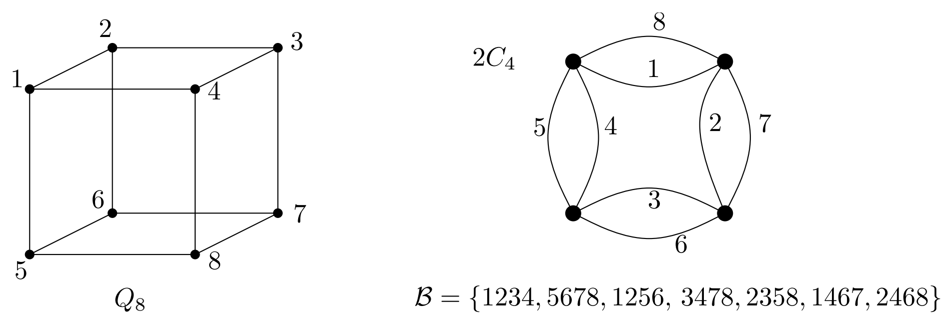


Figure 1.7: Q_8 is not representable over any field, but is frame.

circuits of the affine cube are its 6 faces and 6 diagonal planes. Declaring the circuit-hyperplane $\{1, 3, 5, 7\}$ to be independent defines the matroid Q_8. The 4-point planes of Q_8 are the six faces of the cube, and exactly five of the six diagonal planes, with $\{1, 3, 5, 7\}$ a basis. The matroid Q_8 is not linear over any field ([23], p. 509), but is frame. The graph $2C_4$ with $\mathcal{B} = \{1234, 5678, 1256, 3478, 2468, 2358, 1467\}$ as shown in Figure 1.7 represents Q_8.

Not surprisingly, there are matroids not belonging to either class. The Vamos matroid V_8, is neither linear over any field ([23], p. 511) nor frame. A geometric representation of V_8 is shown in Figure 1.8. Again the elements may be thought of as the vertices of a cube, but now the only circuits of size four are those planes indicated by shading. The Vamos matroid has rank 4 and 8 elements. All sets of size less than 4 are independent, and all but five of the sets of size 4 are independent. The circuits of size 4 are the four "sides" of the cube, 1256, 2367, 3478, and 1458, and just the one diagonal plane 2468. The "top" and "bottom" of the cube, the other diagonal planes, and all other sets of size 4 are independent. Any biased graph representing V_8 would have to have 4 vertices, 8 edges, and just these five dependent 4-sets forming circuits as balanced cycles, handcuffs, or contrabalanced thetas. An exhaustive search of the possibilities (there are not many) shows this is not possible.

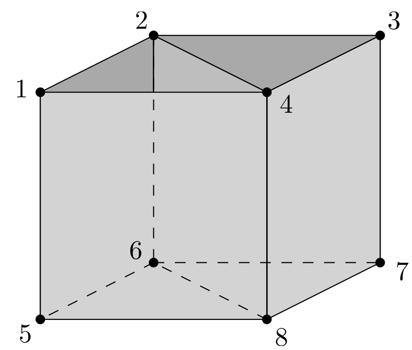


Figure 1.8: The Vamos matroid is neither linear nor frame.

Seeing frame matroids in biased graphs

We have already seen how the circuits of a frame matroid appear in a biased graph representation. We now briefly consider the analogous questions for other important attributes of the matroid. Let $M = F(G, \mathcal{B})$ be a frame matroid on ground set E represented by $(G, \mathcal{B})$. Theorem 1.8 immediately implies the following.

1.9 (Independent sets)**.** *The independent sets of M are those sets of edges inducing a subgraph having no balanced cycle and at most one unbalanced cycle in each of its components.*

In particular, for a subset $X \subseteq E$, a maximal independent set contained in X is of this form. (Recall $b(X)$ is the number of balanced components of $G[X]$.)

1.10 (Rank)**.** *The rank of a set $X \subseteq E$ is $|V(X)| - b(X)$.*

In particular, a basis for M is a maximal set of edges inducing a subgraph having no balanced cycle and at most one unbalanced cycle in each component. The rank of a frame matroid represented by $(G, \mathcal{B})$ is therefore $|V(G)| - b(G)$, where $b(G)$ is the number of balanced components of G.

A *line* is a simple rank 2 matroid. Lines are denoted $U_{2,n}$, where n is the number of elements in the line (colloquially called the *length* of the line). In general, the *uniform* matroid $U_{r,n}$ of rank r on n elements is the matroid in which the circuits are precisely the subsets of size $r + 1$. Since the 4-point line $U_{2,4}$ is an excluded minor for the class, no graphic matroid contains a line having more than three elements. A 3-point line appears in a graph representing a graphic matroid as a cycle of length three. In contrast, frame matroids may contain arbitrarily long lines. There are three biased graphs representing any line. If $F(G, \mathcal{B})$

contains an n-point line, then its elements in $(G, \mathcal{B})$ induce a subgraph on a pair of vertices in which either all edges are links, just one edge is a loop, or just two edges are loops on distinct endpoints, and in any case all cycles are unbalanced. Thus any biased graph representing the line $U_{2,n}$ with $n \geq 4$ may be obtained from a biased graph representing $U_{2,4}$ (Figure 1.5) by adding links and declaring all cycles unbalanced.

Recall that a *hyperplane* in a rank r matroid is a flat of rank $r - 1$. It is elementary that X is a hyperplane of a matroid if and only if the complement of X is a cocircuit (Proposition 2.1.6 in [23]). This fact, along with a straightforward application of the rank function of $F(G, \mathcal{B})$ enables us to determine the form a cocircuit of the matroid takes in $(G, \mathcal{B})$.

1.11 (Cocircuits)**.** *A cocircuit of $F(G, \mathcal{B})$ is a minimal edge set whose removal increases the number of balanced components of the resulting biased graph by one.*

1.2.3 Minors of biased graphs

If M is a frame matroid represented by biased graph $(G, \mathcal{B})$, then there are natural minor operations we may perform on $(G, \mathcal{B})$ that correspond to minor operations in M [37, 39]. For an element $e \in E(M){=}E(G)$, ***delete*** e from $(G, \mathcal{B})$ by deleting e from G and removing from $\mathcal{B}$ every cycle containing e. To *contract* e, there are three cases: If e is a balanced loop, then $(G, \mathcal{B})/e = (G, \mathcal{B}) \setminus e$. If e is a link, contract e in G and declare a cycle C to be balanced if either $C \in \mathcal{B}$ or $E(C) \cup \{e\}$ forms a cycle in $\mathcal{B}$. If e is an unbalanced loop with endpoint u, then $(G, \mathcal{B})/e$ is the biased graph obtained from $(G, \mathcal{B})$ as follows: e is deleted, all other loops incident to u become balanced, and links incident to u become unbalanced loops incident to their other endpoint. Clearly, deletion and contraction preserve the theta property. A *minor* of $(G, \mathcal{B})$ is any biased graph obtained by a sequence of deletions and contractions.

With these definitions, it is readily checked that minor operations on biased graphs agree with matroid minor operations on their frame matroids; that is, for any element $e \in E(G)$, $F(G, \mathcal{B}) \setminus e = F((G, \mathcal{B}) \setminus e)$ and $F(G, \mathcal{B})/e = F((G, \mathcal{B})/e)$. The class of frame matroids is therefore minor-closed.

Chapter 4 is an investigation into excluded minors for the class of frame matroids. We seek to determine those excluded minors of connectivity 2 for the class. We feel that we have almost succeeded. We determine a set $\mathcal{E}$ of 18 excluded minors of connectivity 2 and show that any remaining excluded minor that is not 3-connected has a special form:

Theorem 4.1. *Let M be an excluded minor for the class of frame matroids, and suppose M is not 3-connected. Then either $M \in \mathcal{E}$ or M is the 2-sum of $U_{2,4}$ and a 3-connected non-binary frame matroid.*

While we have determined another thirty or so excluded minors consisting of a 2-sum as described in the second phrase of the theorem, there may be a few more such excluded minors waiting to be discovered, and the proof will undoubtably be longer and more technical than the proof of Theorem 4.1. It may be that Theorem 4.1 is enough to allow us to proceed in our search for excluded minors for the class under the assumption that our quarry are essentially 3-connected.

Taking minors in a group-labelled graph

If $(G, \mathcal{B})$ is group-labelled, say by $\gamma : E(G) \to \Gamma$, then the following procedure may be adopted when performing minor operations so that the minor inherits a Γ-labelling realising the biases of its cycles. Deletion is straightforward: $(G, \mathcal{B}) \setminus e = (G \setminus e, \mathcal{B}')$ where $\mathcal{B}'$ is determined by γ restricted to $E(G) \setminus e$. To contract a link e, first relabel so that e is labelled by the identity, then contract e; all remaining edges keep their new group labels. If e is a balanced loop, then $(G, \mathcal{B})/e = (G, \mathcal{B}) \setminus e$ and the restriction of γ to $E(G) \setminus e$ labels $(G, \mathcal{B})/e$. If e is an unbalanced loop, incident with vertex v, then the underlying graph of $(G, \mathcal{B})/e$ is $G - v$ together with each link uv in $\delta(v)$ now an unbalanced loop incident to u, and all other loops aside from e incident to v now balanced loops incident to v. The labelling given by γ on $E(G) \setminus \delta(v)$, an arbitrary orientation and label $\gamma(e) \neq 1$ given to each unbalanced loop formerly a link in $\delta(v)$, and an arbitrary orientation and label 1 given to each loop remaining incident to v, realises the biases of cycles in $(G, \mathcal{B})/e$. Hence the following proposition is immediate.

Proposition 1.12. *Let Γ be a group. The class of Γ-labellable biased graphs is minor-closed.*

Dowling matroids

We mentioned at the end of Section 1.1.1 on matroid minors that frame matroids linear over a finite field $\mathbb{F}$ play an important role in matroid structure theory. In this section, we briefly describe this class in a little more detail, to get a hint of why frame matroids should have such an important role in matroid structure theory. We first briefly revisit the source of the other fundamental minor-closed classes appearing in the Matroid Structure Theorem, projective geometries.

For a finite field $GF(q)$, denote the r-dimensional vector space over $GF(q)$ by $V(r, q)$. The vector space $V(r, q)$ is a matroid, but has many elements in parallel — namely, every pair of nonzero vectors in a 1-dimensional subspace | and has the zero vector as a loop. The projective geometry obtained by removing all but one nonzero element from each 1-dimensional subspace of $V(r, q)$ yields a canonical simple matroid associated with $V(r, q)$, namely, the projective geometry of dimension $r-1$ over $GF(q)$, denoted $PG(r-1, q)$. (Note that while $PG(r-1, q)$ has dimension $r-1$, as a matroid it has rank r.) As mentioned earlier, just as every simple graph on n vertices is a subgraph of the complete graph on n

vertices K_n, every simple rank r matroid linear over $GF(q)$ may be viewed as a *submatroid* of $PG(r-1, q)$, that is, isomorphic to a matorid obtained by deleting points from $PG(r-1, q)$ ([23], Theorem 6.1.3).

To explain the significance of this requires a little background. In a beautiful paper [14], Kahn and Kung take the notion of free objects in a variety from universal algebra, and ask how it may be applied in matroid theory. Let $\mathcal{T}$ be a minor-closed class of matorids that is also closed under the operation of direct sum. A *sequence of universal models* for $\mathcal{T}$ is a sequence $(T_r)_{r\geq 1}$ of matroids such that

U1. T_r is in $\mathcal{T}$ and has rank r.

U2. If M has rank r and is in $\mathcal{T}$, then M is a submatroid of T_r.

A *variety* of matroids is a class $\mathcal{T}$ closed under minors and direct sums having a sequence of universal models. The projective geometries $PG(r-1, q)$ thus form a sequence of universal models for the variety of matroids linear over $GF(q)$. From the simple fact that every graph on n vertices is a subgraph of K_n, it follows that every rank r graphic matroid $M(G)$ is a submatroid of $M(K_{r+1})$. The sequence of universal models $(M(K_{r+1}))_{r\geq 1}$ for the variety of graphic matroids is in fact the simplest of an infinite family of varieties.

Let Γ be a finite group of order m. The rank r *Dowling geometry* over Γ, $D(r, \Gamma)$, is the frame matroid defined as follows. Let $V = \{v_1, \dots, v_r\}$ be a basis, and let V be the vertex set of a graph. For each pair v_i, v_j with $i < j$, place m edges directed from v_i to v_j, each labelled with a group element distinct from the others (so that for each pair of vertices v_i, v_j, each group element appears as a label on one v_i-v_j edge). Denote this labelling by γ. Finally, place a unbalanced loop incident to each vertex v_i. Let K_r^Γ denote the resulting graph. The Dowling geometry $D(r, \Gamma)$ is the frame matroid $F(K_r^\Gamma, \mathcal{B}_\gamma)$. (As long as Γ is not the trivial group, the labelling γ may be extended to include the unbalanced loops by giving each an arbitrary orientation and non-identity label.) We call K_r^Γ the *Dowling graph* on r vertices over Γ.

A *Dowling matroid over* Γ is any matroid that is a minor of a Dowling geometry $D(r, \Gamma)$. So just as every simple matroid representable over $GF(q)$ is a submatroid of a projective geometry over $GF(q)$, for every finite group Γ, there is a Dowling geometry $D(r, \Gamma)$ such that every simple rank r Dowling matroid is a submatroid of $D(r, \Gamma)$. In the language of Kahn and Kung: for a fixed finite group Γ, the Dowling geometries $D(r, \Gamma)$ provide a sequence of universal models for the variety of Dowling matroids over Γ. If Γ is the trivial group, we obtain the complete graphs as universal models for graphs, since $D(r, \Gamma) \cong M(K_{r+1})$ (this is perhaps most easily seen, after reading Section 1.2.4, as follows: when Γ is trivial, $D(r, \Gamma)$ is represented by K_r, balanced aside from its loops; now apply Proposition 1.25 to unroll the loops and obtain the graph K_{r+1}, which by Proposition 1.25 represents $D(r, \Gamma)$.) Since minor operations in a group-labelled graph representation of a frame matroid agree with the corresponding minor operations in the matroid, the Dowling matroids are precisely the frame matroids represented by a graph labelled by a finite group. The main theorem of [14] states that, remarkably:

Theorem 1.13 (Kahn & Kung, [14])**.** *Apart from three "degenerate cases", the projective geometries $PG(r-1,q)$ and the Dowling geometries $D(r,\Gamma)$ for a fixed finite group Γ are the only varieties of matroids.*

It appears that projective geometries over finite fields and Dowling geometries are very special classes of matroids indeed.

The following theorem tells us when a frame matroid is a submatroid of both a projective geometry and a Dowling geometry.

Theorem 1.14 (Dowling [3])**.** *A Dowling geometry $D(r,\Gamma)$ is linear over a finite field $GF(q)$ if and only if Γ is isomorphic to a subgroup of the multiplicative group $GF(q)^{\times}$ of $GF(q)$.*

Proof. By taking subgraphs, it is sufficient to consider the case $\Gamma = GF(q)^{\times}$. A matrix representation of $D(r,\Gamma)$ over $GF(q)$ is obtained as follows. Take the r columns of an $r\times r$ identity matrix to represent the basis V. Index the columns and rows by V so that column v_i's entry containing 1 appears in row v_i. Add a column, one for each edge e of G, where if e is directed from v_i to v_j then the column representing e has $\gamma(e)$ in coordinate v_i, a 1 in coordinate v_j, and is otherwise 0. Call the resulting matrix A. The Dowling geometry $D(r,\Gamma)$ is represented by the matrix $[I_r \mid A]$. To see that this is so, just observe that a set of columns of A form a circuit in $M[I_r \mid A]$ if and only if their corresponding edges in G form a balanced cycle, handcuffs, or a contrabalanced theta subgraph.

Conversely, suppose M is a rank r frame matroid linear over $GF(q)$, with distinguished basis $B=\{b_1,\dots,b_r\}$ as its frame. Then there is a matroid $N=M\backslash B$ and a matrix A whose vector matroid is M. Via elementary row operations, we may obtain a matrix representing N in which the columns $b_1,\dots,b_r$ representing B form an identity matrix. Indexing the rows by B so entry b_ib_i is 1, and applying further elementary row operations and column scaling if necessary, we may obtain a matrix representing N such that: for every element $e\in E(M)\setminus B$, if e is parallel to element $b_i\in B$, then column e has a non-zero element $\neq 1$ in coordinate b and is zero elsewhere, and otherwise if e is spanned by $b_i,b_j\in B$ with $i<j$, then column e has a nonzero entry in coordinate b_i, 1 in coordinate b_j, and 0 in all other coordinates. The procedure of the previous paragraph, applied in reverse, constructs a $GF(q)^{\times}$-labelled graph representing M. $\square$

Constructing frame matroids not linear over any finite field is therefore as easy as labelling a graph with a finite group that is not isomorphic to the multiplicative subgroup of a finite field, with sufficient complexity that the resulting collection of balanced cycles may not be realised by any $GF(q)^{\times}$-labelling.

Fixing a group Γ, we have the natural minor-closed class of Γ-labelled graphs. The larger class of biased graphs labellable by some group is also a minor-closed class. Naturally, we ask for its set of excluded minors. We quickly find, however, that this is asking for the moon. However, in Chapter 2 we provide a nice topological characterisation of those biased graphs that are group-labellable, and it comes with a corresponding purely graph theoretical reason that a biased graph may not admit a group-labelling. This is the main theorem of Chaper 2.

Theorem 2.1. *Let $(G, \mathcal{B})$ be a biased graph and construct a 2-cell complex K from G by adding a disc with boundary C for every $C \in \mathcal{B}$. Then the following are equivalent.*

1. *$(G, \mathcal{B})$ is group-labellable.*
2. *A cycle $C \in \mathcal{B}$ if and only if C is a contractible curve in K.*
3. *No unbalanced cycle can be moved to a balanced cycle via a sequence of balanced reroutings on closed walks.*

Using Theorem 2.1, we find that if Γ is any infinite group, then the class of Γ-labellable biased graphs has many infinite families of excluded minors.

Corollary 2.3. *For every infinite group Γ and every $t \geq 3$ there are infinitely many excluded minors for the class of Γ-labellable biased graphs with exactly t vertices.*

For any group Γ, let $\mathcal{F}_\Gamma$ be the class of frame matroids represented by a Γ-labelled graph. Then $\mathcal{F}_\Gamma$ is a minor-closed class of frame matroids. As another corollary to Theorem 2.1, we obtain the following rather surprising result.

Theorem 2.4. *For every infinite group Γ and every $t \geq 3$ the class $\mathcal{F}_\Gamma$ has infinitely many excluded minors of rank t.*

Further, we prove that for every infinite group Γ and every $t \geq 3$, the class $\mathcal{F}_\Gamma$ contains infinite antichains of rank t matroids. Either one of these results yields the following.

Corollary. *The class of frame matroids is not well-quasi-ordered by the minor relation.*

On a positive note, we also prove:

Theorem 2.16. *Let Γ be a finite group and n be an integer. Then every infinite set of Γ-labelled graphs of branch-width at most n has two members one of which is isomorphic to a minor of the other.*

This has the corollary:

Corollary. *Let Γ be a finite group, and let n be an integer. The class of Dowling matroids over Γ of branch-width at most n is well-quasi-ordered by the minor relation.*

1.2.4 Biased graph representations

One of the main difficulties when studying excluded minors for classes of matroids is that of representations. In the case M is a matroid linear over a field $\mathbb{F}$, there may be many matrices over $\mathbb{F}$ whose vector matroid is isomorphic to M. Many of these will be *equivalent*, in that one may be obtained from another by a sequence of elementary matrix row operations and column scaling. But in general such a matroid may have many non-equivalent matrix representations, and their existence complicates the study of excluded minors for these classes.

Similarly, if M is a graphic matroid there may be many non-isomorphic graphs representing M. Whitney's 2-isomorphism theorem characterises when two graphs represent the same cycle matroid, as follows ([23], Theorem 5.3.1). Let G be a graph. Since none of the following three operations has any effect on the edge set of a cycle in G, each yields a graph G' with $M(G') \cong M(G)$:

1. *Vertex identification*: Let u, v be a pair of vertices in distinct components of G. Let G' be the graph obtained from G by identifying u and v as a single vertex.

2. *Cleaving a vertex*: Let v be a cut vertex of G. Let G' be a graph such that G is obtained from G' by a vertex identification operation in which two vertices are identified as v.

3. *Twisting on a pair of vertices*: Suppose u, v are vertices of G such that G is obtained from two disjoint graphs G_1, G_2 by identifying vertices $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$ to a vertex u and identifying $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ to a vertex v. Let G' be the graph obtained from G_1 and G_2 by identifying u_1 with v_2 and u_2 with v_1.

We call operations 1, 2, and 3 *Whitney operations*. A graph H is *2-isomorphic* to another graph G if H may be transformed into a graph isomorphic to G by a sequence of Whitney operations.

Theorem 1.15 (Whitney's 2-isomorphism Theorem)**.** *Let G and H be graphs. Then $M(G)$ and $M(H)$ are isomorphic if and only if G and H are 2-isomorphic.*

Similarly, if M is a frame matroid, there may be more than one biased graph representing M. We will encounter many situations in which non-isomorphic biased graphs represent the same frame matroid.

A key piece in the proof of Whitney's 2-isomorphism Theorem is the fact that if G has no loop or isolated vertex and is 3-connected, then $M(G)$ is uniquely represented by G. Slilaty has given an analogous sufficient condition for a frame matroid to be uniquely represented by a biased graph:

Theorem 1.16 ([31])**.** *Let Ω and Ω' be biased graphs with no balanced loops or isolated vertices. If Ω is 3-connected and contains three vertex disjoint unbalanced cycles, at most* *one of which is a loop, then $F(\Omega)$ and $F(\Omega')$ are isomorphic if and only if Ω and Ω' are isomorphic.*

We have made efforts to determine the alternate biased graph representations of $F(\Omega)$ in the cases Ω is only 2-connected or does not have three vertex disjoint unbalanced cycles at most one of which is a loop. Chapter 5 provides a partial result in this direction.

The proof of Theorem 1.16, minus the details, is as follows. If Ω satisfies the conditions of the theorem, then the deletion of any vertex of Ω leaves a connected non-graphic hyperplane. In a biased graph representation of a frame matroid, the complementary cocircuit of such a hyperplane is always the set of vertices incident with a vertex. Hence all vertex-edge incidences are determined by the set of $|V(\Omega)|$ connected non-graphic hyperplanes of $F(\Omega)$, and Ω uniquely represents $F(\Omega)$.

Therefore if Ω does not uniquely represent $F(\Omega)$, it has a vertex whose deletion leaves either a disconnected or a graphic hyperplane. Hence the first step in determining the alternate biased graph representations of a given frame matroid is that of understanding all the biased graph representations of a given graphic matroid. In [36], Zaslavsky characterised the biased graphs whose frame matroids are binary as being of four types. The first three types in fact represent only graphic matroids. The fourth type is that of signed graphs with no two vertex disjoint unbalanced cycles. There are two types of such signed graphs: those with or without a vertex whose deletion destroys all unbalanced cycles. The former represent only graphic matroids. The later are called *tangled*. Figure 1.5 shows that, aside from $U_{2,4}$, the frame excluded minors for the class of graphic matroids are represented only by tangled signed graphs. Hence Zaslavsky's characterisation of biased graphs whose frame matroids are binary is just one step away from a characterisation of those biased graphs representing graphic matroids: only missing is a characterisation of those tangled signed graphs representing graphic matroids. In [29], Slilaty provides this missing piece, with the following decomposition theorem for tangled signed graphs whose frame matroids are graphic:

Theorem 1.17 (Slilaty)**.** *If $(G, \mathcal{B}_\Sigma)$ is a connected tangled signed graph with $F(G, \mathcal{B}_\Sigma)$ graphic, then $(G, \mathcal{B}_\Sigma)$ is either*

1. *a projective planar signed graph whose topological dual is planar, or*
2. *a 1-, 2-, or 3-sum of a tangled signed graph whose frame matroid is graphic and a balanced signed graph with at least 2, 3, or 5 vertices, respectively.*

Pivotto has observed [24] that a result of Shih [27] on lift matroids of signed graphs characterises when two signed graphs represent the same graphic lift matroid. (Lift matroids will be defined and briefly discussed in Section 2.1.1.) For tangled signed graphs, this characterisation equally applies to frame matroids, since the lift and frame matroid of a tangled signed graph coincide. Hence Shih's result gives an alternate characterisation of those tangled signed graphs whose frame matroids are graphic. In Chapter 3, we put Zaslavsky's and Shih's results together to describe those biased graphs that represent graphic frame matroids:

Theorem 3.1 *Let M be a connected graphic matroid, and let Ω be a biased graph representing M. Then Ω is a member of one of six explicit families of biased graphs.*

Theorem 1.17 follows as a corollary of Theorem 3.1; we also show this in Chapter 3.

The description of biased graphs with graphic frame matroids given by Theorem 3.1 is the main tool in our efforts to determine possible biased graph representations of non-graphic frame matroids. In Chapter 5 we use Theorem 3.1 to determine the possible different biased graph representations of a frame matroid having a particular type of biased graph representation. We agree with Slilaty's comment that, "Finding the correct necessary and sufficient or almost necessary and sufficient conditions to guarantee unique representability of bias matroids by biased graphs seems to be a very difficult problem." [31]

In addition to Whitney operations on graphs, there are a few operations known that may be performed on biased graphs to yield a non-isomorphic biased graph representing the same frame matroid. We discuss these in the next section. More interestingly, in the course of our proof of the main theorem of Chapter 4, we discover an operation for group-labelled graphs analogous to the Whitney operation of twisting on a pair of vertices in a graph, which generalises other previously known operations. We call it a *twisted flip*. A twisted flip is illustrated in Figure 1.9, and is performed as follows.

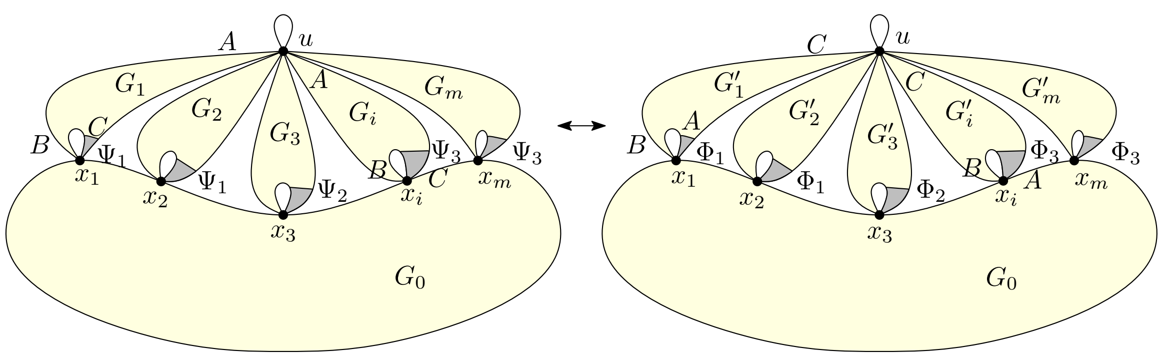


Figure 1.9: A twisted flip: Edges in $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}'$ are shaded; edges marked A in G become incident to x_i in G' and are in $\boldsymbol{\Sigma}'$; edges marked C in G become incident to u in G'.

Let G_0 be a balanced graph with a vertex u not adjacent to any vertex of G_0, and let $x_1, \ldots, x_m$ be vertices of G_0 (not necessarily distinct). Let $G_1, \ldots, G_m$ be signed graphs on disjoint edge sets, with each G_i meeting G_0 in precisely in $\{u, x_i\}$. For each $1 \leq i \leq m$, let Σ_i

be the signature of signed graph G_i, with $\Sigma_i \subseteq \delta(x_i)$. Now let $\{\Psi'_1, \dots, \Psi'_k\}$ be a partition of $\{\Sigma_1, \dots, \Sigma_m\}$ into k sets, and let $\mathbf{\Sigma} = \{\Psi_1, \dots, \Psi_k\}$ be formed from the partition by taking for each $1 \leq j \leq k$, $\Psi_j = \bigcup_{\Sigma_i \in \Psi'_j} \Sigma_i$. Finally, let $G = G_0 \cup G_1 \cup \cdots \cup G_m$, and let $\mathcal{B}_\Sigma$ be the set of those cycles C of G with $|C \cap \Psi_j|$ even for every $1 \leq j \leq k$. Consider biased graph $(G, \mathcal{B}_\Sigma)$ and its associated frame matroid $F(G, \mathcal{B}_\Sigma)$. A biased graph $(G', \mathcal{B}_{\Sigma'})$ with $F(G', \mathcal{B}_{\Sigma'}) \cong F(G, \mathcal{B}_\Sigma)$ is obtained from $(G, \mathcal{B}_\Sigma)$ as follows:

- each edge of the form $e = uz \notin \Sigma_i$ in G_i has its endpoints redefined so $e = x_i z$;
- each edge of the form $e = x_i z \in \Sigma_i$ with $z \neq u$ has its endpoints redefined so $e = uz$;
- for each $1 \leq j \leq k$, let $\Phi'_j = \{e$: the endpoints of e have been redefined so that $e = x_i z$ for some $z \in V(G_i)\} \cup \{e = x_i u \in \Psi_j\}$. Put $\mathbf{\Sigma}' = \{\Phi'_1, \dots, \Phi'_k\}$;
- let $\mathcal{B}_{\Sigma'}$ consist of the those cycles C in the resulting graph G' with $|C \cap \Phi'_j|$ even for every $1 \leq j \leq k$.

Theorem 4.2. *If $(G', \mathcal{B}_{\Sigma'})$ is obtained from $(G, \mathcal{B}_\Sigma)$ by a twisted flip, then their frame matroids are isomorphic.*

Proof. It is straightforward to check that $F(G, \mathcal{B}_\Sigma)$ and $F(G', \mathcal{B}_{\Sigma'})$ have the same set of circuits. □

1.3 Some useful technical tools

In this section, we describe some slightly more technical notions, which are used in more than one of the following chapters. Time and space constraints make it infeasible to throughly describe all the elementary matroid theory we require. Oxley's text [23] is standard and a good reference.

If X, Y are subgraphs of a graph G, an X-Y *path* in G is a path that meets $X \cup Y$ exactly in its endpoints, with one endpoint in X and the other in Y.

1.3.1 Rerouting

Let G be a graph, let P be a path in G, and let Q be a path internally disjoint from P linking two vertices $x, y \in V(P)$. We say the path P' obtained from P by replacing the subpath of P linking x and y with Q is obtained by *rerouting* P along Q.

Observation 1.18. Given two u-v paths P, P' in a graph, P may be transformed into P' by a sequence of reroutings.

Proof. To see this, suppose P and P' agree on an initial segment from u. Let x be the final vertex on this common initial subpath. If $x = v$, then $P = P'$, so assume $x \neq v$. Let y be the next vertex of P' following x that is also in P. Denote the subpath of P' from x to y by Q. Since y is different from x, the path obtained by rerouting P along Q has a strictly longer common initial segment with P' than P. Continuing in this manner, eventually $x = v$, and P has been transformed into P'. $\square$

If subpath R of path P is rerouted along Q, and the cycle $R \cup Q$ is balanced, we refer to this as rerouting *along a balanced cycle* or a *balanced rerouting*. If P is a path with distinct endpoints x, y contained in a cycle C and Q is an x-y path internally disjoint from C, and the cycle $P \cup Q$ is balanced, then the balanced rerouting of P along Q yields a new cycle C'. The following simple fact will be used extensively.

Lemma 1.19. *Let C be a cycle. If C' is obtained from C by rerouting along a balanced cycle, then C and C' have the same bias.*

Proof. Since $C \cup Q$ is a theta subgraph, this follows immediately from the theta property. $\square$

1.3.2 A characterisation of signed graphs

The following characterisation of those biased graphs that are signed graphs is useful. If a theta subgraph in a signed graph has two cycles containing an odd number of edges labelled -1, then parity implies the third cycle has an even number of edges labelled -1. Thus signed graphs have no contrabalanced theta subgraph. Conversely, any biased graph having no contrabalanced theta is a signed graph:

Proposition 1.20. *A biased graph $(G, \mathcal{B})$ is a signed graph if and only if $(G, \mathcal{B})$ contains no contrabalanced theta subgraph.*

Proof. Suppose P_1, P_2, P_3 are three internally disjoint paths forming a theta subgraph in signed graph (G, Σ), with Σ realised by $\sigma : E(G) \to \mathbf{C}_2$. If $P_1 \cup P_2$ and $P_2 \cup P_3$ are unbalanced, then $\sigma(P_1) \neq \sigma(P_2) \neq \sigma(P_3)$, so $\sigma(P_1) = \sigma(P_3)$, and $\sigma(P_1 \cup P_3) = +1$.

To prove the converse, we may assume G is connected; if not, apply the following argument to each component of G. Let T be a spanning tree of G. For each $e \in E(G)$, define $\sigma(e) = +1$ if and only if $e \in T$ or the fundamental cycle $C(e, T)$ of e with respect to T in $T \cup e$ is balanced. Otherwise put $\sigma(e) = -1$. We show that a cycle C is in $\mathcal{B}$ if and only if $\sigma(C) = +1$, by induction on the number of edges in $C \setminus T$.

If all but one edge e of C is contained in T, then the result holds by definition of σ. Suppose $|C \setminus T| = n \geq 2$, and the result holds for all cycles having less than n edges not in T. Choose a minimal path P in $T \setminus C$ linking two vertices x, y in $V(C)$ (such a path exists since C has at least two edges not in T: say $e = uv, f \in C \setminus T$; the u-v path in T avoids f

and so at some vertex leaves C and then at some vertex returns to C). Cycle C is the union of two internally disjoint x-y paths P_1, P_2 and together P, P_1, P_2 form a theta subgraph of G. Let $C_1 = P_1 \cup P$ and $C_2 = P_2 \cup P$. Each of C_1 and C_2 have a number of edges not in T strictly less than n, so by induction $\sigma(C_1) = +1$ if and only if $C_1 \in \mathcal{B}$ and $\sigma(C_2) = +1$ if and only if $C_2 \in \mathcal{B}$. Since, for $i \in \{1, 2\}$, $\sigma(C_i) = +1$ if and only if $\sigma(P_i) = \sigma(P)$, by the theta property and the fact that $(G, \mathcal{B})$ contains no contrabalanced theta, $\sigma(C) = +1$ if and only if $C \in \mathcal{B}$. $\square$

1.3.3 Biased graphs with a balancing vertex

If $\Omega{=}(G, \mathcal{B})$ is a biased graph and $v \in V(G)$ we let $\Omega - v$ denote the biased graph $(G - v, \mathcal{B}')$ where $\mathcal{B}'$ consists of all cycles in $\mathcal{B}$ which do not contain v (it is immediate that $\Omega - v$ still satisfies the theta property). If clear in context, we also write $G - v$ for $\Omega - v$. A vertex u is a *balancing* vertex of a biased graph Ω if $\Omega - u$ is balanced. Cycles in a biased graph with a balancing vertex have a particularly simple structure, and such biased graphs have a particularly nice group-labelling, which we now describe.

Lemma 1.21. *Let $(G, \mathcal{B})$ be a biased graph and suppose u is a balancing vertex in $(G, \mathcal{B})$. Let $\delta(u) = \{e_1, \dots, e_k\}$. For each pair of edges e_i, e_j ($1 \leq i < j \leq k$), either all cycles containing e_i and e_j are balanced or all cycles containing e_i and e_j are unbalanced.*

Proof. Fix i, j, and consider two cycles C and C' containing e_i and e_j. Let $e_i = ux_i$ and $e_j = ux_j$. Write $C = ue_ix_iPx_je_ju$ and $C' = ue_ix_iP'x_je_ju$. Path P may be transformed into P' by a series of reroutings, $P{=}P_0, P_1, \dots, P_l{=}P'$ in $G - u$. Since u is balancing, each rerouting is along a balanced cycle. Hence by Lemma 1.19, at each step $m \in \{1, \dots, l\}$, the cycles $ue_ix_iP_{m-1}x_je_ju$ and $ue_ix_iP_mx_je_ju$ have the same bias. $\square$

For a balancing vertex u, define a relation $\sim$ on $\delta(u)$ by $e_i \sim e_j$ if there is a balanced cycle containing e_i and e_j, or if $i = j$. Clearly $\sim$ is reflexive and symmetric. The relation $\sim$ is also transitive: Suppose $e_i \sim e_j$ and $e_j \sim e_t$. Say $e_i = ux_i$, $e_j = ux_j$, and $e_t = ux_t$. Since there is a balanced cycle containing x_iux_j and a balanced cycle containing x_jux_t, there is an x_i-x_j path avoiding u and an x_j-x_t path avoiding u. Hence there is an x_i-x_t path P avoiding u and a P-x_j path Q avoiding u. Let $P \cap Q = \{y\}$. Together, u, e_i, e_j, e_t, P, and Q form a theta subgraph of G. By Lemma 1.21, $ue_ix_iPyQx_je_ju$ and $ue_jx_jQyPx_te_tu$ are each balanced. By the theta property therefore, $ue_ix_iPx_te_tu$ is balanced. Hence $e_i \sim e_t$. Thus for a balancing vertex u, $\sim$ is an equivalence relation on $\delta(u)$. We call the $\sim$ classes of $\delta(u)$ its *balancing* classes.

Let $(G, \mathcal{B})$ be a biased graph with a balancing vertex u. Assume first that u is not a cut vertex of G. Suppose $|\delta(u)/{\sim}| = k$, and let Σ_1, Σ_2, ..., Σ_k be the balancing classes of

$\delta(u)$. Let $\sigma : E(G) \to \mathbf{C}_2^k$ be the labelling given by

$$\sigma(e) = \begin{cases} (1, 1, \ldots, 1) & \text{if } e \text{ is not incident to } u \\ (-1, 1, \ldots, 1) & \text{if } e \text{ is an unbalanced loop,} \\ (a_1, a_2, \ldots, a_k) & \text{if } e \in \Sigma_i, \text{ where } a_i = -1 \text{ if } e \in \Sigma_i \text{ and } a_i = 1 \text{ otherwise.} \end{cases}$$

If u is a cut vertex of G, then there are biased subgraphs $(G_1, \mathcal{B}_1)$, ..., $(G_m, \mathcal{B}_m)$ where each $(G_i, \mathcal{B}_i)$ has a balancing vertex u_i ($i \in \{1, \ldots, m\}$), such that u_i is not a cut vertex in G_i and $(G, \mathcal{B})$ is obtained by identifying vertices u_1, ..., u_m to a single vertex u. Let $k = \max_i\{|\delta(u_i)/\sim|\}$. Let $\sigma : E(G) \to \mathbf{C}_2^k$ be the labelling obtained by applying the above procedure to each biased subgraph $(G_i, \mathcal{B}_i)$ separately, and extending each labelling to a labelling by $\mathbf{C}_2^k$ appropriately in the obvious way.

Proposition 1.22. *Suppose $(G, \mathcal{B})$ has balancing vertex u. Then the above labelling realises $\mathcal{B}$.*

Proof. This follows easily from the fact that $\sim$ is an equivalence relation. □

k-signed graphs

Biases of cycles in $\mathbf{C}_2^k$-labelled graphs behave much as cycles in signed graphs. Suppose $(G, \mathcal{B}_\sigma)$ is labelled by $\sigma : E(G) \to \mathbf{C}_2^k$ (not necessarily with a balancing vertex). Let $\mathbf{\Sigma} = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_k\}$, where for $1 \leq i \leq k$, $\Sigma_i = \{e \in E(G) : \text{coordinate } i \text{ of } \sigma(e) \text{ is } -1\}$. We again call $\mathbf{\Sigma}$ a *signature* for the graph. A cycle C is balanced in $(G, \mathcal{B}_\sigma)$ if and only if $|C \cap \Sigma_i|$ is even for every $1 \leq i \leq k$, and is unbalanced otherwise. We therefore call a $\mathbf{C}_2^k$-labelled graph a *k-signed graph*. Conversely, given a graph, choosing a signature $\mathbf{\Sigma} = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_k\}$ in turn defines a set of balanced cycles $\mathcal{B}_\Sigma$, in which a cycle C is balanced if and only if $|C \cap \Sigma_i|$ is even for each i, and so defines a k-signed graph. Observe that a 1-signed graph is a signed graph. When dealing with signed and k-signed graphs, it is often more convenient to specify signatures than a labelling.

Observation 1.23. Let $(G, \mathcal{B})$ be a biased graph with a balancing vertex u after deleting its set U of unbalanced loops. Let $\{\Sigma_1, \ldots, \Sigma_k\}$ be the partition of $\delta(u)$ into its balancing classes in $(G, \mathcal{B}) \setminus U$, and let $\mathbf{\Sigma} = \{U, \Sigma_1, \ldots, \Sigma_k\}$. Then $(G, \mathcal{B})$ is a k-signed graph with $\mathcal{B}_\Sigma = \mathcal{B}_{\Sigma \setminus \Sigma_i} = \mathcal{B}$ for every $1 \leq i \leq k$.

Proof. This follows easily from the fact that the relation $\sim$ determining the balancing classes of $\delta(u)$ is an equivalence relation in $(G, \mathcal{B}) \setminus U$, and by relabelling. □

1.3.4 Pinches and roll ups

The operation of *pinching* two vertices in a graph and the operation of *rolling up* a particular set of edges incident to a balancing vertex in a biased graph each yield another biased graph representing the same frame matroid. We describe these operations next.

Pinching and splitting

Let H be a graph. Choose two distinct vertices $u, v \in V(H)$, and let G be the graph obtained from H by identifying u and v as a single vertex w. Then $\delta(w) = \delta(u) \cup \delta(v) \setminus \{e : e = uv\}$ (since an edge with endpoints u and v becomes a loop incident to w); let $\mathcal{B}$ be the set of all cycles in G not meeting both $\delta(u)$ and $\delta(v)$. It is easily verified (for instance, by checking all circuits of the two matroids) that $F(G, \mathcal{B}) \cong M(H)$. We say the biased graph $(G, \mathcal{B})$ is obtained by *pinching* u and v. Biased graph $(G, \mathcal{B})$ is a signed graph: setting $\Sigma = \delta(u)$ gives a signature so that $(G, \mathcal{B}) = (G, \mathcal{B}_\Sigma)$.

The signed graph obtained by pinching two vertices of a graph to a single vertex w has w as a balancing vertex. Conversely, if $(G, \mathcal{B})$ is a signed graph with a balancing vertex u, then $(G, \mathcal{B})$ is obtained as a pinch of a graph H, which we may describe as follows. If $|\delta(u)/{\sim}| > 2$, then $(G, \mathcal{B})$ contains a contrabalanced theta, contradicting Proposition 1.20. Hence $|\delta(u)/{\sim}| \leq 2$, and Proposition 1.22 together with Observation 1.23 gives a $\mathbf{C}_2$-labelling of $(G, \mathcal{B})$ in which all edges not incident to u are labelled $+1$. Let H be the graph obtained from G by *splitting* vertex u; that is, replace u with two vertices, u' and u'', put all edges in $\delta(u)$ labelled -1 incident to u', and all edges in $\delta(u)$ labelled $+1$ incident to u''; put unbalanced loops as $u'u''$ edges and leave balanced loops as balanced loops incident to either u' or u''. It is easily verified that $M(H)$ and $F(G, \mathcal{B})$ have the same set of circuits:

Proposition 1.24. *Let (G, Σ) be a signed graph with a balancing vertex u. If H is obtained from (G, Σ) by splitting u, then $M(H) \cong F(G, \Sigma)$.*

Roll-ups and unrolling

If Ω is a biased graph with a balancing vertex u, then the following *roll-up* operation produces another biased graph with frame matroid isomorphic to $F(\Omega)$. Let $\Sigma = \{e_1, \dots, e_k\}$ be the set of edges of one of the balancing classes in $\delta(u)$. Let Ω' be the biased graph obtained from Ω by replacing each edge $e_i = uv_i \in \Sigma$ with an unbalanced loop incident to its endpoint v_i (Figure 1.10). We say the biased graph Ω' is obtained by a *roll-up of balancing class* Σ of $\delta(u)$.

Likewise, if a biased graph $(G, \mathcal{B}_\Sigma)$ has a vertex u that is balancing after deleting its set U of unbalanced loops, and $\mathbf{\Sigma}$ is a signature realising $\mathcal{B}$ such that $\Sigma \setminus U \subseteq \delta(u)$, then the biased graph $(G', \mathcal{B}_\Sigma)$ obtained by replacing each unbalanced loop incident to $x \neq u$ with a xu link is obtained by *unrolling* the set of unbalanced loops of Ω.

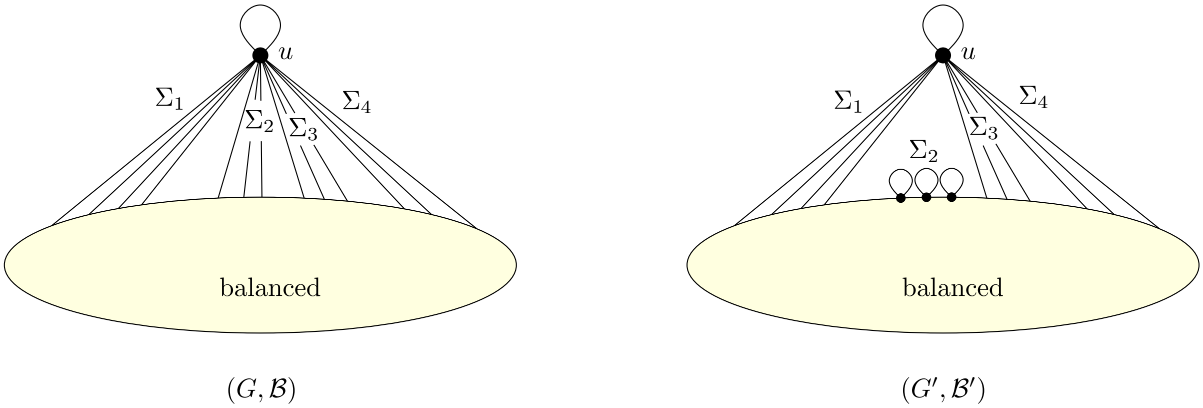


Figure 1.10: A roll-up: $F(G, \mathcal{B}) \cong F(G', \mathcal{B}')$

Suppose Ω_0 is a biased graph with balancing vertex u after deleting its set U of unbalanced loops, and that in $\Omega_0 \setminus U$ there are k balancing classes $\Sigma_1, \dots, \Sigma_k$ in $\delta(u)$. Let Ω be the biased graph obtained from Ω_0 by unrolling U, and write $\Sigma_0 = U$. For each $i \in \{0, 1, \dots, k\}$, let Ω_i be the biased graph obtained from Ω by rolling up balancing class Σ_i. Consider the set $\{\Omega, \Omega_0, \Omega_1, \dots, \Omega_k\}$. We say any member of this set is a *roll-up* of any other. It is straightforward to check that the frame matorids of any two roll-ups have the same set of circuits:

Proposition 1.25. *Let Ω be a biased graph with a balancing vertex after deleting its set of unbalanced loops. If Ω' is a roll-up of Ω, then $F(\Omega') \cong F(\Omega)$.*

Hence $\{\Omega, \Omega_0, \Omega_1, \dots, \Omega_k\}$ is a set of $k + 2$ representations of $F(\Omega_0)$.

Observe that if H is a graph, then for each vertex $v \in V(H)$ the biased graph $(G, \mathcal{B})$ obtained by rolling up all edges in $\delta(v)$ has $F(G, \mathcal{B}) \cong M(H)$. Conversely, if $(G, \mathcal{B})$ is balanced after deleting its set U of unbalanced loops, then U is a signature for G such that $\mathcal{B} = \mathcal{B}_U$. Hence the graph H obtained from G by adding an isolated vertex u and unrolling the edges in U to u has $M(H) \cong F(G, \mathcal{B})$.

Twisted flips

As mentioned in the previous section, the twisted flip operation of Theorem 4.2 generalises both the pinch and roll-up operations. Refer to the description of the twisted flip on page 26 and illustrated Figure 1.9. If $G_0, G_2, \dots, G_m$ are empty, so $G = G_1$, and if $\mathbf{\Sigma} = \emptyset$, then $(G, \mathcal{B})$ is balanced, so every vertex of G is a balancing vertex, and any other vertex may play the role of x_1. Choosing two vertices to play the roles of u and x_1 and applying a twisted flip is equivalent to pinching u and x_1. Conversely, if $(G, \mathcal{B}_\Sigma)$ is a signed graph with balancing vertex x and signature $\Sigma \subset \delta(x)$, then adding an isolated vertex u to G yields a biased graph of the form required for a twisted flip with $G = G_1$ and $G_0, G_2, \dots, G_k$ all empty. Applying a twisted flip results in all vx edges in Σ having endpoint x redefined as u, which is equivalent to splitting vertex x.

If u is a balancing vertex in a biased graph $(G, \mathcal{B})$ after deleting its set of unbalanced loops U, and S is one of the balancing classes in $\delta(u)$, then applying Observation 1.23 yields a signature $\mathbf{\Sigma} \subseteq E(G)$ so $\mathcal{B} = \mathcal{B}_\Sigma$, with the property that S is disjoint from the members of $\mathbf{\Sigma}$. Then a twisted flip operation on $(G, \mathcal{B}_\Sigma)$ is the operation of rolling up balancing class S. If $S = U$, then a twisted flip unrolls S.

A *curling* is a signed graph of a particular form whose associated frame matroid is graphic, one of the six families of graphs given by Theorem 3.1 (a curling is shown in Figure 3.1). A curling results from the operation of a twisted flip on $(G, \mathcal{B}_\Sigma)$ in the case that $\mathbf{\Sigma} = \emptyset$ and there are no unbalanced loops incident to u. Conversely, if $(G, \mathcal{B}_\Sigma)$ is a curling, then applying a twisted flip yields a graph H with $M(H) = F(G, \mathcal{B}_\Sigma))$, as shown in Figure 3.1.

1.3.5 Connectivity

A *separation* of a graph $G{=}(V, E)$ is a pair of edge disjoint subgraphs G_1, G_2 of G with $G = G_1 \cup G_2$. The *order* of a separation is $|V(G_1) \cap V(G_2)|$. A separation of order k is a k-*separation*. If both $V(G_1) \backslash V(G_2)$ and $V(G_2) \backslash V(G_1)$ are non-empty, then the separation is *proper*. If G has no proper separation of order less than k, then G is k-*connected*. The least integer k for which G has a proper k-separation is the *connectivity* of G. (Note that highly connected graphs may contain loops or parallel edges.) A partition (X, Y) of E naturally induces a separation $G[X], G[Y]$ of G, which we also denote (X, Y). We call X and Y the *sides* of the separation. The *connectivity function* of G is the function λ_G that to each partition (X, Y) of E assigns its order. That is, $\lambda_G(X, Y) = |V(X) \cap V(Y)|$.

A *separation* of a matroid $M{=}(E, \mathcal{C})$ is a partition of E into two subsets A, B, and is denoted (A, B); we call A and B the *sides* of the separation. The *order* of a separation (A, B) is $r(A) + r(B) - r(E) + 1$. A separation of order k with both $|A|, |B| \geq k$ is a k-*separation*. If M has no l-separation with $l < k$, then M is k-*connected*. The *connectivity* of M is the least integer k such that M has a k-separation. A matroid is connected if and only if it has no proper 1-separation. The *connectivity function* of M is the function λ_M that assigns to each partition (A, B) of E its order; that is, $\lambda_M(A, B) = r(A) + r(B) - r(M) + 1$.

In general, if (X, Y) is a partition of the edge set of a graph G, the order of separation (X, Y) in G will be different than that in $M(G)$. If each of $G[X]$ and $G[Y]$ is connected, however, then these orders are the same (indeed, this is the sole reason for the "$+1$" in the definition of the order of a separation of a matroid): if each of $G[X]$ and $G[Y]$ is connected, then

$$\begin{aligned}\lambda_{M(G)}(X,Y) &= r(X) + r(Y) - r(M) + 1 \\ &= (|V(X)| - 1) + (|V(Y)| - 1) - (|V| - 1) + 1 \\ &= |V(X) \cap V(Y)| = \lambda_G(X,Y).\end{aligned}$$

A k-*separation* of a biased graph $\Omega=(G,\mathcal{B})$ is a k-separation of its underlying graph G, and the *connectivity* of Ω is that of G. The *connectivity function* λ_Ω of Ω is that of G.

1.3.6 How to find a $U_{2,4}$ minor

A matroid is binary if and only if it has no $U_{2,4}$ minor. Since graphic matroids are binary, finding a $U_{2,4}$ minor in a frame matroid $F(G,\mathcal{B})$ certifies that $F(G,\mathcal{B})$ is not graphic. We will use this in Chapter 3. In Chapter 2, we will see that for a frame matroid M the elements of the complementary cocircuit of a connected non-binary hyperplane correspond, in any biased graph representing M, to the set of edges incident to a vertex. Hence we will often be looking for representations of $U_{2,4}$ minors in biased graphs. The following four lemmas give us four ways to do so.

The biased graphs representing $U_{2,4}$ are shown in Figure 1.11. The biased graph shown in (a) is a signed graph, with signature indicated by dashed edges; biased graphs (b) and (c) are contrabalanced.

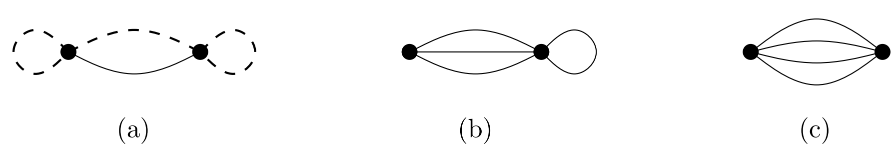


Figure 1.11: The biased graphs representing $U_{2,4}$.

Lemma 1.26 (Slilaty [31])**.** *Suppose $M = F(G,\mathcal{B})$, G is 2-connected, and $(G,\mathcal{B})$ contains two vertex disjoint unbalanced cycles, at most one of which is a loop. Then M is non-binary.*

Proof. Choose two unbalanced cycles C, C', with C not a loop. Choose two C-C' paths P, Q, meeting in at most one endpoint (Figure 1.12). Contracting all but one edge of C' yields the biased graph shown in Figure 1.12(b). Now $C \cup P \cup Q$ is a theta subgraph, say with branch vertices x, y. Let R, R' be the two internally disjoint x-y paths contained in C. By the theta property not both $P \cup Q \cup R$ and $P \cup Q \cup R'$ are balanced; without loss of generality suppose $P \cup Q \cup R'$ is unbalanced. Contracting all edges of R', all but one edge of R, and all but one edge of each of P and Q yields a biased graph representing $U_{2,4}$. □

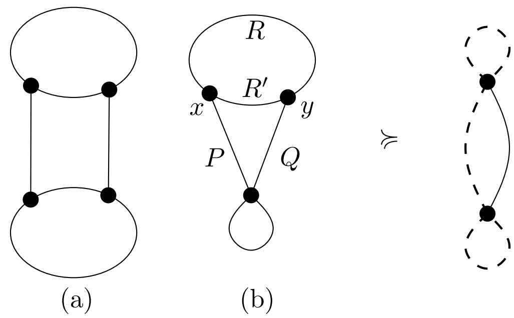


Figure 1.12: Finding a $U_{2,4}$ minor.

Let T be a theta subgraph with branch vertices x, y composed of three internally disjoint x-y paths Q_1, Q_2, Q_3. A *shortcut* of T is path P linking any two of Q_1, Q_2, Q_3 and avoiding the third, such that P also avoids $\{x, y\}$. In Figure 1.13, P is a shortcut linking Q_2 and Q_3.

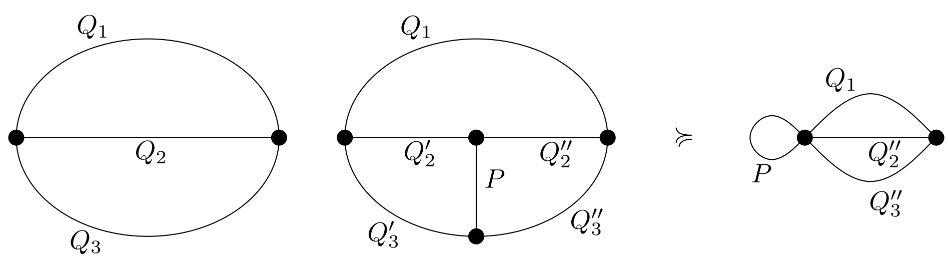


Figure 1.13: A contrabalanced theta with a shortcut has a $U_{2,4}$ minor.

Lemma 1.27. *Suppose $M = F(G, \mathcal{B})$. If $(G, \mathcal{B})$ contains a contrabalanced theta with a shortcut, then M is non-binary.*

Proof. Consider a theta subgraph $T = Q_1 \cup Q_2 \cup Q_3$ with shortcut P linking Q_2 and Q_3 avoiding Q_1 (Figure 1.13). For $i \in \{2, 3\}$, let Q'_i and Q''_i be the two internally disjoint paths in Q_i determined by the endpoint of P meeting Q_i. By the theta property, one of $Q'_2PQ'_3$ or $Q''_2PQ''_3$ is unbalanced. Without loss of generality, suppose is $Q'_2PQ'_3$ unbalanced. Contracting Q'_2 and Q'_3 yields a biased graph representing $U_{2,4}$: each cycle in the biased graph so obtained is unbalanced since each is a contraction of an unbalanced cycle in T. □

We can immediately generalise Lemma 1.27.

Lemma 1.28. *Suppose $M = F(G, \mathcal{B})$ and G is connected. If $(G, \mathcal{B})$ contains a contrabalanced theta and an unbalanced cycle avoiding one of its branch vertices, then M is non-binary.*

Proof. Let Q_1, Q_2, Q_3 be three internally disjoint u-v paths forming a contrabalanced theta T, and let C be an unbalanced cycle avoiding branch vertex u of T. If there is a subpath of

C forming a shortcut of T, then by Lemma 1.27, M has a $U_{2,4}$ minor. Otherwise, C meets an internal vertex of at most one of Q_1, Q_2, or Q_3. Let P be a C-T path (P is trivial if C meets T). For $i \in \{1, 2, 3\}$, let $e_i \in Q_i$ be the edge in T incident with u, and let e_4 be an edge in C that is not in T. Contract all edges in Q_1, Q_2, and Q_3 except e_1, e_2, and e_3. Depending upon how C meets T, we now have one of the biased graphs shown in Figure 1.14. In case (a)

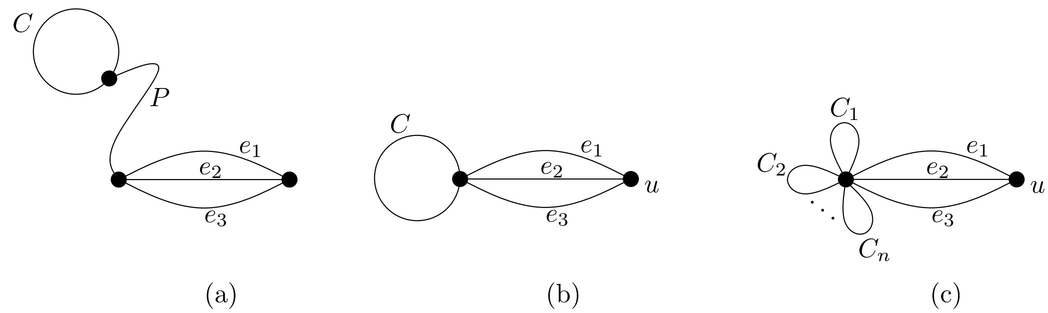


Figure 1.14: If $(G, \mathcal{B})$ contains an odd theta and an unbalanced cycle avoiding one of its branch vertices.

C is disjoint from T or (b) after contraction of the edges in Q_1, Q_2, and Q_3, the remaining edges in C form a single cycle, contract all edges of P and all edges remaining in C but e_4 to obtain a biased graph representing $U_{2,4}$. If after contraction of the edges of Q_1, Q_2, and Q_3 the remaining edges of C form several cycles, then contract all edges but e_4 in the cycle containing e_4 and delete all edges of C left in the remaining cycles. This again yields a biased graph representing $U_{2,4}$. $\square$

Clearly, $v \in V(G)$ is a cut vertex of G if and only if there is a proper 1-separation (G_1, G_2) of G with $V(G_1) \cap V(G_2) = \{v\}$.

Lemma 1.29. *Let $(G, \mathcal{B})$ be a biased graph with $F(G, \mathcal{B})$ connected, and with balancing vertex u. Let k be the number of balancing classes in $\delta(u)$. If $k \geq 4$, then either $(G, \mathcal{B})$ is a signed graph and u is a cut vertex, or $F(G, \mathcal{B})$ is non-binary.*

Proof. If $(G, \mathcal{B})$ is a signed graph, then by Proposition 1.20 $(G, \mathcal{B})$ contains no contrabalanced theta, so u is a cut vertex such that each component of $G - u$ contains endpoints of edges of at most two balancing classes of $\delta(u)$. So suppose $(G, \mathcal{B})$ is not a signed graph. If a component of $G - u$ contains endpoints of edges of four distinct balancing classes of $\delta(u)$, then $(G, \mathcal{B})$ contains as a minor the contrabalanced biased graph consisting of two vertices with four edges between them, which represents $U_{2,4}$ (Figure 1.11(c)). So now assume that no component of $G - u$ contains endpoints of edges of four distinct balancing classes. By Proposition 1.20, $(G, \mathcal{B})$ contains a contrabalanced theta T. Since u is balancing, u is one of its branch vertices, and there are three balancing classes A_1, A_2, A_3 each containing one of the three edges in $T \cap \delta(u)$. Since $|\delta(u)/{\sim}| > 3$, there is a proper 1-separation (G_1, G_2)

of G with $V(G_1) \cap V(G_2) = \{u\}$ where $T \subseteq G_1$ and G_2 contains an edge in a fourth balancing class distinct from A_1, A_2, A_3. Since $F(G, \mathcal{B})$ is connected, G_2 contains an unbalanced cycle C. Contracting all edges of T but those in $\delta(u)$ and all but one edge of C, we obtain the biased graph representing $U_{2,4}$ of Figure 1.11(b). $\square$

Chapter 2

When is a biased graph group-labellable?

In this chapter we prove:

Theorem 2.1. *Let $(G, \mathcal{B})$ be a biased graph. Let K be the 2-cell complex obtained by adding a disc with boundary C for each $C \in \mathcal{B}$. The following are equivalent:*

1. *$(G, \mathcal{B})$ is $\pi_1(K)$-labellable.*
2. *$(G, \mathcal{B})$ is group-labellable.*
3. *No unbalanced cycle can be moved to a balanced cycle via a sequence of balanced reroutings of closed walks.*
4. *A cycle C is contractible in K if and only if $C \in \mathcal{B}$.*

Theorem 2.1 has several significant consequences regarding minor-closed classes of group-labelled graphs, frame matroids, and lift matroids. In particular, there are natural minor-closed classes of group-labelled graphs that have infinite sets of excluded minors on a fixed number of vertices, and there are natural minor-closed classes of frame and lift matroids that have infinite sets of excluded minors of fixed rank. Further, not only are there such infinite antichains of biased graphs, and matroids, that are minor-minimal subject to not belonging to these classes, but each of these classes themselves contain such infinite antichains. We construct antichains using biased graphs whose underlying simple graphs are complete, and with underlying simple graphs that are cycles. Table 2.1 summarizes the types of theorems on antichains we prove using Theorem 2.1. This is done in Sections 2.3-2.6.

In Section 2.7, we show that if Γ is a finite group, then the class of Γ-labelled graphs of bounded branch-width is well-quasi-ordered by the minor relation.

Constructions	- complete graphs - cycles
Antichains	- of excluded minors for group-labelled graphs - contained in classes of group-labelled graphs
Setting	- group-labelled graphs - frame matroids - lift matroids

Table 2.1: Antichain theorems.

2.1 Context and preliminaries

A *quasi-order* on a set $\mathcal{X}$ is a reflexive, transitive relation $\preccurlyeq$. A quasi-order on $\mathcal{X}$ is a *well-quasi-order* if every infinite sequence of elements in $\mathcal{X}$ contains a pair of elements x_i, x_j with $i < j$ and $x_i \preccurlyeq x_j$. An *antichain* in $\mathcal{X}$ is a set of pairwise incomparable elements. A quasi-order on $\mathcal{X}$ is a well-quasi-order if and only if $\mathcal{X}$ contains no infinite strictly decreasing sequence (a sequence $(x_i)_{i\in\mathbb{N}}$ with $x_j \preccurlyeq x_i$ and $x_j \neq x_i$ for all $i < j$) and no infinite antichain ([2], Proposition 12.1.1).

Let $\mathcal{X}$ be a set of biased graphs or matroids, and $\preccurlyeq$ the relation of minor containment. Then clearly $\mathcal{X}$ contains no infinite strictly decreasing sequence. In this case, the statement that $\mathcal{X}$ is well-quasi-ordered by $\preccurlyeq$ is equivalent to the statement that $\mathcal{X}$ contains no infinite antichain. Consider a subset $X \subset \mathcal{X}$ that is closed under minors. If X has infinitely many excluded minors in $\mathcal{X}$, then its set of excluded minors is an infinite antichain in $\mathcal{X}$. Conversely, if Y is an infinite antichain in $\mathcal{X}$, then the set $\{H \in \mathcal{X} : H$ has no minor in $Y\}$ is a minor-closed class having infinitely many excluded minors. Hence if $\mathcal{X}$ is a set of graphs or matroids, the statement that $\mathcal{X}$ contains no infinite antichain is in turn equivalent to the statement that $\mathcal{X}$ contains no minor-closed class having an infinite set of excluded minors.

Thus Robertson and Seymour's Theorem 1.3 is equivalent to the statement that the set of all graphs is well-quasi-ordered by the minor relation. And Geelen, Gerards, and Whittle's Theorem 1.6 is equivalent to the statement that for any finite field $\mathbb{F}$, the set of matroids linear over $\mathbb{F}$ is well-quasi-ordered by the minor relation.

Infinite antichains of biased graphs are not very hard to find. For example, let $2C_n$ denote the graph obtained from a cycle of length n by adding an edge in parallel with every existing edge, and let $\mathcal{B}_n$ be a set of two edge disjoint Hamilton cycles in $2C_n$. Since every theta subgraph consists of a Hamilton cycle and one pair of parallel edges, $(2C_n, \mathcal{B}_n)$ is a biased graph.

Observation 2.2. The set $\{(2C_n, \mathcal{B}_n) : n \geq 3\}$ is an infinite antichain.

Proof. Each of these biased graphs has exactly two balanced cycles, but contracting or deleting an edge gives a biased graph with just one balanced cycle. Further minor operations will never increase the number of balanced cycles. □

In fact, using Theorem 2.1, we are able to construct infinite antichains of biased graphs all on a fixed number of vertices. Given a group Γ, let $\mathcal{G}_\Gamma$ denote the class of all biased graphs which are Γ-labellable. By Proposition 1.12, $\mathcal{G}_\Gamma$ is minor-closed, and we may ask for its set of excluded minors. As a corollary to Theorem 2.1, we obtain the following.

Corollary 2.3. *For every infinite group Γ and every $t \geq 3$ there are infinitely many excluded minors for $\mathcal{G}_\Gamma$ with exactly t vertices.*

Not only do these classes have infinitely many excluded minors, but they also contain infinite antichains of biased graphs.

Theorem 2.4. *Let Γ be an infinite group.*

(a) *For every $t \geq 3$ there exists an infinite antichain of Γ-labelled graphs on t vertices, each of whose underlying simple graph is K_t.*

(b) *For every $t \geq 3$, $t \neq 4$, there exists an infinite antichain of Γ-labelled graphs on t vertices, each of whose underlying simple graph is a cycle of length t.*

These results are in sharp contrast to what Theorem 1.3 tells us about minor-closed classes of graphs. Thus, in some sense, biased graphs behave less like graphs than one might perhaps expect, and more like matrices over fields. The result of Section 2.7, however, shows that, at least for biased graphs of bounded branch-width, if Γ is a finite group, such pathological behaviour is not possible. We prove:

Theorem 2.5. *Let Γ be a finite group and n be an integer. Then every infinite set of Γ-labelled graphs of branch-width at most n has two members one of which is isomorphic to a minor of the other.*

2.1.1 Lift matroids

Our results in this chapter are about group-labelled graphs, and consequences for certain classes of matroids. Biased graphs represent a second class of matroids, namely, *lift* matroids. When translated to the setting of matroids, these results apply equally to lift as to frame matroids.

The class of lift matroids arises as follows. A matroid M is an *elementary lift* of a matroid N if M may be obtained by coextending N by an element e so that e is not a loop or coloop, then deleting e. In other words, M is an elementary lift of N if there is a matroid L such that $M = L \setminus e$ and $N = L/e$ (and $M \not\cong N$). If N is graphic, say represented by the graph G, then a biased graph naturally arises from M, L, and G as follows.

Theorem 2.6 (Zaslavski [39]). *Let M be an elementary lift of a graphic matroid N on ground set E, and let G be a graph with $M(G) \cong N$. Let $\mathcal{B} = \{C : C$ is a cycle of G and $E(C)$ is a circuit in $M\}$. Then $(G, \mathcal{B})$ is a biased graph, and $C \subseteq E$ is a circuit of M if and only if C induces one of the following in $(G, \mathcal{B})$:*

1. *a balanced cycle,*

2. *two unbalanced cycles meeting in at most one vertex, or*

3. *a contrabalanced theta.*

Proof. The cycles of G are precisely the circuits of N, which are precisely the minimal non-empty member of $\{C \setminus e : C$ is a circuit of $L\}$. That $\mathcal{B}$ has the theta property is seen as follows. Suppose C_1, C_2, C_3 are the three cycles in a theta subgraph of G, and $C_1, C_2 \in \mathcal{B}$ but $C_3 \notin \mathcal{B}$. We first claim that a cycle C of G is unbalanced if and only if $C \cup e \in \mathcal{C}(L)$. If $C \cup e$ is a circuit of L, then C is independent in M. Conversely, let C be an unbalanced cycle of $(G, \mathcal{B})$. Then $C \in \mathcal{C}(N)$ and $C \notin \mathcal{C}(M)$. Hence there is a circuit $D \in \mathcal{C}(L)$ such that $C = D \setminus e$. If $e \notin D$, then $C = D$ and $C \in \mathcal{C}(M)$ and C would be balanced. Hence $e \in D$, so $C \cup e = D \in \mathcal{C}(L)$. So we have that C_1, C_2 are circuits of L not containing e, and $C_3 \cup e \in \mathcal{C}(L)$. Now let $f \in C_1 \cap C_3$. Since $e \in (C_3 \cup e) \setminus C_1$, by the strong circuit elimination axiom, there is a circuit C' of L contained in $((C_3 \cup e) \cup C_1) \setminus f$ with $e \in C'$. Then $C' \setminus e \in \mathcal{C}(N)$, so $C' \setminus e$ is a cycle in G contained in $(C_1 \cup C_3) \setminus f$. This implies $C' \setminus e = C_2$. But then $C_2 \subset C'$, a contradiction.

Now let D be a circuit of M. Then D is a circuit of L not containing e. If D is a circuit of N, then D is a balanced cycle in $(G, \mathcal{B})$. Otherwise, D properly contains a circuit C of N with $C \cup e$ a circuit of L. Let $f \in C$. Applying the strong circuit elimination axiom to $C \cup e$ and D, we find a circuit $C' \in \mathcal{C}(L)$ such that $e \in C' \subseteq ((C \cup e) \cup D) \setminus f = (D \cup e) \setminus f$. Now consider circuits $C \cup e$ and C' of L: both contain e, so there is a circuit $D' \in \mathcal{C}(L)$ contained in $(C \cup C') \setminus e$. But as $D' \subseteq (C \cup C') \setminus e \subseteq D$, we have $D' = D = (C \cup C') \setminus e$. Let $C'' = C' \setminus e$. Since C and C'' are both circuits of N, both are unbalanced cycles in $(G, \mathcal{B})$. Finally, consider the arrangement of edges of $D = C \cup C''$ in G. No cycle of G contained in D is balanced, since this would be a circuit of M properly contained in D, a contradiction. Suppose D contains two unbalanced cycles A, A'. Then $A \cup e$ and $A' \cup e$ are circuits of L, so $A \cup A'$ contains a circuit $D'' \in \mathcal{C}(L)$. But $D'' \subseteq A \cup A' \subseteq D$ implies $D'' = D = A \cup A'$. This implies that D is either a pair of unbalanced cycles meeting in at most one vertex or a contrabalanced theta subgraph of $(G, \mathcal{B})$. $\square$

These matroids have been studied by Zaslavski [39] and Pivotto [10, 24]. They also appear as a main tool in Lovász's characterisation of graphs with the property that any two odd length cycles have a common vertex (see Chapter 10 of [9]). As with frame matroids, we say $(G, \mathcal{B})$ *represents* M, and we write $M \cong L(G, \mathcal{B})$.

Minors of lift matroids The minor operations on a biased graph $(G, \mathcal{B})$ described in Section 1.2.3, except for contraction of an unbalanced loop, are consistent with the corresponding minor operations in $L(G, \mathcal{B})$. That is, $L(G, \mathcal{B}) \setminus e = L((G, \mathcal{B}) \setminus e)$ and when e is a link or a balanced loop $L(G, \mathcal{B})/e = L((G, \mathcal{B})/e)$. When we are interested in the lift matroid represented by $(G, \mathcal{B})$, so that the minor operations in the biased graph and lift matroid agree, contraction of an unbalanced loop is defined as follows. If e is an unbalanced loop, then $(G, \mathcal{B})/e$ is the graph G/e, having all cycles balanced. With this modification, we see that also the class of lift matroids is minor-closed.

Let Γ be a group and $e \in E(G)$. If $(G, \mathcal{B})$ is a Γ-labelled graph, then preforming the deletion operation as described in Section 1.2.3 for group-labelled graphs yields a Γ-labelling of $(G, \mathcal{B}) \setminus e$. If e is not an unbalanced loop, then performing the contraction operation as described in Section 1.2.3 for group-labelled graphs yields a Γ-labelling of $(G, \mathcal{B})/e$. If e is an unbalanced loop, then arbitrarily orienting the edges of $(G, \mathcal{B})/e$ and labelling all edges with 1 is a Γ-labelling of $(G, \mathcal{B})/e$. Hence the class of Γ-labelled graphs is minor-closed, whichever minor operation we choose for unbalanced loops.

In this chapter, to avoid the complication of having to consider two different minor operations for unbalanced loops, we prohibit contraction of an unbalanced loop. Results in which certain minors are produced, proved under this restriction, then also hold for minors when either type of contraction of an unbalanced loop is permitted. This simplifies translation of results of this type into the settings of frame and lift matroids.

2.1.2 Branch decompositions

A function $\lambda : 2^E \to \mathbb{Z}$ defined on the the subsets of a finite ground set E is *symmetric* if for all subsets $A \subseteq E$, $\lambda(A) = \lambda(E \setminus A)$. For such a function, for disjoint subsets A, B of E, define $\lambda(A, B) = \min\{\lambda(X) : A \subseteq X \subseteq E \setminus B\}$.

A *branch-decomposition* of a symmetric, submodular function λ on a finite set E is cubic tree T (*i.e.*, all degrees are 1 or 3) together with an injective function from E into the set of leaves of T. The set *displayed by* a subtree of T is the subset of elements of E in that subtree. A set of elements of E is *displayed by* an edge e of T if it is displayed by one of the components of $T \setminus e$. The *width* $\lambda(e)$ of $e \in T$ is the value given by λ of one of the two sets displayed by e (λ is symmetric, so they are equal). The *width* of a branch decomposition is the maximum of the widths of its edges, and the *branch-width* of a symmetric submodular function is the minimum of the widths of all its branch decompositions.

Given a subset A of the edges of a graph $G{=}(V, E)$, let $\lambda_G(A) = \lambda_G(A, E \setminus A)$, where $\lambda_G(A, E \setminus A)$ is the connectivity function of G (defined in Section 1.3.5). The connectivity function λ_G is symmetric and submodular. A *branch decomposition* of G is a branch-decomposition of its connectivity function λ_G; the *branch-width* of G is the branch-width of λ_G. Similarly, for a subset A of the elements of a matroid $M=(E,\mathcal{C})$, set $\lambda_M(A) = \lambda_M(A, E \setminus A)$, where λ_M is the connectivity function of M. As for graphs, the connectivity function λ_M is symmetric and submodular, and we define *branch decompositions* of M and the *branch-width* of M to be those of its connectivity function.

2.1.3 Spikes and swirls

Spikes and swirls are two families of matroids that have been an important source of examples in studies of representability of matroids over fields (used to show that the number of inequivalent matrix representations of a 3-connected matroid linear over $GF(q)$ is not bounded by any constant depending only on q [22]). For each integer $n \geq 3$, a rank n *spike* is obtained by taking n concurrent three-point lines $\{x_i, y_i, z\}$ ($i \in \{1, \dots, n\}$) freely in n-space, then deleting their common point of intersection z. If no choice of n points, one from each pair $\{x_i, y_i\}$, form a circuit-hyperplane, then this is the rank n *free* spike; other spikes have such circuit-hyperplanes. The matroids of Example 3 are spikes.

The rank n *whirl* is obtained from the cycle matroid $M(W_n)$ of the rank n wheel by relaxing its unique circuit-hyperplane. A rank n *swirl* is obtained by adding a point freely to each 3-point line of the rank n whirl, then deleting those points lying on the intersection of two 3-point lines. If no n points form a circuit-hyperplane, this is the rank n *free* swirl; other swirls have such circuit hyperplanes (which necessarily have exactly one point from each of the original n lines used to construct the swirl). Zaslavsky [41] observed that spikes are lift matroids and swirls are frame matroids both coming from biased graphs of the form $(2C_n, \mathcal{B})$ where every cycle in $\mathcal{B}$ is of length n. The family of biased graphs $(2C_n, \mathcal{B}_n)$ of Observation 2.2 yields both an infinite antichain of spikes and an infinite antichain of swirls. The proof is a simply a translation of the proof of Observation 2.2 from biased graphs to the setting of lift and frame matroids: a balanced Hamilton cycle in $2C_n$ is a circuit hyperplane in each of $L(2C_n, \mathcal{B}_n)$ and $F(2C_n, \mathcal{B}_n)$. Each of $L(2C_n, \mathcal{B}_n)$ and $F(2C_n, \mathcal{B}_n)$ have exactly two circuit hyperplanes partitioning their ground sets, but any proper minor of any of them destroys this property.

Spikes and swirls are 3-connected and have branch-width 3. A swirl of rank ≥ 4 is linear over a field $\mathbb{F}$ if and only if its biased graph representation is labellable by the multiplicative group of $\mathbb{F}$, and a spike of rank ≥ 4 is linear over $\mathbb{F}$ if and only if its biased graph representation is labellable by the additive group of $\mathbb{F}$ [41]. We note that the biased graphs $(2C_n, \mathcal{B}_n)$ uniquely represent the swirl $F(2C_n, \mathcal{B}_n)$.

The antichains we exhibit in Sections 2.5 and 2.6 show that spikes and swirls are in fact the tip of the iceberg when it comes to families of infinite antichains of lift and frame matroids.

For every group Γ, let $\mathcal{F}_\Gamma$ (resp. $\mathcal{L}_\Gamma$) denote the class of matroids which can be represented as a frame (lift) matroid of a biased graph which is Γ-labellable. Each of these is a proper minor-closed class of frame (resp. lift) matroids. In general, a matroid in either of these classes may have many different biased graph representations. However, we are able to construct non-group-labellable biased graphs that uniquely represent their associated frame (resp. lift) matroids. This yields the following somewhat surprising result.

Theorem 2.7. *For every infinite group Γ and every $t \geq 3$ the classes $\mathcal{L}_\Gamma$ and $\mathcal{F}_\Gamma$ have infinitely many excluded minors of rank t.*

It is also the case that for every infinite group Γ and every $t \geq 3$, each of $\mathcal{L}_\Gamma$, and $\mathcal{F}_\Gamma$ contain infinite antichains of rank t matroids. We prove this in Section 2.6.

2.2 A Topological Characterisation

In Section 1.3.1 we defined balanced rerouting, and observed in Lemma 1.19 that a balanced rerouting of a cycle preserves the bias of the cycle. We now extend this notion to a notion of balanced rerouting of closed walks, as follows. Let $(G, \mathcal{B})$ be a biased graph. Let W be a closed walk in G, and suppose C is a balanced cycle in $(G, \mathcal{B})$ containing a u-v path $P \subseteq W$. Let Q be the u-v path in C distinct from P, and let W' be the closed walk obtained from W by replacing P with Q. Then W' is obtained from W by a balanced rerouting of P along C. Such a rerouting is a *balanced rerouting* of a closed walk. If G is labelled by $\gamma : E(G) \to \Gamma$ for some group Γ, then since C is balanced, $\gamma(Q) = \gamma(P)$, and so also $\gamma(W') = \gamma(W)$.

A cycle C is *moved* to a cycle C' via a sequence of balanced reroutings if there exists a sequence of closed walks $(W_0, W_1, \ldots, W_n)$, with $W_0{=}C$ and $W_n{=}C'$, such that W_i is obtained from W_{i-1} by a balanced rerouting for each $i \in \{1, 2, \ldots, n\}$. Again, if $(G, \mathcal{B})$ is group-labelled by γ, then $\gamma(W_{i-1}) = \gamma(W_i)$ for each $i \in \{1, \ldots, n\}$, so $\gamma(C) = \gamma(C')$.

We may now prove Theorem 2.1.

Theorem 2.1. *Let $(G, \mathcal{B})$ be a biased graph. Let K be the 2-cell complex obtained by adding a disc with boundary C for each $C \in \mathcal{B}$. The following are equivalent:*

1. *$(G, \mathcal{B})$ is $\pi_1(K)$-labellable.*
2. *$(G, \mathcal{B})$ is group-labellable.*
3. *No unbalanced cycle can be moved to a balanced cycle via a sequence of balanced reroutings of closed walks.*
4. *A cycle C is contractible in K if and only if $C \in \mathcal{B}$.*

Proof of Theorem 2.1. Trivially, (1) implies (2). The discussion above shows that (2) implies (3). To show that (3) implies (4), we show that a contractible unbalanced cycle may be moved to a balanced cycle via a sequence of balanced reroutings of closed walks, violating (3). To show that (4) implies (1), we construct a $\pi_1(K)$-labelling of $(G, \mathcal{B})$.

We may assume that G is connected (otherwise apply the argument to each component). Choose a spanning tree T. Let $(G', \mathcal{B}')$ denote the biased graph obtained from $(G, \mathcal{B})$ by contracting every edge in $E(T)$. Let K' denote the cell complex obtained from K by identifying T to a single point. Since T is contractible, it follows that $\pi_1(K) \cong \pi_1(K')$ (see Proposition 0.17 in [11]).

We now apply a standard result to obtain a description of the fundamental group of K. Arbitrarily orient the edges of G, and for each edge $e \in E(G') = E(G) \setminus E(T)$ let g_e be a group generator. For every cycle $C \in \mathcal{B}$ choose a simple closed walk around C, and let $e_1, \ldots, e_m$ be the sequence of edges of this walk appearing in $E(G')$. This yields a closed walk consisting of a sequence of loops on the single vertex of G', obtained by removing the edges in T from the closed walk around C. For $i \in \{1, \ldots, m\}$, set $\epsilon_i = 1$ if e_i is traversed in the forward direction in this walk and set $\epsilon_i = -1$ if it is traversed backward. Define β_C to be the word $g_{e_1}^{\epsilon_1} g_{e_2}^{\epsilon_2} \cdots g_{e_n}^{\epsilon_n}$. Now let Γ be the group presented by the generating set $\{g_e : e \in E(G')\}$ with the relations given by setting the words in $\{\beta_C : C \in \mathcal{B}\}$ to be the identity. It follows from an application of Van Kampen's theorem (see, for instance, Section 1.2 in [11]) that $\Gamma \cong \pi_1(K') \cong \pi_1(K)$ and that a closed walk W with edge sequence $e_1, \ldots, e_m$ with orientations $\epsilon_1, \ldots, \epsilon_m$, respectively, will be contractible in K' if and only if the product $\prod_{i=1}^m g_{e_i}^{\epsilon_i}$ is the identity in Γ.

We now obtain a Γ-labelling $\gamma : E(G) \to \Gamma$ of G by extending our function $\gamma : E(G') \to \Gamma$. Define, for each $e \in E(T)$, $\gamma(e)$ to be the identity element of Γ. Now let W be a closed walk in G and let W' be the corresponding closed walk in G'. Let $e_1, \ldots, e_m$ be the edge sequence of W', and let $\epsilon_i = 1$ if e_i is traversed forward in W' and $\epsilon_i = -1$ if it is traversed backward. Then

$$W \text{ is contractible in } K \iff W' \text{ is contractible in } K' \iff \prod_{i=1}^{m} g_{e_i}^{\epsilon_i} = 1 \iff \gamma(W) = 1. \tag{2.1}$$

If (4) holds, then the logical equivalences of (2.1) imply $\mathcal{B} = \mathcal{B}_\gamma$. *I.e.*, $(G, \mathcal{B})$ is Γ-labellable, so (1) holds. On the other hand, if (4) fails, then there is a cycle $C \notin \mathcal{B}$ that is contractible in K. By (2.1) a simple closed walk W around C has $\gamma(W) = 1$. The group relations in Γ that reduce the product of the corresponding edge labels to the identity yield a sequence of closed walks moving C to a balanced cycle via a sequence of balanced reroutings, violating (3). □

Call a labelling of $(G, \mathcal{B})$ obtained as in Theorem 2.1 a $\pi_1(K)$-*labelling of* $(G, \mathcal{B})$. The following corollary highlights an interesting property of a $\pi_1(K)$-labelling of a biased graph. While, by definition, all group-labellings of a biased graph $(G, \mathcal{B})$ realising $\mathcal{B}$ share the same set of balanced cycles, they may differ on closed walks. Call a closed walk W *balanced in a labelling* γ if $\gamma(W) = 1$.

Corollary 2.8. *Among all group-labellings of $(G, \mathcal{B})$, a $\pi_1(K)$-labelling as constructed in Theorem 2.1 has the unique minimal collection of balanced closed walks.*

Proof. Let W be a closed walk that is balanced in a $\pi_1(K)$-labelling of $(G, \mathcal{B})$ given by $\gamma : E(G) \to \pi_1(K)$, and let $\phi : E(G) \to \Phi$ be a group-labelling of $(G, \mathcal{B})$ by some group Φ with $\mathcal{B}_\Phi = \mathcal{B}$. Then $\gamma(W) = 1$, and the group relations in $\pi_1(K)$ that reduce the product of the edge labels of W to the identity yield a sequence of closed walks moving W to a balanced cycle via a sequence of balanced reroutings, say $W, W_1, \ldots, W_n$, where $W_n \in \mathcal{B}$. Hence $\phi(W) = \phi(W_1) = \cdots = \phi(W_n) = 1$, so W is a balanced closed walk in the Φ-labelling of G. This shows that every closed walk balanced in a $\pi_1(K)$-labelling is balanced in any other group-labelling of $(G, \mathcal{B})$. The uniqueness and minimality of the collection of balanced closed walks given by a $\pi_1(K)$-labelling immediately follows. □

2.2.1 Group-labelling by arbitrary groups

The $\pi_1(K)$-labelling constructed by Theorem 2.1 is in general a labelling by an infinite group. In practice we often work with graphs labelled by a finite group. Our biased graphs are finite. If $(G, \mathcal{B})$ is group-labellable, is $(G, \mathcal{B})$ always labellable by a finite group? Or are there group-labelled graphs whose collection of balanced cycles are only labellable by an infinite group?

Given a biased graph $(G, \mathcal{B})$ with $\mathcal{B}$ realised by a labelling $\gamma : E(G) \to \Gamma$ for some group Γ, there is a homomorphism $\varphi : \pi_1(K) \to \Gamma$, which we may describe as follows. The fundamental group $\pi_1(K)$ of K constructed in the proof of Theorem 2.1 has presentation in terms of generators g_e, one for each edge e not in a chosen spanning tree of G, and relations among these generators given by simple closed walks around the cycles in $\mathcal{B}$. Let $\rho : E(G) \to \pi_1(K)$ be the $\pi_1(K)$-labelling of $(G, \mathcal{B})$ constructed in the proof of Theorem 2.1. For an element $g \in \pi_1(K)$ expressed as the word $g = g_{e_1}^{a_1} g_{e_2}^{a_2} \cdots g_{e_m}^{a_m}$ for some generators $g_{e_1}, g_{e_2}, \ldots, g_{e_m}$ and integers $a_1, a_2, \ldots, a_m$, define $\varphi : \pi_1(K) \to \Gamma$ by

$$\varphi(g) = \varphi\left(g_{e_1}^{a_1} g_{e_2}^{a_2} \cdots g_{e_m}^{a_m}\right) = \gamma\left(\rho^{-1}(g_{e_1})^{a_1} \rho^{-1}(g_{e_2})^{a_2} \cdots \rho^{-1}(g_{e_m})^{a_m}\right).$$

Theorem 2.9. *The map $\varphi : \pi_1(K) \to \Gamma$ is a group homomorphism.*

Proof. Clearly φ is a homomorphism if φ is well-defined. So suppose $g = g_{n_1}^{a_{n_1}} g_{n_2}^{a_{n_2}} \cdots g_{n_m}^{a_{m_n}} = g_{k_1}^{a_{k_1}} g_{k_2}^{a_{k_2}} \cdots g_{k_p}^{a_{k_p}}$ are two words expressing the element $g \in \pi_1(K)$, where g_{n_1}, g_{n_2}, ..., g_{n_m}, g_{k_1}, g_{k_2}, ..., g_{k_p} are generators and $a_{n_1}, a_{n_2}, \ldots, a_{n_m}, a_{k_1}, a_{k_2}, \ldots, a_{k_p}$ are integers. We show that

$$\varphi\left(g_{n_1}^{a_{n_1}} g_{n_2}^{a_{n_2}} \cdots g_{n_m}^{a_{m_n}} (g_{k_1}^{a_{k_1}} g_{k_2}^{a_2} \cdots g_{k_p}^{a_{k_p}})^{-1}\right) = 1,$$

and so that φ maps both words $g_{n_1}^{a_{n_1}} g_{n_2}^{a_{n_2}} \cdots g_{n_m}^{a_{m_n}}$ and $g_{k_1}^{a_{k_1}} g_{k_2}^{a_2} \cdots g_{k_p}^{a_{k_p}}$ to the same element of Γ. We have

$$\begin{aligned}
&\varphi\left(g_{n_1}^{a_{n_1}} g_{n_2}^{a_{n_2}} \cdots g_{m_n}^{a_{m_n}} (g_{k_1}^{a_{k_1}} g_{k_2}^{a_{k_2}} \cdots g_{k_p}^{a_{k_p}})^{-1} \right) \\
&\quad = \gamma\left(\rho^{-1}(g_{n_1})^{a_{n_1}} \rho^{-1}(g_{n_2})^{a_{n_2}} \cdots \rho^{-1}(g_{n_m})^{a_{nm}} \rho^{-1}(g_{k_p})^{-a_{k_p}} \cdots \rho^{-1}(g_{k_2})^{-a_{k_2}} \rho^{-1}(g_{k_1})^{-a_{k_1}}\right) \\
&\qquad = \gamma\left(\rho^{-1}(g_{n_1})\right)^{a_{n_1}} \gamma\left(\rho^{-1}(g_{n_2})\right)^{a_{n_2}} \cdots \gamma\left(\rho^{-1}(g_{n_m})\right)^{a_{nm}} \gamma\left(\rho^{-1}(g_{k_p})\right)^{-a_{k_p}} \cdots \\
&\qquad\qquad \gamma\left(\rho^{-1}(g_{k_2})\right)^{-a_{k_2}} \gamma\left(\rho^{-1}(g_{k_1})\right)^{-a_{k_1}} .
\end{aligned}$$

In $\pi_1(K)$, $g_{n_1}^{a_{n_1}} g_{n_2}^{a_{n_2}} \cdots g_{m_n}^{a_{m_n}} g_{k_p}^{-a_{k_p}} \cdots g_{k_2}^{-a_{k_2}} g_{k_1}^{-a_{k_1}} = 1$; the relations in $\pi_1(K)$ that reduce this word to the identity correspond to a sequence of reroutings via balanced cycles. The word in Γ given by φ has the same corresponding edge sequence. Since $\mathcal{B}_\gamma = \mathcal{B}$, this word is therefore reduced to the identity by the same sequence of balanced reroutings. $\square$

The existence of this homomorphism may be used to prove the following.

Theorem 2.10. *There exists a group-labellable biased graph whose collection of balanced cycles is not realised by any labelling by any finite group.*

Proof. Let H be the Higman group

$$H = \left\langle a, b, c, d \mid a^{-1}ba = b^2, b^{-1}cb = c^2, c^{-1}dc = d^2, d^{-1}ad = a^2 \right\rangle .$$

The Higman group is infinite with no non-trivial finite quotients [12]. Construct a simplicial 2-complex $\mathcal{K}$ by identifying the points of edges marked a, b, c, and d, respectively, of five pentagons, oriented as shown in Figure 2.1. Let G be the graph consisting of the vertices

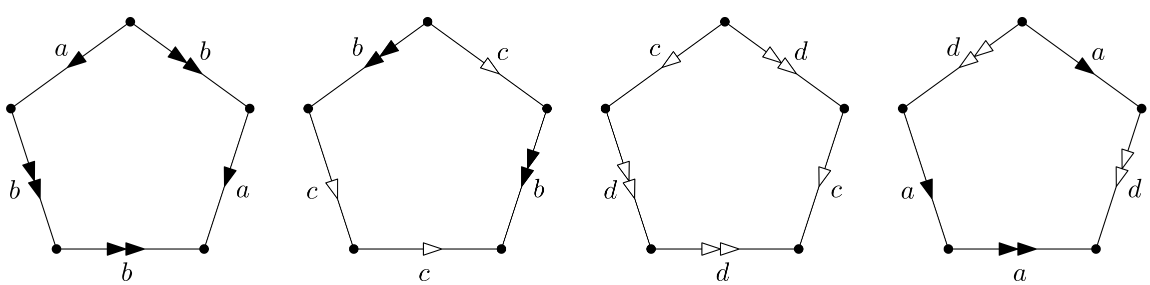


Figure 2.1: Constructing a 2-complex whose fundamental group is the Higman group.

and edges of the barycentric subdivision of $\mathcal{K}$ (Figure 2.2 shows the part of the graph obtained from the first pentagon in Figure 2.1; G is shown in Figure 2.3), and let $\mathcal{B}$ be the set of cycles of G that are contractible in $\mathcal{K}$. By construction, the fundamental group $\pi_1(\mathcal{K})$ is H. Hence the biased graph $(G, \mathcal{B})$ is H-labellable. Let Γ be a group for which there is a labelling $\gamma : E(G) \to \Gamma$ realising $\mathcal{B}$. By Theorem 2.9, there is a homomorphism from H to Γ. Since H has no non-trivial finite quotient, Γ cannot be finite. $\square$

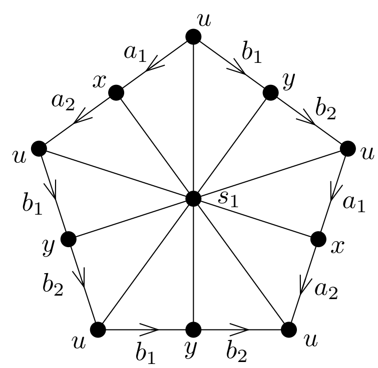


Figure 2.2: Constructing a biased graph from $\mathcal{K}$.

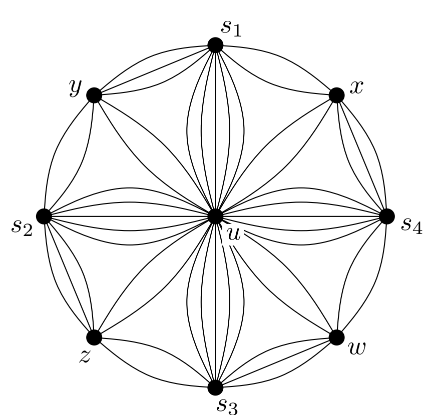


Figure 2.3: The underlying graph of the biased graph constructed in the proof of Theorem 2.10. Vertex u is the vertex resulting from the identifications of the sides of the pentagons; vertices x, y, z, and w are the vertices resulting from the subdivision of edges a, b, c, and d, respectively; vertices s_1, s_2, s_3, and s_4 are those at the barycentres of the four pentagons.

2.3 Constructing minor-minimal non-group-labellable biased graphs

A biased graph $(G, \mathcal{B})$ is minor-minimal subject to not being group-labellable if it is not group-labellable, but deleting or contracting any edge results in a group-labellable biased graph. We also say such biased graphs are *minor-minimal non-group-labellable*. In this section we use Theorem 2.1 to construct infinite families of biased graphs that are minor-minimal not group-labellable.

Let G be a simple graph embedded in the plane and equipped with a t-colouring of its vertices, satisfying:

N1. G is a subdivision of a 3-connected graph.

N2. Every colour appears exactly once on every face (so every face has size t).

N3. Every cycle of G of size $\leq t$ is the boundary of a face.

Now let $\widetilde{G}$ be the graph obtained from G by identifying each colour class to a single vertex, and let $\mathcal{B}$ be the set of all cycles of $\widetilde{G}$ corresponding to boundaries of finite faces of G.

Claim. $(\widetilde{G}, \mathcal{B})$ is a biased graph.

Proof of Claim. Since every cycle in $\mathcal{B}$ is a Hamilton cycle of $\widetilde{G}$, the only way for a theta subgraph of $\widetilde{G}$ to contain two members C, C' of $\mathcal{B}$ would be for this theta subgraph to have two edges in parallel, with C and C' sharing all but this pair of edges. But then this pair of edges would be a parallel pair in G, contradicting the assumption that G is simple. $\square$

Theorem 2.11. *The biased graph $(\widetilde{G}, \mathcal{B})$ constructed above is minor-minimal non-group-labellable. Moreover, every proper minor of $(\widetilde{G}, \mathcal{B})$ is Γ-labellable by any infinite group Γ.*

Proof. Let K be the 2-cell complex obtained from the embedding of G by removing the infinite face. Thus K is a disc and its boundary is a cycle C of G. Let $\widetilde{K}$ be the 2-cell complex obtained from K by identifying each colour class of vertices to a single point. The cycle C is a contractible curve in K, so it is also a contractible curve in $\widetilde{K}$. Since $C \notin \mathcal{B}$, by Theorem 2.1, $(\widetilde{G}, \mathcal{B})$ is not group-labellable.

Now let $e \in E(\widetilde{G})$, and let Γ be an infinite group (written multiplicatively). We construct a Γ-labelling of $(\widetilde{G}, \mathcal{B}) \setminus e$ and a Γ-labelling of $(\widetilde{G}, \mathcal{B})/e$. To prepare, we choose a sequence of group elements we will use for the labellings. Choose $g_0 \in \Gamma \setminus \{1\}$. For $1 \leq k \leq |E(G)| + |V(G)|$ choose $g_k \in \Gamma$ so that g_k cannot be expressed as a word of length $\leq 3t$ using $g_0, g_0^{-1}, \ldots, g_{k-1}, g_{k-1}^{-1}$.

Contraction Write $(\widetilde{G}', \mathcal{B}') = (\widetilde{G}, \mathcal{B})/e$. Since every cycle in $\mathcal{B}$ is Hamiltonian in $\widetilde{G}$, every such cycle not containing e will form handcuffs upon contracting e. So the only cycles in $\mathcal{B}'$ correspond to finite faces of the planar graph G which contain e; thus $|\mathcal{B}'| \leq 2$. To Γ-label $\widetilde{G}'$, we label $E(G) \setminus e$; $\widetilde{G}'$ then inherits its labels from G/e. Let H be the subgraph of G consisting of all its vertices and edges that are on a finite face containing e. Since G is a subdivision of a 3-connected graph, H must either be a cycle or a theta subgraph (depending on whether e lies on the infinite face or not). Let $V(H/e) = \{v_0, \ldots, v_n\}$ and let $E(G) \setminus E(H) = \{e_{n+1}, \ldots, e_m\}$. Let ϕ be the Γ-labelling obtained by arbitrarily orienting the edges of G, and assigning edge labels as follows: For every edge $f \in E(H/e)$, if $f = v_i v_j$, oriented from v_i to v_j, put $\phi(f) = g_i^{-1} g_j$. For every edge $e_k \in E(G) \setminus E(H)$, put $\phi(e_k) = g_k$.

Claim. ϕ realises $\mathcal{B}'$.

Proof of Claim. Let $\widetilde{D}$ be an arbitrary cycle in $\widetilde{G}'$. To show that $\mathcal{B}_\phi = \mathcal{B}'$, we show that either $\widetilde{D}$ is in both $\mathcal{B}'$ and $\mathcal{B}_\phi$, or in neither. Let D be the subgraph of G induced by $E(\widetilde{D})$. Then D is either a cycle or a union of disjoint paths. First suppose that $\widetilde{D}$ contains an edge $e_k \in E(G) \setminus E(H)$, and choose such an edge for which k is maximum. Since $e_k \notin H$, we have $\widetilde{D} \notin \mathcal{B}'$. If W is a simple closed walk in $\widetilde{G}'$ around $\widetilde{D}$ beginning with e_k in the forward direction, then $\phi(W)$ has the form $g_k S$, where S a word of length $< 2(t-1) < 3t$ consisting of group elements in $\{g_0, g_0^{-1}, \ldots, g_{k-1}, g_{k-1}^{-1}\}$. Thus $\phi(W) \neq 1$ and we have $\widetilde{D} \notin \mathcal{B}_\phi$ as desired. So now suppose $E(\widetilde{D}) \subseteq E(H)$. If D is a cycle in H/e, then $\widetilde{D} \in \mathcal{B}'$ and $\widetilde{D} \in \mathcal{B}_\phi$ by definition. If D is not a cycle in H/e, then $\widetilde{D} \notin \mathcal{B}'$ and we must show that $\widetilde{D} \notin \mathcal{B}_\phi$. Let $D_1, \ldots, D_r$ be the components of D, let W be a simple closed walk around $\widetilde{D}$ and assume that W encounters each D_i consecutively. If the subwalk W' of W traversing D_h begins at v_i and ends at v_j, then we have $\phi(W') = g_i^{-1} g_j$. Therefore, choosing k to be the largest value so that v_k is an endpoint of one of the paths $D_1, \ldots, D_r$, we have that $\phi(W)$ may be expressed as a word of length $\leq 2r < 2(t-1) < 3t$ using exactly one copy of g_k or g_k^{-1} with all other terms equal to one of $g_0, g_0^{-1}, \ldots, g_{k-1}, g_{k-1}^{-1}$. Hence $\widetilde{D} \notin \mathcal{B}_\phi$, as desired. $\square$

Deletion Now let $(\widetilde{G}', \mathcal{B}') = (\widetilde{G}, \mathcal{B}) \setminus e$. We consider two cases: either e is incident with the infinite face of G, or not. Suppose first that e is incident with the infinite face of G. Let $V(G) = \{v_1, \ldots, v_n\}$. Orient the edges in $E \setminus e$ arbitrarily, and for every $f \in E \setminus e$ oriented from v_i to v_j define $\phi(f) = g_i^{-1} g_j$.

Claim. ϕ realises $\mathcal{B}'$.

Proof of Claim. As above, let $\widetilde{D}$ be an arbitrary cycle in $\widetilde{G}'$, and let D be the corresponding subgraph of $G \setminus e$. As above, we show that either $\widetilde{D}$ is in both $\mathcal{B}'$ and $\mathcal{B}_\phi$, or in neither. As above, subgraph D must either be a cycle or a disjoint union of paths. If D is a cycle, then by property N3 of G, D must be a face boundary, so $\widetilde{D} \in \mathcal{B}'$ by definition and $\widetilde{D} \in \mathcal{B}_\phi$ by construction. If D is a disjoint union of paths, say given by $D_1, \ldots, D_r$, then $\widetilde{D} \notin \mathcal{B}'$. Choose a closed walk W traversing $\widetilde{D}$ such that W encounters each D_h ($h = 1, \dots, r$) consecutively. If the subwalk W' of W traversing D_h starts at v_i and ends at v_j, then $\phi(W') = g_i g_j^{-1}$. So as before, if k is the largest integer so that v_k is an endpoint of one of the paths $D_1, \dots, D_r$, we find that $\phi(W)$ may be written as a word of length $\leq 2r \leq 2t < 3t$ using only one copy of either g_k or g_k^{-1} with all other terms one of $g_0, g_0^{-1}, \dots, g_{k-1}, g_{k-1}^{-1}$. Hence $\widetilde{D} \notin \mathcal{B}_\phi$, as desired. □

Suppose finally that e is not incident with the infinite face of G. Let R be the new face in $G \setminus e$ formed by the deletion of e. Choose a path P in the plane dual graph of $G \setminus e$ from the infinite face to R, and orient the edges in $E \setminus e$ so that the edges dual to those in P cross the path P consistently (for instance, if P is given a direction, then $E \setminus e$ may be oriented so that each edge dual to one in P crosses P from the left to the right). Now let $V(G) = \{v_1, \dots, v_n\}$ and define a Γ-labelling ϕ as follows. If f is an edge from v_i to v_j and f is not dual to an edge in P, let $\phi(e) = g_i^{-1} g_j$; if e is dual to an edge in P, let $\phi(e) = g_i^{-1} g_0 g_j$.

Claim. ϕ realises $\mathcal{B}'$.

Proof of Claim. Observe that for any closed walk W in $G \setminus e$ we have $\phi(W) = g_0^s$, where s is the number of times the curve W winds around the face R. As above, let $\widetilde{D}$ be a cycle of $\widetilde{G}'$, and let D be the corresponding subgraph of $G \setminus e$. If D is a cycle, then since its length is at most t, it bounds a face in $G \setminus e$ other than R. If this is a finite face, then $\widetilde{D} \in \mathcal{B}'$ and by definition $\widetilde{D} \in \mathcal{B}_\phi$. If this is the infinite face, then $\widetilde{D} \notin \mathcal{B}'$ and since this face winds around R exactly once we have $\phi(W) = g_0$ or $\phi(W) = g_0^{-1}$, so $\widetilde{D} \notin \mathcal{B}_\phi$.

If D is not a cycle, then D is a union of disjoint paths $D_1, \dots, D_r$, and $\widetilde{D} \notin \mathcal{B}'$. We need to show that $\widetilde{D} \notin \mathcal{B}_\phi$. Choose a closed walk W traversing $\widetilde{D}$ encountering each D_h ($h = 1, \dots, r$) consecutively. Let $W = e_1 e_2 \cdots e_s$. Then $s \leq t$, and $\phi(W) = \phi(e_1)\phi(e_2) \cdots \phi(e_s)$ is a word of length $\leq 3s$ — each word $\phi(e_i)$ is a word of the form $g_i^{-1} g_j$, $g_i^{-1} g_0 g_j$, or $g_i^{-1} g_0^{-1} g_j$, and so has length at most 3. Letting k be the largest value so that v_k is an endpoint of one of the paths $D_1, \dots, D_r$, we have that $\phi(W)$ may be written as a word of length $\leq 3t$ using just one copy of either g_k or g_k^{-1} with all other terms in $\{g_0, g_0^{-1}, \dots, g_{k-1}, g_{k-1}^{-1}\}$. As in previous cases, this implies that $\phi(W) \neq 1$, so $\widetilde{D} \notin \mathcal{B}_\phi$, as desired. □

This completes the proof. □

2.4 Excluded Minors — Biased Graphs

In this section we prove the following two theorems.

Theorem 2.12. *For every $t \geq 3$ and ℓ there exists a minor-minimal non-group-labellable biased graph $(G, \mathcal{B})$ on t vertices with ℓ edges linking every pair of vertices, such that every proper minor of $(G, \mathcal{B})$ is Γ-labellable by any infinite group Γ.*

Theorem 2.13. *For $t = 3$ and every $t > 4$ and every $k \geq 2$ there exists a minor-minimal non-group-labellable biased graph $(G, \mathcal{B})$ whose underlying simple graph is a t-cycle, with exactly $2k$ edges linking every pair of adjacent vertices, such that every proper minor of $(G, \mathcal{B})$ is Γ-labellable by any infinite group Γ.*

Theorem 2.12 gives us a large collection of minor-minimal non-group-labellable biased graphs, each of whose underlying simple graph is complete. Theorem 2.13 yields collections of minor-minimal non-group-labellable biased graphs, each of whose underlying simply graph is a cycle. The proofs of Theorems 2.12 and 2.13 are constructive. Each uses Theorem 2.11, along with particular families of coloured planar graphs.

Family 1: $\{F_{2k} : k \geq 1\}$**.** For every positive integer k define F_{2k} to be the 3-coloured planar graph obtained as follows. To a cycle of length $2k$ embedded in the plane with vertices alternately coloured 0 and 1, add two additional vertices of colour a, one in each face. Add edges so that each new vertex is adjacent to all vertices of the cycle (Figure 2.4).

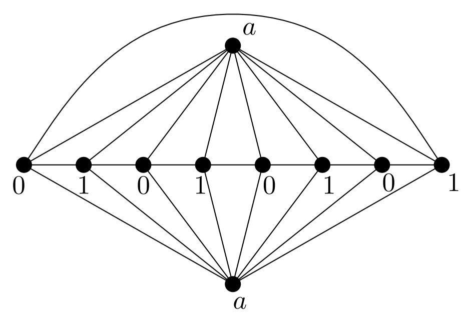


Figure 2.4: F_8.

Family 2: $\{H_{2k} : k \geq 1\}$**.** For every positive integer k we define H_{2k} to be the planar graph constructed as follows. Begin with $2k$ nested 8-cycles embedded in the plane, each joined to the previous and the next by a perfect matching. Colour this portion of the graph by colouring the innermost cycle b, 0, b, 1, b, 0, b, 1, and extend this colouring so that every 4-cycle contains exactly one vertex of each of the colours $\{a, b, 0, 1\}$ (this extension is unique). Finally, add a vertex v_1 in the inner 8-cycle of colour a joined to all vertices on this cycle not of colour b and similarly, add a vertex v_2 in the infinite face coloured b and adjacent to all vertices not of colour a on this face (Figure 2.5).

Theorem 2.12 follows immediately from Lemma 2.14 and Theorem 2.11.

Lemma 2.14. *For every $t \geq 3$ and ℓ there exists a t-coloured planar graph satisfying (N1), (N2), and (N3), in which every pair of distinct colours appear as endpoints of at least ℓ edges.*

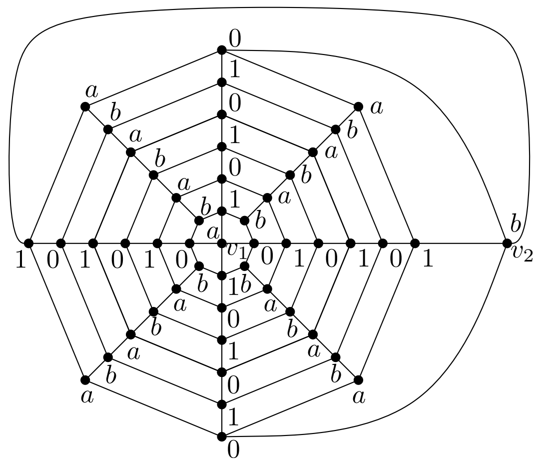


Figure 2.5: H_6.

Proof. We consider two cases depending on the parity of t.

Case 1: t **odd.** For $t = 3$ the coloured graphs F_{2k} with $k \geq \min\{\ell/2, 2\}$ satisfy (N1), (N2), and (N3), and have every pair of distinct colours appearing as endpoints of at least ℓ edges. For $t > 3$, we construct a graph as required by modifying F_{2k}, taking k as large as necessary to achieve what is required in each step. Choose s so that $t = 2s+1$, and colour from the set $\{a\} \cup \{1, 2, \ldots, 2s\}$. First, choose a sequence $(x_1, x_2, \ldots, x_{2k})$ of elements from $\{1, 2, \ldots, 2s\}$ having the following properties:

(i) every x_i has the same parity as i,

(ii) every pair of colours in $\{1, 2, \ldots, 2s\}$ of differing parities appear consecutively in the sequence at least ℓ times, and

(iii) every element in $\{1, 2, \ldots, 2s\}$ appears at least $\ell(s-1)$ times in the sequence.

Now modify the colouring of F_{2k} by replacing the sequence of colours $(0, 1, 0, 1, \ldots, 0, 1)$ with our chosen sequence of colours $(x_1, x_2, \ldots, x_{2k})$. Subdivide $s-1$ times each edge having one end of colour a and the other an odd (respectively, even) colour i; colour the new vertices with distinct odd (resp. even) colours in $\{1, \ldots, 2s\} \backslash \{i\}$, subject to the following restrictions. Let v_1 and v_2 be the two vertices of colour a. For every edge $v_1 w$ in F_{2k}, where w is coloured x_i, assign colour $x_i + 2$ (modulo $2s$) to the neighbour of v_1 in the subdivided edge $v_1 w$. Thus, we have that v_1 has at least ℓ neighbours of each colour $\{1, \ldots, 2s\}$. We ensure that every pair of distinct colours of the same parity appear on opposite ends of at least ℓ edges by placing the following restriction on the choice of colours of the vertices sharing a face with v_2. For each $j \in \{1, \ldots, 2s\}$, let $w_1, \ldots, w_{\ell(s-1)}$ be a set of $\ell(s-1)$ vertices coloured j in F_{2k}. Let u_i be the degree-2 neighbour of w_i in the subdivided $w_i v_2$

edge. For every $n \in \{1, \ldots, s-1\}$ assign colour $j + 2n$ to vertices $u_{(n-1)\ell+1}, \ldots, u_{n\ell}$. The resulting t-coloured planar graph has the desired properties.

Case 2: t even. For $t = 4$ the coloured graphs H_{2k} with $k \geq \min\{\ell/4, 2\}$ satisfy (N1), (N2), and (N3), and have every pair of distinct colours appearing as endpoints of at least ℓ edges. For $t > 4$, we construct a graph as required by modifying H_{2k}, taking k as large as necessary to achieve what is required in each step. Choose s so that $t = 2s+2$, and colour from the set $\{a, b\} \cup \{1, 2, \ldots, 2s\}$. First, choose a sequence $(x_1, x_2, \ldots, x_{2k})$ of elements from $\{1, 2, \ldots, 2s\}$ with the following properties:

(i) every x_i has the same parity as i,

(ii) every pair of colours in $\{1, 2, \ldots, 2s\}$ of differing parities appear consecutively in the sequence at least ℓ times, and

(iii) every element in $\{1, 2, \ldots, 2s\}$ appears at least ℓs times in the sequence.

Consider the coloured graph H_{2k}. Let P_1, P_3 be the paths of length $2k-1$ that are coloured alternately 0 and 1 beginning with a vertex incident to v_1 coloured 1, and let P_2, P_4 be the paths of length $2k-1$ that are coloured alternatively 0 and 1 beginning with a vertex coloured 0 incident to v_1. Modify the colouring of H_{2k} by replacing the colours along each of P_1 and P_3 with the sequence of colours $(x_1, \ldots, x_{2k})$, starting at the vertex coloured 1, and replacing the colours along each of P_2 and P_4 with the sequence of colours $(x_2, \ldots, x_{2k}, x_1)$, starting at the vertex coloured 0. Note that in this manner we have replaced each vertex previously coloured 0 with an even colour, and each vertex previously coloured 1 with an odd colour. Now modify the graph as follows. Aside from the four edges incident with the central vertex v_1 (coloured a) and the four edges incident with outer vertex v_2 (coloured b), for every other edge

- ai or bi with i even: subdivide the edge $s-1$ times and give each new vertex a distinct even colour from $\{1, \ldots, 2s\} \setminus \{i\}$;
- ai or bi with i odd: subdivide the edge $s-1$ times and give each new vertex a distinct odd colour from $\{1, \ldots, 2s\} \setminus \{i\}$.

These subdivisions ensure that every colour appears exactly once on every face.

Similarly to the previous case, we choose this assignment of colours subject to some restrictions to ensure that for every pair of colour classes there are at least ℓ edges joining vertices of different colours. We now describe these restrictions. To help with bookkeeping, we partition the set of all pairs of colour classes into eight types: even-even, odd-odd, even-odd, a-even, a-odd, b-even, b-odd, and a-b (where each pair of colour classes belongs to the obvious type described by its name). Property (ii) of our chosen sequence $(x_1, x_2, \ldots, x_{2k})$

ensures that we have at least ℓ edges between all even-odd pairs of colour classes. Our coloured graph H_{2k} has $4(2k-1)$ edges with one endpoint coloured a and the other endpoint coloured b. These edges remain in our modified graph; since k is taken large enough to accommodate the sequence $(x_1, x_2, \dots, x_{2k})$ required by property (ii), we certainly have at least ℓ edges with one endpoint coloured a and the other coloured b. We ensure that this also holds for all remaining pairs of colour classes by colouring the new vertices on the subdivided edges ai and bi as follows. The $2k$ subdivided edges ai with i in P_1 have i even: colour the new vertices on these subdivided edges so that there are at least ℓ edges with one endpoint of colour a and the other of colour i for each even $i \in \{1, \dots, 2s\}$. The $2k$ subdivided edges bi with i in P_1 have i odd: colour the new vertices on these subdivided edges so that there are at least ℓ edges with one endpoint of colour b and the other of colour i for each odd $i \in \{1, \dots, 2s\}$. In this way we ensure that there are at least ℓ edges between all a-even and at least ℓ edges between all b-odd pairs of colour classes. The subdivided edges ai with i in P_2 have i odd; subdivided edges bi with i in P_2 have i even. Colouring the new vertices on these subdivided edges so that there are at least ℓ edges with one endpoint of colour a and the other of colour i for each odd $i \in \{1, \dots, 2s\}$, and at least ℓ edges with one endpoint of colour b and other other of colour i for each even $i \in \{1, \dots, 2s\}$ ensures that there are at least ℓ edges between all a-odd and all b-even pairs of colour classes. Remaining are pairs of colour classes of types even-even and odd-odd. There are $4k$ subdivided edges of the forms ai, bi with i in P_3 or P_4: colouring these new vertices so that every pair of integers in $\{1, \dots, 2s\}$ of the same parity appear as endpoints of at least ℓ edges, we ensure that there are at least ℓ edges between all even-even and all odd-odd pairs of colour classes. Keeping in mind that we may take k as large as necessary, this colouring is clearly possible. The resulting t-coloured planar graph has the desired properties. □

As we proved Theorem 2.12, we likewise prove Theorem 2.13 by constructing the biased graphs whose existence the Theorem asserts. Theorem 2.13 follows immediately from Lemma 2.15 and Theorem 2.11.

Lemma 2.15. *For $t = 3$ and every $t > 4$, and every $k \geq 2$, there exists a t-coloured planar graph satisfying (N1), (N2), and (N3), in which*

1. *on each face the cyclic ordering of colours is given by either $(0, 1, \dots, t-1)$ or its reverse, and*
2. *there are exactly $4k$ faces.*

Observe that if G is a graph with the properties described in Lemma 2.15, then property (1) guarantees that the graph $\widetilde{G}$ constructed by Theorem 2.11 from G has underlying simple graph C_t, and that property (2) guarantees exactly $2k$ edges between every pair of adjacent vertices of $\widetilde{G}$.

Proof of Lemma 2.15. For $t = 3$, replacing colour a with colour 2 in F_{2k} yields a 3-coloured planar graph as required. If $t \geq 5$: If $t = 5$ and $k = 2$, the graph in Figure 2.6(a) has the desired properties. If $t = 8$ and $k = 2$, the graph in Figure 2.6(b) has the desired properties.

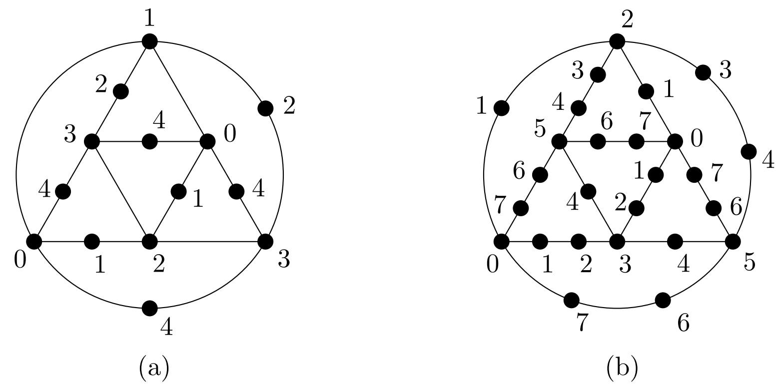


Figure 2.6: t-coloured planar graphs with exactly 8 faces, in the cases (a) $t = 5$, and (b) $t = 8$.

For the remaining cases, we have $t \geq 5$, and $(t, k) \notin \{(5, 2), (8, 2)\}$. Choose s, p and q with $p, q \in \{0, 1\}$ so that $t = 3s + p + q$. Modify the colouring of F_{2k} by recolouring every vertex of colour a with colour $s + p$ and every vertex of colour 1 with colour $2s + p + q$. Subdivide every edge with endpoints of colours 0 and $s + p$ exactly $s + p - 1$ times, every edge with ends of colours $s + p$ and $2s + p + q$ exactly $s + q - 1$ times, and every edge with ends of colours 0 and $2s + p + q$ exactly $s - 1$ times. Now colour the vertices of degree two so that around every face the cyclic order of colours clockwise or counterclockwise is $0, 1, \ldots, 3s + p + q - 1$. In this way, each triangle is subdivided as in Figure 2.7.

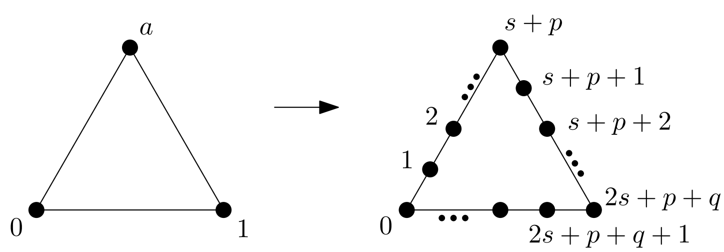


Figure 2.7: Modifying F_{2k} when $t \geq 5$, and $(t, k) \notin \{(5, 2), (8, 2)\}$.

The graph so constructed is a subdivision of a 3-connected planar graph with exactly $4k$ faces, with each face coloured as shown in Figure 2.7; *i.e.*, the graph satisfies (N1), (N2), and both properties required in the statement of the lemma. To complete the proof, we just need to verify that the graph satisfies (N3), namely, that every cycle of length at most t is a facial boundary.

Observe that every cycle that is not a facial boundary contains at least four vertices of degree ≥ 3, so will have length at least $4\lfloor \frac{t}{3} \rfloor$. This value is greater than t for all $t \geq 6$, except for $t = 8$. In the cases $t = 5$ and $t = 8$, by assumption $k \geq 3$. It is straightforward to check that when $k \geq 3$, any cycle that is not a facial boundary has length at least 6 if $t = 5$, and length at least 9 if $t = 8$. Hence our graph also satisfies (N3). $\square$

2.5 Excluded Minors — Matroids

In this section we translate results of Sections 2.4 to the setting of matroids. The two theorems proved in Section 2.4 show that there are infinitely many infinite families of biased graphs that are minor-minimal subject to not being labellable by any group. For each such biased graph $(G, \mathcal{B})$ there is a frame matroid $F(G, \mathcal{B})$ and a lift matroid $L(G, \mathcal{B})$. However, it is not immediate that any such matroid must be an excluded minor for the class $\mathcal{F}_\Gamma$, or $\mathcal{L}_\Gamma$, respectively. This is because it may be the case that there is a biased graph $(G', \mathcal{B}')$ that is Γ-labellable, with $F(G', \mathcal{B}') \cong F(G, \mathcal{B})$, and similarly for $L(G, \mathcal{B})$. However, if we can show that $(G, \mathcal{B})$ uniquely represents the frame matroid $F(G, \mathcal{B})$, then indeed, $F(G, \mathcal{B})$ would be an excluded minor for $\mathcal{F}_\Gamma$ (and similarly for $L(G, \mathcal{B})$ and $\mathcal{L}_\Gamma$).

2.5.1 Excluded minors — frame matroids

Let M be a frame matroid represented by the biased graph $\Omega = (G, \mathcal{B})$. For determining other possible biased graphs representing M not isomorphic to Ω, the following observation is key.

Proposition 2.16 (Slilaty, [31]). *If Ω is a connected biased graph with no balanced loops, then the complementary cocircuit of a connected non-binary hyperplane of $F(\Omega)$ consists precisely of the set of edges incident to a vertex.*

Proof. Call a set of edges whose removal results in a balanced biased graph a *balancing* set. Since a cocircuit of $F(\Omega)$ is a minimal set of edges whose removal increases the number of balanced components by one, a cocircuit D can be written as a disjoint union $D = S \cup B$ where $S = \emptyset$ or S is a separating edge set of Ω and $B = \emptyset$ or B is a minimal balancing set of an unbalanced component of $\Omega \setminus S$. If a biased graph has two components with nonempty edge sets, then its matroid cannot be connected (so a connected hyperplane in $F(\Omega)$ has at most one component in Ω with edges). Hence the complementary cocircuit of a connected hyperplane of Ω must be either the set of edges incident to a vertex or a minimal balancing set of Ω. The frame matroid of a balanced biased graph is graphic, and so binary. Hence, if X is a connected hyperplane and $E(\Omega) \setminus X$ is a minimal balancing set, then X is binary. Hence if X is a connected and nonbinary hyperplane of $F(\Omega)$, then $E(\Omega) \setminus X$ must be the set of edges incident to a vertex of Ω. $\square$

Proposition 2.16 motivates the following definition. For a vertex x, let $D_G(x) = \{e \in E(G) : e \text{ is incident to } x\}$ denote the set of edges incident to x. Call a vertex $x \in V(G)$

committed if the complement $E(G) \setminus D_G(x)$ of its set of incident edges is a connected non-binary hyperplane of $F(G, \mathcal{B})$. Now suppose Ω' is a biased graph with $F(\Omega') \cong F(\Omega)$. By Proposition 2.16, for every committed vertex $x \in V(\Omega)$, there is a vertex $x' \in V(\Omega')$ with precisely the same set of incident edges (more pedantically: the set of edges incident to x and the set of edges incident to x' both represent precisely the same set of elements of M). We will explore this further in Chapter 5. For now, the following observation is sufficient.

Observation 2.17. If every vertex of $(G, \mathcal{B})$ is committed, then $(G, \mathcal{B})$ uniquely represents $F(G, \mathcal{B})$.

Proof. Since all vertex-edge incidences are determined by a set of $|V(G)|$ connected non-binary hyperplanes of $F(G, \mathcal{B})$, this collection of hyperplanes uniquely determines G. If $\mathcal{B}'$ is any collection of cycles of G, then $F(G, \mathcal{B}') \cong F(G, \mathcal{B})$ if and only if $\mathcal{B}' = \mathcal{B}$. □

Lemma 2.18. *Let Γ be an infinite group. For each $t \geq 3$ there exists an infinite set of excluded minors for $\mathcal{G}_\Gamma$, such that each member of the set has underlying simple graph K_t, at least 4 edges between every pair of vertices, and no balanced cycle of length $< t$.*

Proof. By Theorem 2.12, there is an infinite set of biased graphs $\{\Omega_1, \Omega_2, \ldots\}$, in which each Ω_i is on t vertices and has at least 4 edges linking each pair of vertices, such that each biased graph in the set is an excluded minor for $\mathcal{G}_\Gamma$. By the construction in the proof of Theorem 2.12, such a set exists in which every member has no balanced cycle of length $< t$. □

Theorem 2.19. *Let Γ be an infinite group, and fix $t \geq 3$. The class $\mathcal{F}_\Gamma$ has infinitely many excluded minors of rank t.*

Proof. Let $\mathcal{K}_t = \{\Omega_1, \Omega_2, \ldots\}$ be an infinite set of excluded minors for $\mathcal{G}_\Gamma$ as given by Lemma 2.18. Let $\Omega \in \mathcal{K}_t$. Obviously Ω is not balanced, so $F(\Omega)$ has rank t. Let $x \in V(\Omega)$. The biased graph $\Omega - x$ has underlying simple graph K_{t-1}, at least 4 edges linking every pair of vertices, and every cycle of length two unbalanced. Hence $F(\Omega - x)$ is connected and contains $U_{2,4}$ as a minor; *i.e.*, $E(\Omega - x)$ is a connected non-binary hyperplane of $F(\Omega)$. This implies x is committed. By Observation 2.17 therefore, Ω uniquely represents $F(\Omega)$. Since $\Omega \notin \mathcal{G}_\Gamma$, we conclude $F(\Omega) \notin \mathcal{F}_\Gamma$. However, for any element $e \in E(\Omega)$, both $\Omega \setminus e$ and Ω / e are Γ-labellable. Hence each of $F(\Omega) \setminus e = F(\Omega \setminus e)$ and $F(\Omega)/e = F(\Omega/e)$ belong to $\mathcal{F}_\Gamma$. □

If a matroid has branch-width $\leq k$, then it is in some sense "thin", decomposing into small pieces along a set of non-crossing separations of order $\leq k$. A collection of biased graphs given by Lemma 2.18 contains members of branch-width as large as a graph on t may have. Correspondingly, a collection of frame matroids give by Theorem 2.19 contains members whose branch-width is a large as possible in a rank t non-uniform matroid. However, there are also infinite antichains of frame matroids all of fixed rank and all of branch-width 3.

Lemma 2.20. *Let Γ be an infinite group. For each $t \geq 3$, $t \neq 4$, there exists an infinite set of excluded minors for $\mathcal{G}_\Gamma$, such that each member of the set has underlying simple graph C_t, at least 4 edges between every pair of adjacent vertices, and no balanced cycle of length $< t$.*

Proof. By Theorem 2.13, there is an infinite set of excluded minors for $\mathcal{G}_\Gamma$, $\{\Omega_1, \Omega_2, \ldots\}$, in which for each $i \geq 1$ biased graph Ω_i has underlying simple graph C_t and exactly $2(i+1)$ edges linking each pair of adjacent vertices. By the construction in the proof of Theorem 2.13, such a set exists in which every member has no balanced cycle of length $< t$. □

Let us call a biased graph whose underlying simple graph is a cycle of length t, having ≥ 4 edges between every pair of adjacent vertices, and in which every balanced cycle is Hamiltonian, a ***fat*** t***-cycle***. Note that if Ω is a fat t-cycle, then $F(\Omega)$ contains many rank t swirls as restrictions, each obtained by deleting all but two edges between every pair of adjacent vertices. We therefore call the frame matroid $F(\Omega)$ of a fat t-cycle Ω a ***superswirl***.

Lemma 2.21. *A superswirl $\mathcal{W} = F(\Omega)$, for some fat t-cycle Ω, is uniquely represented by Ω.*

Proof. Let $x \in V(\Omega)$. The underlying simple graph of $\Omega - x$ is a path of length $t-2$. Moreover, $\Omega - x$ has at least 4 edges between every pair of adjacent vertices, and every cycle of length two is unbalanced. Hence $F(\Omega - x)$ is connected and contains $U_{2,4}$ as a minor. Since adding any edge incident to x to $E(\Omega - x)$ increases its rank, $E(\Omega - x)$ is a flat of $\mathcal{W}$. Hence $E(\Omega - x)$ is a connected non-binary hyperplane of $\mathcal{W}$, so x is committed. By Observation 2.17 therefore, Ω uniquely represents $F(\Omega)$. □

It is well known that swirls have branch-width 3. So do superswirls:

Lemma 2.22. *Superswirls have branch-width 3.*

Proof. Let $\mathcal{W} = F(\Omega)$ be a superswirl, represented by fat t-cycle Ω, on ground set E. Call each set consisting of all edges linking a pair of adjacent vertices of Ω a *rod*. Let $R_1, \ldots, R_t$ be the rods of Ω, in cyclic order, so that for $1 \leq i \leq t$, $|V(R_i) \cap V(R_{i+1})| = 1$ (adding subscripts modulo t).

The following branch decomposition T of $\mathcal{W}$ has width 3. Let T be a cubic tree constructed from a path P of length t, with vertex set r, $u_1, \ldots, u_t$, beginning at r and ending at u_t. Let r be the root of T. Incident to each vertex u_i, $1 \leq i \leq t$, add a pendent edge e_i. Add an extra edge incident to the last vertex of the path u_t to create a cubic tree. Now subdivide $|R_i| - 1$ times each pendent edge e_i and add a new pendent edge incident to each new vertex to create $|R_i|$ leaves with vertex u_i as their common parent; mark the leaves so created with the elements in R_i. Now let e_i be the edge of the subdivided edge incident to u_i. Then for each $1 \leq i \leq t$, the set displayed by the subtree $T \setminus e_i$ not containing r is precisely the set of elements in R_i. For each $1 \leq i \leq t$, the width of edge e_i is 3, and the width of any other edge on a path from u_i to a leaf in R_i is at most 3. The width of an edge in P is also 3. Hence the maximum width of an edge in T is 3; *i.e.*, T has width 3. Inspection of the connectivity function

$$\begin{aligned}\lambda_{\mathcal{W}}(X) &= r(X) + r(E \setminus X) - r(M) + 1 \\ &= |V(X)| - b(X) + |V(E \setminus X)| - b(E \setminus X) - |V(G)| + 1\end{aligned}$$

shows that any other branch decomposition will have width at least 3. $\square$

Theorem 2.23. *Let Γ be an infinite group, and fix $t \geq 3$, $t \neq 4$. The class $\mathcal{F}_\Gamma$ has infinitely many excluded minors of rank t and branch-width 3.*

Proof. Let $\mathcal{S}_t = \{\Omega_1, \Omega_2, \ldots\}$ be an infinite set of fat t-cycles that are excluded minors for $\mathcal{G}_\Gamma$ — such a set exists by Lemma 2.20. Let $\Omega \in \mathcal{S}_t$. Since Ω is not balanced, $F(\Omega)$ has rank t. By Lemma 2.22, $F(\Omega)$ has branch-width 3. By Lemma 2.21, Ω uniquely represents $F(\Omega)$. Since $\Omega \notin \mathcal{G}_\Gamma$, we conclude $F(\Omega) \notin \mathcal{F}_\Gamma$. However, for any element $e \in E(\Omega)$, both $\Omega \setminus e$ and Ω / e are Γ-labellable. Hence each of $F(\Omega) \setminus e = F(\Omega \setminus e)$ and $F(\Omega)/e = F(\Omega/e)$ belong to $\mathcal{F}_\Gamma$. $\square$

In other words, for each $t \geq 3$, $t \neq 4$, there are infinite antichains of superswirls of rank t, each of which is an excluded minor for $\mathcal{F}_\Gamma$.

2.5.2 Excluded minors — lift matroids

Suppose $(G, \mathcal{B})$ is a connected biased graph and e is an unbalanced loop incident to a vertex $v \in V(G)$. Let $(G', \mathcal{B}')$ be the biased graph obtained from $(G, \mathcal{B})$ by deleting e, adding a new isolated vertex v_e to G, then adding e as an unbalanced loop incident to v_e. Let $(G'', \mathcal{B}'')$ be a biased graph obtained from $(G', \mathcal{B}')$ by identifying vertex v_e with any vertex of G'. We refer to these operations as *placement* of an unbalanced loop. Clearly, $L(G, \mathcal{B}) \cong L(G', \mathcal{B}') \cong L(G'', \mathcal{B}'')$. Without loss of generality therefore, in this section we assume that whenever e is an unbalanced loop in a biased graph $(G, \mathcal{B})$, e is incident to a vertex v_e that is otherwise not adjacent to any other vertex of G.

As with the previous section's frame matroids, we are able to show that some of the biased graphs constructed in Section 2.4 uniquely represent their lift matroids. These biased graphs have many edges linking every pair of adjacent vertices, and all of the 2-cycles formed by these parallel edges are unbalanced. Such a 2-vertex biased graph is a lift representation of the m-point line $U_{2,m}$, where m is the number of links between the two vertices. For $m \geq 4$, up to placement of unbalanced loops, there are two biased graphs representing $U_{2,m}$. We call them K_2^m and $K_2^{(m-1)+}$. In both, all cycles are unbalanced. Let biased graph

K_2^m be a two-vertex graph consisting of m edges in parallel. Let $K_2^{(m-1)+}$ consist of $m-1$ links between two vertices, together with an unbalanced loop.

Let us call a biased graph $(G, \mathcal{B})$ *lift-unique* if the only biased graphs with lift matroid isomorphic to $L(G, \mathcal{B})$ are obtained from $(G, \mathcal{B})$ by renaming the vertices or placement of unbalanced loops.

Lemma 2.24. *Let $(G, \mathcal{B})$ be a loopless biased graph on $n \geq 3$ vertices for which every pair of vertices are joined by at least four edges, and all cycles of length two are unbalanced. Then $(G, \mathcal{B})$ is lift-unique.*

Proof. Let $E = E(G)$ and define a relation $\sim$ on E by the rule that $e \sim f$ if there exists a restriction of $L(G, \mathcal{B})$ isomorphic to $U_{2,4}$ which contains both e and f. It follows easily from the description of $(G, \mathcal{B})$ that $\sim$ is an equivalence relation and its equivalence classes are precisely the parallel classes of G, which we denote by $E_1, E_2, \ldots, E_{\binom{n}{2}}$.

Suppose that $(G', \mathcal{B}')$ is another biased graph on E with the same lift matroid; *i.e.*, $L(G', \mathcal{B}') \cong L(G, \mathcal{B})$. If $|E_i| = m$ then the restriction of our matroid to E_i is isomorphic to $U_{2,m}$ and thus in G', the edges in E_i induce one of K_2^m or $K_2^{(m-1)+}$. It follows from the fact that $\sim$ is an equivalence relation that for $i \neq j$ the edge sets E_i and E_j induce graphs on distinct two-vertex sets. Suppose, for a contradiction, that in G' there is a loop e. Then e is contained in every equivalence class E_i, $1 \leq i \leq \binom{n}{2}$, a contradiction. Thus G' is loopless, and $E_1, \ldots, E_{\binom{n}{2}}$ are also its parallel classes.

Let e, f, g be three edges which form a triangle in G and let e' be parallel with e. Then one of $\{e, f, g\}$, $\{e', f, g\}$, $\{e, e', f, g\}$ is a circuit in $L(G, \mathcal{B})$. It follows from this, and the fact that G' is loopless with the same parallel classes as G, that the edges e, f, g must also form a triangle in G'. In particular, this implies that two edges e, f are adjacent in G if and only if they are adjacent in G'. Therefore, the line graphs of G and G' are isomorphic. For $n \geq 5$ the maximum cliques in the line graph of K_n correspond precisely to sets of edges incident with a common vertex, and it follows that for $n \geq 5$ the biased graph $(G', \mathcal{B}')$ may be obtained from $(G, \mathcal{B})$ by renaming the vertices. For $n = 3$ there is also nothing left to prove, so we are left with the case $n = 4$. The maximum cliques of the line graph of K_4 are given by either triangles or sets of edges incident with a common vertex. Since three edges form a triangle in G if and only if they form a triangle in G' we conclude that again in this case, the biased graph $(G', \mathcal{B}')$ may be obtained from $(G, \mathcal{B})$ by renaming the vertices. We conclude that $(G, \mathcal{B})$ is lift-unique. □

Theorem 2.25. *For every infinite group Γ and every $t \geq 3$ the class $\mathcal{L}_\Gamma$ has infinitely many excluded minors of rank t.*

Proof. Fix $t \geq 3$ and let Γ be an infinite group. By Theorem 2.12 we may choose an infinite set of biased graphs on t vertices $\{(G_i, \mathcal{B}_i) \mid i \in \{1, 2, \ldots\}\}$ with $|E(G_{i+1})| > |E(G_i)|$ so that every pair of vertices is joined by at least 4 edges in every G_i, every $(G_i, \mathcal{B}_i)$ is not Γ-labellable, and every proper minor of each $(G_i, \mathcal{B}_i)$ is Γ-labellable. By the constructions used in the proof of Theorem 2.12 we may assume each $(G_i, \mathcal{B}_i)$ is loopless. Since $\mathrm{rank}(L(G_i, \mathcal{B}_i)) = |V(G_i)|$, each $L(G_i, \mathcal{B}_i)$ has rank t. By Lemma 2.24, each $(G_i, \mathcal{B}_i)$ is lift-unique. We conclude $L(G_i, \mathcal{B}_i) \notin \mathcal{L}_\Gamma$. Since for any $e \in E(G_i)$, we have $L(G_i, \mathcal{B}_i) \setminus e = L((G_i, \mathcal{B}_i) \setminus e)$ and $L(G_i, \mathcal{B}_i)/e = L((G_i, \mathcal{B}_i)/e)$, every proper minor of $L(G_i, \mathcal{B}_i)$ is in $\mathcal{L}_\Gamma$. □

Analogously with what we see in Theorem 2.19, the infinite antichains of lift matroids exhibited by Theorem 2.25 have branch-width as large as is permitted by their rank. As with frame matroids, analogously with Theorem 2.23, we may also fix a positive integer t and ask for an infinite antichain of lift matroids all of rank t and having branch-width 3. Analogous to Theorem 2.23, for lift matroids we have:

Theorem 2.26. *Let Γ be an infinite Abelian group, and fix $t \geq 3$, $t \neq 4$. The class $\mathcal{L}_\Gamma$ has infinitely many excluded minors of rank t and branch-width 3.*

Like the superswirls of Theorem 2.23, the excluded minors for $\mathcal{L}_\Gamma$ we exhibit to prove Theorem 2.26 come from fat t-cycles. Observe that if Ω is a fat t-cycle, then $L(\Omega)$ contains many rank t spikes as restrictions, each obtained by deleting all but two edges between every pair of adjacent vertices. We therefore call the lift matroid $L(\Omega)$ of a fat t-cycle Ω a *superspike*.

Let $\mathcal{P} = L(\Omega)$ be a superspike, represented by fat t-cycle Ω, on ground set E. Call each set consisting of all edges linking a pair of adjacent vertices of Ω the *legs* of Ω (and of $\mathcal{P}$).

It is well known that spikes have branch-width 3. So do superspikes. The connectivity function $\lambda_{\mathcal{P}}$ of a superspike $\mathcal{P}$ represented by fat t-cycle Ω is give by

$$\begin{aligned}\lambda_{\mathcal{P}}(X) &= r(X) + r(E \setminus X) - r(M) + 1 \\ &= |V(X)| - c(X) + u(X) + |V(E \setminus X)| - c(E \setminus X) + u(E \setminus X) - |V(\Omega)| + 1\end{aligned}$$

where for a subset A, $c(A)$ is the number of components of $\Omega[A]$ and $u(A) = 1$ if $\Omega[A]$ contains an unbalanced cycle and is otherwise 0.

Lemma 2.27. *Superspikes have branch-width 3.*

Proof. Let $\mathcal{P} = L(\Omega)$ be a superspike, represented by fat t-cycle Ω. Let T be the branch decomposition of Ω given in Lemma 2.22, with the legs of Ω in place of the rods of the superswirl. Then T has width 3. If X is any union of legs, then $|V(X)| + |V(E \setminus X) - |V(\Omega)| = 2c(X)$, $c(X) = c(E \setminus X)$, and $u(X) = u(E \setminus X) = 1$, so

$$\begin{aligned}\lambda_{\mathcal{P}}(X) &= r(X) + r(E \setminus X) - r(M) + 1 \\ &= |V(X)| - c(X) + u(X) + |V(E \setminus X)| - c(E \setminus X) + u(E \setminus X) - |V(\Omega)| + 1 \\ &= 2c(X) - 2c(X) + 3 = 3.\end{aligned}$$

Inspection of the connectivity function shows that any other branch decomposition will have width at least 3. $\square$

Let $L_1, \ldots, L_t$ be the legs of Ω, in cyclic order, so that for $1 \leq i \leq t$, $|V(R_i) \cap V(R_{i+1})| = 1$ (adding subscripts modulo t). Let Ω' be a fat t-cycle obtained from Ω by permuting the cyclic order of its legs $L_1, \ldots, L_t$. Since this does not change the edge set of any balanced cycle, pair of unbalanced cycles meeting in at most one vertex, or contrabalanced theta, $L(\Omega') = L(\Omega)$.

Lemma 2.28. *Suppose $\mathcal{P} = L(\Omega)$ is a superspike represented by fat t-cycle Ω. If Ω' is a biased graph with $L(\Omega') \cong \mathcal{P}$, then Ω' is obtained by permuting the cyclic order of the legs of Ω.*

Proof. The argument contained in the first two paragraphs of the proof of Lemma 2.24 shows that in any biased graph representing $L(\Omega)$, each leg L_i must be a parallel class of links between two vertices. Since $\mathcal{P}$ is 3-connected, Ω' is 2-connected. Together, these facts imply that every biased graph representation of $\mathcal{P}$ is a fat t-cycle, and moreover that any such fat t-cycle may be obtained from any other by a permutation of the cyclic order of the legs of $\mathcal{P}$. $\square$

We now show that if Γ is Abelian and $\mathcal{P}$ is a superspike in $\mathcal{L}_\Gamma$, then every biased graph representing $\mathcal{P}$ is in $\mathcal{G}_\Gamma$.

Lemma 2.29. *Let $\mathcal{P} = L(G, \mathcal{B})$ be a superspike represented by $(G, \mathcal{B})$, and suppose $(G, \mathcal{B})$ is Γ-labellable for some Abelian group Γ. If $(G', \mathcal{B}')$ is a biased graph with $L(G', \mathcal{B}') \cong \mathcal{P}$, then $(G', \mathcal{B}')$ is Γ-labellable.*

Proof. By Lemma 2.28, $(G', \mathcal{B}')$ may be obtained from $(G, \mathcal{B})$ by permuting the cyclic order of its legs, say by $\sigma : \{1, \ldots, t\} \to \{1, \ldots, t\}$. Let E be the common ground set of $\mathcal{P}$, G and G'. Let $\gamma : E \to \Gamma$ be a labelling of $(G, \mathcal{B})$ with $\mathcal{B}_\gamma = \mathcal{B}$. Without loss of generality, we may assume all edges of G are oriented consistently, with all edges in L_i sharing their head with the tail of all edges in L_{i+1} (adding indices modulo t). Similarly orient the edges of $(G', \mathcal{B}')$ consistently with all edges in each leg $L_{\sigma(i)}$ sharing their head with the tail of all edges in $L_{\sigma(i+1)}$.

The cycles of $(G, \mathcal{B})$ are in one-to-one correspondence with the cycles of $(G', \mathcal{B}')$, via the bijection $C \leftrightarrow C'$ if and only if $E(C) = E(C')$, and every cycle in each of $(G, \mathcal{B})$ and $(G', \mathcal{B}')$ is either of length 2 or length t. All cycles of length 2 are independent in $\mathcal{P}$, and so are unbalanced in both $(G, \mathcal{B})$ and $(G', \mathcal{B}')$. Every cycle of length t is either a basis or a circuit-hyperplane of $\mathcal{P}$ and so is accordingly either unbalanced or balanced in both $(G, \mathcal{B})$ and $(G', \mathcal{B}')$. Hence as collections of edge sets of cycles, $\mathcal{B} = \mathcal{B}'$. Since Γ is Abelian, $\mathcal{B}_\gamma = \mathcal{B} = \mathcal{B}'$. $\square$

We may now prove Theorem 2.26 by exhibiting an infinite set of rank t superspikes, each of which is an excluded minor for $\mathcal{L}_\Gamma$.

Proof of Theorem 2.26. Fix $t \geq 3$, $t \neq 4$. Let $\mathcal{S}_t = \{\Omega_1, \Omega_2, \ldots\}$ be an infinite set of fat t-cycles that are excluded minors for $\mathcal{G}_\Gamma$ — such a set exists by Lemma 2.20. Let $\Omega \in \mathcal{S}_t$. Since Ω is not balanced, $L(\Omega)$ has rank t. By Lemma 2.27, $L(\Omega)$ has branch-width 3. By Lemmas 2.28 and 2.29, $L(\Omega) \notin \mathcal{L}_\Gamma$. For every element $e \in E(\Omega)$, both $L(\Omega) \setminus e = L(\Omega \setminus e)$ and $L(\Omega)/e = L(\Omega/e)$ belong to $\mathcal{L}_\Gamma$. $\square$

As with superswirls in the class of frame matroids, in other words, for each $t \geq 3$, $t \neq 4$, there are infinite antichains of superspikes of rank t, each of which is an excluded minor for $\mathcal{L}_\Gamma$.

2.6 Infinite antichains in $\mathcal{G}_\Gamma$, $\mathcal{F}_\Gamma$, $\mathcal{L}_\Gamma$

Theorems 2.12 and 2.13 exhibit infinite antichains of biased graphs, each of whose members is on a fixed number of vertices t, by constructing infinite families of biased graphs on t vertices each of which is minor-minimal subject to being not group-labellable. Since every proper minor of these are labellable by any infinite group Γ, these in turn, could be used to construct infinite sets of excluded minors for classes $\mathcal{F}_\Gamma$ and $\mathcal{L}_\Gamma$ of frame and lift matroids, all of a fixed rank. Let Γ be an infinite group. In this section we show that not only do the classes $\mathcal{G}_\Gamma$, $\mathcal{F}_\Gamma$, and $\mathcal{L}_\Gamma$ have infinite sets of excluded minors, respectively on fixed numbers of vertices and of fixed ranks, but that also $\mathcal{G}_\Gamma$ contains infinite antichains of biased graph on a fixed number of vertices, and that each of $\mathcal{F}_\Gamma$ and $\mathcal{L}_\Gamma$ themselves contain infinite antichains of matroids of fixed rank.

Observation 2.30. For every infinite group Γ and every $n \geq 2$ the biased graph $(2C_n, \mathcal{B}_n)$ is Γ-labellable.

Proof. Orient the edges so that each of the two balanced cycles is a directed cycle, and label all edges in one of the balanced cycles with 1. Let $e_1, \ldots, e_n$ be the edges in the other balanced cycle, in cyclic order. Choose a sequence of group elements $g_1, \ldots, g_{n-1}$ so that no subsequence of these elements has product equal to 1 (this may be done greedily). For $1 \leq i \leq n-1$, label edge e_i with element g_i, and label e_n withl $g_{n-1}^{-1} \cdots g_1^{-1}$. This Γ-labelling realises $\mathcal{B}_n$. $\square$

Together, Observations 2.2 and 2.30 exhibit, for every infinite group Γ, an infinite antichain of biased graphs in $\mathcal{G}_\Gamma$. It is not difficult to show that this infinite antichain of Γ-labelled graphs yields an infinite antichain of swirls, $F(2C_n, \mathcal{B}_n)$ in $\mathcal{F}_\Gamma$, and an infinite antichain of spikes, $L(2C_n, \mathcal{B}_n)$ in $\mathcal{L}_\Gamma$.

We now prove Theorem 2.4, showing that there are also infinite antichains in $\mathcal{G}_\Gamma$ having all members on a fixed number of vertices. The proof uses an argument similar to that of the proof of Theorem 2.11.

Theorem 2.4. *Let Γ be an infinite group.*

(a) *For every $t \geq 3$ there exists an infinite antichain of Γ-labelled graphs on t vertices, each of whose underlying simple graph is K_t.*

(b) *For every $t \geq 3$, $t \neq 4$, there exists an infinite antichain of Γ-labelled graphs on t vertices, each of whose underlying simple graph is a cycle of length t.*

Proof. In case (a), let $\{G_1, G_2, \ldots\}$ be an infinite set of t-coloured planar graphs given by Lemma 2.14, such that no two graphs in the set have the same number of edges. In case (b), let $\{G_1, G_2, \ldots\}$ be an infinite set of t-coloured planar graphs as given by Lemma 2.15, such that graph G_k has exactly $4(k+1)$ faces.

For both case (a) and case (b), proceed as follows. For every k, let $\widetilde{G}_k$ be the graph obtained from G_k by identifying each colour class to a single vertex, and let $\mathcal{B}_k$ be the set of cycles which are faces of the planar embedding of G_k.

Claim. For each $k \geq 1$, $(\widetilde{G}_k, \mathcal{B}_k) \in \mathcal{G}_\Gamma$.

To see this, let $V(G_k) = \{v_1, \ldots, v_n\}$, and choose a sequence of group elements $g_1, \ldots, g_n$ such that each g_i cannot be represented as a product of distinct elements from the set $\{g_1, g_1^{-1}, \ldots, g_{i-1}, g_{i-1}^{-1}\}$ (in any order). Orient $E(G_k)$ arbitrarily, and let $E(\widetilde{G}_k)$ inherit this orientation. For every edge e, if e is oriented from v_i to v_j define $\phi(e) = g_i^{-1} g_j$.

We show that $\mathcal{B}_\phi = \mathcal{B}_k$. Let C be an arbitrary cycle in $\widetilde{G}_k$. Since C has length at most t, if C is also a cycle in G_k, then C bounds a face. Hence $C \in \mathcal{B}_k$ and, by construction $C \in \mathcal{B}_\phi$. Otherwise, the set of edges $E(C)$ forms a collection of paths in G_k, say $D_1, \ldots, D_r$, and $C \notin \mathcal{B}_k$. Choose a closed walk W around the cycle C in $\widetilde{G}_k$; assume that W encounters the paths $D_1, \ldots, D_r$ consecutively. By construction, $\phi(W)$ may be expressed as a product of distinct group elements from $S = \{g_i : v_i \text{ is an end of some } D_j\}$ and S^{-1}. Our choice of group elements labelling $E(G)$ ensures that this product is not the identity. Hence $C \notin \mathcal{B}_\phi$, as desired. Thus $\phi : E(\widetilde{G}) \to \Gamma$ labels $(\widetilde{G}, \mathcal{B}_k)$.

Claim. $\{(\widetilde{G}_k, \mathcal{B}_k) : k \in \mathbb{N}\}$ is an antichain.

Suppose that $(\widetilde{G}_i, \mathcal{B}_i)$ contains a biased graph isomorphic to $(\widetilde{G}_j, \mathcal{B}_j)$ as a minor and $i \neq j$. Since these graphs have the same number of vertices, it must be that $(\widetilde{G}_j, \mathcal{B}_j)$ is isomorphic to $(\widetilde{G}_i, \mathcal{B}_i) \setminus R$ for some nonempty set of edges R. Choose an edge $e \in E(\widetilde{G}_i) \setminus R$ that lies on a common face in the planar embedding of G_i with an edge in R. Edge e is in at most one balanced cycle in $(\widetilde{G}_i, \mathcal{B}_i) \setminus R$, but every edge in $(\widetilde{G}_j, \mathcal{B}_j)$ is contained in exactly two balanced cycles, a contradiction. $\square$

By Lemma 2.24, each of the biased graphs $(\widetilde{G}_k, \mathcal{B}_k)$ in the proof of Theorem 2.4(a) is lift-unique as long as it has at least four edges between each pair of vertices. Since four edges linking a pair of vertices with all cycles unbalanced is a frame representation of $U_{2,4}$, in this case every vertex in each of these biased graphs is committed. By Observation 2.17 then, each uniquely represents $F(\widetilde{G}_k, \mathcal{B}_k)$. Hence the following corollary is immediate.

Corollary 2.31. *For every infinite group* Γ *and every* $t \geq 3$*, there exist infinite antichains of rank* t *matroids in* $\mathcal{L}_\Gamma$ *and in* $\mathcal{F}_\Gamma$*.*

From Theorem 2.4(b), for any fixed $t \geq 5$ we obtain infinite antichains of rank t matroids of branch-width 3, in both $\mathcal{L}_\Gamma$ and $\mathcal{F}_\Gamma$:

Corollary 2.32. *Let* Γ *be an infinite group.*

(a) *For* $t = 3$ *and all* $t \geq 4$*, there is an infinite antichain of superswirls of rank* t *in* $\mathcal{F}_\Gamma$*.*

(b) *For each* $t \geq 5$*, there is an infinite antichain of superspikes of rank* t *in* $\mathcal{L}_\Gamma$*.*

Proof. For any t satisfying the conditions of (a) (respectively, (b)), let $\mathcal{S}_t = \{\Omega_1, \Omega_2, \ldots\}$ be an infinite set of fat t-cycles as constructed in the proof of Theorem 2.4(b). Since none of the biased graphs in $\mathcal{S}_t$ are balanced, each frame matroid $F(\Omega_k)$ and each lift matroid $L(\Omega_k)$ with $\Omega_k \in \mathcal{S}_t$ has rank t. Each biased graph $\Omega_k \in \mathcal{S}_t$ is constructed by identifying colour classes of a t coloured planar graph G_k as given by Lemma 2.15. Observe that the circuit-hyperplanes of each of $F(\Omega_k)$ and $L(\Omega_k)$ are precisely the balanced cycles of Ω_k. Suppose for some $i < j$, $F(\Omega_j)$ (respectively $L(\Omega_j)$) contains $F(\Omega_i)$ (resp. $L(\Omega_i)$) as a minor. Since both $F(\Omega_i)$ and $F(\Omega_j)$ (resp. $L(\Omega_i)$ and $L(\Omega_j)$) have rank t, it must be that for some subset $R \subset E(\Omega_j)$, $F(\Omega_i) \cong F(\Omega_j) \setminus R$ (resp. $L(\Omega_i) \cong L(\Omega_j) \setminus R$). Choose an edge $f \in R$ and an edge $e \in E(\Omega_j) \setminus R$ such that e and f lie on a common face in the planar embedding of G_j. Then e is contained in at most one circuit-hyperplane of $F(\Omega_j) \setminus R$ (resp. $L(\Omega_j) \setminus R$), but every element of $F(\Omega_i)$ (resp. $L(\Omega_i)$) is contained in exactly two circuit-hyperplanes, a contradiction. $\square$

2.7 Finitely group-labelled graphs of bounded branch-width

It is a conjecture of Geelen and Gerards [8] that for a fixed finite abelian group Γ, the class of Γ-labelled graphs is well-quasi-ordered. Huynh states in his thesis [13] (page 21) that Geelen, Gerards, and Whittle have now proved this, and that the paper is in preparation. Here we observe that the argument in Section 4 of [6], adding orientations and group labels to edges, essentially proves that for a fixed finite group Γ, the class of Γ-labelled graphs of bounded branch-width is well-quasi-ordered. An additional definition and some additional reasoning extends the proof.

Theorem 2.5. *Let Γ be a finite group and n be an integer. Then every infinite set of Γ-labelled graphs of branch-width at most n has two members one of which is isomorphic to a minor of the other.*

The argument into which we insert the phrase "group-labelled" is a proof of Robertson and Seymour's well-quasi-ordering of graphs of bounded branch-width.

Theorem 2.33 (Robertson and Seymour)**.** *Let n be an integer. Then each infinite set of graphs with branch-width at most n has two members one of which is isomorphic to a minor of the other.*

As we have shown in Section 2.6, this does not hold for the class of group-labelled graphs. Here we present Geelen, Gerhards, and Whittle's proof of Theorem 2.33 given in [6], along with an additional argument showing that Theorem 2.33 holds for Γ-labelled graphs when Γ is fixed and finite.

2.7.1 Linked branch decompositions and a lemma about trees

We now describe two important tools used in the proof of Theorem 2.33 in [6].

Let f and g be two edges in a branch decomposition T of λ; let F be the set displayed by the component of $T \setminus f$ not containing g, and let G be the set displayed by the component of $T \setminus g$ not containing f. Let P be the shortest path in T containing f and g. Then the widths of the edges of P are upper bounds for $\lambda(F, G)$. Say f and g are *linked* if $\lambda(F, G)$ is equal to the minimum width of an edge on P. Say a branch decomposition is *linked* if all its edge pairs are linked.

Theorem 2.34 ([6])**.** *An integer-valued symmetric submodular function with branch-width n has a linked branch decomposition of width n.*

The other main tool in the proof of Theorem 2.33 in [6] is derived from Robertson and Seymour's "Lemma about trees", which extends Kruskal's result that forests are well-quasi-ordered by topological containment. A *rooted tree* is a finite directed tree in which all but one vertex has indegree 1. A *rooted forest* is a countable collection of vertex disjoint rooted trees. Its vertices of indegree 0 are its *roots* and those of outdegree 0 are *leaves*. Edges leaving a root are *root edges*; edges entering a leaf are *leaf edges*. An n*-edge marking* of a graph G is a map $E(G) \to \{0, \ldots, n\}$. If λ is an n-edge marking of a rooted forest F and $e, f \in E(F)$, say e is λ*-linked to* f if F contains a directed path P starting at e, ending at f, such that $\lambda(g) \geq \lambda(e) = \lambda(f)$ for each edge g of P.

A *binary forest* is a rooted orientation of a cubic forest (all degrees 1 or 3) with a distinction between left and right out-edges; *i.e.*, (F, l, r) is a *binary forest* if F is a rooted forest in which the roots have out-degree 1 and l and r are functions on the nonleaf edges of F such that the head of each nonleaf each e of F has exactly two out-edges, $l(e)$ and $r(e)$.

Lemma 2.35 (Lemma on cubic trees). *Let (F, l, r) be an infinite binary forest with n-edge marking λ. Let $\preccurlyeq$ be a quasi-order on $E(F)$ with no infinite strictly decreasing sequences, such that $e \preccurlyeq f$ whenever f is λ-linked to e. If the leaf edges of F are well-quasi-ordered by $\preccurlyeq$ but the root edges of F are not, then F contains an infinite sequence $(e_0, e_1, \ldots)$ of nonleaf edges such that*

1. $\{e_0, e_1, \ldots\}$ *is an antichain with respect to* $\preccurlyeq$,
2. $l(e_0) \preccurlyeq \cdots \preccurlyeq l(e_{i-1}) \preccurlyeq l(e_i) \preccurlyeq \cdots$, *and*
3. $r(e_0) \preccurlyeq \cdots \preccurlyeq r(e_{i-1}) \preccurlyeq r(e_i) \preccurlyeq \cdots$.

2.7.2 Rooted Γ-labelled graphs

Let Γ be a finite group (not necessarily abelian). Let G be a Γ-labelled graph.

A *rooted Γ-labelled graph* is a pair (G, X) where G is a Γ-labelled graph and X is a subset of $V(G)$. For an edge $e{=}uv \in E(G)$, the rooted Γ-labelled graph $(G, X)\backslash e$ is $(G\backslash e, X)$. If e is labelled 1, the rooted Γ-labelled graph $(G, X)/e$ is $(G/e, X')$ where $X' = X$ if both $u, v \notin X$ and $X' = (X \setminus \{u, v\}) \cup \{w\}$ if u or v is in X and $e{=}uv$ is contracted into w. A *minor* of a rooted Γ-labelled graph (G, X) is any rooted Γ-labelled graph obtained by a sequence of the operations: edge deletion, deletion of isolated vertices not in X, relabellings, or contractions of edges labelled 1. As for rooted graphs, the minor ordering on rooted Γ-labelled graphs is a quasi-order. Note that this definition does not permit contraction of an unbalanced loop, since no relabelling will label an unbalanced loop with the group identity.

Let (G, X) and (H, Y) be two rooted Γ-labelled graphs with $|X| = |Y|$. The labellings of each graph define the sets of balanced cycles for each. Any relabelling of either (G, X) or (H, Y) results in a biased graph with exactly the same set of balanced cycles; each of (G, X) and (H, Y) is a representative of its equivalence class of Γ-labelled graphs under relabelling (as discussed in Section 1.2.1). Up to relabelling, there are only a bounded number of Γ-labelled graphs that may be obtained by identifying the vertices in X one-to-one with the vertices in Y (at most $|X|! \cdot |\Gamma|^{|X|}$: there are $|X|!$ ways to choose the pairs of vertices $x \in X$ and $y \in Y$ to identify, and for each $x \in X$ there are $|\Gamma|$ relabellings at x that may be done prior to identification).

For a subset $A \subseteq E$ of edges in a graph $G{=}(V, E)$, let $\Lambda_G(A) = V(A) \cap V(E \setminus A)$ be the set of vertices incident with both an edge in A and an edge in $E \setminus A$. Then $\lambda_G(A) = |\Lambda_G(A)|$.

Lemma 2.36. *Let $E_1 \subseteq E_2$ be subsets of the edge set E of a Γ-labelled graph G. For $i = 1, 2$, let G_i be the Γ-labelled subgraph of G induced by E_i. If $\lambda_G(E_1) = \lambda_G(E_2) \leq \lambda_G(E_1, E \setminus E_2)$, then $(G_1, \Lambda_G(E_1))$ is a minor of $(G_2, \Lambda_G(E_2))$.*

Proof. By Menger's theorem, the graph induced by $E_2 \setminus E_1$ contains a collection of $\lambda_G(E_1)$ vertex disjoint paths linking $\Lambda_G(E_1)$ to $\Lambda_G(E_2)$. Deleting from $(G_2, \Lambda_G(E_2))$ all edges in

$E_2 \setminus E_1$ that are not in these paths (and resulting isolated vertices), relabelling at each vertex of these paths so that each edge in each path is labelled by 1, and contracting the edges of these paths yields $(G_1, \Lambda_G(E_1))$. $\square$

2.7.3 Proof of Theorem 2.5

Proof of Theorem 2.5. Let Γ be a finite group, and let $\mathcal{G}$ be the set of all Γ-labelled graphs having branch-width at most n. Suppose for a contradiction that $\mathcal{G}$ is not well-quasi-ordered by minor containment.

For each $G \in \mathcal{G}$, let T_G be a linked branch decomposition of G with width at most n. We may choose T_G such that at least one leaf corresponds to no edge in G (otherwise, subdivide an edge of the tree and add a pendant edge to make it cubic). Fix an unmarked leaf r and orient T_G so it becomes a rooted cubic tree with r as root. For an edge e of T_G, let E^e be the set of edges of G displayed by the component of $T_G \setminus e$ not containing the root of T_G. Define G^e to be the Γ-labelled subgraph of G induced by E^e. Put $X^e = \Lambda_G(E^e)$ and $\lambda(e) = \lambda_G(E^e)$.

Let (F, l, r) be the rooted binary forest composed of the rooted cubic trees T_G for $G \in \mathcal{G}$. Define a quasi-order $\preccurlyeq$ on $E(F)$ as follows:

> If e, f are edges of F and the rooted Γ-labelled graph (G^e, X^e) is isomorphic to a minor of the rooted Γ-labelled graph (G^f, X^f), then put $e \preccurlyeq f$.

We now check that these objects satisfy the conditions of Lemma 2.35. By Lemma 2.36, and the fact that every T_G is linked, $e \preccurlyeq f$ whenever f is λ-linked to e. Clearly $\preccurlyeq$ has no strictly descending sequences (as each T_G is finite). The leaf edges of F are well-quasi-ordered by $\preccurlyeq$, since each of them corresponds to a rooted Γ-labelled graph with at most one edge (by relabelling one can be labelled identically to the other). The root edges are not well-quasi-ordered by $\preccurlyeq$, since each root edge corresponds to the rooted Γ-labelled graph $(G, \emptyset)$ with $G \in \mathcal{G}$. So (F, l, r), λ, and $\preccurlyeq$ do indeed satisfy the conditions of Lemma 2.35. Hence there exists an infinite sequence $(e_0, e_1, \ldots)$ of nonleaf edges of F such that

1. $\{e_0, e_1, \ldots\}$ is an antichain with respect to $\preccurlyeq$,
2. $l(e_0) \preccurlyeq \cdots \preccurlyeq l(e_{i-1}) \preccurlyeq l(e_i) \preccurlyeq \cdots$, and
3. $r(e_0) \preccurlyeq \cdots \preccurlyeq r(e_{i-1}) \preccurlyeq r(e_i) \preccurlyeq \cdots$.

As every $G \in \mathcal{G}$ has branch-width at most n, each $X^{l(e_i)}$ and each $X^{r(e_i)}$ has at most n elements. Taking an infinite subsequence of $(e_0, e_1, \ldots)$, we may assume that the sets $X^{l(e_i)}$ all have the same size and also that the sets $X^{r(e_i)}$ all have the same size.

By (2), for each $i \in \{0, 1, 2, \ldots\}$, we can mark each vertex in $X^{l(e_i)}$ by a different *left mark* from $\{1, \ldots, n\}$ such that for each $i < j$, $G^{l(e_i)}$ can be obtained as a minor of $G^{l(e_j)}$

in such a way that a vertex in $X^{l(e_j)}$ goes to the vertex in $X^{l(e_i)}$ with the same left mark. Similarly, by (3), we can assign a different *right mark* from $\{1,\ldots,n\}$ to the vertices in each of $X^{r(e_1)}, X^{r(e_2)},\ldots$ so that for all $i<j$, $G^{r(e_i)}$ can be obtained as a minor of $G^{r(e_j)}$ in such a way that a vertex in $X^{r(e_j)}$ goes to the vertex in $X^{r(e_i)}$ with the same right mark. Vertices in $X^{l(e_i)}\cap X^{r(e_i)}$ receive both a right and a left mark. Since the left and right marks all come from the same finite set $\{1,\ldots,n\}$, we may take an infinite subsequence and reindex so that for all $i<j$ in $(e_0,e_1,\ldots)$,

(i) the set of left/right mark pairs assigned to the vertices in $X^{l(e_i)}\cap X^{r(e_i)}$ is the same as the set of left/right mark pairs assigned to the vertices in $X^{l(e_j)}\cap X^{r(e_j)}$,

(ii) the set of left marks assigned to X^{e_i} equals the set of left marks assigned to X^{e_j}, and

(iii) the set of right marks assigned to X^{e_i} equals the set of right marks assigned to X^{e_j}.

Denote the underlying graph of Γ-labelled graph G by $\overline{G}$. For each nonleaf edge e of F, $\overline{G^e}$ can be obtained from $\overline{G^{l(e)}}$ and $\overline{G^{r(e)}}$ by identifying the vertices in $X^{l(e)}\cap X^{r(e)}$. Hence by the definition of the left and right marks, for all $i<j$, (i) implies that $\overline{G^{e_i}}$ can be obtained as a minor of $\overline{G^{e_j}}$ such that each vertex in $X^{l(e_j)}\cap X^{r(e_j)}$ goes to a vertex in $X^{l(e_i)}\cap X^{r(e_i)}$ with the same left/right mark pair. Combined with (ii) and (iii), this implies that $(\overline{G^{e_i}}, X^{e_i})$ is a minor of $(\overline{G^{e_j}}, X^{e_j})$.

This completes the proof in [6] of Theorem 2.33: that $(\overline{G^{e_i}}, X^{e_i})$ is a minor of $(\overline{G^{e_j}}, X^{e_j})$ implies in that proof that $e_i \preccurlyeq e_j$, contradicting (1). Thus it is at this point in the argument that we require an additional definition along with some further reasoning in order to extend the proof to Γ-labelled graphs.

For each i, let $\gamma^{e_i}:\Gamma\to E^{e_i}$ be the labelling of G^{e_i}. We denote a graph G together with a specified labelling $\gamma: E(G)\to\Gamma$ by (G,γ); a rooted Γ-labelled graph (G,X) with a specified labelling γ is denoted (G,γ,X). Put $|X^{l(e_i)}\cap X^{r(e_i)}|=k$ (this number is the same for each i).

Since $\left(G^{l(e_i)},X^{l(e_i)}\right) \preccurlyeq \left(G^{l(e_j)},X^{l(e_j)}\right)$ and $\left(G^{r(e_i)},X^{r(e_i)}\right) \preccurlyeq \left(G^{r(e_j)},X^{r(e_j)}\right)$, there are subsets $R^{l(e_j)}, S^{l(e_j)}\subseteq E^{l(e_j)}$ and an associated relabelling $\gamma^{l(e_j)}_{\eta^{l(e_j)}}$ such that $G^{l(e_j)}\backslash R^{l(e_j)}/S^{l(e_j)} = G^{l(e_i)}$. Similarly, there are subsets $R^{r(e_j)}, S^{r(e_j)}\subseteq E^{r(e_j)}$ and an associated relabelling $\gamma^{r(e_j)}_{\eta^{r(e_j)}}$ such that $G^{r(e_j)}\backslash R^{r(e_j)}/S^{r(e_j)} = G^{r(e_i)}$. Contained in $S^{l(e_j)}$ is a set of k disjoint paths linking each vertex in $X^{l(e_j)}\cap X^{r(e_j)}$ with a vertex in $X^{l(e_i)}$ having the same left mark; similarly in $S^{r(e_j)}$, there is a set of such paths linking vertices of the same right marks. In $(G^{l(e_j)},\gamma^{l(e_j)}_{\eta^{l(e_j)}})$ (resp. $(G^{r(e_j)},\gamma^{r(e_j)}_{\eta^{r(e_j)}})$) every edge of these paths is labelled 1.

We now define for each i an auxiliary rooted Γ-labelled graph (H_i,Y_i) as follows. Let H_i be the Γ-labelled graph obtained from $(G^{l(e_i)},\gamma^{l(e_i)})\cup(G^{r(e_i)},\gamma^{r(e_i)})$ by adding a path of length two between each pair of vertices in $X^{l(e_i)}\cap X^{r(e_i)}$ having the same left/right mark pair, with each edge oriented out from the inner vertex of the path and labelled 1. Let Y_i

consist of the inner vertex on each of these paths, and let A_i be the set of edges of these added paths. Each vertex $y \in Y_i$ has a neighbour $x \in X^{l(e_i)}$ and $z \in X^{r(e_i)}$, which have been given the same left/right mark pair. Mark each vertex in Y_i with the left/right mark pair of its neighbours. Edge $yx \in A_i$ is the ***left*** edge incident with y, edge yz is its ***right*** edge. Clearly, contracting A_i yields $(G^{e_i}, X^{e_i}) \preccurlyeq (H_i, Y_i)$.

For each pair of indices $i < j$, do the following. Apply the relabellings $\gamma^{l(e_j)}_{\eta^{l(e_j)}}$ to $G^{l(e_j)}$ and $\gamma^{r(e_j)}_{\eta^{r(e_j)}}$ to $G^{r(e_j)}$. This relabels all edges in $S^{l(e_j)}$ and $S^{r(e_j)}$ with 1. Apply these same relabellings in H_j, so that the labelling on $E^{l(e_j)}$ in H_j agrees with $(G^{l(e_j)}, \gamma^{l(e_j)}_{\eta^{l(e_j)}})$ and the labelling on $E^{r(e_j)}$ in H_j agrees with $(G^{r(e_j)}, \gamma^{r(e_j)}_{\eta^{r(e_j)}})$. Now relabel each of $G^{l(e_i)}$ and $G^{r(e_i)}$ so that their labellings agree with that on $E^{l(e_i)} \subseteq E^{l(e_j)}$ in $(G^{l(e_j)}, \gamma^{l(e_j)}_{\eta^{l(e_j)}})$ and on $E^{r(e_i)} \subseteq E^{r(e_j)}$ in $(G^{r(e_j)}, \gamma^{r(e_j)}_{\eta^{r(e_j)}})$; let us denote these relabellings by $\gamma^{l(e_j)}_{\eta^{l(e_j)}}|E^{l(e_i)}$ and $\gamma^{r(e_j)}_{\eta^{r(e_j)}}|E^{r(e_i)}$ respectively. Now relabel H_i accordingly, so that on $E^{l(e_i)}$ its labelling is the same as that of $(G^{l(e_i)}, \gamma^{l(e_j)}_{\eta^{l(e_j)}}|E^{l(e_i)})$ and the same as that of $(G^{r(e_i)}, \gamma^{r(e_j)}_{\eta^{r(e_j)}}|E^{r(e_i)})$ on $E^{r(e_i)}$.

Now it will be the case that $(G^{e_i}, X^{e_i}) \preccurlyeq (G^{e_j}, X^{e_j})$ if in H_i and H_j, for each pair of vertices $y \in Y_i$, $z \in Y_j$ having the same left/right mark pair, the label on the left edge of y is the same as the label on the left edge of z, and the label on the right edge of y is the same as the label on the right edge of z. Since Γ is finite, this will occur for some pair of incidences $i < j$. But then $e_i \preccurlyeq e_j$, contradicting (1). $\square$

Chapter 3

Biased graph representations of graphic matroids

We exhibit six families of biased graphs whose associated frame matroids are graphic. We then show that if M is a graphic matroid, then every biased graph representing M is in one of these families. This is Theorem 3.1, the main result of this chapter.

Theorem 3.1. *Let M be a connected graphic matroid and $(G, \mathcal{B})$ a biased graph with $M = F(G, \mathcal{B})$. Then $(G, \mathcal{B})$ is one of the following:*

1. *balanced,*
2. *a curling,*
3. *a pinch,*
4. *a fat theta,*
5. *a 4-twisting,*
6. *an odd twisted fat k-wheel.*

3.1 Six families of biased graphs whose frame matroids are graphic

We describe each of the families enumerated in Theorem 3.1.

1. Balanced. Our first family of biased graphs is that of balanced biased graphs — *i.e.*, graphs. Trivially, if $(G, \mathcal{B})$ is balanced then $F(G, \mathcal{B}) = M(G)$.

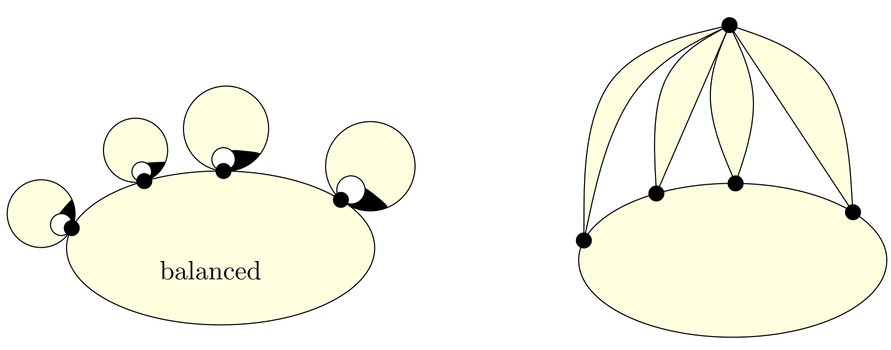
2. Curlings. A *curling* is a signed graph $(G, \mathcal{B})$ such that

1. G has at least two unbalanced blocks,
2. in the tree of blocks of G, a block is unbalanced if and only if it is a leaf block, and
3. the vertex of attachment of each unbalanced block U to $G \setminus U$ is a balancing vertex of $G[U]$.

A curling is shown in Figure 3.1(a).

Proposition 3.2. *If $(G, \mathcal{B})$ is a curling, then $F(G, \mathcal{B})$ is graphic.*

Proof. We may obtain a graph H with $M(H) = F(G, \mathcal{B})$ as follows. Label the unbalanced blocks $U_1, \ldots, U_n$, label the union of the balanced blocks B, and let u_i be the vertex of attachment in U_i ($i = 1, \ldots, n$). Since each u_i is balancing in $G[U_i]$, applying Proposition 1.22 to each $G[U_i]$, there is a signing of G such that all negative edges in each U_i are incident with u_i. Now for each unbalanced block U_i, delete the vertex of attachment u_i, and replace u_i with two vertices u_{i+} and u_{i-}. For every edge $e = u_i v$ with $v \in V(B)$, put $e = u_{i+}v$; for every positive edge $e = u_i v$ with $v \in V(U_i)$, put $e = u_{i+}v$; and for every negative edge $e = u_i v$ with $v \in V(U_i)$, put $e = u_{i-}v$. Now identify each of the new vertices u_{i-} to a single vertex u_- (Figure 3.1(b)). The circuits of $F(G, \mathcal{B})$ are balanced cycles contained in one of $B, U_1, \ldots, U_n$, and handcuffs consisting of an unbalanced cycle in each of two unbalanced blocks U_i, U_j ($1 \leq i \leq j \leq n$) and a path connecting them. Each balanced cycle in $(G, \mathcal{B})$ remains cycle in H, and a pair of handcuffs in $(G, \mathcal{B})$ containing unbalanced cycles in U_i and U_j, say, becomes a cycle in H through u_- traversing U_i and U_j, from u_- to u_{i+} and u_{j+}, respectively. Hence $M(H) = F(G, \mathcal{B})$. □



(a) (b)

Figure 3.1: Obtaining a graph H with $M(H) = F(G, \mathcal{B})$ when $(G, \mathcal{B})$ is a curling.

Note that rolling up all edges incident to a vertex in a graph (Proposition 1.25) yields a curling in which each unbalanced block consists of a single loop.

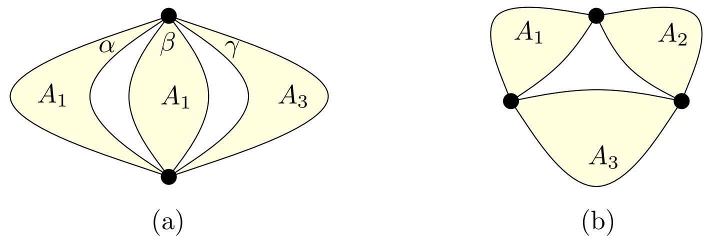


Figure 3.2: Obtaining a graph H with $M(H) \cong F(G, \mathcal{B})$ when $(G, \mathcal{B})$ is a fat theta.

3. Pinches. A *pinch* is a biased graph that may be obtained from a graph by pinching two vertices. By Proposition 1.24, these are precisely the signed graphs with a balancing vertex, and these are graphic.

4. Fat thetas. A *fat theta* is a biased graph that is the union of three balanced subgraphs A_1, A_2, A_3 mutually meeting at just a single pair of vertices, in which a cycle C is balanced if and only if $C \subseteq A_i$ for some $i \in \{1, 2, 3\}$ (Figure 3.2(a)).

Proposition 3.3. *If $(G, \mathcal{B})$ is a fat theta, then $F(G, \mathcal{B})$ is graphic.*

Proof. A graph H with $M(H) = F(G, \mathcal{B})$ is obtained by taking a copy of each of A_1, A_2, A_3, labelling the copies of u and v in each with u_i, v_i ($i = 1, 2, 3$), and identifying vertices u_1 with v_2, u_2 with v_3, and u_3 with v_1 (Figure 3.2, at right). □

5. 4-twistings. A *4-twisting* is a signed graph of the form shown at top left in Figure 3.3. It is the union of four balanced subgraphs (called *lobes*, not necessarily all non-empty) A, B, C, D, whose pairwise intersections are contained in $\{x, y, z\} \subseteq V(G)$. The 4-twisting $(G, \mathcal{B})$ is obtained by choosing up to three vertices x_i, y_i, z_i ($i \in \{A, B, C, D\}$) in each of four graphs A, B, C, D and identifying x_A, x_B, x_C, x_D to a single vertex x, y_A, y_B, y_C, y_D to a single vertex y, and z_A, z_B, z_C, z_D to a single vertex z. Its signature Σ consists of the edges in A incident with x_A, the edges in B incident with z_B, and the edges in D incident to y_D. The four lobes with this signing are shown at bottom in Figure 3.3.

Proposition 3.4. *If $(G, \mathcal{B})$ is a 4-twisting, then $F(G, \mathcal{B})$ is graphic.*

Proof. Figure 3.3 shows a graph H whose cycle matroid $M(H)$ is isomorphic to $F(G, \mathcal{B})$. The graph H is obtained from the lobes A, B, C, D of the 4-twisting by identifying x_A, y_D, z_B to a single vertex t, x_D, y_A, z_C to a single vertex u, x_C, y_B, z_D to a single vertex v, and x_B, y_C, z_A to a single vertex w. Every circuit in $F(G, \mathcal{B})$ is either a balanced cycle or a pair of tight handcuffs in $(G, \mathcal{B})$. It is straightforward to check that every such circuit is a cycle in H, and conversely that every cycle in H is either a balanced cycle or a pair of tight handcuffs in $(G, \mathcal{B})$. □

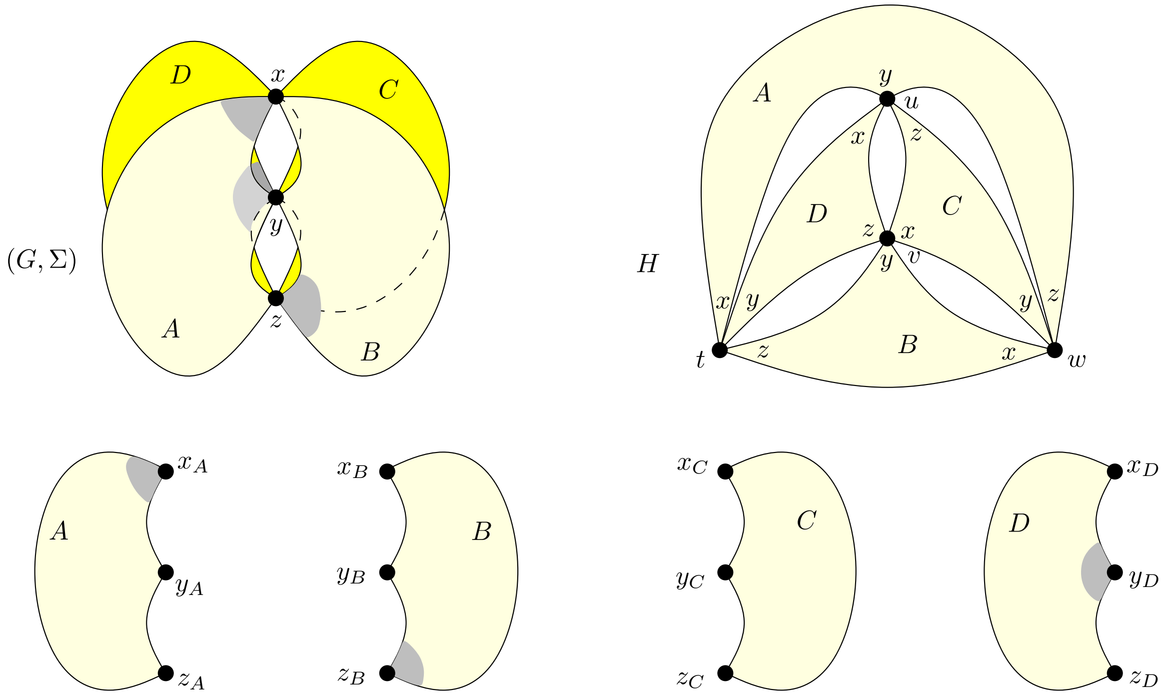


Figure 3.3: A 4-twisting (G, Σ) and a graph H with $M(H) \cong F(G, \Sigma)$.

6. Odd twisted fat k-wheels. A k-wheel is a graph on $k+1$ vertices consisting of a k-cycle $w_1 w_2 \cdots w_k$ and a central vertex z incident to each vertex w_i ($1 \le i \le k$) in the cycle. Let $k \ge 3$, $G_1, G_2, \dots, G_k$ be graphs, and let $x_i, y_i, z_i \in V(G_i)$ for each $i \in \{1, 2, \dots, k\}$. A *fat k-wheel* is obtained from a k-wheel by replacing each triangle $z w_i w_{i+1}$ ($1 \le i \le k$, indices modulo k) with graph G_i, identifying x_i with w_i, y_i with w_{i+1}, and z_i with z. An *odd fat k-wheel* is a fat k-wheel with k odd. A *twisted fat k-wheel* is a signed graph $(G, \mathcal{B}_\Sigma)$ obtained from graphs G_i, $i \in \{1, \dots, k\}$, by identifying vertices y_{i-1}, z_i, x_{i+1} to a vertex w_i' ($1 \le i \le k$, indices modulo k). Its signature Σ consists of all edges in $\delta(x_1)$ and $\delta(y_k)$. An *odd twisted fat k-wheel* is a twisted fat k-wheel with k odd (Figure 3.4).

Proposition 3.5. *If $(G, \mathcal{B}_\Sigma)$ is an odd twisted fat k-wheel, then $F(G, \mathcal{B}_\Sigma)$ is graphic.*

Proof. Let H be the fat k-wheel obtained from the graphs $G_1, G_2, \dots, G_k$ of which $(G, \mathcal{B}_\Sigma)$ is composed. It is straightforward to check that the circuits of $M(H)$ and $M(G, \mathcal{B}_\Sigma)$ coincide. $\square$

Observe that 4-twistings and odd twisted fat k-wheels have no two vertex disjoint unbalanced cycles. The rest of this chapter is devoted to a proof of Theorem 3.1.

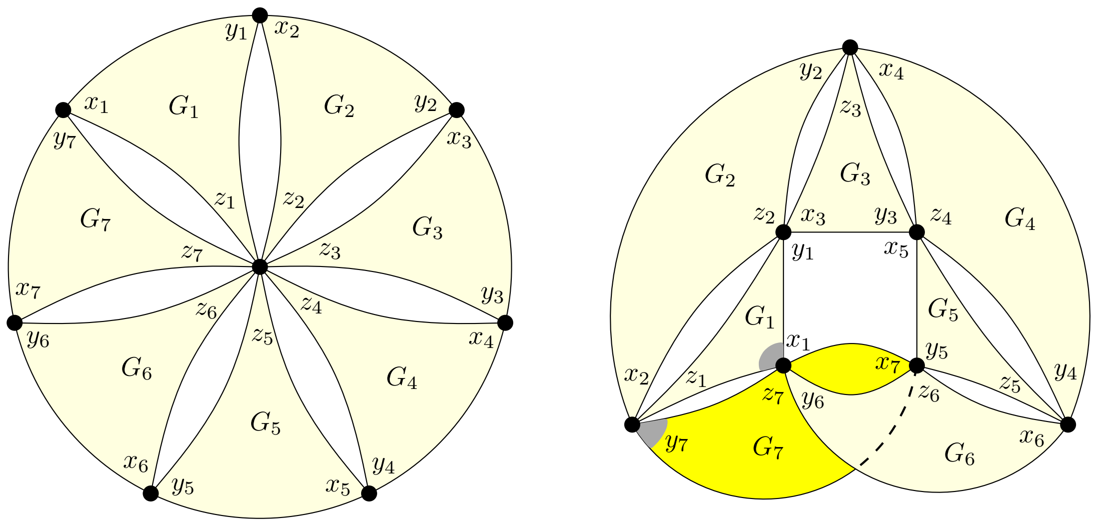


Figure 3.4: A fat 7-wheel (left) and a twisted fat 7-wheel (right).

3.2 Proof of Theorem 3.1

We first show that curlings are the only biased graphs having two vertex disjoint unbalanced cycles with connected graphic frame matroid.

Lemma 3.6. *Let $(G, \mathcal{B})$ be a biased graph containing two vertex disjoint unbalanced cycles, with $F(G, \mathcal{B})$ is connected. Then $F(G, \mathcal{B})$ is graphic if and only if $(G, \mathcal{B})$ is a curling.*

Proof. If $(G, \mathcal{B})$ is a curling, then by Proposition 3.2 $F(G, \mathcal{B})$ is graphic. Conversely, suppose $(G, \mathcal{B})$ is not a curling. Then one of the following properties does not hold:

1. $(G, \mathcal{B})$ is signed graphic;
2. G has at least two unbalanced blocks;
3. in the tree of blocks of G, a block is unbalanced if and only if it is a leaf-block; or
4. the vertex of attachment of each unbalanced block U to $G \setminus U$ is a balancing vertex of $G[U]$.

We show that in any case $F(G, \mathcal{B})$ is non-binary, and so not graphic.

Suppose (1) does not hold. Then by Proposition 1.20, $(G, \mathcal{B})$ contains a contrabalanced theta subgraph T. By assumption $(G, \mathcal{B})$ contains an unbalanced cycle avoiding one of the branch vertices of T, and so by Lemma 1.28 is non-binary. Now suppose (2) does not hold; *i.e.* G has only one unbalanced block. An unbalanced loop is a block, so if $(G, \mathcal{B})$ has an unbalanced loop e, then $(G, \mathcal{B}) \setminus e$ is balanced, a contradiction since $(G, \mathcal{B})$ contains two disjoint unbalanced cycles. Hence $(G, \mathcal{B})$ has no unbalanced loop, and so by Lemma 1.26 $F(G, \mathcal{B})$ is not binary.

Now suppose (3) fails. Since $F(G, \mathcal{B})$ is connected, a leaf block cannot be balanced. Hence there is an unbalanced cycle C in a block B that is not a leaf block. Since an unbalanced loop is itself a leaf block, C is not a loop. Since B is not a leaf block, there is a path in the block tree of G, in which B has degree 2, linking two leaf blocks L_1, L_2. Each of L_1 and L_2 contain an unbalanced cycle, let us call them C_1 and C_2. Clearly, C_1 and C_2 are disjoint. Since B is a block, and C is not an unbalanced loop, there are disjoint paths P_1, P_2 linking C with C_1, C_2, respectively. Say $P_1 \cap C = \{x\}$ and $P_2 \cap C = \{y\}$. The cycle C is composed of two internally disjoint x-y paths, let us call them Q_1 and Q_2. Contracting all but one edge in each of Q_1, Q_2, C_1, and C_2, and all edges in each of P_1 and P_2, and deleting all remaining edges, we obtain a biased graph representing $U_{2,4}$. Hence $F(G, \mathcal{B})$ is not binary.

So suppose now that (1), (2), and (3) hold, but (4) fails: there is an unbalanced leaf block U in which the vertex of attachment u is not a balancing vertex. Then there is an unbalanced cycle C, not a loop, in U avoiding u (again, an unbalanced loop is itself a leaf block, and its vertex of attachment is a balancing vertex for it). Let C' be an unbalanced cycle in $(G, \mathcal{B})$ disjoint from C (which exists by hypothesis). If $C' \subset U$, then by Lemma 1.26 $(G, \mathcal{B})$ is non-binary. Otherwise, since U is a block, there is a pair of paths P_1, P_2 linking C with u that are otherwise disjoint. Say $P_1 \cap C = \{x\}$ and $P_2 \cap C = \{y\}$. The cycle C is composed of two internally disjoint x-y paths, let us call them Q_1 and Q_2. Choose a u-C' path P_3. Now contract all but one of the edges in each of Q_1, Q_2, P_1, P_2, and C', and all the edges in P_3. This yields the biased graph shown in Figure 1.12 labelled (b). By the argument in the proof of Lemma 1.26, contracting one of Q_1 or Q_2 yields a biased graph representing $U_{2,4}$. $\square$

We use the following result of Shih [27], on the relationship between two graphs when one has its cycle space as a subspace of the cycle space of the other (see also [10], or [24] Theorem 4.1). To aid with the statement of Shih's theorem below, we use the terms *pinch*, *4-twisting*, and *twisted fat k-wheel* in the statement of the theorem to refer to the underlying graph of the signed graphs we have defined as pinches, 4-twistings, and twisted fat k-wheels, respectively. We denote the cycle space of a graph G by $\mathcal{C}(G)$.

Theorem 3.7 (Shih [27]). *Let H and G be graphs with $E(H) = E(G)$. Then $\mathcal{C}(H)$ is a codimension-1 subspace of $\mathcal{C}(G)$ if and only if G is obtained from H as a pinch, 4-twisting, or twisted fat k-wheel, and Whitney operations.*

For us, Shih's theorem has the following useful corollary (we now revert to our usual use of the terms pinch, 4-twisting, and twisted fat k-wheel).

Corollary 3.8. *Suppose $(G, \mathcal{B})$ is a connected signed graph containing an unbalanced cycle and no two vertex disjoint unbalanced cycles, and that $F(G, \mathcal{B})$ is graphic. Then $(G, \mathcal{B})$ is a pinch, a 4-twisting, or an odd twisted fat k-wheel.*

Proof. Since $F(G,\mathcal{B})$ is graphic, there is a graph H with $E(H) = E(G)$ and $F(G,\mathcal{B}) = M(H)$. We may assume H is connected, since otherwise a Whitney operation yields a connected graph with the same cycle space. Let $C \in \mathcal{C}(H)$. Then C is an edge-disjoint union of cycles of H, say $C = C_1 \cup \cdots \cup C_n$. Since $F(G,\mathcal{B}) = M(H)$, each cycle C_i is a circuit of $F(G,\mathcal{B})$. Since $(G,\mathcal{B})$ contains no two vertex disjoint unbalanced cycles and no contrabalanced theta, each C_i appears in $(G,\mathcal{B})$ as either a balanced cycle or a pair of unbalanced cycles meeting in exactly one vertex. Hence C is an edge-disjoint union of cycles in G, ***i.e.*** $C \in \mathcal{C}(G)$. Hence $\mathcal{C}(H) \subseteq \mathcal{C}(G)$. Since $|V(G)| = \mathrm{rank}(F(G,\mathcal{B})) = |V(H)|-1$, the codimension of $\mathcal{C}(H)$ in $\mathcal{C}(G)$ is

$$\dim(\mathcal{C}(G)) - \dim(\mathcal{C}(H)) = |E(G)| - |V(G)| + 1 - (|E(H)| - |V(H)| + 1) = 1.$$

Hence by Theorem 3.7, G is obtained from H as the underlying graph of a pinch, 4-twisting, or twisted fat k-wheel, and Whitney operations. In each case, signing G according to our definition of each yields a signed graph $(G,\mathcal{B}_\Sigma)$. That the signature $\Sigma \subseteq E(G)$ in each of the cases that G is a pinch, 4-twisting, or twisted fat k-wheel realises $\mathcal{B}$ is immediate upon consideration of independent sets in $F(G,\mathcal{B})$ that form cycles in G. Since a twisted fat k-wheel with k even either contains two vertex disjoint unbalanced cycles or is otherwise a pinch, the result follows. $\square$

We may now show that our six families of biased graphs provide all frame representations of connected graphic matroids.

Proof of Theorem 3.1. Let $(G,\mathcal{B})$ be a biased graph with $F(G,\mathcal{B})$ connected and graphic. Then certainly G is connected. Assume $(G,\mathcal{B})$ is not balanced. If $(G,\mathcal{B})$ contains two vertex disjoint unbalanced cycles, then by Lemma 3.6 $(G,\mathcal{B})$ is a curling. So assume $(G,\mathcal{B})$ has no two vertex disjoint unbalanced cycles. Suppose first also that $(G,\mathcal{B})$ is a signed graph. Then by Corollary 3.8, $(G,\mathcal{B})$ is in this case a pinch, a 4-twisting, or an odd twisted fat k-wheel.

So now suppose $(G,\mathcal{B})$ is not a signed graph. Then $(G,\mathcal{B})$ contains a contrabalanced theta subgraph T, say with branch vertices u, v. If there were an unbalanced cycle avoiding one of u or v, then by Lemma 1.28, $F(G,\mathcal{B})$ would contain $U_{2,4}$ as a minor, a contradiction. Hence each of u and v are balancing vertices. Consider the number k of balancing classes of $\delta(u)$. There are at least three distinct classes, but if $k \geq 4$ then by Lemma 1.29, $F(G,\mathcal{B})$ is not binary, a contradiction. Hence $\delta(u)$ has exactly three balancing classes, say B_1, B_2, B_3. Let Q_1, Q_2, Q_3 be the three u-v paths whose union is T. Each path Q_i contains exactly one edge in B_i ($i \in \{1,2,3\}$). By Lemma 1.27, there can be no path linking two internal vertices of any two of Q_1, Q_2, or Q_3. Hence $\{u,v\}$ is a two vertex cut, and $E(G)$ may be partitioned into three sets A_1, A_2, A_3, such that each $A_i \supseteq Q_i$ ($i \in \{1,2,3\}$) and such that a cycle C is balanced if and only if $C \subseteq A_i$ for some $i \in \{1,2,3\}$; ***i.e.*** $(G,\mathcal{B})$ is a fat theta. $\square$

To see that Theorem 1.17 follows as a corollary of Theorem 3.1, suppose (G, B_Σ) is a connected tangled signed graph with $F(G, \mathcal{B}_\Sigma)$ graphic. By Theorem 3.1, $(G, \mathcal{B}_\Sigma)$ is either a 4-twisting or an odd twisted fat k-wheel. Suppose $(G, \mathcal{B}_\Sigma)$ is a 4-twisting, say with lobes A, B, C, D meeting just in $\{x, y, z\}$ (as shown in Figure 3.3). Let Ω be the signed graph obtained from (G, Σ) by replacing each lobe $i \in \{A, B, C, D\}$ with a triangle T_i on vertices x, y, z. Let the edge sets of $T_A = \{a_1, a_2, a_3\}$, $T_B = \{b_1, b_2, b_3\}$, $T_C = \{c_1, c_2, c_3\}$, and $T_D = \{d_1, d_2, d_3\}$, where for each $k \in \{a, b, c, d\}$ edge $k_1 = xy$, $k_2 = yz$, and $k_3 = xz$. Let the signature of Ω be $\{a_1, a_3, b_2, b_3, d_1, d_2\}$. The resulting signed graph Ω is show at left in Figure 3.5, with edges in the signature dashed. We illustrate embeddings in the projective

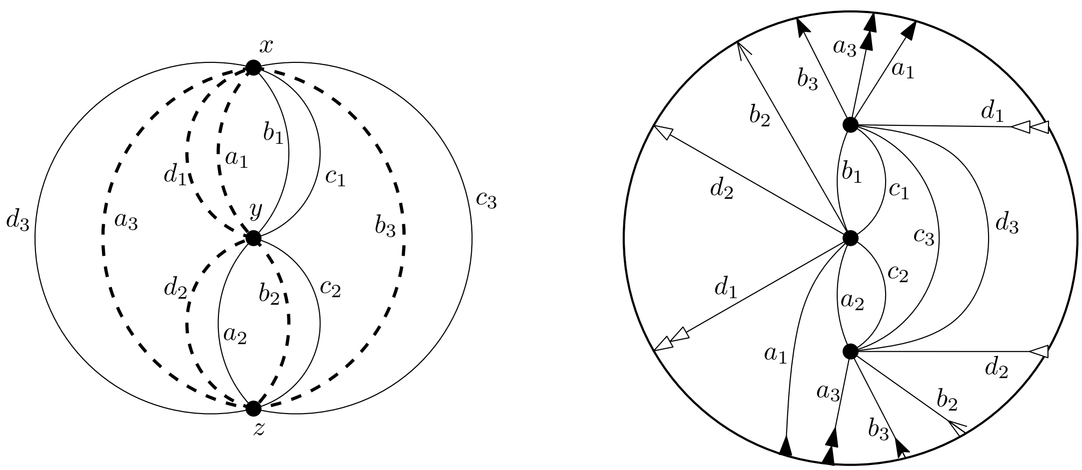


Figure 3.5: Embedding a 4-twisting in the projective plane.

plane using the standard representation of the projective plane as a closed disc with any two antipodal points on its boundary being identified. The embedding of Ω in the projective plane shown at right in Figure 3.5 shows that Ω is a projective planar signed graph whose topological dual is planar. Moreover, $(G, \mathcal{B}_\Sigma)$ is obtained by taking 1-, 2-, or 3-sums of Ω with balanced signed subgraphs of the lobes A, B, C, D of the form described by Theorem 1.17, with the following exception. If a 3-sum with a balanced signed graph with just 4 vertices is required, then this may be done first: the result is also a projective planar signed graph whose topological dual is planar.

The case that (G, Σ) is an odd twisted fat k-wheel is similar. An embedding of a twisted 7-wheel in the projective plane is shown at right in Figure 3.6. For $k \neq 7$, an embedding as required may be obtained from this embedding by lengthening (or shortening) the ladder consisting of the two paths of vertices w_{2n+1} and w_{2n} respectively ($n \in \{1, \ldots, (k-1)/2\}$) with rungs between them, in the obvious way; replace edge w_1w_7 with w_1w_{2n+1}, edge w_1w_6 with w_1w_{2n}, and w_2w_7 with w_2w_{2n+1} (Figure 3.7).

In any case, the topological dual of the embedded projective planar signed graph must be planar, else $F(G, \mathcal{B}_\Sigma)$ would contain one of $M^*(K_{3,3})$ or $M^*(K_5)$ as a minor, a contradic-

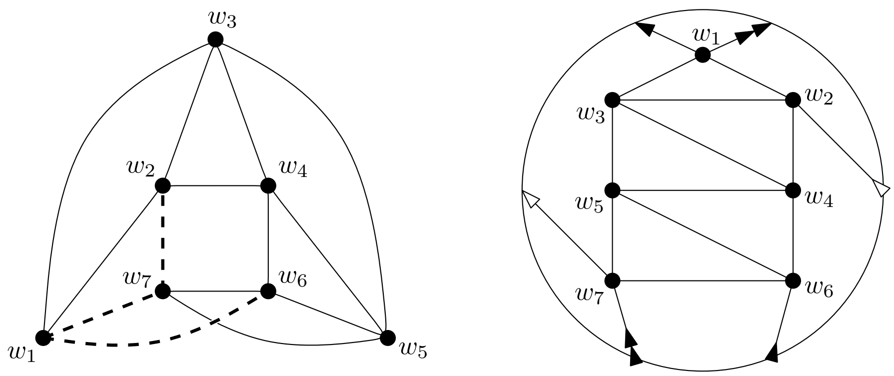


Figure 3.6: Embedding a twisted 7-wheel in the projective plane.

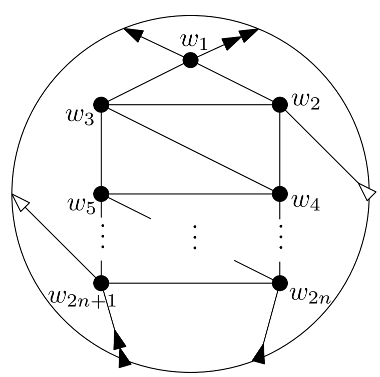


Figure 3.7: Embedding an odd twisted k-wheel in the projective plane.

tion.

The underlying graphs of 4-twistings and odd fat twisted k-wheels also appear in Mohar, Robertson, and Vitray's characterisation of graphs that may be embedded in the projective plane such that their geometric dual is planar ([20], Corollary 4.4). They show that G is such a graph if and only if

1. G is planar,
2. G is the underlying graph of a 4-twisting with each lobe $H \in \{A, B, C, D\}$ plane such that the vertices x_H, y_H, z_H identified to produce G are on the boundary of a common face of the plane embedding of H, or
3. G is the underlying graph of an odd fat twisted k-wheel with each subgraph $G_1, \ldots, G_k$ plane such that the vertices $x_i, y_i, z_i \in V(G_i)$ $(i \in \{1, \ldots, k\})$ identified to produce G are on the boundary of a common face in the plane embedding of G_i.

This statement is also a corollary of Theorem 3.1. Suppose G can be embedded in the projective plane such that its geometric dual is planar. Then taking Σ to be the set of edges crossing the boundary, $F(G, \mathcal{B}_\Sigma)$ is graphic: none of $U_{2,4}$, F_7, nor F_7^* is signed graphic projective planar so $F(G, \mathcal{B}_\Sigma)$ has none of these as a minor, and since its geometric dual is planar, $F(G, \mathcal{B}_\Sigma)$ has no $M^*(K_{3,3})$ nor $M^*(K_5)$ as a minor. Since $(G, \mathcal{B}_\Sigma)$ is a signed graph with no two vertex disjoint unbalanced cycles, by Theorem 3.1, $(G, \mathcal{B}_\Sigma)$ is either balanced, a pinch, a 4-twisting, or an odd twisted fat k-wheel. If $(G, \mathcal{B}_\Sigma)$ is balanced, then G has an embedding in the projective plane such that no edges cross the boundary, so G is planar. If $(G, \mathcal{B}_\Sigma)$ is a pinch, then there is an embedding of G in the projective plane such that all edges crossing the boundary are incident to the balancing vertex u of G. Splitting u, there is an embedding of the resulting graph H in the projective plane with no edges crossing the boundary: take the embedding obtained from G by, when splitting u, replacing u with u_+ and u_- and then sliding u_- just across the boundary so that no edges of the resulting embedding cross the boundary. A planar embedding of G is now obtained by sliding u_- just inside the boundary until it meets u_+: now identify u_+ and u_-. So again in this case, G is planar. If $(G, \mathcal{B}_\Sigma)$ is 4-twisting or odd fat twisted k-wheel, then $(G, \mathcal{B}_\Sigma)$ is of the form (2) or (3) respectively.

Chapter 4

On excluded minors of connectivity 2 for the class of frame matroids

In this chapter, we investigate the excluded minors of connectivity 2 for the class of frame matroids. We determine a set $\mathcal{E}$ of 18 particular excluded minors for the class, and show that any other excluded minor of connectivity 2 for the class has a special form. We prove:

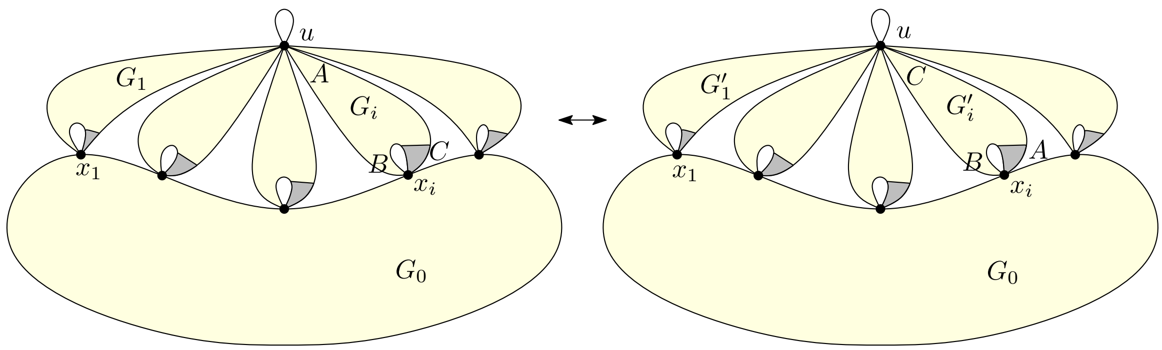
Theorem 4.1. *Let M be an excluded minor for the class of frame matroids, and suppose M is not 3-connected. Then either M is isomorphic to a matroid in $\mathcal{E}$ or M is the 2-sum of a 3-connected non-binary frame matroid and $U_{2,4}$.*

The chapter is organised as follows. We first discuss some of the key concepts we need for our investigation. In Section 4.2 we discuss 2-sums of frame matroids and of biased graphs, and provide a characterisation of when a 2-sum of two frame matroids is frame. This is enough for us to determine the first nine excluded minors on our list, and to drastically narrow our search for more. These tasks are accomplished in Section 4.3. In particular, we investigate some key properties any excluded minor not yet on our list must have. In Section 4.4 we complete the proof of Theorem 4.1, determining the remaining excluded minors in our list.

Theorem 4.1 give a strong structural description of excluded minors that are not 3-connected. However, the investigation remains incomplete — the final case remaining is to determine those excluded minors of the form captured in the second part of the statement of Theorem 4.1. It is anticipated that the analysis required to complete this final case will be longer and more technical than that required here. We expect the result to be at least a doubling of the number of excluded minors on our list, but that the list will remain finite.

Twisted flips. As mentioned in the introduction, in the course of proving Theorem 4.1, we discover an operation analogous to a Whitney twist in a graph, which we call a *twisted flip*. Given a k-signed graph $(G, \mathcal{B}_\Sigma)$ of a particular structure, a twisted flip produces a (generally) non-isomorphic k-signed graph $(G', \mathcal{B}'_\Sigma)$ with $F(G, \mathcal{B}_\Sigma) \cong F(G', \mathcal{B}'_\Sigma)$. The operations of pinching two vertices in a graph, and its inverse operation of splitting a balancing vertex of a signed graph, as well as the operations of rolling up the edges of a balancing class of a balancing vertex or unrolling unbalanced loops, are each special cases of a twisted flip.

A twisted flip may be applied to k-signed graphs having the following structure. Let G be a graph, let $u \in V(G)$, let $G_0, \ldots, G_m$ be edge disjoint connected subgraphs of G, and let $\mathbf{\Sigma} = \{\Sigma_1, \ldots, \Sigma_k\}$ be a collection of subsets of $E(G)$ satisfying the following (see Figure 4.1(a)).



(a) $(G, \mathcal{B}_{\mathbf{\Sigma}})$ (b) $(G', \mathcal{B}_{\mathbf{\Sigma}'})$

Figure 4.1: A twisted flip: Edges in $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are shaded; edges marked A in G become incident to x_i in G' and are in $\mathbf{\Sigma}'$; edges marked C in G become incident to u in G'.

1. $E(G) \setminus \bigcup_{i=0}^{m} E(G_i)$ is empty or consists of loops at u.
2. $E(G_0) \cap \Sigma_i = \emptyset$ for $1 \leq i \leq k$.
3. For every $1 \leq i \leq m$ there is a vertex x_i so that $V(G_i) \cap \left(\bigcup_{j \neq i} V(G_j)\right) \subseteq \{u, x_i\}$.
4. For every $1 \leq i \leq m$ there exists a unique s_i, $1 \leq s_i \leq k$, so that $E(G_i) \cap \Sigma_j = \emptyset$ for $j \neq s_i$.
5. Every edge in $E(G_i) \cap \Sigma_{s_i}$ is incident with x_i.

Consider the resulting biased graph $(G, \mathcal{B}_\Sigma)$ and its associated frame matroid $F(G, \mathcal{B}_\Sigma)$. We obtain a biased graph $(G', \mathcal{B}_{\Sigma'})$ — which in general is not isomorphic to $(G, \mathcal{B}_\Sigma)$ — with $F(G', \mathcal{B}_{\Sigma'}) \cong F(G, \mathcal{B}_\Sigma)$ from $(G, \mathcal{B}_\Sigma)$ as follows.

- Redefine the endpoints of each edge of the form $e{=}yu \notin \Sigma_{s_i}$ so that $e{=}yx_i$ (note that an edge $e{=}x_iu \notin \Sigma_{s_i}$ thus becomes a loop $e{=}x_ix_i$).
- Redefine the endpoints of each edge of the form $e{=}yx_i \in \Sigma_{s_i}$ with $y \neq u$ so that $e{=}yu$.

- For each $1 \leq j \leq k$, let $\Sigma'_j = \{e$: the endpoints of e have been redefined so that $e{=}yx_i$ for some $y \in V(G_i)\} \cup \{e : e{=}x_i u \in \Sigma_j\}$. Put $\boldsymbol{\Sigma}' = \{\Sigma'_1, \ldots, \Sigma'_k\}$.

The biased graph obtained from this process is $(G', \mathcal{B}_{\Sigma'})$.

Theorem 4.2. *If $(G', \mathcal{B}_{\Sigma'})$ is obtained from $(G, \mathcal{B}_\Sigma)$ by a twisted flip, then $F(G', \mathcal{B}_{\Sigma'}) \cong F(G, \mathcal{B}_\Sigma)$.*

Proof. It is straightforward to check that $F(G, \mathcal{B}_\Sigma)$ and $F(G', \mathcal{B}_{\Sigma'})$ have the same set of circuits. □

4.1 On connectivity

4.1.1 Excluded minors are connected, simple and cosimple

In this section we discuss some basic concepts on connectivity, and prove the following basic result.

Theorem 4.3. *Every excluded minor for the class of frame matroids is connected, simple, and cosimple.*

Theorem 4.3 consists of the following two observations.

Observation 4.4. If M is an excluded minor for the class of frame matroids, then M is connected.

The direct sum of two matroids M and N is denoted $M \oplus N$. Evidently, if Ω and Ψ are biased graphs, then the disjoint union $\Omega \mathbin{\dot\cup} \Psi$ of Ω and Ψ represents $F(\Omega) \oplus F(\Psi)$.

Proof of Observation 4.4. Suppose to the contrary that M is an excluded minor, and that M has a 1-separation (A, B). Then M is the direct sum of its restrictions to each of A and B. By minimality, each of $M|A$ and $M|B$ are frame. Let Ω and Ψ be biased graphs representing $M|A$ and $M|B$ respectively, and let $\Omega \mathbin{\dot\cup} \Psi$ denote the biased graph which is the disjoint union of Ω and Ψ. Then $M = M|A \oplus_1 M|B = F(\Omega) \oplus_1 F(\Psi) = F(\Omega \mathbin{\dot\cup} \Psi)$, so M is frame, a contradiction. □

Observation 4.5. Let M be an excluded minor for the class of frame matroids. Then M is simple and cosimple.

Proof. Suppose M has a loop e. By minimality, there is a biased graph $(G, \mathcal{B})$ representing $M \setminus e$. Adding a balanced loop labelled e incident to any vertex of G yields a biased graph representing M, a contradiction. Similarly, if M has a coloop f, consider a biased graph $(G, \mathcal{B})$ representing M/f. Adding a new vertex w, choosing any vertex $v \in V(G)$, and adding edge $f = vw$ to G yields a biased graph representing M, a contradiction.

Now suppose M has a two-element circuit $\{e, f\}$. Let $(G, \mathcal{B})$ be a biased graph representing $M \setminus e$. If f is a link in G, say $f = uv$, then let G' be the graph obtained from G by adding e in parallel with f so e also has endpoints u and v, and let $\mathcal{B}' = \mathcal{B} \cup \{C \setminus e \cup f : e \subset C \in \mathcal{B}\}$. If f is an unbalanced loop in G, say incident to $u \in V(G)$, then let G' be the graph obtained from G by adding e as an unbalanced loop also incident with u, and let $\mathcal{B}' = \mathcal{B}$. Then $M = F(G', \mathcal{B}')$, a contradiction.

Similarly, if e and f are elements in series in M, let $(G, \mathcal{B})$ be a biased graph representing M/e. If f is a link in G, say $f = uv$, then let G' be the graph obtained from G by deleting f, adding a new vertex w, and putting $f = uw$ and $e = wv$; let $\mathcal{B}' = \{C : C \in \mathcal{B}$ or $C/e \in \mathcal{B}\}$. If f is an unbalanced loop in G, say incident to $u \in V(G)$, let G' be the graph obtained from G by deleting f, adding a new vertex w, and adding edges e and f in parallel, both with endpoints u, w; let $\mathcal{B}' = \mathcal{B}$ (so $\{e, f\}$ is an unbalanced cycle). Again, then $M = F(G', \mathcal{B}')$, a contradiction. $\square$

4.1.2 Separations in biased graphs and frame matroids

Let M be a frame matroid on E and let $\Omega=(G, \mathcal{B})$ be a biased graph representing M. The following facts regarding the relationship between the order of a separation (X, Y) in M and the order of (X, Y) in Ω will be used extensively throughout the rest of Chapter 4.

Just as a separation (A, B) in a graph G has, in general, different orders in G and $M(G)$ (Section 1.3.5), so a separation in a biased graph Ω generally has a different order in its frame matroid $F(\Omega)$. For example, if $F(\Omega)$ is a circuit, then every partition (X, Y) of E is a 2-separation of $F(\Omega)$, but in a biased graph Ω representing M it may certainly be that $\lambda_\Omega(X, Y) > 2$. If the sides of a separation are connected in the biased graph however, then this difference is at most one. To see this, let (X, Y) be a partition of E. If Ω is balanced, then M is graphic, and the orders of the separation agree. So assume Ω is unbalanced. Consider the following calculation of the order of (X, Y) in M.

$$\begin{aligned}
\lambda_M(X, Y) &= r(X) + r(Y) - r(M) + 1 \\
&= |V(X)| - b(X) + |V(Y)| - b(Y) - |V| + 1 \\
&= |V(X) \cap V(Y)| - b(X) - b(Y) + 1 \\
&= \lambda_\Omega(X, Y) - b(X) - b(Y) + 1.
\end{aligned} \tag{4.1}$$

This immediately implies that if M is connected, then Ω must be connected: If there is a partition (X, Y) of E with $\lambda_\Omega(X, Y) = 0$, then $\lambda_M(X, Y) \leq 1$. (Recall that a matroid M is not connected if and only if M has a 1-separation.)

Moreover, if $\Omega[X]$ and $\Omega[Y]$ are both connected we have:

1. If both $\Omega[X]$ and $\Omega[Y]$ are unbalanced, then $\lambda_M(X, Y) = \lambda_\Omega(X, Y) + 1$,

2. if precisely one of $\Omega[X]$ or $\Omega[Y]$ is balanced while the other is unbalanced, then $\lambda_M(X,Y) = \lambda_\Omega(X,Y)$, and

3. if both $\Omega[X]$ and $\Omega[Y]$ are balanced, then $\lambda_M(X,Y) = \lambda_\Omega(X,Y) - 1$.

Observe that it may occur that a frame matroid M represented by a connected biased graph Ω may be disconnected. Let $M{=}F(\Omega)$, where Ω is connected, and suppose (X,Y) is a 1-separation of M, and both $\Omega[X]$ and $\Omega[Y]$ are connected. If Ω is balanced, then M is graphic, and $\lambda_M(X,Y) = \lambda_\Omega(X,Y) = 1$. Otherwise, we have $\lambda_M(X,Y) = 1$ and by (4.1), one of the following holds:

- $\lambda_\Omega(X,Y) = 1$, and precisely one of $\Omega[X]$ or $\Omega[Y]$ is balanced;
- $\lambda_\Omega(X,Y) = 2$, and each of $\Omega[X]$ and $\Omega[Y]$ are balanced.

4.2 2-sums of frame matroids and matroidals

In this section we prove Theorem 4.7 below, which gives necessary and sufficient conditions for a 2-sum of two frame matroids to be frame.

The *2-sum* of two matroids M_1 and M_2 *on* elements $e_1 \in E(M_1)$ and $e_2 \in E(M_2)$, denoted $M_1 \,{}^{e_1}\!\oplus_2^{e_2} M_2$, is the matroid on ground set $(E(M_1) \cup E(M_2)) \setminus \{e_1, e_2\}$ with circuits: the circuits of M_i avoiding e_i for $i = 1, 2$, together with $\{(C_1 \cup C_2) \setminus \{e_1, e_2\} : C_i$ is a circuit of M_i containing e_i for $i = 1, 2\}$. The following result (independently of Bixby, Cunningham, and Seymour) is fundamental.

Theorem 4.6 ([23],Theorem 8.3.1)**.** *A connected matroid M is not 3-connected if and only if there are matroids M_1, M_2, each of which is a proper minor of M, such that M is a 2-sum of M_1 and M_2.*

If M is a matroid whose automorphism group is transitive on $E(M)$, then we write simply $M \oplus_2^f N$ to indicate the 2-sum of M and N taken on some element $e \in E(M)$ and element $f \in E(N)$; if also N has transitive automorphism group we may simply write $M \oplus_2 N$.

Matroidals. A *matroidal* is a pair (M, L) consisting of a matroid M together with a distinguished subset L of its elements. A matroidal $\mathcal{M}{=}(M, L)$ is *frame* if there is a biased graph Ω with $M = F(\Omega)$ in which every element in L is an unbalanced loop. We say a biased graph in which all elements in $L \subseteq E(\Omega)$ are unbalanced loops is *L-biased*. Thus $\mathcal{M}{=}(M, L)$ is a frame matroidal if and only if there exists an L-biased graph Ω with $F(\Omega) = M$. In this case we say Ω *represents* $\mathcal{M}$. Note that this is equivalent to asking that there be an extension N of M, having a distinguished basis B with the property that every element is spanned by a pair of elements in B, such that L is contained in B.

We may now state the main result of this section:

Theorem 4.7. *Let M_1, M_2 be connected matroids and for $i = 1, 2$ let $e_i \in E(M_i)$. The matroid $M_1 \,{}^{e_1}\!\oplus_2^{e_2}\, M_2$ is frame if and only if one of the following holds.*

1. *One of M_1 or M_2 is graphic and the other is frame.*

2. *Both matroidals $(M_1, \{e_1\})$ and $(M_2, \{e_2\})$ are frame.*

We prove a more general statement than Theorem 4.7, giving necessary and sufficient conditions for a 2-sum of two frame matroidals to be frame. This more general result will be required in Section 4.3. The statement and its proof will be given after the following necessary preliminaries.

4.2.1 2-summing biased graphs

Let Ω_1, Ω_2 be biased graphs and let $e_i \in E(\Omega_i)$ for $i = 1, 2$. There are two ways in which we may perform a biased graph 2-sum operation on Ω_1 and Ω_2 to obtain a biased graph representing the 2-sum of the two matroids, $F(\Omega_1) \,{}^{e_1}\!\oplus_2^{e_2}\, F(\Omega_2)$.

1. Suppose e_i is an unbalanced loop in Ω_i incident with vertex v_i, for $i \in \{1, 2\}$. The *loop-sum* of Ω_1 and Ω_2 *on* e_1 and e_2 is the biased graph obtained from the disjoint union of $\Omega_1 - e_1$ and $\Omega_2 - e_2$ by identifying vertices v_1 and v_2. Every cycle in the loop-sum is contained in one of Ω_1 or Ω_2; its bias is defined accordingly.

2. Suppose Ω_1 is balanced, and that e_i is a link in Ω_i incident with vertices u_i, v_i, for $i \in \{1, 2\}$. The *link-sum* of Ω_1 and Ω_2 *on* e_1 and e_2 is the biased graph obtained from the disjoint union of $\Omega_1 - e_1$ and $\Omega_2 - e_2$ by identifying u_1 with u_2 and v_1 with v_2. A cycle in the link-sum is balanced if it is either a balanced cycle in Ω_1 or Ω_2 or if it may be written as a union $(C_1 \setminus e_1) \cup (C_2 \setminus e_2)$ where for $i \in \{1, 2\}$, C_i is a balanced cycle in Ω_i containing e_i. (It is straightforward to verify that the theta rule is satisfied by this construction.)

Proposition 4.8. *Let Ω_1, Ω_2 be biased graphs and let $e_i \in E(\Omega_i)$ for $i \in \{1, 2\}$. If Ω is a loop-sum or link-sum of Ω_1 and Ω_2 on e_1 and e_2, then $F(\Omega) = F(\Omega_1) \,{}^{e_1}\!\oplus_2^{e_2}\, F(\Omega_2)$.*

Proof. It is easily checked that for both the loop-sum and link-sum, the circuits of $F(\Omega)$ and of $F(\Omega_1) \,{}^{e_1}\!\oplus_2^{e_2}\, F(\Omega_2)$ coincide, regardless of the choice of pairs of endpoints of e_1 and e_2 that are identified in the link-sum. □

4.2.2 Decomposing along a 2-separation

By Theorem 4.6, a matroid M of connectivity 2 decomposes into two of its proper minors such that M is a 2-sum of these smaller matroids. If in addition M is frame, then every minor of M is frame, and we would like to be able to express the 2-sum in terms of a loop-sum or link-sum of two biased graphs representing these minors. This motivates the following definitions. Let M be a connected frame matroid on E and let Ω be a biased graph representing M. A 2-separation (A, B) of M is a *biseparation of* Ω. There are four types of biseparation that play key roles. Define a biseparation to be type 1, 2(a), 2(b), 3(a), 3(b), or 4, respectively, if it appears as in Figure 4.2, where each component of $\Omega[A]$ and $\Omega[B]$ is connected; components of each side of the separation marked "b" are balanced, those marked "u" are unbalanced. We refer to a biseparation of type 2(a) or 2(b) as type 2, and a biseparation of type 3(a) or 3(b) as type 3.

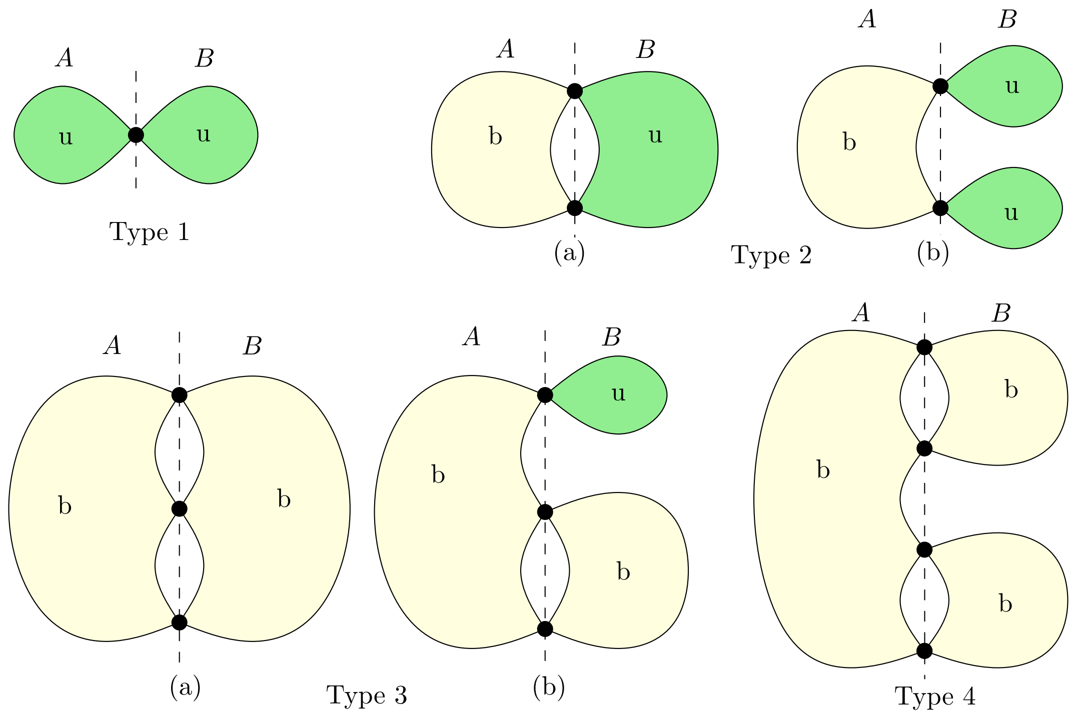


Figure 4.2: Four types of biseparations.

Proposition 4.9. *Let M be a connected frame matroid such that $M = M_1 \, {}^{e_1}\!\oplus_2^{e_2} M_2$ for two matroids M_1, M_2. Let Ω be a biased graph representing M, and let $E(M_i) \setminus \{e_i\} = E_i$ for $i \in \{1, 2\}$. If (E_1, E_2) is type 1 (resp. type 2), then there exist biased graphs Ω_i with $E(\Omega_i) = E(M_i)$, $i \in \{1, 2\}$, such that Ω is the loop-sum (resp. link-sum) of Ω_1 and Ω_2 on e_1 and e_2.*

Proof. If (E_1, E_2) is type 1, then for $i \in \{1, 2\}$ let Ω_i be the biased graph obtained from Ω by replacing $\Omega[E_{i+1}]$ with an unbalanced loop e_i incident to the vertex in $V(E_1) \cap V(E_2)$ (adding indices modulo 2). Then Ω is the loop-sum of Ω_1 and Ω_2 on e_1 and e_2. If (E_1, E_2) is type 2, then for $i \in \{1, 2\}$ let Ω_i be the biased graph obtained from Ω by replacing $\Omega[E_{i+1}]$

with a link joining the two vertices in $V(E_1) \cap V(E_2)$. Then Ω is the link-sum of Ω_1 and Ω_2 on e_1 and e_2. $\square$

Taming biseparations

In light of Proposition 4.9, we want to show that for every 2-separation of a frame matroid M, there exists a biased graph representing M for which the corresponding biseparation is type 1 or 2. We first show that there is always such a representation in which the biseparation is type 1, 2, or 3. In preparation for the more general form of Theorem 4.7 we wish to prove, we now consider matroidals. We say a matroidal $\mathcal{M}=(M, L)$ is connected if M is connected.

Lemma 4.10. *Let $\mathcal{M}=(M, L)$ be a connected frame matroidal. For every 2-separation (A, B) of M, there exists an L-biased representation of $\mathcal{M}$ for which (A, B) is type 1, 2, or 3.*

Proof. Choose an L-biased representation Ω of (M, L) for which Ω is not balanced (any balanced representation can be turned into an unbalanced one by a pinch or roll-up operation, so this is always possible). Let $S = V(A) \cap V(B)$ in Ω. Let $\{A_1, \dots, A_h\}$ be the partition of A and $\{B_1, \dots, B_k\}$ the partition of B so that every $\Omega[A_i]$ is a component of the biased graph $\Omega[A]$ and every $\Omega[B_j]$ is a component of the biased graph $\Omega[B]$. Call the graphs $\Omega[A_1], \dots, \Omega[A_h], \Omega[B_1], \dots, \Omega[B_k]$ *parts*. For every $1 \leq i \leq h$ (resp. $1 \leq j \leq k$) let $\delta_A^i = 1$ ($\delta_B^j = 1$) if $\Omega[A_i]$ is balanced ($\Omega[B_j]$ is balanced) and $\delta_A^i = 0$ ($\delta_B^j = 0$) otherwise. Then $\lambda_M(A, B) = 2 = 1 + |S| - \sum_{i=1}^{h} \delta_A^i - \sum_{j=1}^{k} \delta_B^j$. Since each vertex in S is in exactly one $\Omega[A_i]$ and exactly one $\Omega[B_j]$, doubling both sides of this equation and rearranging, we obtain

$$2 = \sum_{i=1}^{h} \left(|S \cap V(A_i)| - 2\delta_A^i\right) + \sum_{j=1}^{k} \left(|S \cap V(B_j)| - 2\delta_B^j\right).$$

If a part is balanced, it must contain at least two vertices in S (else M is not connected by the discussion in Section 4.1.2), so every term in the sums on the right hand side of the above equation is nonnegative. In particular, letting t be the number of vertices in S contained in a part, a balanced part will contribute $t - 2$ to the sum, and an unbalanced part will contribute t. Call a part *neutral* if it is balanced and contains exactly two vertices in S. Since the total sum is two, the possibilities for the parts of $\Omega[A]$ and $\Omega[B]$ are:

(a) two unbalanced parts each with one vertex in S and all other parts neutral,

(b) one unbalanced part with two vertices in S and all other parts neutral,

(c) one balanced part with three vertices in S, one unbalanced part with one vertex in S, and all other parts neutral,

(d) two balanced parts with three vertices in S and all other parts neutral, or

(e) one balanced part with four vertices in S and all other parts neutral.

These possibilities are illustrated in Figure 4.3.

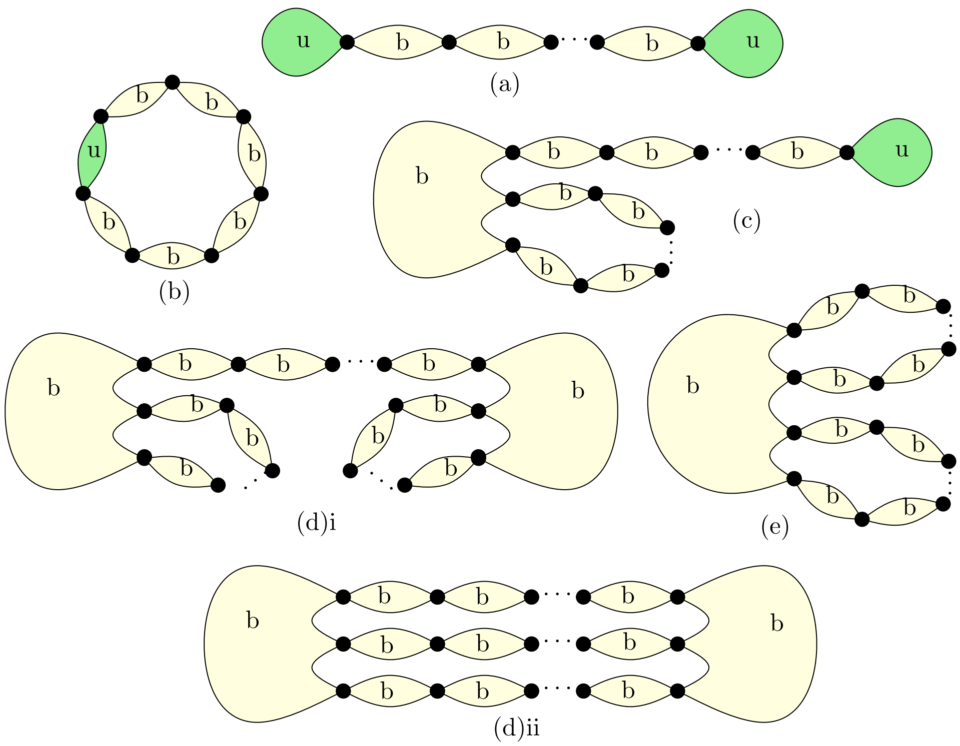


Figure 4.3: The possible decompositions of Ω into the parts of $\Omega[A]$ and $\Omega[B]$.

Observe that every component of $M \setminus B$ (resp. $M \setminus A$) is contained in some part $\Omega[A_i]$ ($\Omega[B_j]$), and every part of $\Omega[A]$ (resp. $\Omega[B]$) is a union of components of $M \setminus B$ (resp. $M \setminus A$). Hence every circuit of M is either contained in a single part, or traverses every part. It is an elementary property of 2-separations that if $A_1, \ldots, A_l$ and $B_1, \ldots, B_m$ are the components of $M \setminus B$ and $M \setminus A$ respectively, and (X, Y) is any partition of $A_1, \ldots, A_l, B_1, \ldots, B_m$, then $(\bigcup X, \bigcup Y)$ is a 2-separation of M (this can be verified by straightforward rank calculations). Hence if $\Omega[D]$ is a neutral part, (D, D^c) is a 2-separation of M. Since $\Omega[D]$ is balanced and connected, the biseparation (D, D^c) of Ω is type 2.

Suppose there are exactly t neutral parts. Repeatedly applying Proposition 4.9, we obtain a biased graph Ω' with links $e'_1, \ldots, e'_t$ together with balanced biased graphs $\Omega_1, \ldots, \Omega_t$ each with a distinguished edge $e_i \in E(\Omega_i)$ so that Ω is obtained as a repeated link-sum of Ω' with each Ω_i on edges e_i and e'_i. It follows from the fact that every circuit of M is either contained in a single part or traverses every part that elements $e'_1, \ldots, e'_t$ are all in series in $F(\Omega')$. We use this fact to find another biased graph representing M in which the biseparation (A, B) is type 1, 2, or 3. First, in Ω' contract edges $e'_1, \ldots, e'_{t-1}$: let $\Omega'' = \Omega'/\{e'_1, \ldots, e'_{t-1}\}$. Now subdivide link e'_t to form a path P with edge set $\{e'_1, \ldots, e'_t\}$

to obtain a new biased graph Ψ, in which a cycle containing P is balanced if and only if the corresponding cycle in Ω'' containing e'_t is balanced. Since elements $e'_1, \ldots, e'_t$ are in series in $F(\Omega')$, $F(\Psi) \cong F(\Omega')$. For the same reason, any biased graph Ψ' obtained from Ψ by permuting the order in which edges $e'_1, \ldots, e'_t$ occur in P has $F(\Psi') \cong F(\Psi)$. Let Φ' be the biased graph obtained from Ψ by arranging the edges of P in an order so that an initial segment of the path has all of the edges e'_i whose corresponding neutral parts of Ω are in A, followed by the edges e'_i whose corresponding neutral parts are in B. Now let Φ be the biased graph obtained by repeatedly link-summing each Ω_i on edge e'_i, $i \in \{1, \ldots, t\}$. Then $F(\Phi) \cong F(\Omega)$. Since every unbalanced loop in Ω remains an unbalanced loop in Φ, Φ is an L-biased representation of $\mathcal{M}$. Since at least one of $\Phi[A]$ or $\Phi[B]$ is connected, and $\Phi[A]$ and $\Phi[B]$ meet in at most three vertices, in Φ biseparation (A, B) is type 1, 2, or 3. □

Taming type 3

We now do away with type 3 biseparations.

Theorem 4.11. *Let $\mathcal{M}$=(M, L) be a connected frame matroidal. For every 2-separation (A, B) of M, there exists an L-biased representation of $\mathcal{M}$ for which (A, B) is type 1 or 2.*

Proof. By Lemma 4.10 we may choose an L-biased graph Ω representing M in which biseparation (A, B) is type 1, 2, or 3. Suppose it is type 3. Let $\{x, y, z\} = V(A) \cap V(B)$ in Ω. We first claim that all cycles crossing (A, B) through the same pair of vertices $\{x, y\}$, $\{y, z\}$, or $\{z, x\}$ have the same bias. To see this, let C and C' be two cycles crossing (A, B) at $\{x, y\}$. We may assume without loss of generality that $\delta(z) \cap C \subseteq A$. Let $C \cap A = P$ and $C \cap B = Q$, and let $C' \cap A = P'$ and $C' \cap B = Q'$. By Observation 1.18, P may be transformed to P' by a sequence of reroutings in $P \cup P'$. Since every rerouting in this sequence is along a balanced cycle, by Lemma 1.19, C and $P' \cup Q$ have the same bias. Similarly, Q may be transformed into Q' via a sequence of reroutings along balanced cycles in $Q \cup Q'$, so $P' \cup Q$ and $P' \cup Q' = C'$ have the same bias. *I.e.*, C and C' are of the same bias.

There are three types of cycles crossing the 2-separation: those crossing at $\{x, y\}$, those crossing at $\{x, z\}$, and those crossing at $\{y, z\}$; by the claim, all cycles of the same type have the same bias. Let us denote the sets of these cycles by $\mathcal{C}_{xy}$, $\mathcal{C}_{xz}$ and $\mathcal{C}_{yz}$, respectively.

We claim that at least one of these sets contains an unbalanced cycle. For suppose the contrary. If the biseparation of Ω is type 3(a), then Ω is balanced with $|V(A) \cap V(B)| = 3$; but then (A, B) is not a 2-separation of $F(\Omega)$, a contradiction. If the biseparation is type 3(b), then M is not connected, a contradiction.

Suppose first that just one of our sets of cycles, say $\mathcal{C}_{xy}$, contains an unbalanced cycle C. Suppose further that in one of $\Omega[A]$ or $\Omega[B]$ there is a z-C path P avoiding x and that in the other side there is a z-C path Q avoiding y. Then $C \cup P \cup Q$ is a theta subgraph of

Ω containing exactly two balanced cycles, a contradiction. So no such pair of paths exist. Hence either:

1. both $\Omega[A]$ and $\Omega[B]$ contain a z-C path, but either every z-C path in both meets x or every z-C path in both meets y, or,

2. one of $\Omega[A]$ or $\Omega[B]$ has no z-C path.

In case 1, either x or y is a cut vertex of Ω, and we find that $F(\Omega)$ is not connected, a contradiction. Hence we have case 2. Suppose without loss of generality that $\Omega[B]$ does not contain a z-C path. We have a biseparation of type 3(b). Let us denote by B_1 the balanced component and by B_2 the unbalanced component of biased graph $\Omega[B]$. Let Φ be the biased graph obtained as follows. Detach B_2 from $\Omega[A]$, and form a signed graph $(G, \mathcal{B}_\Sigma)$ from $\Omega[A]$ by identifying vertices x and y, and setting $\Sigma = \delta(y) \cap A$. Now identify vertex x in B_1 with vertex z in $(G, \mathcal{B}_\Sigma)$, and identity vertex y in B_1 with vertex z in B_2 (Figure 4.4). Assign biases to cycles in Φ in $\Phi[A]$ according to their bias in $(G, \mathcal{B}_\Sigma)$ and in $\Phi[B]$ according to their bias in Ω. It is straightforward to verify that the circuits of $F(\Phi)$ and $F(\Omega)$ coincide, so $F(\Phi) \cong M$. The biseparation (A, B) in Φ is type 1, and since edges representing elements in L remain unbalanced loops in Φ, Φ is an L-bias representation of M as required.

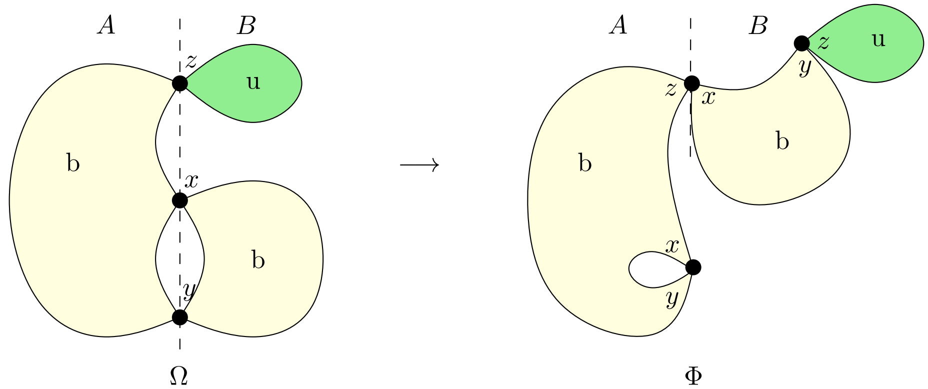


Figure 4.4: Finding a representation in which the biseparation is type 1.

So now assume that at least two of the three sets $\mathcal{C}_{xy}, \mathcal{C}_{yz}$ and $\mathcal{C}_{xz}$ contain an unbalanced cycle. Then our biseparation is type 3(a). If just two of these sets contain an unbalanced cycle — say $\mathcal{C}_{xz}$ does not — then M is graphic, represented by the graph obtained from Ω by splitting vertex y (Figure 4.5). Now pinching vertices x and z yields an L-biased graph representing M in which biseparation (A, B) is type 1.

The remaining case is that all three of $\mathcal{C}_{xy}, \mathcal{C}_{xz}$, and $\mathcal{C}_{yz}$ contain unbalanced cycles, so every cycle crossing (A, B) is unbalanced. In this case every circuit of M contained in A or B is a balanced cycle and every circuit meeting both A and B is either a pair of tight

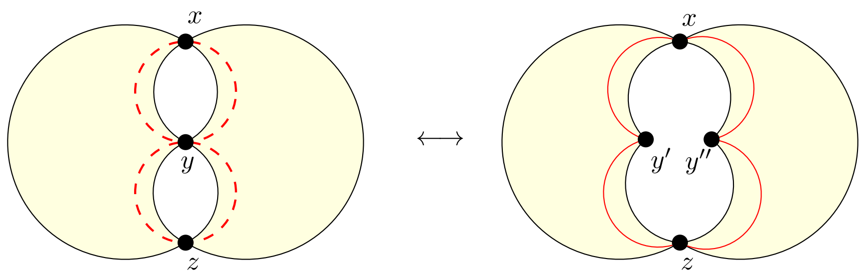


Figure 4.5: If just $\mathcal{C}_{xy}$ and $\mathcal{C}_{yz}$ contain unbalanced cycles, then $F(\Omega)$ is graphic.

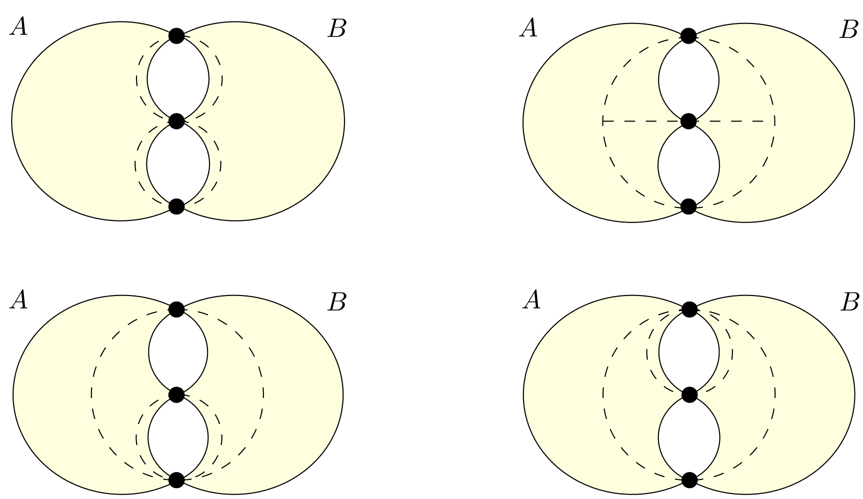


Figure 4.6: Circuits of $F(\Omega)$ meeting both sides of the 2-separation.

handcuffs meeting at a vertex in $V(A) \cap V(B)$, or an contrabalanced theta (Figure 4.6). Let Ω' be the signed graph obtained from Ω as follows. Split vertices y and z, replacing y with two new vertices y' and y'', putting all edges $uy \in A$ incident with y' and all edges $vy \in B$ incident with y'' and similarly replacing z with two new vertices z' and z'', putting all edges $uz \in A$ incident with z' and all edges $vz \in B$ incident with z''. Now identify vertices y' and z' and identify vertices y'' and z'', and put the edges in $\delta(z) \cap A$ and in $\delta(z) \cap B$ in Σ (Figure 4.7). It is easily checked that a subset $C \subseteq E$ is a circuit in $F(\Omega)$ if and only if C is a circuit in $F(\Omega')$, so $F(\Omega') \cong F(\Omega)$. Since in this case L is empty, Ω' is an L-biased graph representing M, as required. $\square$

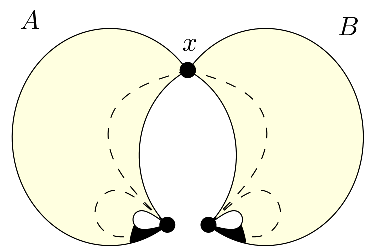


Figure 4.7: Dashed edges correspond to the tight handcuff shown in the upper left graph of Figure 4.6; all edges in the shaded regions are in Σ.

4.2.3 Proof of Theorem 4.7

With this we are ready to prove the main result of this section.

Lemma 4.12. *Let $\mathcal{M}_1=(M_1, L_1)$ and $\mathcal{M}_2=(M_2, L_2)$ be connected frame matroidals on E_1, E_2, respectively. If for $i = 1, 2$, $e_i \in E_i \setminus L_i$, then $(M_1 \, {}^{e_1}\!\oplus_2^{e_2} \, M_2, L_1 \cup L_2)$ is frame if and only if one of the following holds.*

1. *$L_i = \emptyset$ and M_i is graphic for one of $i = 1$ or $i = 2$.*

2. *$(M_i, L_i \cup \{e_i\})$ is frame for both $i = 1, 2$.*

Proof. The "if" direction follows immediately from Proposition 4.8. Conversely, consider a frame matroidal resulting from a 2-sum, $(M_1 \, {}^{e_1}\!\oplus_2^{e_2} \, M_2, L_1 \cup L_2)$. By Theorem 4.11 there is a $(L_1 \cup L_2)$-biased representation Ω of the 2-sum in which the biseparation $(E_1 \setminus e_1, E_2 \setminus e_2)$ is type 1 or 2. By Proposition 4.9, there are biased graphs Ω_1 on E_1 and Ω_2 on E_2 such that Ω is a link- or loop-sum on e_1 and e_2. If Ω is a link-sum, then 1 holds. If Ω is a loop-sum, then both Ω_i are $(L_i \cup \{e_i\})$-biased representations of M_i, so both matroidals $(M_i, L_i \cup \{e_i\})$ are frame ($i \in \{1, 2\}$). $\square$

Lemma 4.12 immediately implies Theorem 4.7.

Proof of Theorem 4.7. Apply Lemma 4.12 with $L_1 = L_2 = \emptyset$. $\square$

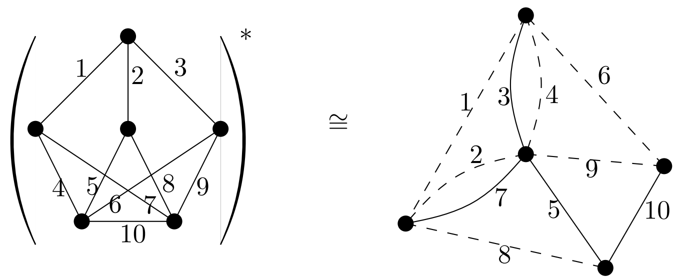


Figure 4.8: The unique biased graph representing $M^*(K'_{3,3})$; dashed edges form the signature Σ.

4.3 Excluded minors

In this section we use Theorem 4.7 to construct a family $\mathcal{E}_0$ of 9 excluded minors with connectivity 2. We then show that any excluded minor of connectivity 2 that is not in $\mathcal{E}_0$ has a special structure.

4.3.1 The excluded minors $\mathcal{E}_0$

The graph obtained from $K_{3,3}$ by adding an edge e' linking two non-adjacent vertices is denoted $K'_{3,3}$; we also denote the corresponding element of $M^*(K'_{3,3})$ by e'. Let

$$\begin{aligned}\mathcal{E}_0 = &\{U_{2,4} \oplus_2 M^*(H) : H \in \{K_5, K_{3,3}, K'_{3,3}\}\} \\ &\cup \{M^*(H_1) \oplus_2 M^*(H_2) : H_1, H_2 \in \{K_5, K_{3,3}, K'_{3,3}\}\},\end{aligned}$$

where the 2-sum is taken on e' whenever H, H_1 or H_2 is $K'_{3,3}$.

There are three biased graphs representing $U_{2,4}$, two biased graphs representing $M^*(K_5)$, and just one biased graph representation of $M^*(K_{3,3})$ [38]. These are shown in Figure 1.5.

Lemma 4.13. *The unique biased graph representing $M^*(K'_{3,3})$ is that shown in Figure 4.8.*

Proof. $M^*(K'_{3,3})$ is a 3-connected single element coextension of $M^*(K_{3,3})$, which is uniquely represented. Hence a biased graph representing $M^*(K'_{3,3})$ has five vertices, no two elements in series, and the property that contracting edge e' yields the biased graph representing $M^*(K_{3,3})$. The biased graph shown in Figure 4.8 is the only such biased graph. □

Theorem 4.14. *The matroids in $\mathcal{E}_0$ are excluded minors for the class of frame matroids.*

Proof. Let $M_1 \oplus_2 M_2 \in \mathcal{E}_0$, with M_1 one of $U_{2,4}$, $M^*(K_5)$, $M^*(K_{3,3})$, or $M^*(K'_{3,3})$ and M_2 one of $M^*(K_5)$, $M^*(K_{3,3})$, or $M^*(K'_{3,3})$. Since neither M_1 nor M_2 is graphic and M_2 has no representation with a loop, by Theorem 4.7 $M_1 \oplus_2 M_2$ is not frame. Since every proper minor of $U_{2,4}$, $M^*(K_5)$, and $M^*(K_{3,3})$ is graphic, and for every $e \neq e'$, both $M^*(K'_{3,3}) \setminus e$ and $M^*(K_{3,3})/e$ are graphic, every proper minor of $M_1 \oplus_2 M_2$ is a 2-sum of a graphic matroid and a frame matroid. Hence by Theorem 4.7, every proper minor of $M_1 \oplus_2 M_2$ is frame. $\square$

4.3.2 Other excluded minors of connectivity 2

We now investigate excluded minors of connectivity 2 that are not in $\mathcal{E}_0$. We show that any such excluded minor has the following structure. For a matroid M and subset $L \subseteq E(M)$, the matroid obtained by taking a 2-sum of a copy of $U_{2,4}$ on each element in L is denoted $M \overset{L}{\oplus}_2 U_{2,4}$.

Theorem 4.15. *Let M be an excluded minor for the class of frame matroids. If M has connectivity 2 and is not in $\mathcal{E}_0$, then $M = N \overset{L}{\oplus}_2 U_{2,4}$ for a 3-connected frame matroid N.*

We prove Theorem 4.15 via three lemmas, each of which requires some explanation.

A collection $\mathcal{N}$ of connected matroids is *1-rounded* if it has the property that whenever a connected matroid M has a minor $N \in \mathcal{N}$, then every element $e \in E(M)$ is contained in some minor N' of M with $N' \in \mathcal{N}$. The following is a result of Seymour ([23] Section 11.3).

Theorem 4.16. *The collection $\{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3})\}$ is 1-rounded.*

Lemma 4.17. *Let M_1, M_2 be nontrivial matroids and suppose $M_1 \,{}^{e_1}\!\oplus_2^{e_2} M_2$ is an excluded minor for the class of frame matroids, for some $e_1 \in E(M_1)$ and $e_2 \in E(M_2)$. Then either $M_1 \,{}^{e_1}\!\oplus_2^{e_2} M_2 \in \mathcal{E}_0$ or both M_1 and M_2 are non-binary frame matroids.*

Proof. By minimality, M_1 and M_2 are both frame. By Theorem 4.7, neither M_1 nor M_2 is graphic. Thus each contains an excluded minor for the class of graphic matroids, namely, one of $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, or $M^*(K_{3,3})$. By Theorem 4.16, for $i \in \{1,2\}$, matroid M_i contains a minor N_i isomorphic to one of $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, $M^*(K_{3,3})$, or $M^*(K'_{3,3})$ with $e_i \in E(N_i)$; we may assume that if $N_i \cong M^*(K'_{3,3})$ then e_i is edge e'. Since neither F_7 nor F_7^* are frame, neither N_1 nor N_2 is isomorphic to F_7 or F_7^*. If $N_1 \,{}^{e_1}\!\oplus_2^{e_2} N_2 \in \mathcal{E}_0$, then by minimality, for $i \in \{1,2\}$, $M_i \cong N_i$ and $M_1 \,{}^{e_1}\!\oplus_2^{e_2} M_2 \cong N_1 \,{}^{e_1}\!\oplus_2^{e_2} N_2$. Otherwise, $N_1 \cong N_2 \cong U_{2,4}$, so both M_1 and M_2 are non-binary. $\square$

Our next lemma requires two basic facts. The first is a result of Bixby; the second was proved independently by Brylawski and Seymour.

Proposition 4.18 ([23], Proposition 11.3.7)**.** *Let M be a connected matroid having a $U_{2,4}$ minor and let $e \in E(M)$. Then M has a $U_{2,4}$ minor using e.*

Proposition 4.19 ([23], Proposition 4.3.6)**.** *Let N be a connected minor of a connected matroid M and suppose that $e \in E(M) \setminus E(N)$. Then at least one of $M \setminus e$ and M/e is connected and has N as a minor.*

Lemma 4.20. *Let $M_1 \;{}^{e_1}\!\oplus_2^{e_2}\; M_2$ be an excluded minor for the class of frame matroids with both M_1 and M_2 non-binary. Then one of M_1 or M_2 is isomorphic to $U_{2,4}$.*

Proof. Suppose for a contradiction that neither M_1 nor M_2 is isomorphic to $U_{2,4}$. By Propositions 4.18 and 4.19 we may choose an element $f \in E(M_1) \setminus \{e_1\}$ so that a matroid M_1' obtained from M_1 by either deleting or contracting f is connected and has $U_{2,4}$ as a minor. Since $M_1' \;{}^{e_1}\!\oplus_2^{e_2}\; M_2$ is a minor of $M_1 \;{}^{e_1}\!\oplus_2^{e_2}\; M_2$, by minimality $M_1' \;{}^{e_1}\!\oplus_2^{e_2}\; M_2$ is frame. By Theorem 4.7, $(M_2, \{e_2\})$ is frame. Similarly, $(M_1, \{e_1\})$ is frame. Hence by Thereom 4.7, $M_1 \;{}^{e_1}\!\oplus_2^{e_2}\; M_2$ is frame, a contradiction. □

The final lemma we need to prove Theorem 4.15 tells us that in our current setting, 2-separations having one side just a 3-circuit cannot interact with each other. The restriction of a matroid M to a subset A of its ground set E is denoted $M|A$; the complement of a subset $A \subseteq E$ is denoted A^c.

Lemma 4.21. *Let M be a connected matroid on E with $|E| \geq 6$ and assume that for every 2-separation (A, A^c) of M, one of $M|A$ or $M|A^c$ is a circuit of size 3. If (A, A^c) and (B, B^c) are 2-separations with both $M|A$ and $M|B$ a circuit of size 3, then either $A = B$ or $A \cap B = \emptyset$.*

Proof. Suppose for a contradiction that $\emptyset \neq A \cap B \neq A$. We consider two cases depending on the size of $A \cap B$. If $|A \cap B| = 1$ then let $A \cap B = \{e\}$ and consider the separation $(A \setminus \{e\}, A^c \cup \{e\})$. Since $B \setminus \{e\}$ spans e, $r(A^c) = r(A^c \cup \{e\})$. But this implies that $(A \setminus \{e\}, A^c \cup \{e\})$ is a 2-separation, a contradiction as neither side has size three.

Next suppose $|A \cap B| = 2$. Then, summing the orders of the separations $(A \cap B, A^c \cup B^c)$ and $(A \cup B, A^c \cap B^c)$, by submodularity, we have

$$\begin{aligned}
\lambda_M(A \cap B, A^c \cup B^c)) &+ \lambda_M(A \cup B, A^c \cap B^c) \\
&= r(A \cap B) + r(A^c \cup B^c) + r(A \cup B) + r(A^c \cap B^c) - 2r(M) + 2 \\
&\leq r(A) + r(A^c) + r(B) + r(B^c) - 2r(M) + 2 \\
&= \lambda_M(A, A^c) + \lambda_M(B, B^c) = 4.
\end{aligned}$$

As M is connected, each of $\lambda_M(A \cap B, A^c \cup B^c))$ and $\lambda_M(A \cup B, A^c \cap B^c)$ is at least two, so this implies that $(A \cap B, A^c \cup B^c)$ is a 2-separation, again a contradiction. □

Proof of Theorem 4.15. Let M be an excluded minor for the class frame matroids, and suppose M has connectivity 2 and $M \notin \mathcal{E}_0$. By Lemma 4.17, whenever M is written as a 2-sum, each term of the sum is non-binary, and by Lemma 4.20 one of these terms is isomorphic to $U_{2,4}$. Hence every 2-separation (A, A^c) of M has one of $M|A$ or $M|A^c$ a circuit of size 3. By Lemma 4.21 the 3-circuits corresponding to these $U_{2,4}$ minors are pairwise disjoint. Therefore we may write $M = N \overset{L}{\oplus}_2 U_{2,4}$, where N is a 3-connected matroid. □

4.3.3 Excluded minors for the class of frame matroidals

Theorem 4.15 says that every excluded minor of connectivity 2 for the class of frame matroids that is not in $\mathcal{E}_0$ can be expressed in the form $N \overset{L}{\oplus}_2 U_{2,4}$, where N is a 3-connected frame matroid. In this section we equate the problem of representing a matroid of this form as a biased graph to frame matroidals. We begin with the following key result.

Theorem 4.22. *Let N be a matroid and let $L \subseteq E(N)$. Then $N \overset{L}{\oplus}_2 U_{2,4}$ is frame if and only if the matroidal (N, L) is frame.*

Proof. Let $L = \{e_1, \dots, e_k\}$ and repeatedly apply Lemma 4.12:

$$\begin{aligned}
N \overset{L}{\oplus}_2 U_{2,4} \text{ is frame} &\iff \Big(\big(N \overset{\{e_1\dots e_{k-1}\}}{\oplus_2} U_{2,4}\big) {}^{e_k}\!\oplus_2 U_{2,4}\,,\, \emptyset\Big) \text{ is frame} \\
&\iff \Big(\big(N \overset{\{e_1\dots e_{k-2}\}}{\oplus_2} U_{2,4}\big) {}^{e_{k-1}}\!\oplus_2 U_{2,4}\,,\, \{e_k\}\Big) \text{ is frame} \\
&\iff \Big(\big(N \overset{\{e_1\dots e_{k-3}\}}{\oplus_2} U_{2,4}\big) {}^{e_{k-2}}\!\oplus_2 U_{2,4}\,,\, \{e_{k-1}, e_k\}\Big) \text{ is frame} \\
&\quad\vdots \\
&\iff \big(N, \{e_1, \dots, e_k\}\big) \text{ is frame.}
\end{aligned}$$

□

So that we may work directly with matroidals, we now define minors of matroidals. Any matroidal (N, K) obtained from a matroidal (M, L) by a sequence of the operations of deleting or contracting an element not in L or removing an element from L is a *minor* of (M, L). Clearly, the class of frame matroidals is minor-closed, and so we may ask for its set of excluded minors. We have the following immediate corollary of Theorem 4.22.

Corollary 4.23. *Let N be a matroid and let $L \subseteq E(N)$. Then $N \overset{L}{\oplus}_2 U_{2,4}$ is an excluded minor for the class of frame matroids if and only if (N, L) is an excluded minor for the class of frame matroidals.*

Our search for the remaining excluded minors of connectivity 2 for the class of frame matroids is therefore equivalent to the problem of finding excluded minors for the class of frame matroidals.

There are three ways to represent the 3-circuit $U_{2,3}$ as a biased graph: a balanced triangle, a contrabalanced theta on two vertices, or as a pair of loose handcuffs consisting of a link and two unbalanced loops; no biased graph representation of $U_{2,3}$ has all three elements as unbalanced loops. Evidently therefore, $U_{2,3} \overset{E(U_{2,3})}{\oplus_2} U_{2,4}$ is not frame. Let us denote this matroid N_9. *I.e.*,

$$N_9 = U_{2,3} \overset{E(U_{2,3})}{\oplus_2} U_{2,4}.$$

Proposition 4.24. *N_9 is an excluded minor for the class of frame matroids.*

Proof. By Corollary 4.23, the above statement is equivalent to the statement that $(U_{2,3}, E(U_{2,3}))$ is an excluded minor for the class of frame matroidals. There is no biased graph representing $U_{2,3}$ in which all three elements are unbalanced loops, so the matroidal $(U_{2,3}, E(U_{2,3}))$ is not frame. For every two element subset $L \subseteq E(U_{2,3})$ the matroidal $(U_{2,3}, L)$ is frame: a link between two vertices together with an unbalanced loop on each endpoint, where the two unbalanced loops represent the two elements in L is an L-biased graph representing $U_{2,3}$. $\square$

The matroid N_9 is the only excluded minor for the class of frame matroids of the form $N \overset{L}{\oplus}_2 U_{2,4}$ with $|L| \geq 3$:

Theorem 4.25. *Let N be a 3-connected matroid, let $L \subseteq E(N)$, and suppose that $M = N \overset{L}{\oplus}_2 U_{2,4}$ is an excluded minor for the class of frame matroids. If $|L| \geq 3$ then $M \cong N_9$.*

Proof. Let $L = \{e_1, \ldots, e_k\}$. By Corollary 4.23, (N, L) is an excluded minor for the class of frame matroidals. By minimality then, $(N, \{e_2, \ldots, e_k\})$ is frame. Let Ω be a $\{e_2, \ldots, e_k\}$-biased graph representing $(N, \{e_2, \ldots, e_k\})$. In Ω, edges e_2, e_3 are unbalanced loops and e_1 is a link. Since N is 3-connected, Ω is 2-connected. Hence there are disjoint paths P, Q linking the endpoints of e_1 and the vertices incident to e_2 and e_3. Contracting P and Q yields $U_{2,3}$ as a minor with $E(U_{2,3}) = \{e_1, e_2, e_3\}$. By minimality and Proposition 4.24 therefore, $N \cong U_{2,3}$ and $L = \{e_1, e_2, e_3\}$. $\square$

4.4 Proof of Theorem 4.1

We are now ready to exhibit all of the the matroids in the set $\mathcal{E}$ and to prove the main result of the chapter.

Theorem 4.1. *Let M be an excluded minor for the class of frame matroids, and suppose M is not 3-connected. Then either M is isomorphic to a matroid in $\mathcal{E}$ or M is the 2-sum of a 3-connected non-binary frame matroid and $U_{2,4}$.*

The set $\mathcal{E}$ of excluded minors in the statement of Theorem 4.1 contains $\mathcal{E}_0$ and N_9. In this section we exhibit the remaining matroids in $\mathcal{E}$, and show that any other excluded minor of connectivity 2 is a 2-sum of a 3-connected non-binary matroid and $U_{2,4}$. We do this using matroidals. We show that the nine matroidals $\mathcal{M}_0, \ldots, \mathcal{M}_8$ illustrated in Figure 4.9 are excluded minors for the class of frame matroidals. Each matroidal $\mathcal{M}_i$=(M_i, L_i), $i \in \{0, \ldots, 8\}$, is given as the frame matroid $M_i = F(\Omega_i)$ represented by a biased graph $\Omega_i = (G, \mathcal{B})$, where the graph G is shown in Figure 4.9 and collections $\mathcal{B}$ are as listed. Each matroidal's set L_i is the set $\{e_1, e_2\}$, consisting of the pair of elements represented by edges e_1, e_2 in each graph.

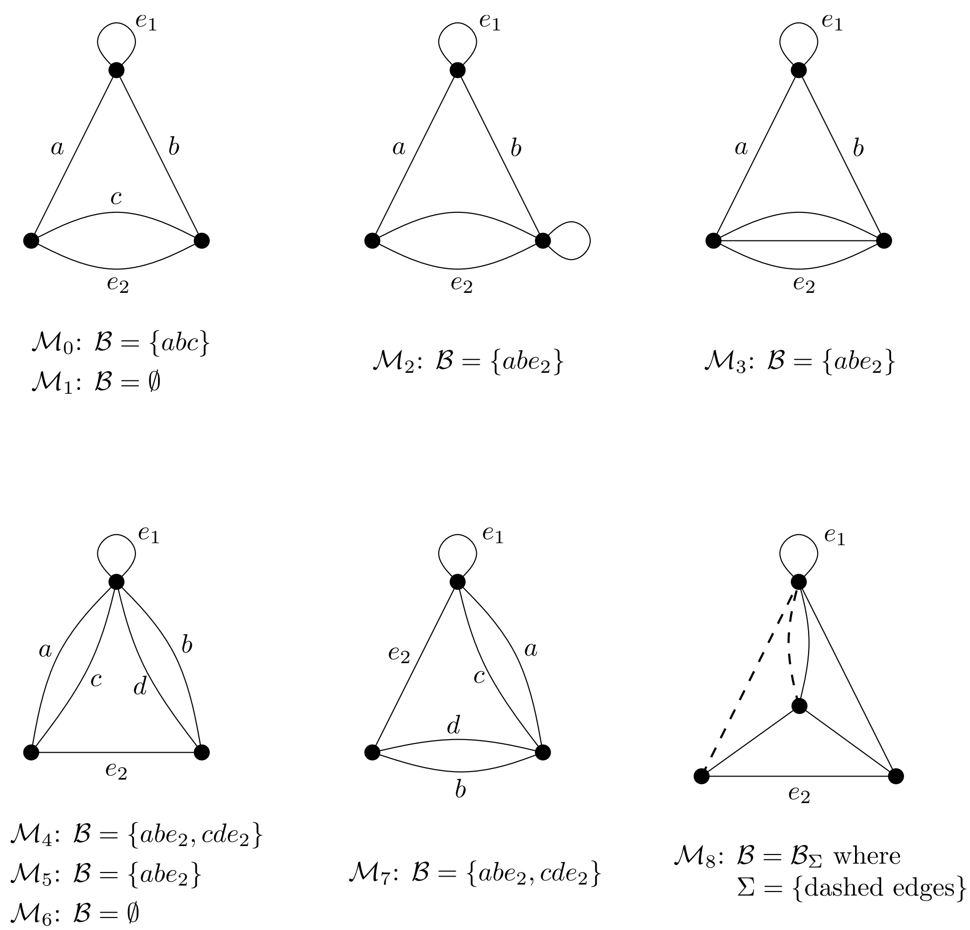


Figure 4.9: Excluded minors for the class of frame matroidals with $|L| > 1$.

Note that the excluded minor N_9 is given by the matroidal $\mathcal{M}_0=(M_0, L_0)$: $M_0 \overset{L_0}{\oplus}_2 U_{2,4} \cong N_9$ (it is straightforward to verify that the circuits of $M_0 \overset{L_0}{\oplus}_2 U_{2,4}$ and those of N_9 coincide). In fact, $U_{2,3}$ gives rise to four excluded minors for the class of frame matroidals, each yielding N_9 as corresponding excluded minor for the class of frame matroids, as follows. Write $E = E(U_{2,3})$, choose a subset $S \subseteq E$, and let $N = U_{2,3} \overset{S}{\oplus}_2 U_{2,4}$. Set $L = E \setminus S$. Then $N \overset{L}{\oplus}_2 U_{2,4} \cong N_9$ and matoridal (N, L) is an excluded minor for the class of frame matroidals. The four choices for the size of S each thereby yield an excluded minor for the class of frame matroidals.

Matroidals $\mathcal{M}_1, \ldots, \mathcal{M}_8$ give rise to excluded minors for the class of frame matroids that we have not yet encountered. Let $\mathcal{E}_1 = \{N \overset{L}{\oplus}_2 U_{2,4} : (N, L) \in \{\mathcal{M}_0, \ldots, \mathcal{M}_8\}\}$. Let $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$.

The hard work in proving Theorem 4.1 is in showing that $\{\mathcal{M}_1, \ldots, \mathcal{M}_8\}$ is the complete list of excluded minors for the class of frame matroidals having $|L| = 2$. This is the content of Lemma 4.26.

Lemma 4.26. *Let N be a 3-connected matroid and let $L \subseteq E(N)$ with $|L| = 2$. If (N, L) is an excluded minor for the class of frame matroidals, then it is isomorphic to one of $\mathcal{M}_1, \ldots, \mathcal{M}_8$.*

Before proving Lemma 4.26, let us show that it implies Theorem 4.1.

Proof of Theorem 4.1. Let M be an excluded minor for the class of frame matroids, and suppose M is not 3-connected. By Theorem 4.15, either M is isomorphic to a matroid in $\mathcal{E}_0$ or $M = N \overset{L}{\oplus}_2 U_{2,4}$ for a 3-connected frame matroid N and a nonempty set L. So suppose $M \notin \mathcal{E}_0$. By Theorem 4.25, if $|L| \geq 3$ then $M \cong N_9$. If $|L| = 1$ then M is a 2-sum of N and $U_{2,4}$, and by Lemma 4.17 N is a non-binary. Finally, if $|L| = 2$, then by Corollary 4.23 (N, L) is an excluded minor for the class of frame matroidals. By Lemma 4.26, M is isomorphic to a matroid in $\mathcal{E}_1$. □

4.4.1 The excluded minors $\mathcal{E}_1$

Let us substantiate our claim that the matroids in $\mathcal{E}_1$ are excluded minors for the class of frame matroids.

Say a matroid M *series reduces* to a matroid M' if M' may be obtained from M by repeatedly contracting elements contained in a nontrivial series class. Series reduction of matroids is useful because matroidals consisting of a rank 2 matroid with a distinguished subset L of size 2 are aways frame:

Lemma 4.27. *Let (N, L) be a matroidal. If N has rank 2 and $|L| = 2$, then (N, L) is frame.*

Proof. We may assume N has no loops. Let $L = \{e_1, e_2\}$. Since N has rank 2, N is obtained from some uniform matroid $U_{2,m}$ by adding elements in parallel. We may assume that either

$e_1, e_2 \in E(U_{2,m})$ or that $e \in E(U_{2,m})$ and e_1 and e_2 are in the same parallel class. Let Ω be the contrabalanced biased graph representing $U_{2,m}$ with $V(\Omega) = \{u, v\}$, e_1 a loop incident to u, e_2 a loop incident to v if $e_2 \in E(U_{2,3})$, and all other elements represented by u-v edges. Let Ω' be the biased graph obtained by adding each element $f \in E(N) \setminus E(U_{2,m})$ in the same parallel class as an element $e \neq e_1$ as a u-v edge and declaring circuit ef balanced, and adding each element in $E(N) \setminus E(U_{2,3})$ in the same parallel class as e_1 as an unbalanced loop incident to u. Then Ω' is an L-biased representation of N. □

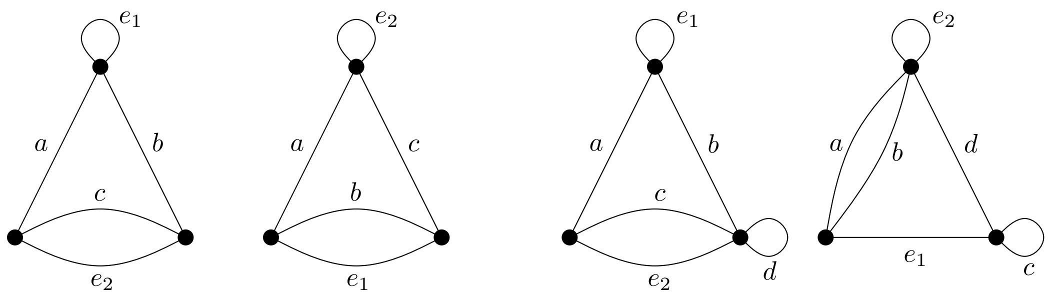
This tool in hand, we may now prove:

Proposition 4.28. *The matroidals $\mathcal{M}_0, \ldots, \mathcal{M}_7$ are excluded minors for the class of frame matroidals.*

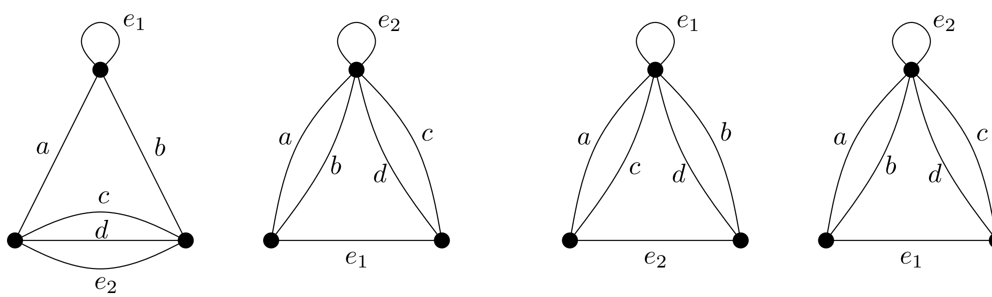
Proof. That $\mathcal{M}_0$ is an excluded minor follows immediately from Corollary 4.23, Proposition 4.24, and the fact that $M_0 \overset{L_0}{\oplus_2} U_{2,4} \cong N_9$. So suppose for a contradiction that for some $i \in \{1, \ldots, 7\}$, Ω is a L_i-biased graph representing $\mathcal{M}_i = (M_i, L_i) \in \{\mathcal{M}_1, \ldots, \mathcal{M}_7\}$. Let e_1, e_2 be the elements in L_i. For $j \in \{1, 2\}$, let v_j be the vertex of Ω incident to e_j. Since $\{e_1, e_2\}$ is not a circuit, $v_1 \neq v_2$. Since each of $M_1, \ldots, M_7$ has rank 3, $|V(\Omega)| = 3$; let u be the third vertex of Ω. Since none of $M_1, \ldots, M_7$ has a circuit of size three containing e_1 and e_2, there cannot be an edge linking v_1 and v_2. But then u is a cut-vertex of Ω, a contradiction since all of $M_1, \ldots, M_7$ are 3-connected.

We now show that every proper minor of each of $\mathcal{M}_1, \ldots, \mathcal{M}_7$ is frame. The biased graphs shown in Figure 4.9 show that each matroidal $(M_i, L_i \setminus e_2)$ is frame ($i \in \{1, \ldots, 7\}$). The biased graphs shown in Figure 4.10 show that also each matroidal $(M_i, L_i \setminus e_1)$ is frame. Any matroidal (N, L) obtained from one of $\mathcal{M}_1, \ldots, \mathcal{M}_7$ by contracting an element other than e_1 or e_2 has matroid N of rank 2, and so is frame by Lemma 4.27. Finally, suppose that (N, L) is a matroidal obtained from one of $\mathcal{M}_1, \ldots, \mathcal{M}_7$ by deleting an element e other than e_1, e_2. In all cases, the resulting matroid series reduces to a matroid N' of rank 2 with both $e_1, e_2 \in E(N')$ by the contraction of a single edge s (this is easy to see by considering the biased graph representations of Figure 4.10: in each case, one of the biased graphs representing $M_i \setminus e$ obtained by deleting an edge $e \in \{a, b, c, d\}$ has a vertex incident to just two edges). By Lemma 4.27 therefore, there is an L-biased representation Ω' of the series reduced matroid N'. Now let Ω be a biased graph obtained from Ω' by placing an edge representing s in series with the other edge t in its series class in $M_i \setminus e$ — that is, if t is a link, subdivide t to produce a path consisting of edges s and t, and if t is a loop, say incident to v, add a vertex w, add s as a v-w link, and place t as a loop incident to w. Evidently this corresponds to a coextention of N' to recover N, and Ω is an L-biased representation of N. □

Proposition 4.29. *The matroidal $\mathcal{M}_8$ is an excluded minor for the class of frame matroidals.*



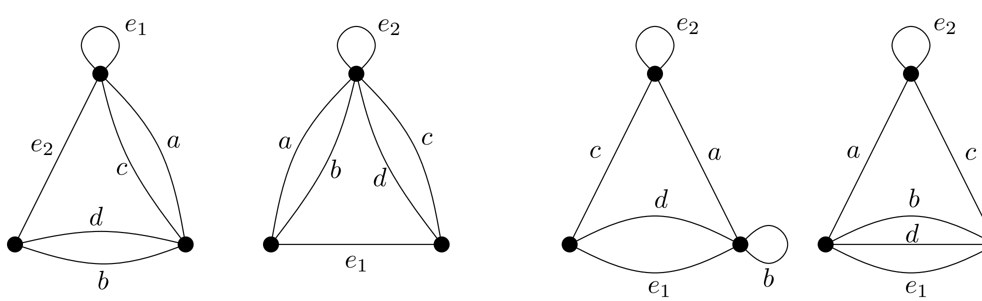
$\mathcal{M}_1$: $\mathcal{B} = \emptyset$ $\quad$ $\mathcal{M}_1$: $\mathcal{B} = \emptyset$ $\quad$ $\mathcal{M}_2$: $\mathcal{B} = \{abe_2\}$ $\quad$ $\mathcal{M}_2$: $\mathcal{B} = \{bde_1\}$



$\mathcal{M}_3$: $\mathcal{B} = \{abe_2\}$ $\quad$ $\mathcal{M}_3$: $\mathcal{B} = \emptyset$ $\quad$ $\mathcal{M}_4$: $\mathcal{B} = \{abe_2, cde_2\}$ $\quad$ $\mathcal{M}_4$: $\mathcal{B} = \{ace_1, bde_1\}$

$\mathcal{M}_5$: $\mathcal{B} = \{abe_2\}$

$\mathcal{M}_6$: $\mathcal{B} = \emptyset$



$\mathcal{M}_7$: $\mathcal{B} = \{abe_2, cde_2\}$ $\quad$ $\mathcal{M}_7$: $\mathcal{B} = \{ace_1\}$ $\quad$ $\mathcal{M}_5$: $\mathcal{B} = \{ace_1\}$ $\quad$ $\mathcal{M}_6$: $\mathcal{B} = \{ace_1\}$

Figure 4.10: Alternate representations of $\mathcal{M}_1, \ldots, \mathcal{M}_7$.

Proof. The matroid of $\mathcal{M}_8$ is the rank 4 wheel, *i.e.* the cycle matroid $M(W_4)$ where W_4 is the five vertex simple graph having one vertex of degree 4 and four vertices of degree 3 (Figure 4.11). Its distinguished subset $L = \{e_1, e_2\}$ consists of two nonadjacent edges both

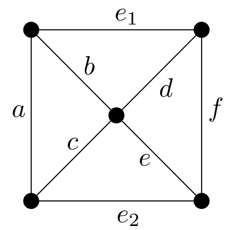


Figure 4.11: W_4.

of which have both ends of degree three. (Pinch the two ends of e_1 to obtain the biased graph representation shown in Figure 4.9.) We first show that $\mathcal{M}_8$ is not frame. Suppose for a contradiction that $M(W_4) \cong F(\Omega)$ for some L-biased graph Ω. Then e_1 and e_2 are both unbalanced loops in Ω; say e_i is incident to vertex u_i ($i \in \{1, 2\}$). There is a unique circuit C of size 4 in $M(W_4)$ containing $\{e_1, e_2\}$; say $C = e_1 e_2 f f'$. Elements f, f' must form a path of length 2 in Ω linking u_1 and u_2, say with interior vertex v. Since Ω is not balanced, $|V(\Omega)| = 4$; let v' be the fourth vertex of Ω. Note that since e_1, e_2 are not in a circuit of size 3 and not in any other circuit of size 4, all four remaining edges (other than e_1, e_2, f, f') must be incident to v'. Since $M(W_4)$ has no elements in series or in parallel, there must be an edge with ends u_1, v' and another edge with ends u_2, v'. This yields another 4-circuit in $F(\Omega)$ containing e_1 and e_2, a contradiction.

We now show that every proper minor of $\mathcal{M}_8$ is frame. For $i \in \{1, 2\}$, an $(L \setminus e_i)$-biased graph is obtained by pinching the ends of e_{3-i} in the graph W_4, so the matroidal $(M(W_4), L \setminus e_i)$ is frame. Now consider a matroidal obtained from $\mathcal{M}_8$ by deleting or contracting an element not in L. Up to symmetry there are only two such edges to consider, say elements d and f as shown in Figure 4.11. The biased graphs of Figure 4.12 show that deleting or contracting either of d or f yields a frame matroidal. These L-biased graphs may be obtained as follows.

- Contracting f in W_4 yields a graph in which e_1 and e_2 are incident to a common vertex. Rolling up the edges incident to that vertex yields an $\{e_1, e_2\}$-biased graph, so $(M(W_4)/f, \{e_1, e_2\})$ is frame.

- In $M(W_4)/d$ elements $\{e, f\}$ are parallel. In $M(W_4)/d \setminus f$, elements e and e_2 are in series, so $M(W_4)/d$ is represented by the graph obtained from W_4/d by replacing e_2 with the pair of parallel edges e, f and replacing the pair e, f with e_2. This yields a graph in which e_1 and e_2 are incident to a common vertex v. Now rolling up the edges in $\delta(v)$ yields an $\{e_1, e_2\}$-biased graph representing $M(W_4)/d$, so $(M(W_5)/d, \{e_1, e_2\})$ is frame.

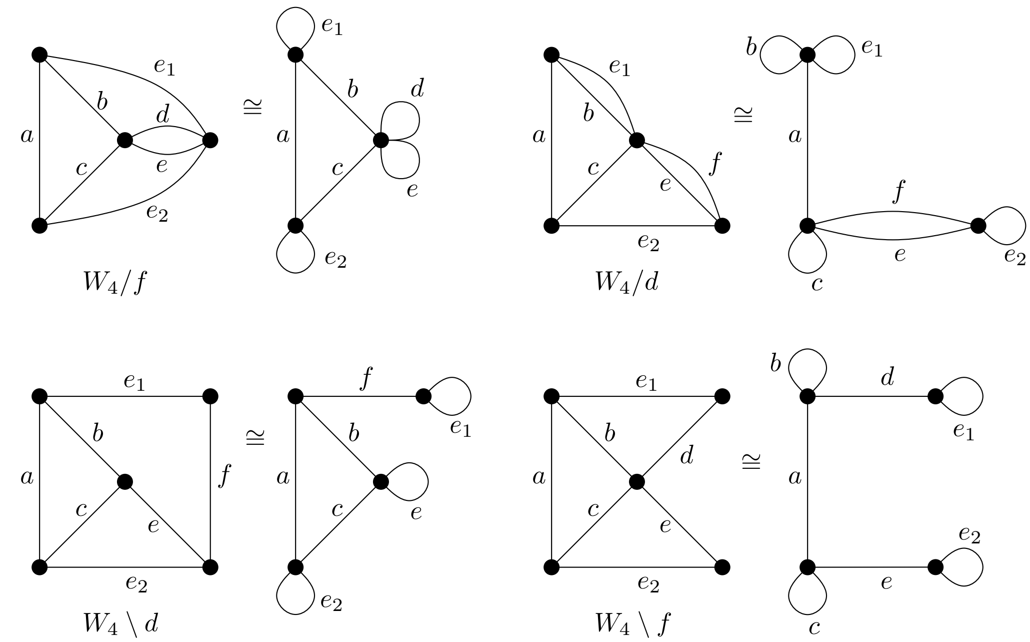


Figure 4.12: Any proper minor of W_4 is $\{e_1, e_2\}$-biased.

- In $M(W_4) \setminus d$ elements e_1 and f are in series, so the biased graph Ω obtained from $W_4 \setminus d$ by swapping edges e_1 and f represents $M(W_4) \setminus d$. Since e_1 and e_2 are incident to a common vertex v in Ω, the biased graph obtained by rolling up the edges in $\delta(v)$ is an $\{e_1, e_2\}$-biased graph representing $M(W_4) \setminus d$.
- Similarly, $M(W_4) \setminus f$ has series classes $\{e_1, d\}$ and $\{e_2, e\}$. Hence swapping edges e_1 and d, and swapping edges e_2 and e, we obtain a biased graph representing $M(W_4) \setminus f$ in which e_1 and e_2 are incident to a common vertex. Rolling up the edges e_1, e_2, b, c incident to that vertex yields an $\{e_1, e_2\}$-biased graph representing $M(W_4) \setminus f$. $\square$

4.4.2 Finding matroidal minors using configurations

To prove Lemma 4.26, we suppose $\mathcal{N}=(N, L)$ is an excluded minor for the class of frame matroidals with $|L| = 2$ that is not one of $\mathcal{M}_1, \ldots, \mathcal{M}_8$. We then work with a biased graph Ψ representing N to derive the contradiction that (N, L) contains one of $\mathcal{M}_0, \ldots, \mathcal{M}_8$ as a minor. When doing so, we are looking for biased graphs representing one of $\mathcal{M}_0, \ldots, \mathcal{M}_8$. Some of $\mathcal{M}_0, \ldots, \mathcal{M}_8$ share the same underlying graphs or have an underlying graph contained in the underlying graph of another (Figures 4.9 and 4.10). Since which of $\mathcal{M}_0, \ldots, \mathcal{M}_8$ we find as a minor of $\mathcal{N}$ is irrelevant, it is enough to determine the underlying graph of a minor of Ψ along with just enough information about the biases of its cycles to see that Ψ must contain one of $\mathcal{M}_0, \ldots, \mathcal{M}_8$ as a minor. We formalize this as follows.

A *configuration* $\mathcal{C}$ consists of a graph G with two distinguished edges e_1, e_2, together with a set $\mathcal{U}$ of cycles of G, which we call *unbalanced*. The configurations we find are those named $\mathcal{C}_1, \ldots, \mathcal{C}_4, \mathcal{C}'_4, \mathcal{C}''_4, \mathcal{C}_5, \ldots, \mathcal{C}_8$ in Figure 4.13, and $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}'_2$, and $\mathcal{D}_3$ in Figure 4.14. We say that a biased graph $\Omega=(G,\mathcal{B})$ *realises* configuration $\mathcal{C}=(G,\mathcal{U})$ if $\mathcal{B} \cap \mathcal{U} = \emptyset$. The following two lemmas guarantee that finding one of these configurations in Ψ implies that $\mathcal{N}$ contains one of $\mathcal{M}_0, \ldots, \mathcal{M}_8$ as a minor.

Lemma 4.30. *Let Ω be a biased graph that realises one of the configurations $\mathcal{C}_1, \ldots, \mathcal{C}_4$, $\mathcal{C}'_4$, $\mathcal{C}''_4$, $\mathcal{C}_5, \ldots, \mathcal{C}_8$. Then $(F(\Omega), \{e_1, e_2\})$ contains one of $\mathcal{M}_0, \ldots, \mathcal{M}_8$ as a minor.*

Proof. We show that in each case, Ω has a minor containing $\{e_1, e_2\}$ isomorphic to one of the biased graphs Ω_i representing the matroid M_i of a matroidal $\mathcal{M}_i$ ($i \in \{0, \ldots, 8\}$). This implies that $F(\Omega)$ has M_i as a minor containing $\{e_1, e_2\}$, and so that $(F(\Omega), \{e_1, e_2\})$ contains $\mathcal{M}_i$ as a minor. Recall that the biased graphs Ω_i defining M_i ($i \in \{0, \ldots, 8\}$) are those shown in Figure 4.9.

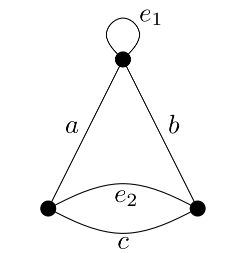
The only two realisations of $\mathcal{C}_1$ are the biased graphs Ω_0 and Ω_1 representing the matroids M_0 of $\mathcal{M}_0$ and M_1 of $\mathcal{M}_1$. A biased graph realising $\mathcal{C}_2$ (resp. $\mathcal{C}_3$) will have a subgraph realising $\mathcal{C}_1$ unless it is isomorphic to Ω_2 (resp. Ω_3). A biased graph realising $\mathcal{C}_4$ has either 0, 1, or 2 balanced cycles, and so is isomorphic to one of Ω_4, Ω_5, or Ω_6, respectively. If Ω is a biased graph realising $\mathcal{C}'_4$ or $\mathcal{C}''_4$ then Ω has a unique balancing vertex different from e_1 after deleting its unbalanced loops; unrolling its unbalanced loops we obtain a biased graph Φ realising $\mathcal{C}_4$ with $F(\Phi) \cong F(\Omega)$.

Suppose Ω realises $\mathcal{C}_5$. Let a, b be the two parallel edges forming the unbalanced cycle. We may assume by possibly interchanging a and b that the unique triangle containing a is unbalanced. Contracting a and deleting b yields a $\mathcal{C}'_4$ configuration.

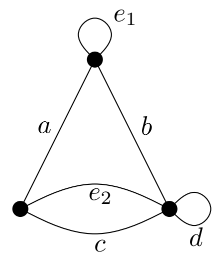
Suppose Ω realises $\mathcal{C}_6$. Then by the theta property there is an unbalanced cycle either of length 3 or length 4 containing e_2. In either case, this unbalanced cycle together with unbalanced cycle de_2 has a minor that is a $\mathcal{C}_1$ configuration.

If Ω realises $\mathcal{C}_7$, then — since by the theta property one of a or b is in an unbalanced triangle — contracting one of edges a or b we obtain a $\mathcal{C}_2$ configuration.

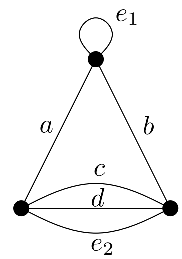
Finally suppose that Ω realises $\mathcal{C}_8$. If triangle efe_2 is unbalanced, then deleting c, d and contracting one of the edges now in series yields configuration $\mathcal{C}_1$. So suppose triangle efe_2 is balanced. If one of c or d — say d — fails to be contained in a balanced triangle, then deleting c and contracting d yields configuration $\mathcal{C}_4$. The remaining possibility is that efe_2 is balanced and both c and d are contained in a balanced triangle. Then Ω may be embedded in the plane as drawn in Figure 4.13 with precisely facial cycles efe_2, ace, and bdf balanced. The theta property implies that every cycle of length > 1 in this graph is unbalanced if in the embedding its interior contains the face bounded by unbalanced cycle cd, and is otherwise balanced. Hence $\Omega \cong \Omega_8$. $\square$



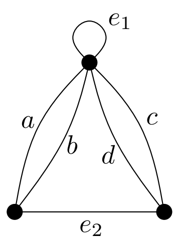
$\mathcal{C}_1$
$\mathcal{U} = \{e_1, abe_2, ce_2\}$



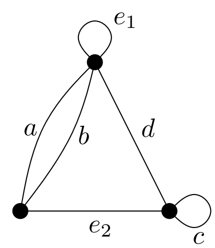
$\mathcal{C}_2$
$\mathcal{U} = \{e_1, ce_2, d\}$



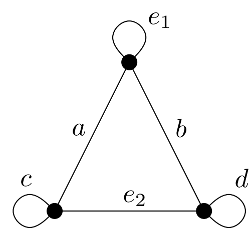
$\mathcal{C}_3$
$\mathcal{U} = \{e_1, cd, de_2, ce_2\}$



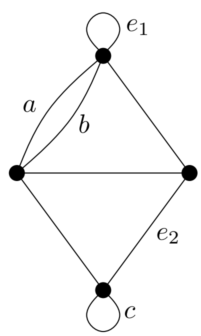
$\mathcal{C}_4$
$\mathcal{U} = \{e_1, ab, cd\}$



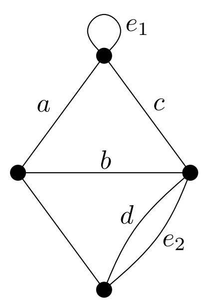
$\mathcal{C}_4'$
$\mathcal{U} = \{e_1, ab, c\}$



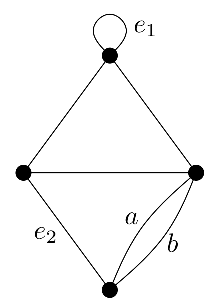
$\mathcal{C}_4''$
$\mathcal{U} = \{e_1, c, d\}$



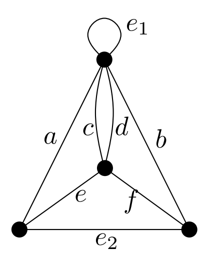
$\mathcal{C}_5$
$\mathcal{U} = \{e_1, ab, c\}$



$\mathcal{C}_6$
$\mathcal{U} = \{e_1, abc, de_2\}$



$\mathcal{C}_7$
$\mathcal{U} = \{e_1, ab\}$



$\mathcal{C}_8$
$\mathcal{U} = \{e_1, cd, abe_2\}$

Figure 4.13: Configurations used to find $\mathcal{M}_0, \ldots, \mathcal{M}_8$.

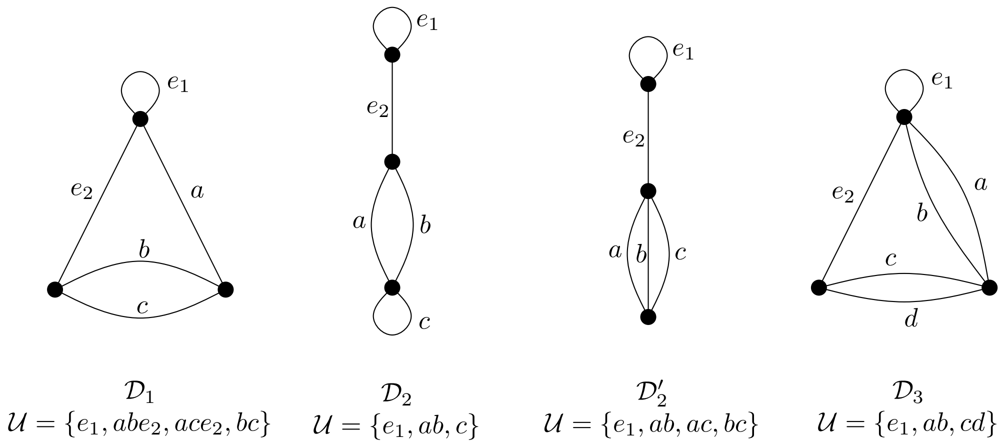


Figure 4.14: More configurations.

Lemma 4.31. *Let Ω be a biased graph which realises one of the configurations $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}'_2$, or $\mathcal{D}_3$. Then $(F(\Omega), \{e_1, e_2\})$ contains one of $\mathcal{M}_0$, $\mathcal{M}_1$, or $\mathcal{M}_7$ as a minor.*

Proof. If Ω realises $\mathcal{D}_1$ then $F(\Omega) \cong M_1$, so $(F(\Omega), \{e_1, e_2\})$ is isomorphic to $\mathcal{M}_1$. If Ω realises either $\mathcal{D}_2$ or $\mathcal{D}'_2$ then $F(\Omega) \cong M_0$, so $(F(\Omega), \{e_1, e_2\})$ is isomorphic to $\mathcal{M}_0$. If Ω realises $\mathcal{D}_3$, then either Ω contains a $\mathcal{D}_1$ configuration or $F(\Omega) \cong M_7$ so $(F(\Omega), \{e_1, e_2\})$ is isomorphic to $\mathcal{M}_7$. $\square$

Two of the excluded minors for the class of frame matroidals have graphic matroids, namely $\mathcal{M}_4$ and $\mathcal{M}_8$: M_4 is the cycle matroid of K_4 and M_8 is the rank 4 wheel. The following lemma will help us locate either $\mathcal{M}_4$ or $\mathcal{M}_8$ as a minor in a purported excluded minor $\mathcal{N}=(N, L)$ in which N is graphic.

Lemma 4.32. *Let G be a simple 3-connected graph, and let $\{e_1, e_2\} \subseteq E(G)$, with $e_1=s_1t_1$ and $e_2=s_2t_2$, with s_1, s_2, t_1, t_2 pairwise distinct. Then either G has a K_4 minor containing $\{e_1, e_2\}$ in which e_1 and e_2 do not share an endpoint, or G has W_4 as a minor containing $\{e_1, e_2\}$ in which e_1 and e_2 are opposite each other in the rim of W_4 (i.e., e_1 and e_2 do not share an endpoint and each of e_1 and e_2 have both endpoints of degree three).*

Proof. Let $\mathrm{co}(H)$ denote the graph obtained from a graph H by suppressing vertices of degree 2. It is well known that if G is a 3-connected graph, then for every $e \in E(G)$, either $\mathrm{co}(G \setminus e)$ or G/e is 3-connected (for instance, it is a special case of Proposition 8.4.6 in [23]). In the following, if in $G \setminus e$ edge e_i, $i \in \{1, 2\}$, has an endpoint of degree two, then $\mathrm{co}(G \setminus e)$ is obtained by contracting the edge other than e_i incident to that vertex.

Let G be a minimal counter-example to the statement of the lemma. If there is an edge $e \in E(G)$ such that $\mathrm{co}(G \setminus e)$ or G/e is 3-connected such that e_1 and e_2 are not incident to a common vertex, then by minimality this graph has a minor of one of the required forms. But then so would G have had that minor, a contradiction. Hence for every edge $e \notin \{e_1, e_2\}$, if

$\mathrm{co}(G \setminus e)$ is 3-connected then e_1 and e_2 are adjacent in $\mathrm{co}(G \setminus e)$, and if G/e is 3-connected then e_1 and e_2 are adjacent in G/e.

Suppose there is an edge $e \in E(G)$ that does not have any of s_1, t_1, s_2, t_2 as an endpoint. Then $\mathrm{co}(G \setminus e)$ has e_1 and e_2 nonadjacent, and so is not 3-connected. Hence G/e is 3-connected. But neither are e_1 and e_2 adjacent in G/e, contradicting the previous paragraph. Therefore every edge of G has an endpoint incident to e_1 or e_2. Now suppose $e \in E(G)$ does not have both endpoints in $\{s_1, t_1, s_2, t_2\}$; say $e = xs_1$ with $x \notin \{s_1, t_1, s_2, t_2\}$. Then G/e does not have e_1 and e_2 adjacent, and so is not 3-connected. Hence $\mathrm{co}(G \setminus e)$ is 3-connected, and so has e_1 and e_2 adjacent. This implies that the degree of s_1 is three, and the three edges incident to s_1 are e, e_1, and f, where the other endpoint of f is one of s_2 or t_2. It follows that $|V(G)| \leq 5$. (Every vertex $x \notin \{s_1, t_1, s_2, t_2\}$ has neighbourhood of size ≥ 3 contained in $\{s_1, t_1, s_2, t_2\}$. Further, each vertex in the neighbourhood of x has degree three, which, together with its edge to x and its incident edge in $\{e_1, e_2\}$, includes an edge whose other endpoint is also in $\{s_1, t_1, s_2, t_2\}$. These edges resulting from the existence of $x \notin \{s_1, t_1, s_2, t_2\}$ accounted for thus far leave just one vertex u in $\{s_1, t_1, s_2, t_2\}$ for which it is possible that u has an additional incident edge, yet the existence of a vertex $y \notin \{x, s_1, t_1, s_2, t_2\}$ requires three such vertices.)

If $|V(G)| = 4$, then $G \cong K_4$ and we are done. So suppose $|V(G)| = 5$; let $V(G) = \{x, s_1, t_1, s_2, t_2\}$. The fact that the degree of every vertex is at least three, together with the above constraints on edges incident to a neighbour of x forces the existence of either a K_4 or W_4 minor of the required form. This contradiction completes the proof. $\square$

4.4.3 Proof of Lemma 4.26

If a biased graph Ω has a minor realising a configuration, we say Ω *contains* the configuration. Let us call the configurations $\mathcal{C}_1, \ldots, \mathcal{C}_4, \mathcal{C}_4', \mathcal{C}_4'', \mathcal{C}_5, \ldots, \mathcal{C}_8, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_2', \mathcal{D}_3$ *bad* configurations. Thus by Lemmas 4.30 and 4.31, if Ω represents M, and Ω contains a bad configuration, then the matroidal $(M, \{e_1, e_2\})$ has one of $\mathcal{M}_0, \ldots, \mathcal{M}_8$ as a minor.

Proof of Lemma 4.26. Let $\mathcal{N}=(N, L)$ be an excluded minor for the class of frame matroidals with N 3-connected and $L=\{e_1, e_2\}$, and suppose $\mathcal{N}$ is not isomorphic to one of $\mathcal{M}_1, \ldots, \mathcal{M}_8$. Observe that $\mathcal{N}$ cannot have $\mathcal{M}_0$ as a minor, since then minimality would imply that $\mathcal{N} \cong \mathcal{M}_0$; since e_1 and e_2 are in series in M_0 this would contradict the fact that N is 3-connected. In light of this and Lemmas 4.30 and 4.31 it suffices to derive the contradiction that a biased graph Ω representing N contains a bad configuration.

First suppose that N is graphic. Let H be a graph with $N = M(H)$. As N is 3-connected, H is simple and 3-connected. Hence neither e_1 nor e_2 is a loop in H. If edges e_1 and e_2 share an endpoint $v \in V(H)$, then rolling up the edges incident to v yields a biased graph in which both e_1 and e_2 are unbalanced loops, a contradiction. Hence e_1 and e_2 do not share an endpoint. By Lemma 4.32 therefore, H has a minor H' isomorphic to either K_4 with e_1 and e_2 nonadjacent, or isomorphic to W_4 with e_1 and e_2 nonadjacent and neither incident to the vertex of degree 4. In the former case $\mathcal{N}$ contains $\mathcal{M}_4$ as a minor, and in the latter $\mathcal{M}_8$ as a minor, both contradictions.

So N is not graphic. Let $\Omega{=}(G, \mathcal{B})$ be a biased graph representing $(N, \{e_1\})$. Since N is 3-connected:

(C1) Ω is 2-connected, and

(C2) if (A, B) is a separation of N with $|A| \geq 2$ and $\Omega[A]$ is balanced, then $|V(A) \cap V(B)| \geq 3$.

Let v be the vertex to which e_1 is incident. We consider two cases, depending on whether e_1 and e_2 are adjacent in Ω.

Case 1: e_1 and e_2 are not adjacent

Let u, w be the endpoints of e_2. We consider three subcases depending on the behaviour of unbalanced cycles in $\Omega - v$.

Subcase (i): $\Omega - v$ has no unbalanced cycle of length > 1

If $\Omega - v$ contains unbalanced loops, then unrolling them yields an $\{e_1\}$-biased graph representing N (Proposition 1.25) in which v is a balancing vertex. We may assume therefore that $\Omega - v$ is balanced. Consider the balancing equivalence classes in $\delta(v)$. There cannot be just one balancing class in $\delta(v)$, since then e_1 would not be contained in any circuit of N. If there are only two balancing classes, then by Proposition 1.20 Ω is a signed graph. But then splitting v yields a graph H with $M(H) = N$ (Proposition 1.24), so N is graphic, a contradiction. Hence there are at least three balancing classes in $\delta(v)$.

Claim. Ω contains a $\mathcal{C}_4$ configuration.

Proof of claim. Construct an auxiliary graph G from the underlying graph of $\Omega - e_1$, as follows. Let $\{S_1, \ldots, S_t\}$ be the partition of $\delta(v)$ into its balancing classes. Add a set of new vertices $X = \{x_1, \ldots, x_t\}$, and, for each $i \in \{1, \ldots, t\}$, redefine the endpoints of each edge $f{=}xv \in S_i$ so that f has endpoints x, x_i. Add a new vertex y to G that is adjacent to every vertex which is a neighbour of either u or w.

We claim that G contains three vertex disjoint paths between X and $\{u, w, y\}$. For if not, then by Menger's Theorem there exists a pair of subgraphs $G_1, G_2 \subseteq G$ whose edges partition $E(G)$ so that $X \subseteq V(G_1)$ and $\{u, w, y\} \subseteq V(G_2)$ and $V(G_1) \cap V(G_2) = Z$ with $|Z| = 2$. If Z contains at most one vertex of X then the subgraph of Ω induced by $E(G_2)$ is balanced, contradicting (C2). Hence Z contains two vertices of X. But this implies v is a cut vertex of Ω, contradicting (C1). This establishes the existence of our paths.

So we may now assume that in Ω there exist three internally disjoint paths, P_1 and P_2 from v to u and P_3 from v to w such that the three edges of these paths in $\delta(v)$ are in distinct balancing classes. If there exists a path Q from $P_1 \cup P_2$ to P_3 which is disjoint from $\{u, v\}$, then a minor of $P_1 \cup P_2 \cup P_3 \cup Q \cup \{e_1, e_2\}$ contains a $\mathcal{C}_4$ configuration.

If there is no such path Q, then there is a partition (A, B) of $E(\Omega)$ with $V(A) \cap V(B) = \{u, v\}$, $P_1, P_2 \subseteq \Omega[A]$ and $P_3 \subseteq \Omega[B]$. Choose such a partition with B minimal. By (C2), $\Omega[B]$ contains two edges from $\delta(v)$ in distinct equivalence classes, and by our choice of B, neither of these edges is incident with u. Also by our choice of B, the subgraph $\Omega[B]-\{u, v\}$ is connected. It follows that Ω contains a $\mathcal{C}_4$ configuration. □

This completes the proof in subcase 1(i).

Subcase (ii): $\Omega - v$ **has an unbalanced cycle of length** > 1 **but none containing** e_2

Since two vertex disjoint paths linking the endpoints of e_2 and an unbalanced cycle would, by the theta property, yield an unbalanced cycle containing e_2, in this case $\Omega - v$ is not 2-connected. We investigate the block structure of $\Omega - v$ to show that Ω contains a bad configuration.

Suppose Ψ is a leaf block of $\Omega - v$, containing cut-vertex x. By (C1) there is at least one edge between v and $\Psi - x$. By (C2), either Ψ is unbalanced or there exists an unbalanced cycle C containing v with length > 1 with $C - v \subseteq \Psi$. With the goal of finding a bad configuration in mind, edges of Ψ may be deleted or contracted to yield, in the former case, an unbalanced loop at x and a link vx, or in the latter case, two vx links forming an unbalanced cycle.

Let Φ be the block of $\Omega - v$ containing e_2. Suppose first that Φ is not a leaf block of $\Omega - v$. Then Φ contains two distinct cut-vertices x, x'. Choose a path in this block linking x and x' and containing e_2. Applying the argument of the previous paragraph to two leaf blocks of $\Omega - v$, we find that Ω contains one of the configurations $\mathcal{C}_4$, $\mathcal{C}_4'$, or $\mathcal{C}_4''$.

So suppose now that the block Φ of $\Omega - v$ is a leaf block. After deleting unbalanced loops Φ is balanced, else Φ (and so $\Omega - v$) would contain an unbalanced cycle containing e_2. Let x be the cut vertex of $\Omega - v$ contained in Φ, and let S be the set of edges in $\delta(v)$ incident with a vertex of $\Phi - x$. Consider the biased graph Φ' obtained from Φ by deleting its unbalanced loops and adding vertex v together with the edges in S. Vertex v is a balancing vertex of Φ'; let $\{S_1, \dots, S_t\}$ be the partition of S into the balancing classes of $\delta(v)$ in Φ'. Let S_0 be the set of loops in Φ not incident to x.

Now construct an auxiliary graph similar to that appearing in subcase 1(i). Let G be the graph obtained from Φ by adding vertices $x_0, x_1, \dots, x_t$, and for $1 \leq i \leq t$ and every edge $zv \in S_i$ add an edge zx_i; for each unbalanced loop incident to a vertex z add an edge zx_0. Finally, add a vertex y that is adjacent to each vertex which is a neighbour of either u or w. We claim that in G there exist three vertex disjoint paths linking

$\{x, x_0, \ldots, x_t\}$ to $\{u, w, y\}$. For suppose otherwise. Then by Menger's Theorem there exists a pair of subgraphs $G_1, G_2 \subseteq G$ whose edges partition $E(G)$ with $\{x, x_0, \ldots, x_t\} \subseteq V(G_1)$ and $\{u, w, y\} \subseteq V(G_2)$ and $|V(G_1) \cap V(G_2)| = 2$. Let $Z = V(G_1) \cap V(G_2)$. If both vertices in Z are in $\{x_0, x_1, \ldots, x_t\}$, then $\Omega - v$ would have no path linking x and u, contradicting the fact that Φ is a block of $\Omega - v$. Now either $x_0 \notin Z$ or $x_0 \in Z$. If $x_0 \notin Z$, then in Ω the biased subgraph induced by $E(G_2 - y)$ is a balanced subgraph meeting the rest of Ω in just two vertices, contradicting (C2). But if $x_0 \in Z$, then the biased subgraph induced by $E(G_2 - y)$ meets the rest of Ω in just one vertex, contradicting (C1). Hence the paths exist as claimed.

We may assume that one of these three paths begins at vertex x (otherwise choose a path from x to $\{u, w, y\}$ and modify a path appropriately). In Ω this gives us three internally disjoint paths $P_1, P_2, P_3 \subseteq \Phi'$ such that:

1. P_1, P_2 start at v or at a vertex of Φ incident with an unbalanced loop and end at $\{u, w\}$.

2. at least one of P_1, P_2 starts at v, and if both start at v their first edges are in distinct blancing classes.

3. P_3 starts at x and ends at $\{u, w\}$.

4. at least one of P_1, P_2, P_3 ends at u and one at w.

Choose an unbalanced cycle C of length > 1 in $\Omega - v$ and choose two vertex disjoint paths R, R' linking C and $\{v, x\}$. Note that C is not contained in Φ (as Φ without its unbalanced loops is balanced), and so R, R' meet Φ only at x. First suppose that both P_1 and P_2 end at u or both end at w. Consider the subgraph H consisting of $C \cup R \cup R'$ together with $P_1 \cup P_2 \cup P_3$ and the edges e_1, e_2. If both P_1, P_2 begin at v then H contains a $\mathcal{C}_4$ configuration. Otherwise, one of these paths begins at a vertex incident with an unbalanced loop f. Adding f to subgraph H, we find that H contains $\mathcal{C}_4'$ configuration. So now suppose that P_1 ends at u while both P_2 and P_3 end at w. Since Φ is a block of $\Omega - v$, $\Phi - w$ contains a path Q from $P_1 - v$ to $P_3 - \{v, w\}$. If Q contains a vertex in P_2, then again the subgraph H consisting of $C \cup R \cup R'$ together with $P_1 \cup P_2 \cup P_3 \cup \{e_1, e_2\}$ and possibly an unbalanced loop incident to an end of P_1 or P_2, contains either a $\mathcal{C}_4$ or $\mathcal{C}_4'$ configuration. Otherwise, H contains either configuration $\mathcal{C}_5$ (if one of P_1 or P_2 does not begin at v but is incident to an unbalanced loop) or $\mathcal{C}_8$ (if both P_1 and P_2 begin at v).

Subcase (iii): $\Omega - v$ has an unbalanced cycle containing e_2

Let C be an unbalanced cycle containing e_2. Choose two paths P_1, P_2 linking v and C, disjoint except at v, say meeting C at vertices x_1, x_2, respectively. Let R be the x_1-x_2 path in C containing e_2; let R' be the x_1-x_2 path in C avoiding e_2. If the cycle $P_1 \cup P_2 \cup R$ is unbalanced, then Ω contains configuration C_1. So let us now assume that this does not occur for any unbalanced cycle containing e_2 — *i.e.*, for every unbalanced cycle C of $\Omega - v$

containing e_2 and every such pair P_1, P_2 of v-C paths meeting only at v, the cycle formed by $P_1 \cup P_2$ and the path R in C traversing e_2 is balanced. Choose such subgraphs C, P_1 and P_2, with P_1 meeting C at x_1 and P_2 meeting C at x_2, so that the length of the path R' in C avoiding e_2 is minimum.

Suppose R' does not consist of a single edge. First suppose also that there exists a separation (Ω_1, Ω_2) of Ω with $V(\Omega_1) \cap V(\Omega_2) = \{x_1, x_2\}$ with $R' \subseteq \Omega_1$ and $P_1 \cup P_2 \cup R \subseteq \Omega_2$. By choosing such a separation with Ω_1 minimal, we may further assume that $\Omega_1 - \{x_1, x_2\}$ is connected and that there are no x_1x_2 edges in Ω_1. By (C2), Ω_1 is not balanced. If there is an unbalanced cycle in $\Omega_1 - x_1$, then Ω contains a $\mathcal{C}_2$ configuration. Otherwise x_1 is a balancing vertex in Ω_1. Since Ω_1 contains no x_1-x_2 edge and $\Omega_1 - \{x_1, x_2\}$ is connected, there is then an unbalanced cycle in $\Omega_1 - x_2$; again we find a $\mathcal{C}_2$ configuration. So now assume that no such separation exists: there is a path Q from the interior of R' to $(P_1 \cup P_2 \cup R) \setminus \{x_1, x_2\}$. If Q first meets $P_1 \cup P_2 \setminus \{x_1, x_2\}$, then we find our choice of P_1 and P_2 did not minimise R', a contradiction. Hence Q avoids $(P_1 \cup P_2) \setminus \{x_1, x_2\}$ and meets R. Subgraph $Q \cup C$ is a theta. If the cycle in $Q \cup C$ containing e_2 different from C is unbalanced, then again we did not choose C, P_1, and P_2 so as to minimise the length of R', a contradiction. Therefore that cycle is balanced, and so the cycle C' in $C \cup Q$ not containing e_2 is unbalanced. Choose an edge $e \in Q$. Contracting all edges of C' but e, all but one edge of $R' \setminus C'$, all but edge e_2 of $R \setminus C'$, and all but one edge of each of P_1 and P_2, we find configuration $\mathcal{C}_2$.

So the path R' must consist of a single x_1x_2 edge f. Suppose first that $\{x_1, x_2\}$ does not separate v from $C \setminus \{x_1, x_2\}$ and choose a path Q from $(P_1 \cup P_2) \setminus \{x_1, x_2\}$ to $C \setminus \{x_1, x_2\}$. We claim that by the theta property, there exists a cycle in $P_1 \cup P_2 \cup Q \cup C$ containing both e_2 and Q which is unbalanced, and in any case this yields a $\mathcal{C}_1$ configuration. To see this, recall that the cycle $P_1 \cup P_2 \cup C \setminus f$ is balanced. There are, up to symmetry and assuming Q leaves from P_1, two cases to consider: (a) Q is a P_1-C path such that the cycle D in $P_1 \cup Q \cup C$ containing e_2 and Q contains f, or (b) does not contain f. In case (a), if D is balanced, then the cycle in $P_1 \cup P_2 \cup (C \setminus f) \cup Q$ containing Q and e_2 is unbalanced, and we find $\mathcal{C}_1$ contained in this cycle together with C and e_1. If D is unbalanced, then we find $\mathcal{C}_1$ in $P_1 \cup Q \cup C \cup \{e_1\}$. In case (b), if D is balanced we find $\mathcal{C}_1$ by deleting the subpath of P_1 between $P_1 \cap Q$ and x_1. If D is unbalanced, we find $\mathcal{C}_1$ in $P_1 \cup Q \cup C \cup \{e_1\}$.

Hence $\{x_1, x_2\}$ separates v from C. Choose a separation (Ω_1, Ω_2) of Ω with $V(\Omega_1) \cap V(\Omega_2) = \{x_1, x_2\}$ for which $C \subseteq \Omega_2$ and $v \in \Omega_1$, with Ω_1 minimal. Then $\Omega_1 - \{x_1, x_2\}$ is connected and Ω_1 has no x_1x_2 edge. If Ω_1 contains an unbalanced cycle C' of length > 1, then choosing a pair of vertex disjoint paths Q, Q' linking C' and $\{x_1, x_2\}$ and an application of the theta property yield an unbalanced cycle containing e_2 that is not C. But then $C' \cup Q \cup Q' \cup C \cup \{e_1\}$ contains a $\mathcal{C}_1$ configuration. Hence Ω_1 contains no unbalanced cycle of length > 1; suppose Ω_1 contains an unbalanced loop $e \neq e_1$, say incident to v'. Since N is 3-connected, $v' \neq v$. Since Ω is 2-connected, there is a path Q from v' to $(P_1 \cup P_2) \setminus v$. But now in $C \cup P_1 \cup P_2 \cup Q \cup \{e\}$ we find configuration $\mathcal{C}_2$.

So $\Omega_1 - e_1$ is balanced. Now suppose that $V(\Omega_2) = \{x_1, x_2\}$. If there is a loop in Ω_2 we have a $\mathcal{C}_2$ configuration. If there are at least three edges in Ω_2 we have a $\mathcal{C}_3$ configuration (no two such edges form a balanced cycle since N is 3-connected). So in this case Ω_2 consists only of the two edges e_2 and f (which form unbalanced cycle C). Since $\Omega_1 - e_1$ is balanced, an unbalanced cycle in $\Omega - f$ containing e_2, together with the theta property, would yield a $\mathcal{C}_1$ configuration. Hence $\Omega - \{e_1, f\}$ is balanced. But this implies that e_1 and f are in series in N, a contradiction since N is 3-connected. So $|V(\Omega_2)| \geq 3$.

We now claim that Ω_2 contains an unbalanced cycle that does not contain both x_1 and x_2. Let Ψ_0 be a component of $\Omega_2 - \{x_1, x_2\}$ and let Ψ be the subgraph of Ω_2 consisting of Ψ_0 together with all edges between x_i and $V(\Psi_0)$, for $i \in \{1, 2\}$. By (C2), Ψ is unbalanced. Moreover, we may assume x_1 is a balancing vertex in Ω_2, since if not we have the desire cycle. Consider the balancing classes of $\delta_{\Omega_2}(x_1)$. Since Ψ is not balanced, there are two edges in Ψ in distinct balancing classes, and since $\Psi - x_2$ is connected, this yields an unbalanced cycle in Ω_2 not containing x_2, as desired.

Without loss of generality, choose an unbalanced cycle $D \subseteq \Omega_2$ that does not contain x_1. If D and C share at most one vertex, we see that $P_1 \cup P_2 \cup C \cup D$ contains a $\mathcal{C}_2$ configuration. So $|V(C) \cap V(D)| \geq 2$. Let Q be the maximal subpath of C which contains e_2 and has no interior vertex in the set $V(D) \cup \{x_1, x_2\}$. By assumption at least one end of Q must be in $V(D)$. If both ends of Q are in $V(D)$ then Ω_2 contains an unbalanced cycle D' containing e_2 but not x_1. There are two vertex disjoint paths linking D' and $\{x_1, x_2\}$ and these, together with $P_1 \cup P_2 \cup \{e_1, f\}$, contain a $\mathcal{C}_6$ configuration. So finally assume (by possibly interchanging x_1 and x_2) that one end of Q is x_1 and the other is in $V(D)$. The D-x_2 path in C avoids Q; this path together with D, Q, f, P_1, P_2, and e_1 contains a $\mathcal{C}_7$ configuration.

This completes the proof of Case 1.

Case 2: e_1 and e_2 are adjacent

As before, let v be the endpoint of e_1. Let u be the other endpoint of e_2. Let T_0 be the standard block-cutpoint graph of $\Omega - v$. If u is a cut vertex of $\Omega - v$ then set $T = T_0$. Otherwise, let T be the tree obtained by adding vertex u to T_0 together with an edge between u and the unique block of $\Omega - v$ containing u. View tree T as rooted at u. Every block Ψ of $\Omega - v$ is a vertex of T and there is a unique path in T from Ψ to u. The next vertex of T on this path from Ψ is a vertex of Ω, the *parent* of Ψ. Note that the parent of a block of $\Omega - v$ is always either a cut vertex of $\Omega - v$ or is u.

Claim. If x is the parent of a block Ψ of $\Omega - v$, then one of the following holds:

1. Ψ contains no unbalanced cycle of length > 1.

2. x is balancing in Ψ and there are exactly two balancing classes in $\delta_\Psi(x)$.

Proof of Claim. Let Ψ' be the graph obtained from Ψ by deleting all loops. If Ψ' is balanced, (1) holds. Otherwise, suppose x is not a balancing vertex of Ψ' and choose an unbalanced cycle C of $\Psi' - x$ and two paths P_1, P_2 from x to C that are disjoint except at x. Let y_1, y_2 be the respective ends of P_1, P_2 on C, and let Q, Q' be the two paths in C meeting just at y_1 and y_2. By the theta property, one of $P_1 \cup P_2 \cup Q$ or $P_1 \cup P_2 \cup Q'$ is unbalanced. Hence $P_1 \cup P_2 \cup C$ together with an x-u path, e_2, and e_1, contains a $\mathcal{D}_2$ configuration.

So x is balancing in Ψ'. If $\delta_{\Psi}(x)$ contains three balancing classes, Ω contains a $\mathcal{D}_2'$ configuration. Hence there are exactly two balancing classes in $\delta_{\Psi}(x)$. If Ψ contains an unbalanced loop not at vertex x, then an unbalanced cycle in Ψ', together with this loop, an x-u path, e_2, and e_1, contains a $\mathcal{D}_2$ configuration. $\square$

Call a block of $\Omega - v$ as described in statement (1) of our claim a *type 1* block, and a block as in statement (2), a *type 2* block.

Claim. Every type 2 block of $\Omega - v$ is a leaf of T.

Proof of Claim. Suppose there exists a type 2 block Ψ of $\Omega - v$ that is not a leaf of T. Let Φ be a leaf block of $\Omega - v$ with parent y such that the unique path in T from Φ to u contains Ψ. If Φ contains an unbalanced cycle, then Ω contains a $\mathcal{D}_2$ configuration. So Φ is balanced. Let Φ^+ be the biased subgraph of Ω given by Φ together with v and all edges between v and $\Phi - y$. By (C2), Φ^+ is unbalanced, so there is an unbalanced cycle C in Φ^+ containing v. Together with a C-y path in Φ, an unbalanced cycle C' in Ψ, a y-$(C' - x)$ path and an x-u path in $\Omega - v$, we have a biased graph containing a $\mathcal{D}_3$ configuration. $\square$

Along with the structure we have determined of $\Omega - v$ comes knowledge of the biases of all cycles of $\Omega - v$. We wish to extend this knowledge to Ω.

Let Ω_0 be the balanced biased subgraph of Ω consisting of each type 1 block of $\Omega - v$ without its unbalanced loops. By our second claim, Ω_0 is a connected balanced biased subgraph of $\Omega - v$. Let $\Psi_1, \ldots, \Psi_m$ be the type 2 blocks of $\Omega - v$. For each Ψ_i, let x_i be its parent vertex in T, and define Ω_i to be the subgraph of Ω consisting of Ψ_i together with v and all edges between v and $\Psi_i - x_i$. Let U be the set of all loops in $\Omega - v$. The subgraphs $E(\Omega_0), E(\Omega_1), \ldots, E(\Omega_m)$ are edge disjoint and together contain all edges in $E(\Omega)$ except for loops and some edges incident to v.

For every $1 \le i \le m$, vertex x_i is balancing in Ψ_i; let $\{A_i, B_i\}$ be the partition of $\delta_{\Psi_i}(x_i)$ into its two balancing classes. Suppose Ω_i contains an unbalanced cycle C disjoint from x_i. Choose two internally disjoint paths P_1, P_2 linking x_i and C for which $E(P_1) \cap A_i \neq \emptyset$ and $E(P_2) \cap B_i \neq \emptyset$. Then $C \cup P_1 \cup P_2 \cup \{e_1, e_2\}$ together with an x_i-u path in $\Omega - v$ contains a $\mathcal{D}_3$ configuration. Hence every Ω_i has x_i as a balancing vertex. By Lemma 1.19 the balancing classes in each $\delta_{\Omega_i}(x_i)$ are $\{A_i, B_i\}$.

Consider two edges $f, f' \in A_i$ or $f, f' \in B_i$ for some $1 \le i \le m$. Let C (resp. C') be a cycle containing e_2 and f (resp. f'). The path $C - e_2$ ($C' - e_2$) is the union of a u-x_i path

P (P') and an x_i-v path Q (Q'). Applying Lemma 1.19 separately to $P \cup P'$ and $Q \cup Q'$, we conclude that C and C' have the same bias. Now suppose that for some $1 \leq i \leq m$, there is an unbalanced cycle containing e_2 and an edge in A_i and another unbalanced cycle containing e_2 and an edge in B_i. Choose a cycle $C \subseteq \Psi_i$ that contains one edge in each of A_i and B_i, a path P in Ω_i from v to $C - x_i$, and a u-x_i path Q in Ω_0. It now follows that $P \cup Q \cup C \cup \{e_1, e_2\}$ contains a $\mathcal{D}_1$ configuration. Hence two such unbalanced cycles do not exist, and by possibly interchanging the names assigned to the sets A_i, B_i, we may assume that for every $1 \leq i \leq m$, every cycle in Ω containing e_2 and an edge of A_i is balanced. By the theta property then, for every $1 \leq i < j \leq m$, every cycle of Ω containing an edge in A_i and an edge in A_j is balanced.

We now define a signature for Ω that realises $\mathcal{B}$. We use a simpler biased graph Ω' to model the biases of cycles in Ω to do so. Let Ω' be the biased graph obtained from Ω as follows. For every $1 \leq i \leq m$ replace Ω_i with two edges a_i, b_i with endpoints x_i and v, with $a_i \in A_i$ and $b_i \in B_i$, and let the bias of each cycle of Ω' be inherited from a corresponding cycle in Ω in the obvious way. Now $\Omega' - v$ has no unbalanced cycle of length > 1, so by Observation 1.23, Ω' is a k-signed graph. Moreover, by Observation 1.23 there is a signature $\mathbf{\Sigma}' = \{U, \Sigma'_1, \ldots, \Sigma'_k\}$ that realises the biases of cycles of Ω', where each set $\Sigma'_j \subseteq \delta_{\Omega'}(v)$ and U is the set of unbalanced loops of Ω'. Further, Observation 1.23 allows us to assume that e_2 is not a member of any set in the signature $\mathbf{\Sigma}'$. Since for every $1 \leq i \leq m$, e_2 and a_i are in the same balancing class, none of the edges a_i is in a member of the signature. Hence every b_i is contained in some member Σ'_j of $\mathbf{\Sigma}'$. Define a signature $\mathbf{\Sigma} = \{U, \Sigma_1, \ldots, \Sigma_k\}$ for Ω as follows. For every $1 \leq i \leq m$, if $b_i \in \Sigma'_j$ put all edges in B_i in Σ_j. If $e{=}vz \in \Sigma'_j$ is a edge incident to v and a vertex $z \in \Omega_0$, put e in Σ_j. The structural description we have of Ω and the biases of its cycles implies $\mathcal{B}_{\Sigma} = \mathcal{B}$. By Theorem 4.2, the biased graph Γ obtained by performing a twisted flip on Ω has $F(\Gamma) \cong F(\Omega)$ (Figure 4.15). But in Γ both e_1 and e_2 are represented as unbalanced loops, so (N, L) is frame, a contradiction. $\square$

4.5 Some excluded minors of connectivity 2 not in $\mathcal{E}$

In previous sections we have exhibited a list $\mathcal{E}$ of 18 excluded minors of connectivity 2 for the class of frame matroids. By Theorem 4.1, any remaining excluded minor of connectivity 2 is a 2-sum of a 3-connected non-binary frame matroid and $U_{2,4}$. We have determined more than thirty excluded minors of this form, but are not yet sure if this list is complete. Here, we exhibit these excluded minors.

To that end, let N be a 3-connected non-binary frame matroid, let $e \in E(N)$, and suppose $N\,{}^{e}\!\oplus_2 U_{2,4}$ is an excluded minor for the class of frame matroids. By Corollary 4.23, $N\,{}^{e}\!\oplus_2 U_{2,4}$ is an excluded minor because there is no biased graph representation of N in

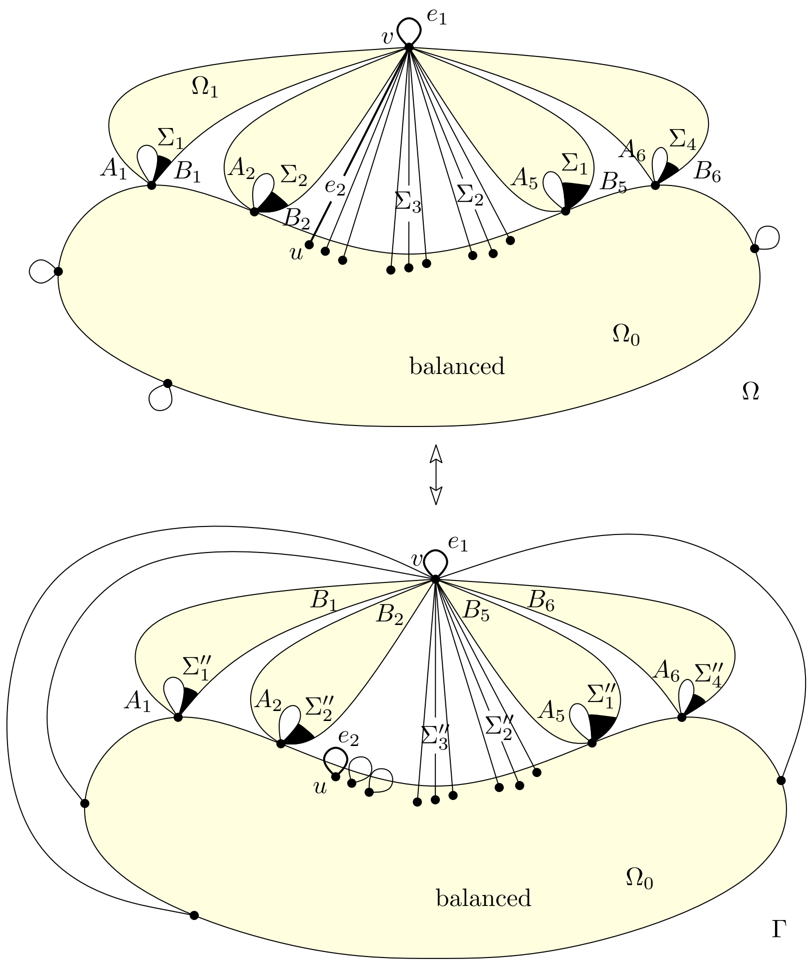


Figure 4.15: A twisted flip: $F(\Omega) \cong F(\Gamma)$.

which e is an unbalanced loop, but after the deletion or contraction of any other element of N such a representation does exist — *i.e.*, $(N, \{e\})$ is an excluded minor for the class of frame matroidals. Each of the biased graphs shown in Figures 4.16 and 4.17 represents a matroid with this property. Thus if $(G, \mathcal{B})$ is a biased graph shown in Figure 4.16 or 4.17, then $F(G, \mathcal{B})\,{}^{e}\!\oplus_2 U_{2,4}$ is an excluded minor for the class of frame matroids.

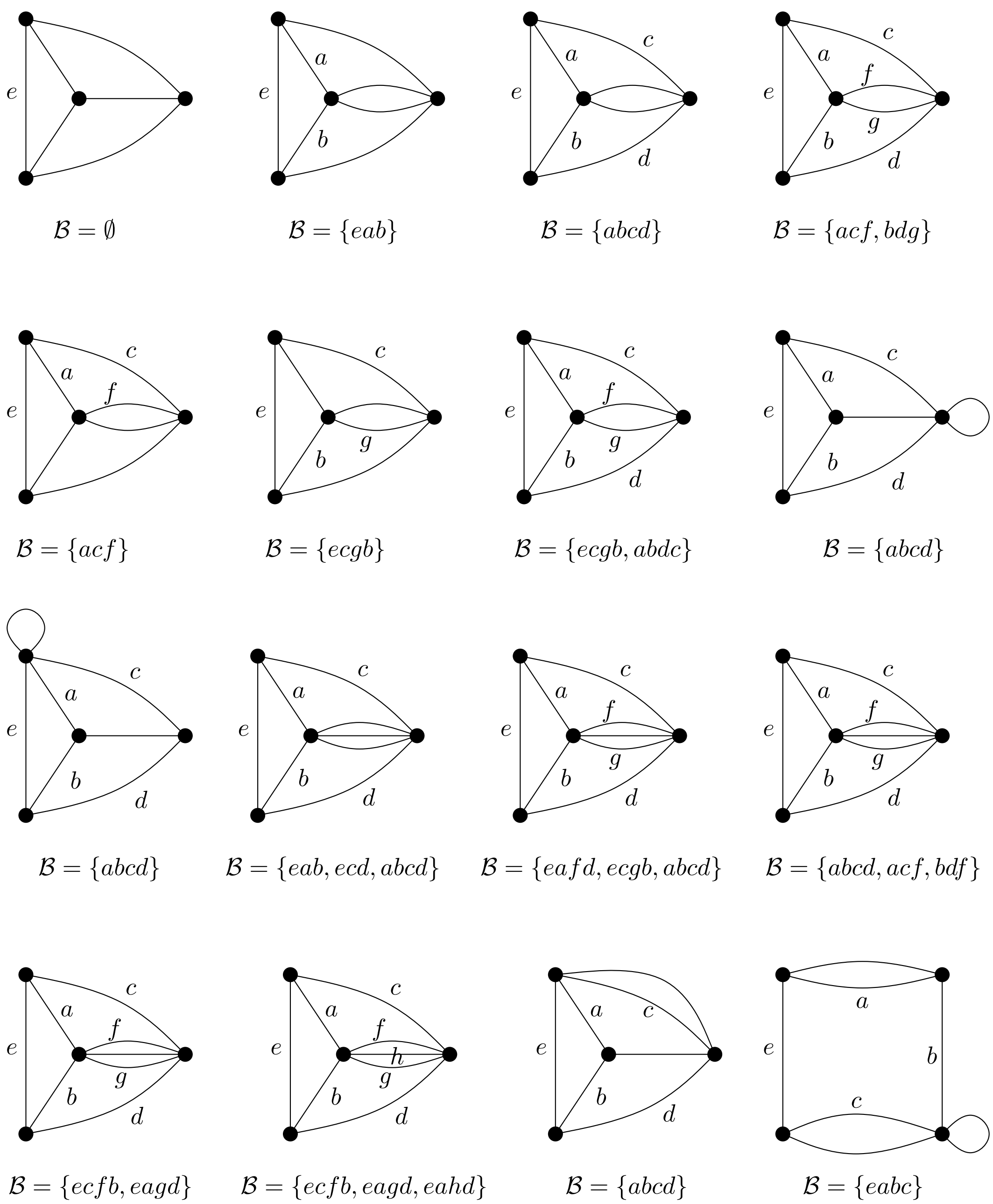


Figure 4.16: Minimally not $\{e\}$-biased graphs.

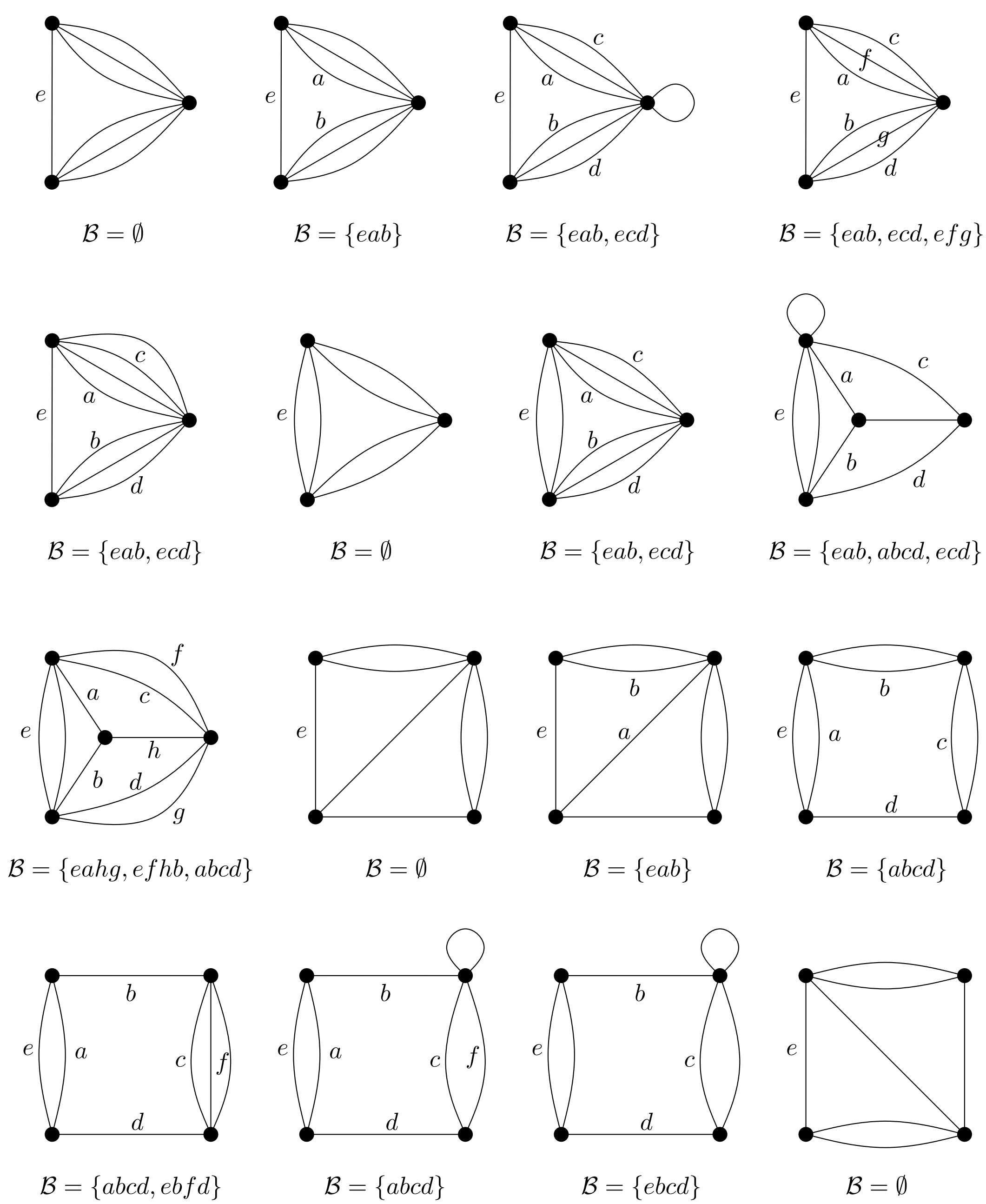


Figure 4.17: Minimally not $\{e\}$-biased graphs.

Chapter 5

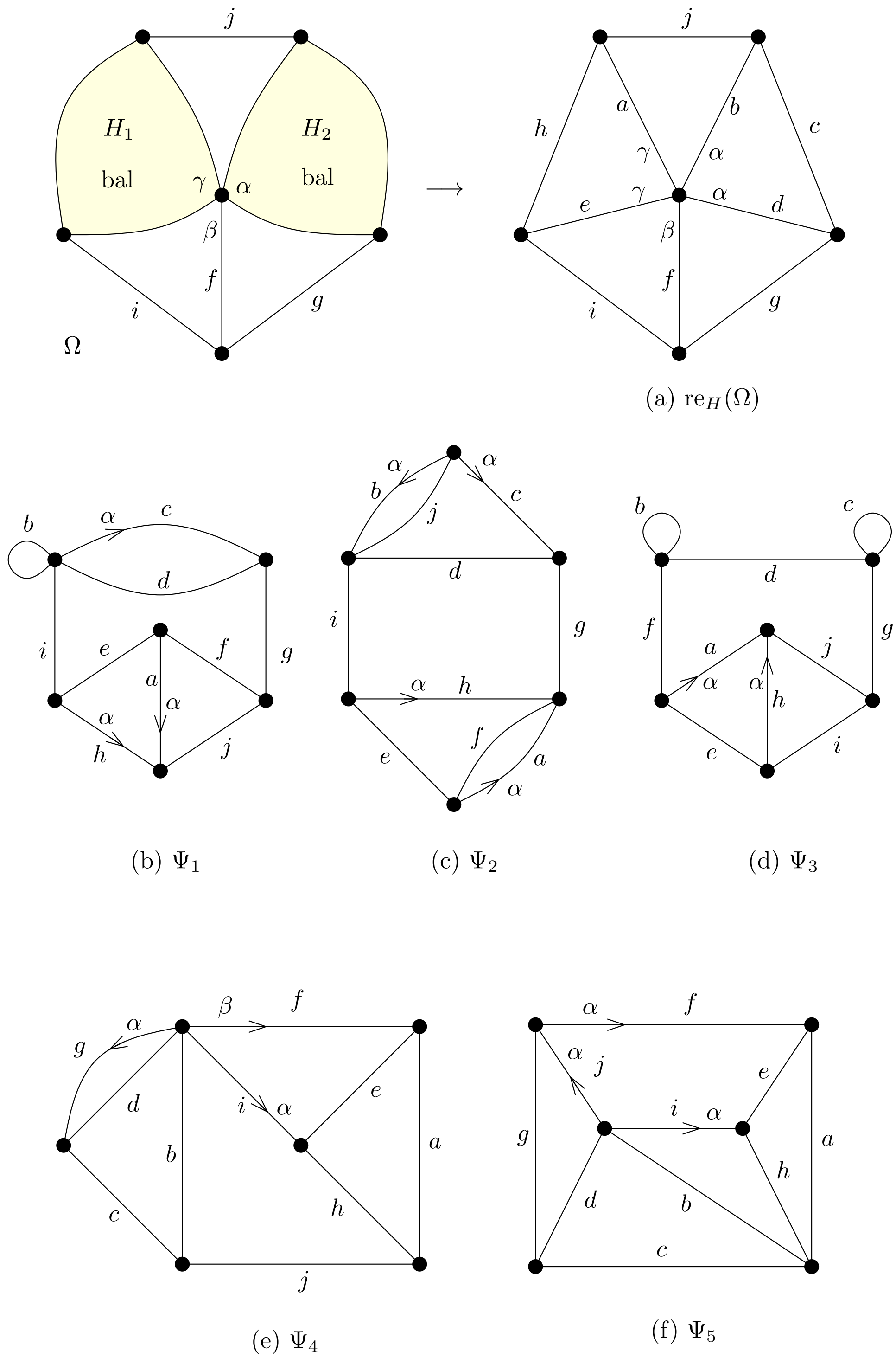
Representations of frame matroids having a biased graph representation with a balancing vertex

Given a 3-connected biased graph Ω with a balancing vertex, and with frame matroid $F(\Omega)$ non-graphic and 3-connected, we determine all biased graphs Ω' with $F(\Omega') \cong F(\Omega)$.

5.1 Introduction

Suppose Ω is a biased graph with a balancing vertex. By Proposition 1.25, any biased graph Ω' obtained as a roll-up of a balancing class of $\delta(u)$ has $F(\Omega') \cong F(\Omega)$. Here is an example of another way to obtain different biased graphs representing $F(\Omega)$. Consider the biased graph Ω at top left in Figure 5.1: Ω is a 3-signed graph with signature $\Sigma = \{\alpha, \beta, \gamma\}$. Alternatively, we may consider α, β, and γ to be elements of a free group, labelling edges oriented out from the balancing vertex. Biased graph Ω has a balanced subgraph H, consisting of two large subgraphs H_1, H_2 each of which meets the rest of the graph in exactly three vertices. Replacing each of these balanced subgraphs with a balanced triangle, we obtain the biased graph shown in Figure 5.1(a); we call this biased graph an H-*reduction* of Ω, and denote it $\mathrm{re}_H(\Omega)$. As with Ω, $\mathrm{re}_H(\Omega)$ is a 3-signed graph with signature $\Sigma = \{\alpha, \beta, \gamma\}$, or $\mathrm{re}_H(\Omega)$ may equivalently be thought of as labelled by elements α, β, γ of a free group, with all edges oriented out from the balancing vertex.

Each of the biased graphs Ψ_i ($i \in \{1, \ldots, 5\}$) shown in (b)-(f) of Figure 5.1 has $F(\Psi_i) \cong F(\mathrm{re}_H(\Omega))$. These are group-labelled by elements α, β, γ of a free group, with edge orientations as shown. Consider the biased graphs Ω_i ($i \in \{1, \ldots, 5\}$) shown in (b)-(f) of Figure



(a) $\mathrm{re}_H(\Omega)$ (b) Ψ_1 (c) Ψ_2 (d) Ψ_3 (e) Ψ_4 (f) Ψ_5

Figure 5.1: Biased graphs representing $F(\mathrm{re}_H(\Omega))$.

5.2. As with the biased graphs Ψ_i, each Ω_i is group-labelled by elements α, β, γ of a free group, with edge orientations as shown. For each $i \in \{1, \dots, 5\}$, $F(\Omega_i) \cong F(\Omega)$. This is readily verified by checking that their collections of circuits agree. Each biased graph Ω_i is obtained from the biased graph Ψ_i by replacing the balanced triangles or handcuffs formed by aeh and bcd with either a copy of H_1 or H_2, respectively, or, for $j \in \{1, 2\}$, with a biased graph H'_j obtained from H_j by a pinch or roll-up operation. We say each biased graph Ω_i is an H-*enlargement* of Ψ_i. (Each of these operations is defined more precisely in Section 5.2.3.)

The result of this chapter is that when Ω is 3-connected with $F(\Omega)$ non-graphic and 3-connected, this construction, along with roll-ups, yields all biased graph representations of $F(\Omega)$.

Theorem 5.1. *Let Ω be a 3-connected biased graph with a balancing vertex and with $F(\Omega)$ 3-connected and non-graphic. Suppose Ω' is a biased graph with $F(\Omega') \cong F(\Omega)$. Then either Ω' is a roll-up of Ω, or there is a subgraph H of Ω and a pair of biased graphs Ψ and Ψ', on at most six vertices with $F(\Psi) \cong F(\Psi')$, such that Ψ is an H-reduction of Ω and Ω' is an H-enlargement of Ψ'.*

The following two corollaries follow immediately from the proof of Theorem 5.1.

Corollary 5.2. *Let Ω be a 3-connected biased graph with a balancing vertex and with $F(\Omega)$ 3-connected and non-graphic. Up to roll-ups, the number of biased graph representations of $F(\Omega)$ is at most 26.*

Corollary 5.3. *Suppose $F(\Omega)$ is a 4-connected non-graphic frame matroid represented by a biased graph Ω having a balancing vertex. Then up to roll-ups, Ω uniquely represents $F(\Omega)$.*

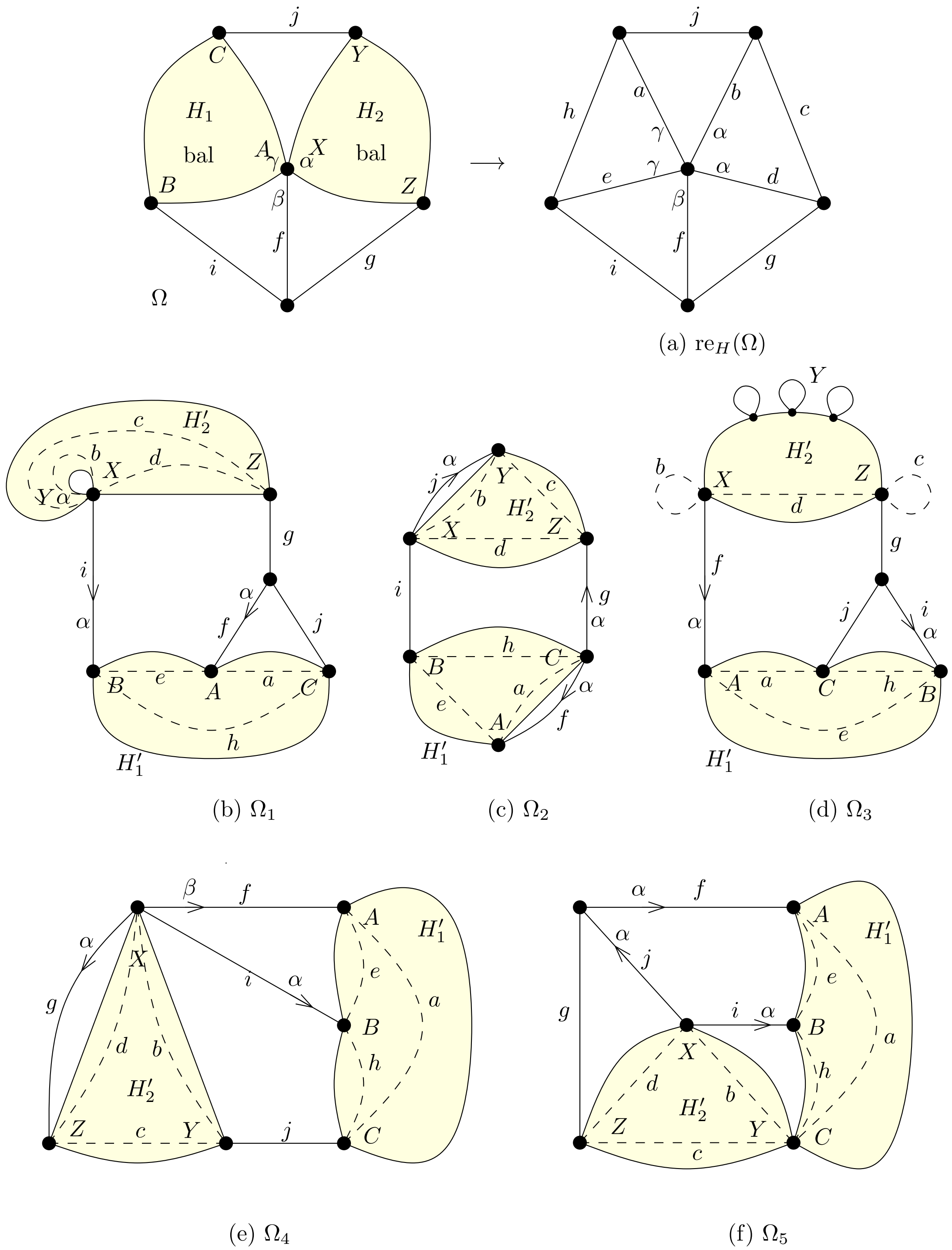
The rest of this chapter is devoted to a proof of Theorem 5.1.

5.2 Preliminaries

5.2.1 Cocircuits and hyperplanes in biased graphs

We collect a few facts about cocircuits, hyperplanes, and how they relate to the set of edges incident to a vertex in a biased graph. Recall that the set of links incident with a vertex v is denoted $\delta(v)$. The set of all edges incident to v we denote by $\delta(v)^+$ — *i.e.*, $\delta(v)^+ = \delta(v) \cup \{e : e \text{ is a loop incident to } v\}$.

Lemma 5.4. *Let Ω be a 2-connected biased graph containing an unbalanced cycle. For each $v \in V(\Omega)$, $\delta(v)^+$ is a cocircuit of $F(\Omega)$ if and only if v is not balancing.*



(a) $\mathrm{re}_H(\Omega)$

(b) Ω_1

(c) Ω_2

(d) Ω_3

(e) Ω_4

(f) Ω_5

Figure 5.2: Biased graphs representing $F(\Omega)$.

Proof. Suppose $v \in V(\Omega)$ is not a balancing vertex. Let $n = |V(\Omega)|$. Since the graph $G - v$ is connected and contains an unbalanced cycle, $r(E \setminus \delta(v)^+) = n - 1 = r(F(\Omega)) - 1$. If $e \in \delta(v)^+$, then $r(E \setminus \delta(v)^+ \cup \{e\}) = n$. Hence $E \setminus \delta(v)^+$ is a hyperplane, so $\delta(v)^+$ is a cocircuit. If $v \in V(\Omega)$ is balancing, then $G - v$ is balanced. Since $G - v$ is connected with $n - 1$ vertices, $r(E \setminus \delta(v)^+) = (n-1) - 1 = n - 2$. Thus $E \setminus \delta(v)^+$ is not a hyperplane. $\square$

Thus $E \setminus \delta(v)^+$ is a hyperplane if and only if v is not balancing.

Lemma 5.5. *Let Ω be a 3-connected biased graph. If $F(\Omega - v)$ is disconnected, then $F(\Omega - v)$ is a graphic hyperplane, and $\Omega - v$ is a pinch.*

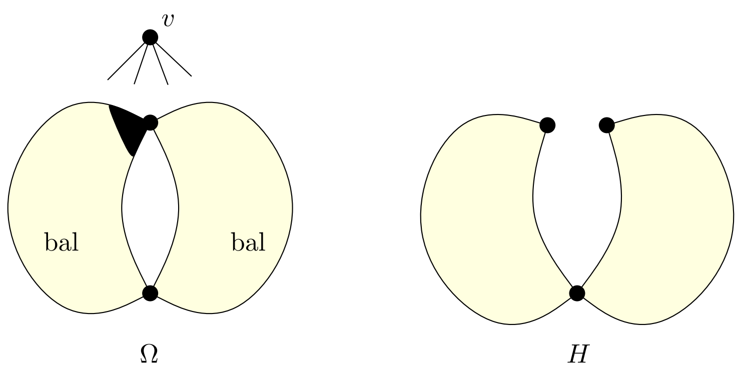


Figure 5.3: Ω is 3-connected, but $F(\Omega - v)$ is disconnected.

To prove Lemma 5.5, we use the following result.

Theorem 5.6 (Pagano [30])**.** *If Ω is a connected biased graph but $F(\Omega)$ is not connected, then there is a 1-separation (X, Y) of $F(\Omega)$ with $\Omega[X]$ and $\Omega[Y]$ connected.*

Proof of Lemma 5.5. Let (X, Y) be a separation of $F(\Omega - v)$ with $\Omega[X]$ and $\Omega[Y]$ connected (since $G - v$ is connected, the separation exists by Theorem 5.6). We first show that each of $(\Omega - v)[X]$ and $(\Omega - v)[Y]$ are balanced. Suppose without loss of generality, $(\Omega - v)[X]$ is unbalanced. Let $C \subseteq (\Omega - v)[X]$ be unbalanced, $e \in Y$. Since $G - v$ is 2-connected, there are two disjoint C-e paths P, P'. Together with e and C, paths P and P' form a theta subgraph T of $G - v$. Either all three cycles in T are unbalanced, or one of the cycles in T crossing the separation is balanced. In either case, we find a circuit in $F(\Omega - v)$ containing an element of X and an element of Y, a contradiction.

Hence both $(\Omega - v)[X]$ and $(\Omega - v)[Y]$ are balanced. Observe that a balanced cycle crossing the separation would be a circuit of $F(\Omega - v)$, so all cycles crossing $V(X) \cap V(Y)$ are unbalanced.

We now show that $|V(X) \cap V(Y)| = 2$. Suppose for a contradiction that $|V(X) \cap V(Y)| > 2$. Let $x, y, z \in V(X) \cap V(Y)$. Let P be an x-y path in $(\Omega - v)[X]$, and let P' be an x-y path in $(\Omega - v)[Y]$. Let Q be a P-z path in $(\Omega - v)[X]$, and let Q' be a P'-z path in $(\Omega - v)[Y]$ (where

Q or Q' are allowed to be trivial). Then $P \cup P' \cup Q \cup Q'$ either contains a theta subgraph T of $G - v$ in which all three cycles cross the separation, or two cycles meeting at a single vertex both of which cross the separation. In the first case, T is an countrabalanced theta and so a circuit of $F(\Omega - v)$ meeting both X and Y, a contradiction. In the second case, we have found a pair of unbalanced cycles meeting at a single vertex, and so a circuit of $F(\Omega - v)$ meeting both X and Y, again a contradiction.

Hence $\Omega - v$ is as shown in Figure 5.3: each of $(\Omega - v)[X]$ and $(\Omega - v)[Y]$ are balanced and all cycles crossing $V(X) \cap V(Y)$ are unbalanced. In other words, $\Omega - v$ is a signed graphic pinch: the biases of the cycles in $\Omega - v$ are obtained by orienting all edges arbitrarily, choosing a vertex $x \in V(X) \cap V(Y)$ and labelling with -1 all edges in $\delta(x) \cap X$, and labelling all other edges with $+1$. Splitting x, we obtain a graph H with $F(H) \cong F(\Omega - v)$.

Since $\Omega - v$ is unbalanced, by Lemma 5.4 $\delta(v)^+$ is a cocircuit, so $E(\Omega - v)$ is a hyperplane of $F(\Omega)$. □

5.2.2 Committed vertices

The following observation, which we stated and proved in Section 2.5.1, is key.

Proposition 2.16 (Slilaty, [31]). *If Ω is a connected biased graph with no balanced loops, then the complementary cocircuit of a connected non-binary hyperplane of $F(\Omega)$ consists precisely of the set of edges incident to a vertex.*

Let $M = F(\Omega)$ be a frame matroid, represented by Ω. Call a vertex $x \in V(\Omega)$ *committed* if $E \setminus \delta(x)^+$ is a connected non-binary hyperplane of $F(\Omega)$. The motivation for this definition is the following. Suppose Ω' is a biased graph with $F(\Omega') \cong F(\Omega)$. By Proposition 2.16, for every committed vertex $x \in V(\Omega)$, there is a vertex $x' \in V(\Omega')$ with precisely the same set of incident edges (more pedantically: the set of edges incident to x and the set of edges incident to x' both represent precisely the same set of elements of M).

The following lemma says that in a 3-connected biased graph, to determine that a vertex is committed it is enough to find a $U_{2,4}$ minor in the complement of the set of its incident edges.

Lemma 5.7. *Let Ω be 3-connected biased graph. Then $x \in V(\Omega)$ is committed if and only if $F(\Omega - x)$ is not binary.*

Proof. If $F(\Omega - x)$ is binary, then by definition x is not committed. Conversely, suppose x is not committed, *i.e.*, $E \setminus \delta(x)^+$ fails to be a connected non-binary hyperplane of $F(\Omega)$. If $F(\Omega - x)$ fails to be connected, then by Lemma 5.5 it is graphic. If $F(\Omega - x)$ fails to be a hyperplane, then by Lemma 5.4 x is balancing, so $F(\Omega - x)$ is graphic. The remaining possibility is that $E \setminus \delta(x)$ is a connected binary hyperplane of $F(\Omega)$. In any case, $F(\Omega - x)$ is binary. □

The following two lemmas are the keys to Theorem 5.1.

Lemma 5.8. *Let Ω be a biased graph with $F(\Omega)$ 3-connected. Suppose (X, Y) is a 3-separation of $F(\Omega)$ with $V(X) \cap V(Y) = \{u, v, w\}$, and suppose the biased subgraph H of Ω induced by X is balanced, $V(H) \setminus \{u, v, w\} \neq \emptyset$, and that every vertex $x \in V(H) \setminus \{u, v, w\}$ is committed. Let Ω' be a biased graph with $F(\Omega') \cong F(\Omega)$. Then the biased subgraph $H' \subseteq \Omega'$ induced by X is either*

1. *balanced and isomorphic to H,*

2. *obtained from H by pinching two vertices in $\{u, v, w\}$, or*

3. *obtained from H by rolling up all edges in H incident to exactly one of u, v, or w.*

Proof. Let the components of $H \setminus \{u, v, w\}$ be $H_1, \ldots, H_k$. Let U_i, V_i, W_i be the set of neighbours of u, v, w, respectively, in H_i, for $i \in \{1, \ldots, k\}$. Since $F(\Omega)$ is 3-connected, each component H_i contains a vertex in each of U_i, V_i and W_i. Let $A = E(H) \cap \delta(u)$, and let A_i be the set of edges in A whose second endpoint is in H_i. We first show that for each $i \in \{1, \ldots, k\}$, every edge in A_i is in Ω' either incident to a common vertex or is an unbalanced loop. If $|A_i| = 1$, the claim holds, so consider two edges e, f in a set A_i. There is a path in H_i linking the endpoints of e and f in H_i. This path together with e, f, and u is a balanced cycle D in Ω, so $E(D)$ is a circuit in $F(\Omega)$. Since every vertex in $D - u$ is committed, this implies that in Ω' either both e and f are incident to a common vertex or are both unbalanced loops. Similarly, define B to be the set of edges in $E(H) \cap \delta(v)$ and $C = E(H) \cap \delta(w)$, and define B_i (resp. C_i) to be the set of edges in B (resp. C) whose second endpoint is in H_i. The analogous argument shows that for each $i \in \{1, \ldots, k\}$, either all edges in B_i (resp. C_i) are incident to a common vertex or are all unbalanced loops.

Now for each $i \in \{1, \ldots, k\}$, let H_i' be the biased subgraph of Ω' induced by the elements of $F(\Omega)$ in $H_i \cup A_i \cup B_i \cup C_i$. Since every vertex $x \in V(H) \setminus \{u, v, w\}$ is committed, for each vertex $x \in V(H_i)$ there is a unique vertex $x' \in V(H_i')$ with $\delta(x') = \delta(x)$. Let U_i', V_i', W_i' be the sets of vertices x' of H' whose corresponding vertices x of H are in U_i, V_i, W_i, respectively. Suppose first that none of A_i, B_i, or C_i consist of unbalanced loops in Ω': each edge in A_i has an endpoint in U_i' and a common second endpoint u', each edge B_i has an endpoint in V_i' and a common second endpoint v', and each edge in C_i has an endpoint in W_i' and a common second endpoint w'. Now it may be that in Ω' all three of u', v', w' are distinct, or that some two of v', u', w' are the same vertex. It cannot be that $u' = v' = w'$: if so, let P be a u-v path and Q be a P-w path in H; then $E(P \cup Q)$ is independent in $F(\Omega)$ but would be dependent in $F(\Omega')$.

We now claim that at most one of A_i, B_i, or C_i are unbalanced loops in Ω'. For suppose to the contrary that the edges representing the elements in both A_i and B_i are unbalanced loops in Ω'. There is a u-v path P in H_i; $E(P)$ is independent in $F(\Omega)$, but a circuit in $F(\Omega')$, a contradiction. Similarly, not both A_i and C_i, nor both B_i and C_i, may be unbalanced loops.

Now suppose that in Ω' the edges in A_i are unbalanced loops, the edges in B_i are incident to a common vertex v', and the edges in C_i are incident to a common vertex w'. We claim that $v' \neq w'$. For supposing $v' = w'$, then, as in the previous paragraph, choosing a u-v path P and a P-w path Q in H_i yields a set $E(P \cup Q)$ independent in $F(\Omega)$ but dependent in $F(\Omega')$, a contradiction. Similarly, if a set B_i (resp. C_i) consists of unbalanced loops in Ω', then the common endpoint u' of the edges in A_i and the common endpoint w' of the edges in C_i (resp. v' of edges in B_i) are distinct in Ω'.

Hence each biased subgraph H_i' has the form of one of the biased graphs (a)-(g) shown in Figure 5.4. It is now easy to see that if for some $i \neq j$, H_i' and H_j' are not both of the

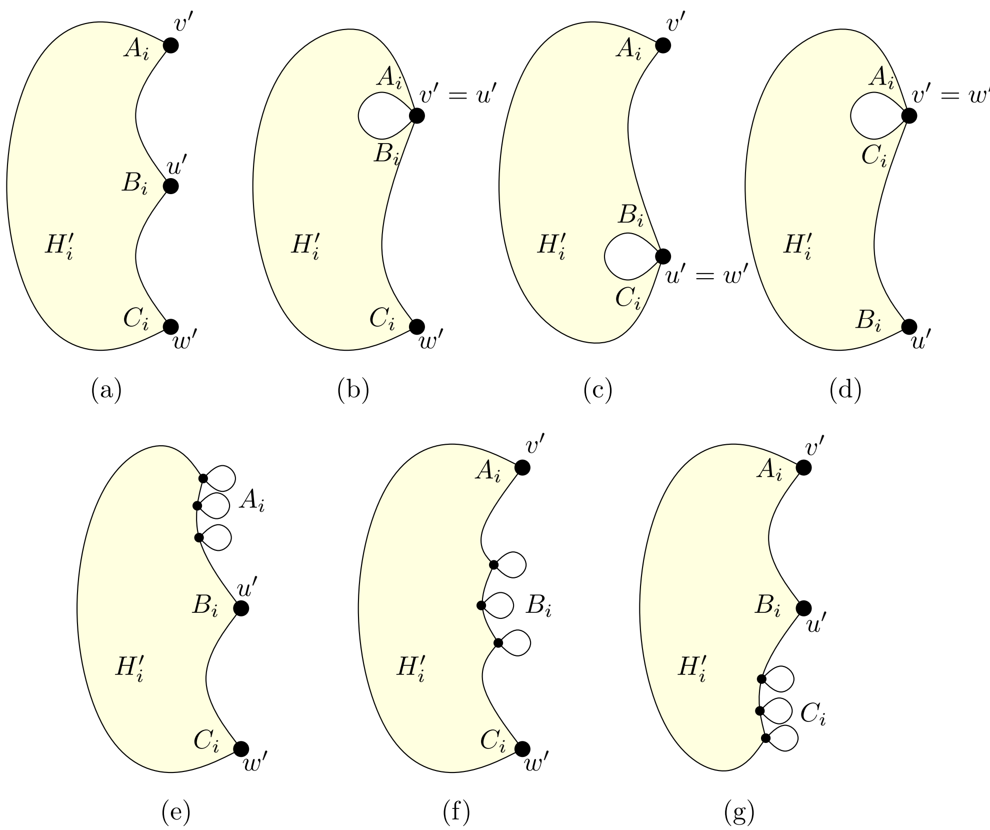


Figure 5.4: Possible biased graph representations of $F(H)$ when H is a balanced or pinched biased subgraphs all of whose internal vertices are committed.

same form (a)-(g), then $F(\Omega') \ncong F(\Omega)$. Hence $H' = \bigcup_i H_i'$ itself has the form of one of these biased graphs. The conclusion now follows: If H' is of the form shown in Figure 5.4 (a), then H' is balanced and isomorphic to H. If H' is of the form (b)-(d), then H' is obtained from H by pinching two of $\{u, v, w\}$, and if H' is one of (e)-(g), then H' is obtained from H by rolling up the edges of H incident to one of u, v, or w. $\square$

Lemma 5.9. *Let Ω be a biased graph with $F(\Omega)$ 3-connected. Suppose (X, Y) is a 3-separation of $F(\Omega)$ with $V(X) \cap V(Y) = \{u, v\}$, and suppose that the biased subgraph H*

of Ω induced by X is a pinch with signature $\Sigma = \{\alpha, \beta\} \subseteq \delta(u)$, that $V(H) \setminus \{u, v\} \neq \emptyset$, and that every vertex $x \in V(H) \setminus \{u, v\}$ is committed. Let H'' be the graph obtained by splitting u, with $\delta(u_1) \cup \delta(u_2) = \delta(u)$. Let Ω' be a biased graph with $F(\Omega') \cong F(\Omega)$. Then the biased subgraph $H' \subseteq \Omega'$ induced by X is either

1. *balanced and isomorphic to H'',*
2. *obtained from H'' by pinching two vertices in $\{u_1, u_2, v\}$, or*
3. *obtained from H'' by rolling up all edges in H'' incident to exactly one of u_1, u_2, or v.*

Proof. By Proposition 1.24, $F(H'') \cong F(H)$. The proof is that of Lemma 5.8, with H'' taking the place of H and u_1, u_2, v taking the place of u, v, w, respectively. □

5.2.3 H-reduction and H-enlargement

Let $F(\Omega)$ be a frame matroid, represented by biased graph Ω with a balancing vertex u. Recall from Chapter 1 that if Ω_0 is a biased graph with balancing vertex u, with k balancing classes $\Sigma_1, \dots, \Sigma_k$ in $\delta(u)$, then by Proposition 1.25 $\{\Omega_0, \Omega_1, \dots, \Omega_k\}$ is a collection of $k+1$ representations of $F(\Omega)$, where each biased graph Ω_i is obtained from Ω_0 by rolling up balancing class Σ_i. By definition, any member of this set is a roll-up of any other (so each Ω_i is a roll-up of itself).

Let Ω be a biased graph with a balancing vertex u, with $F(\Omega)$ 3-connected. Suppose (X, Y) is a 3-separation of $F(\Omega)$. Let $S = V(X) \cap V(Y)$, and let H be the biased subgraph of Ω induced by X. Suppose that $V(H) \setminus S \neq \emptyset$, that every vertex $x \in V(H) \setminus S$ is committed, and that one of the following holds:

1. $S = \{u, v, w\}$ for some $v, w \in V(\Omega)$, and H is balanced, or
2. $S = \{u, v\}$ for some $v \in V(\Omega)$ and H is a pinch with signature $\Sigma \subseteq \delta(u)$.

An H-*reduction* is one of the following operations. In case (1), replace H in Ω with a balanced triangle on $\{u, v, w\}$. In case (2), replace H in Ω with an unbalanced cycle consisting of two u-v edges and an unbalanced loop on u. Likewise, if $H_1, \dots, H_k$ are pairwise edge disjoint biased subgraphs of Ω each satisfying the conditions for an H_i-reduction, then — writing $H = \{H_1, \dots, H_k\}$ — the biased graph obtained by performing an H_i reduction for each $i \in \{1, \dots, k\}$ is also called an H-*reduction*. We call each such balanced or pinched subgraph H_i a *lobe* of Ω. An H-reduction of Ω containing no lobes is denoted by $\mathrm{re}_H(\Omega)$.

If $F(\Psi)$ is isomorphic to $F(\mathrm{re}_H(\Omega))$ — say via replacement of lobes $H_1, \dots, H_k$ — then a biased graph Ω' with $F(\Omega') \cong F(\Omega)$ may be obtained from Ψ as follows. For $i \in \{1, \dots, k\}$, let C_i be the 3-circuit of $F(\mathrm{re}_H(\Omega))$ that replaced lobe H_i in Ω. If C_i is a balanced triangle or a pair of handcuffs in Ψ, then replace C_i in Ψ with a biased subgraph H'_i of one of the three forms given by Lemma 5.8 or 5.9:

1. If C_i is a balanced triangle in Ψ, replace C_i by a balanced biased subgraph H_i', where H_i' is a copy of the balanced subgraph H_i or, in the case H_i is a pinch, a copy of the graph H_i'' obtained by splitting u.

2. If C_i a pair of tight handcuffs, replace C_i with a biased graph H_i' obtained from H_i or H_i'' by pinching two of its vertices in $\{u, v, w\}$ or $\{u_1, u_2, v\}$, respectively.

3. If C_i is a pair of loose handcuffs, replace C_i with a biased graph H_i' obtained from H_i or H_i'' by a roll-up of edges incident to a vertex in $\{u, v, w\}$ or $\{u_1, u_2, v\}$, respectively.

In each case, the replacement is done by deleting $E(C_i)$ from Ψ and identifying each vertex of Ψ previously incident to an edge in C_i with a vertex of H_i' appropriately. Which pairs of vertices to identify are chosen as follows. Suppose 3-circuit abc in Ψ is to be replaced by a biased graph H' of one of the forms given by Lemma 5.8 or 5.9. As in the proofs of Lemma 5.8 and 5.9, let $A = \delta(u) \cap E(H)$, $B = \delta(v) \cap E(H)$, and $C = \delta(w) \cap E(H)$ if H is balanced in Ω, or if H is a pinch in Ω, let $A = \delta(u_1) \cap E(H'')$, $B = \delta(u_2) \cap E(H'')$, and $C = \delta(v) \cap E(H'')$, where H'' is obtained by splitting vertex u and u_1, u_2 are the resulting new vertices of H''. Let

$$v_A = \begin{cases} u & \text{if } H \text{ is balanced} \\ u_1 & \text{if } H \text{ is a pinch,} \end{cases} \qquad v_B = \begin{cases} v & \text{if } H \text{ is balanced} \\ u_2 & \text{if } H \text{ is a pinch,} \end{cases} \qquad v_C = \begin{cases} w & \text{if } H \text{ is balanced} \\ v & \text{if } H \text{ is a pinch.} \end{cases}$$

Each edge in the 3-circuit abc in $F(\text{re}_H(\Omega))$ models a path in Ω linking pairs of vertices in $\{v_A, v_B, v_C\}$, with a corresponding to a v_A-v_B path, b a v_B-v_C path, and c a v_C-v_A path. Indeed, circuit abc in $\text{re}_H(\Omega)$ may be obtained as a minor of Ω from such paths. If in Ψ, edges a and b share a common endpoint x_{ab}, edges b and c share common endpoint x_{bc}, and edges a and c share endpoint x_{ac}, then construct Ω' by identifying vertex v_B with x_{ab}, vertex v_C with x_{bc}, and vertex v_A with x_{ac}. Observe that in the case abc is a tight handcuff, two of x_{ab}, x_{bc}, x_{ac} are the same vertex, thus H' is a pinch in Ω', as desired. If abc is a pair of loose handcuffs, and so has two edges, say a and c, that do not share an endpoint, then again identify vertex v_B with x_{ab} and vertex v_C with x_{bc}, and roll-up the edges in A. We call the biased graph Ω' resulting from carrying out this procedure for each 3-circuit C_i, $i \in \{1, \dots, k\}$, that is not a contrabalanced theta, an H-*enlargement* of Ψ.

Figures 5.1 and 5.2 provide an example of this process. Figure 5.1 shows five biased graphs whose frame matroids are isomorphic to the frame matroid of the H-reduction of the biased graph Ω shown at top left in the figure, where $H = H_1 \cup H_2$. Figure 5.2 shows the five H-enlargements of these biased graphs, and so five non-isomorphic biased graphs representing $F(\Omega)$.

We now show that H-enlargements of biased graphs whose frame matroids are isomorphic to the frame matroid of an H-reduction of Ω, for some subgraph H of Ω, are all we need in order to find all biased graph representations of $F(\Omega)$.

Lemma 5.10. *Let Ω be a biased graph with $F(\Omega)$ 3-connected. Suppose for $i \in \{1, \dots, k\}$, (X_i, Y_i) is a 3-separation of $F(\Omega)$ and $\bigcap_i X_i = \emptyset$. Let $H_i = \Omega[X_i]$ be the biased subgraph induced by X_i, and let $S_i = V(X_i) \cap V(Y_i)$. Suppose for each $i \in \{1, \dots, k\}$, either $|S_i| = 3$ and H_i is balanced, or $|S_i| = 2$ and H_i is a pinch with its balancing vertex contained in S_i. Suppose further that $V(H_i) \setminus S_i$ is non-empty, and that every vertex $x \in V(H_i) \setminus S_i$ is committed. Let $H = \bigcup_{i=1}^{k} H_i$. If Ω' is a biased graph representing $F(\Omega)$, then Ω' is an H-enlargement of a biased graph Ψ with $F(\Psi) \cong F(\mathrm{re}_H(\Omega))$.*

Proof. Biased graph $\mathrm{re}_H(\Omega)$ is a minor of Ω, say $\Omega \setminus S/T = \mathrm{re}_H(\Omega)$. Then $F(\Omega) \setminus S/T = F(\mathrm{re}_H(\Omega))$, and $F(\Omega') \setminus S/T = F(\Psi)$, where $\Psi = \Omega' \setminus S/T$. Since $F(\Omega') \cong F(\Omega)$, we have $F(\mathrm{re}_H(\Omega)) \cong F(\Psi)$, so this produces a biased graph Ψ with $F(\Psi) \cong F(\mathrm{re}_H(\Omega))$. For each $i \in \{1, \dots, k\}$, in $F(\mathrm{re}_H(\Omega))$, there is a circuit of size three $C_i = e_1 e_2 e_3$ resulting from the minor operations which brought Ω to $\mathrm{re}_H(\Omega)$; *i.e.*, for some $H_i \subseteq \Omega$, $C_i = H_i \setminus S/T$ in $\mathrm{re}_H(\Omega)$. By Lemma 5.8 or 5.9, the set of edges X_i in Ω' induces a biased subgraph H_i' of one of types 1, 2, or 3, as described in Lemma 5.8 or 5.9. In $F(\Psi)$, C_i forms a circuit of size 3. Replacing C_i with (1) a balanced subgraph isomorphic to H_i or H_i'' if C_i is a balanced triangle, (2) a pinch of two vertices in $\{u, v, w\}$ of H_i or $\{u_1, u_2, v\}$ of H_i'' if C_i a pair of tight handcuffs, or (3) a roll-up of H_i from one of u, v, or w or of H_i'' from one of u_1, u_2, or v, if C_i is a pair of loose handcuffs, yields Ω'. □

Hence whenever Ω contains a collection of edge disjoint lobes $H = \{H_1, \dots, H_k\}$, all biased graphs representing $F(\Omega)$ are obtained by H-enlargements of biased graphs with frame matroids isomorphic to $F(\mathrm{re}_H(\Omega))$. To find all biased graphs representing $F(\Omega)$ therefore, we just need find all biased graphs with frame matroids isomorphic to $F(\mathrm{re}_H(\Omega))$.

5.3 Proof of Theorem 5.1

We are now ready to prove Theorem 5.1.

Theorem 5.1 *Let Ω be a 3-connected biased graph with a balancing vertex and with $F(\Omega)$ 3-connected and non-graphic. Suppose Ω' is a biased graph with $F(\Omega') \cong F(\Omega)$. Then either Ω' is a roll-up of Ω, or there is a subgraph H of Ω and a pair of biased graphs Ψ and Ψ', on at most six vertices with $F(\Psi) \cong F(\Psi')$, such that Ψ is an H-reduction of Ω and Ω' is an H-enlargement of Ψ'.*

The proof is long, having to deal with several cases, but the strategy is not difficult. Given a biased graph Ω with a balancing vertex u, we show that either up to roll-ups Ω uniquely represents $F(\Omega)$ or Ω has a biased subgraph H so that the H-reduction of Ω has at most six vertices. Since $\mathrm{re}_H(\Omega)$ is small, we may determine all biased graphs Ψ with $F(\Psi) \cong F(\mathrm{re}_H(\Omega))$. In each case, we exhibit these biased graphs in Section 5.4. Then by Lemma 5.10, all representations of $F(\Omega)$ are given by H-enlargements of these biased graphs. Here an an outline of the proof:

1. If u is the only uncommitted vertex of Ω, we show that up to roll-ups Ω uniquely represents $F(\Omega)$.
2. If Ω has a second uncommitted vertex v, then we consider two cases, according to whether Ω has an unbalanced loop incident to u, or not.
   (a) If Ω has an unbalanced loop incident to u, we show that there are at most two balancing classes A, B in $\delta(u)$ in $\Omega - v$. We then consider two sub-cases to determine all biased graphs whose frame matroids are isomorphic to the frame matroid of an H-reduction of Ω.
   (b) If there is no unbalanced loop incident to u, we show that there are at most three balancing classes in $\delta(u)$ in $\Omega - v$. We consider three sub-cases, according to the number of balancing classes in $\delta(u)$ in $\Omega - v$ and in Ω:
      i. $\delta(u)$ has 3 balancing classes in $\Omega - v$ and exactly 3 balancing class in Ω;
      ii. $\delta(u)$ has 3 balancing classes in $\Omega - v$ and > 3 balancing classes in Ω;
      iii. $\delta(u)$ has < 3 balancing classes in $\Omega - v$.

      In each case, we specify a biased subgraph H of Ω and exhibit the set of biased graphs whose frame matroids are isomorphic to $F(\mathrm{re}_H(\Omega))$.

Let Ω=$(G, \mathcal{B})$ be a 3-connected biased graph with a balancing vertex u, with $F(\Omega)$ 3-connected and non-graphic. Set $E = E(G)$ and $V = V(G)$.

5.3.1 All but the balancing vertex are committed

If u is the only uncommitted vertex of Ω, things are straightforward:

Proposition 5.11. *Let Ω be a biased graph with balancing vertex u, and with $F(\Omega)$ 3-connected and non-graphic. If all vertices $v \in V \setminus \{u\}$ are committed, then any biased graph Ω' with $F(\Omega') \cong F(\Omega)$ is obtained as a roll-up of Ω.*

Proof. Let $A_1, A_2, \ldots, A_k$ be the balancing classes of $\delta(u)$. Since $F(\Omega)$ is non-graphic, $k \geq 3$. Since $F(\Omega)$ is 3-connected, there is at most one loop l incident to u, which is unbalanced. Since every vertex but u is committed, every biased graph representing $F(\Omega)$ has a biased subgraph isomorphic to $\Omega - u$. Let Ω' be a biased graph with $F(\Omega') \cong F(\Omega)$. Then for every vertex $v \in V(\Omega - u)$ there is a vertex $v' \in V(\Omega')$ with $\delta(v') = \delta(v)$. Moreover, each element represented by a u-v edge in A_i, $i \in \{1, \ldots, k\}$, is represented in Ω' by either an edge incident to v' or an unbalanced loop incident to v'. Since $F(\Omega)$ is non-graphic, every biased graph representing $F(\Omega)$ has $|V(\Omega)|$ vertices. Hence every biased graph representing $F(\Omega)$ may be obtained from $G-u$ by adding a vertex u', and adding the edges in $A_1, \dots, A_k$, and l, such that the resulting biased graph has frame matroid isomorphic to $F(\Omega)$. Again, since every vertex of Ω but u is committed, for each edge $e = uv$ in a set A_i, in Ω' one of the endpoints of e is v', and our only choice is whether e has u' as its other endpoint or e is an unbalanced loop incident to v'.

Since $l \notin \delta(v)$ for any $v \neq u$, l cannot be incident to any vertex v' corresponding to a vertex $v \neq u$ in Ω, and so must be incident only to u' in Ω, and so remains an unbalanced loop in Ω'. Now suppose an element e represented by an edge uv in A_i, for some $i \in \{1, \dots, k\}$, is represented by an unbalanced loop incident to v' in Ω'. Let $f = uw$ be an edge in A_j, $j \in \{1, \dots, k\}$. There is a v-w path P in $\Omega - u$, and a corresponding v'-w' path P' with $E(P') = E(P)$ in Ω'. If $j \neq i$, then $E(P) \cup \{e, f\}$ is independent in $F(\Omega)$, and so f is not an unbalanced loop in Ω'; f is therefore a u'-w' edge in Ω'. If $j = i$, then $E(P) \cup \{e, f\}$ is a circuit of $F(\Omega)$, which implies f must be an unbalanced loop incident to w' in Ω'. □

5.3.2 Ω has ≥ 2 uncommitted vertices

So assume Ω has a second uncommitted vertex $v \neq u$. We have several cases to consider, according to whether or not there is an unbalanced loop at u, the number of balancing classes in Ω and in $\Omega - v$, and their sizes.

(a) Ω has an unbalanced loop on u

We first consider the case that there is an unbalanced loop l incident to u.

Lemma 5.12. *There are at most two balancing classes in $\delta(u)$ in $\Omega - v$.*

Proof. Suppose for a contradiction that there are three balancing classes in $\delta(u)$ in $\Omega - v$. Since $\Omega - v$ is connected, contracting all edges not incident to u then deleting all but one edge in each of three balancing classes yields, together with the unbalanced loop incident to u, a biased graph representing $U_{2,4}$. Hence $F(\Omega - v)$ is non-binary, and so by Lemma 5.7 v is committed, a contradiction. □

Throughout, to help keep track of biases of cycles, we label edges by elements $\alpha, \beta, \dots$ of a free group generated by the elements labelling edges in a biased graph. Edges are assumed to have been given an arbitrary orientation and label 1 unless stated otherwise. Edges in $\delta(u)$ are assumed to be oriented out from u unless stated otherwise.

Let A, B be the two balancing classes of $\delta(u)$ remaining in $\Omega - v$. Label edges in A with α and edges in B with β. Since $F(\Omega)$ is non-graphic, there is at least one u-v edge in a balancing class distinct from A and B (otherwise by Proposition 1.20 Ω is signed graphic and by Proposition 1.24 is graphic). Since $F(\Omega)$ is 3-connected, no two u-v edges are in the same balancing class. Let us label with γ, ϵ, ζ, ..., the u-v edges not in balancing classes A or B. Let C be the set of edges in $\delta(v) \setminus \delta(u)$, and let F be the set of u-v edges not in $A \cup B$. Let $Y = \{l\} \cup \{e : e \text{ is a } u\text{-}v \text{ edge}\}$, and let $X = E \setminus Y$. If X is empty, then $F(\Omega) \cong U_{2,m+1}$, where m is the number of u-v edges in Ω, and Theorem 5.1 trivially holds. So assume $X \neq \emptyset$. If either A or B has no edge with an endpoint different from v, then (X, Y) is a 2-separation of $F(\Omega)$, a contradiction. Hence (X, Y) is a 3-separation of $F(\Omega)$ and X contains an edge in each of A and B. Note that since Ω is 3-connected, $|C| > 1$ (else u together with the endpoint of the single edge in C different from v separate v from the rest of Ω). Let $H = \Omega[X]$, and let $W = V \setminus \{u, v\}$.

i. ≥ 2 u**-**v **edges not in** $A \cup B$

Suppose first that there are at least two u-v edges in balancing classes distinct from A and B (Figure 5.5).

Claim. Every vertex in W is committed.

Proof of Claim. For every $x \in W$, $\Omega[W \setminus x]$ is connected. A u-v path via $\Omega[W \setminus x]$ together with two edges in F and l yields a $U_{2,4}$ minor in $F(\Omega \setminus x)$. □

Hence all vertices of $H - \{u, v\}$ are committed. Then since $V(H) \setminus \{u, v\}$ is non-empty, we may apply Lemma 5.9. The biased graph $\mathrm{re}_H(\Omega)$ obtained by replacing H with a pair of tight handcuffs abc is shown in Figure 5.5. Let d, e be the u-v edges in balancing classes A, B, respectively, if present in Ω. Since $F(\mathrm{re}_H(\Omega)) \setminus \{l, d, e\} \cong U_{2,m}$, where $m = |F| + 3$,

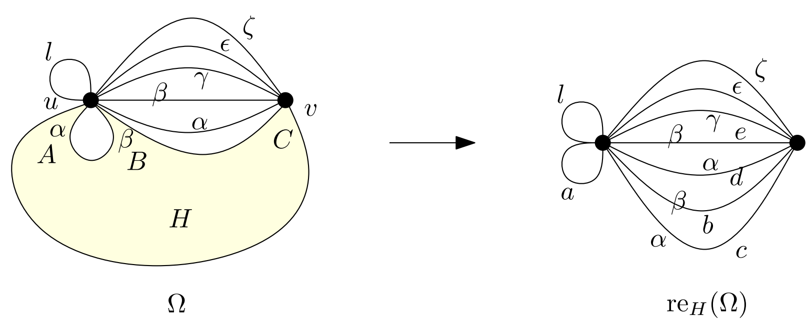


Figure 5.5: $F(\mathrm{re}_H(\Omega)) \setminus \{l, d, e\} \cong U_{2,m}$. Loops l and a are unbalanced; the indicated labelling on the remaining edges of $\mathrm{re}_H(\Omega)$ realises the biases of its cycles.

all biased graphs representing $F(\mathrm{re}_H(\Omega))$ are obtained from a biased graph representing $U_{2,m}$ by adding l so that in $F(\mathrm{re}_H(\Omega))$ element l is parallel with a, element d in parallel with c, and e in parallel with b. Two examples, along with their H-enlargements, are shown in Figure 5.6.

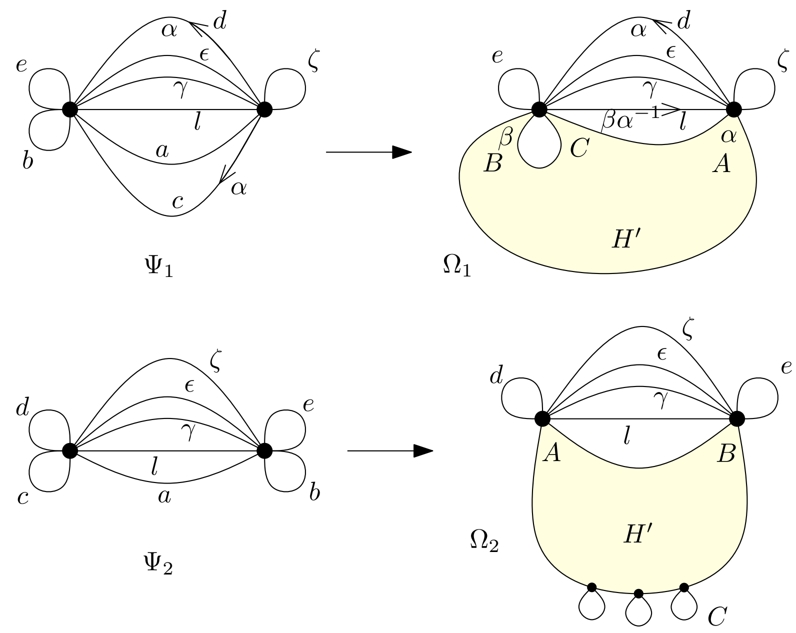


Figure 5.6: H-enlargements of Ψ_1, Ψ_1: $F(\Psi_1) \cong F(\Psi_2) \cong F(\mathrm{re}_H(\Omega))$; $F(\Omega_1) \cong F(\Omega_2) \cong F(\Omega)$.

ii. Just one u-v edge not in $A \cup B$

So suppose now that there is only one edge in F. If each of A and B have size at least two, then we again find that all vertices of $H - \{u, v\}$ are committed: for all $x \in W$ there remains a contrabalanced theta in $\Omega - x$, which together with l yields a $U_{2,4}$ minor in $F(\Omega - x)$. Hence we may apply the same procedure as the preceding paragraph. All biased graphs isomorphic to $F(\mathrm{re}_H(\Omega))$ are obtained by adding at most three edges (representing l, and if present, a u-v edge in A or B) to a biased graph representing $U_{2,4}$.

So suppose now there is only one edge in F and that $|A| = 1$ while $|B| > 1$ (Figure 5.7 (a)). Let $z \in W$ be the other endpoint of the single edge in A. Since for every $x \in W \setminus \{z\}$,

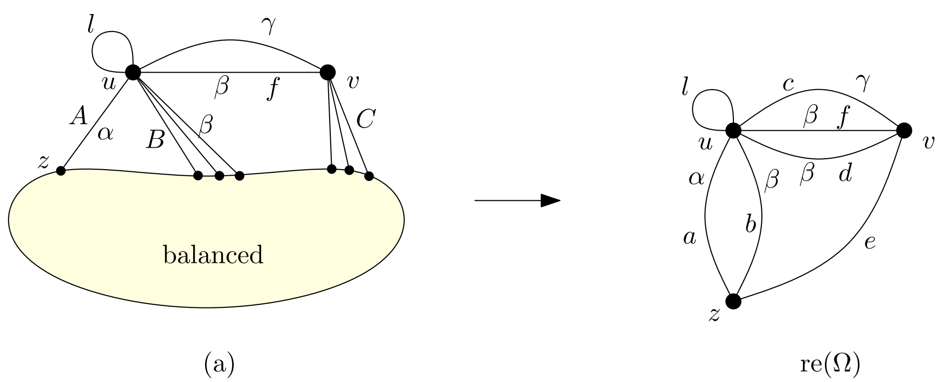


Figure 5.7: There is only one edge in F and $|A| = 1$ while $|B| > 1$.

there remains an countrabalanced theta in $\Omega - x$, which together with l yields a $U_{2,4}$ minor in $F(\Omega - x)$, every vertex $x \in W \setminus \{z\}$ is committed. Let H be the balanced biased subgraph formed by $\Omega[W]$ together with the edges in $B \cap X$ and C. Replacing H by a balanced triangle bde, we find that $\mathrm{re}_H(\Omega)$ is the biased graph at right in Figure 5.7. A u-v edge f in B may or may not be present in Ω; note that such an element is in parallel with d in $F(\mathrm{re}_H(\Omega))$. Hence if Ψ' is a biased graph representing $F(\mathrm{re}_H(\Omega) \setminus f)$, a biased graph Ψ representing $F(\mathrm{re}_H(\Omega))$ is uniquely obtained by adding an edge f to Ψ' so that f is in parallel with d in $F(\Psi)$. Let us denote by L_1 the biased graph obtained from $\mathrm{re}_H(\Omega)$ by deleting a u-v edge in B, if present.

Since all circuits of $F(L_1)$ except abl, cdl, and bde are size four, and $F(L_1)$ has rank three and is non-graphic, we obtain all biased graphs representing $F(L_1)$ by considering all possible ways circuits abl, cdl, and bde may be represented in a biased graph on three vertices. In this way we find that the biased graphs shown in Figure 5.33 are all the biased graphs with frame matroid isomorphic to $F(L_1)$. Hence in the case there is only one edge in F, $|A| = 1$, and $|B| > 1$, every biased graph Ω' with $F(\Omega') \cong F(\Omega)$ is an H-enlargement of a biased graph shown in Figure 5.33 after possibly adding an edge representing the element that is a u-v edge in B, if present in Ω.

Finally, suppose there is only one edge in F and that $|A| = |B| = 1$ (Figure 5.8, left). Let z, w be the endpoints in W of the single edge in A, B, respectively.

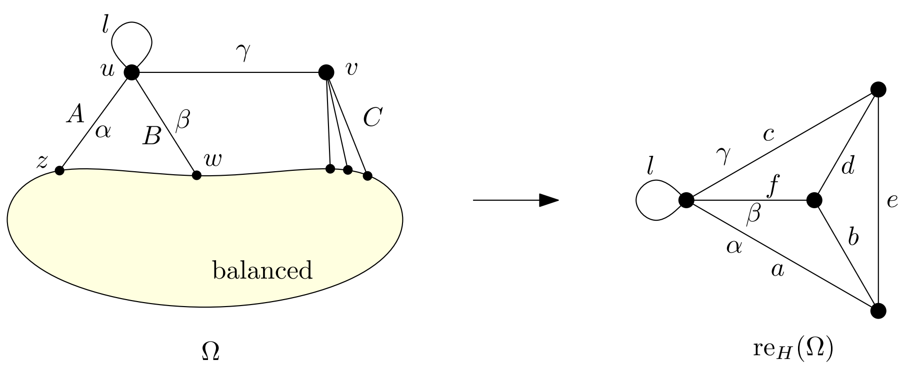


Figure 5.8: If there is only one edge in F and $|A| = |B| = 1$.

Claim. Let $x \in V(\Omega) \setminus \{u, v, w, z\}$. Then x is committed.

Proof of claim. Since $\Omega - x$ is connected, and there are both α-labelled and β-labelled edges in $\Omega - x$, $\Omega - x$ contains a countrabalanced theta. Together with l, this yields a $U_{2,4}$ minor in $F(\Omega - x)$. □

Hence $H = \Omega - u$ is a balanced subgraph of Ω having all vertices except its vertices at which it meets $\delta(u)$ committed, and $(\delta(u)^+, E(H))$ is a 3-separation of $F(\Omega)$. Replacing H by balanced triangle bde, we obtain biased graph $\mathrm{re}_H(\Omega)$ shown at right in Figure 5.8, let us call it L_2. Since $F(L_2)$ is a single element coextension of $F(L_1)$, we may obtain every biased graph representing $F(L_2)$ by uncontracting an element f of every biased graph representing $F(L_1)$ in every possible way such that the resulting biased graph has frame matroid isomorphic to $F(L_2)$. In this way we obtain the biased graphs of Figures 5.34 and 5.35.

This exhausts the possibilities for 3-connected biased graphs with a balancing vertex and an unbalanced loop.

(b) Ω has no unbalanced loop on u

We now consider the case that there is no unbalanced loop incident to u.

Lemma 5.13. *There are at most three balancing classes in $\delta(u)$ in $\Omega - v$.*

Proof. Suppose for a contradiction that there are four balancing classes in $\delta(u)$ in $\Omega - v$. Since $\Omega - v$ is connected, contracting all edges not incident to u then deleting all but one edge in each of four balancing classes yields a biased graph representing $U_{2,4}$. Hence $F(\Omega - v)$ contains a $U_{2,4}$ minor, so by Lemma 5.7 v is committed, a contradiction. □

We consider several cases, according to the number of balancing classes of $\delta(u)$ in Ω and in $\Omega - v$, and their sizes. We consider the following three sub-cases, which are broken down into further subcases:

i. $\delta(u)$ has three balancing classes in $\Omega - v$, and just three balancing classes in Ω;

ii. $\delta(u)$ has three balancing classes in $\Omega - v$, and more than three balancing classes in Ω;

iii. $\delta(u)$ has less than three balancing classes in $\Omega - v$.

i. $\delta(u)$ has 3 balancing classes in $\Omega - v$, and just 3 balancing classes in Ω

The fact that v is uncommitted forces a special structure on $\Omega - v$. Recall that a *fat theta* is a biased graph that is the union of three balanced subgraphs A_1, A_2, A_3 mutually meeting at just a single pair of vertices, in which a cycle C is balanced if and only if $C \subseteq A_i$ for some $i \in \{1, 2, 3\}$ (Figure 5.9; Section 3.1).

Lemma 5.14. *If there are three balancing classes of $\delta(u)$ in $\Omega - v$, then $\Omega - v$ is a fat theta.*

Proof. Lemmas 1.27 and 5.7 immediately imply $\Omega - v$ is a fat theta. □

Lemma 5.15. *At most one balancing class of $\delta(u)$ in Ω has size one.*

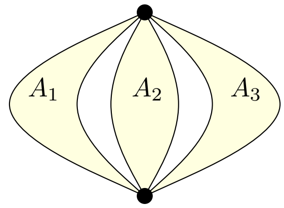


Figure 5.9: A fat theta: A_1, A_2, and A_3 are balanced subgraphs; a cycle is unbalanced if and only if it meets two of A_1, A_2, and A_3.

Proof. Suppose for a contradiction that there are two balancing classes in Ω of size one, let us call them A and B, with edge $a \in A$ and edge $b \in B$. Since edge a is not in any balanced cycle, every circuit of $F(\Omega)$ containing a is either a countrabalanced theta or a pair of handcuffs. A countrabalanced theta must contain an edge from each of the three balancing classes, and so contains b. A pair of handcuffs contain two unbalanced cycles meeting at u; if the cycle containing a does not contain b labelled edge, then the other cycle must contain b. Hence every circuit containing a contains b. Similarly, every circuit containing b contains a. Hence a and b are in series, contradicting the fact that $F(\Omega)$ is 3-connected. $\square$

We consider two sub-cases, according to whether or not Ω has a balancing class that consists of just a single edge.

A. A balancing class of size 1

Suppose that $\delta(u)$ has exactly three balancing classes in Ω, and three balancing classes in $\Omega - v$, and there is a balancing class of $\delta(u)$ in Ω of size one.

If Ω has rank three, then Ω is the biased graph of Figure 5.10. In this case, by Proposition

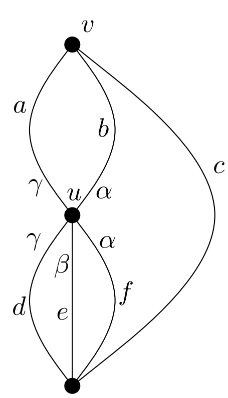


Figure 5.10: Ψ_0; biases are given by the group-labelling.

5.17 (Section 5.4, page 156) all biased graphs with frame matroids isomorphic to $F(\Omega)$ are shown in Figure 5.36. So assume rank$(F(\Omega)) > 3$. Since $\Omega - v$ is a fat theta and we have exactly one balancing class of size one, and exactly three balancing classes in both Ω and $\Omega - v$, Ω has the form of one of the biased graphs shown in Figure 5.11 (a) or (b), where each of H_1, H_2, H_3 are balanced and connected (subgraphs H_1, H_2, and H_3 are obtained by extending, in the obvious way, the partition of edges of the fat theta $\Omega - v$ into three balanced subgraphs meeting precisely at its 2-cut $\{u, w\}$).

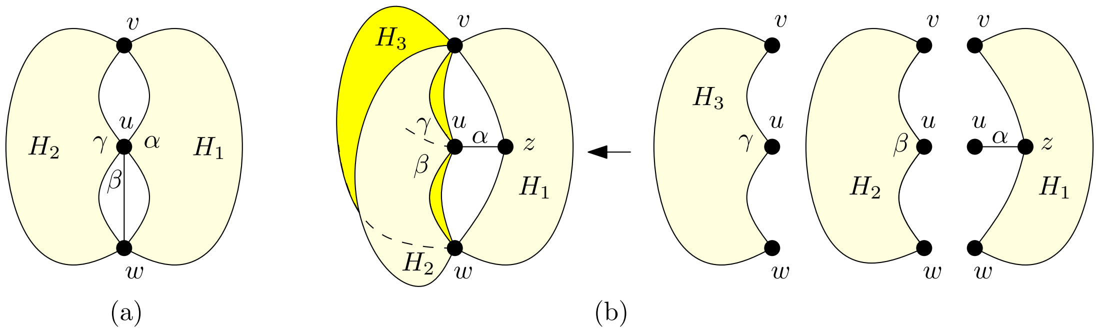


Figure 5.11: If Ω has a balancing class of size one. Biased graph (b) is obtained by identifying the vertices labelled v, those labelled u, and those labelled w in each of H_1, H_2, H_3.

We need to know precisely which vertices of Ω are committed. Our next lemma provides the answer.

Lemma 5.16. *Let Ω be a 3-connected biased graph with $F(\Omega)$ non-graphic and 3-connected, with a balancing vertex u and a second uncommitted vertex $v \neq u$, with no loop incident to u. Suppose there are exactly three balancing classes in Ω and in $\Omega - v$, and that Ω has a balancing class of size one. Then Ω has the form of one of the biased graphs shown in Figure 5.12, where all internal vertices $t \in V(\Omega) \setminus \{u, v, w, x, y, z\}$ of each of the lobes H_1, H_2, H_3 are committed.*

Proof. Clearly none of u, v, w are committed in either of biased graphs (a) or (b) of Figure 5.11. Suppose first Ω has the form of biased graph (a). Let $x \in V(\Omega) \setminus \{u, v, w\}$. As long as in $\Omega - x$ there are u-w paths $P \subseteq H_1$ and $P' \subseteq H_2$, and u-v paths $Q \subseteq H_1$ and $Q' \subseteq H_2$, there is a $U_{2,4}$ minor in $M|_{E \setminus \delta(x)}$, and so x is committed. Suppose $x \in V(H_1)$, say, is not committed. Then the deletion of x must destroy either all u-w paths in H_1 or all u-v paths in H_1. *I.e.*, x is a cut vertex in H_1. Connectivity implies now that either x is incident to v and there are no other vertices in H_1 incident to v, or that x is incident to w and there are no other vertices in H_1 incident to w.

Now suppose Ω has the form of biased graph (b). Let $x \in V(H_1) \setminus \{u, v, w\}$. As long as in $\Omega - x$ there is either a u-w or a u-v path contained in H_1, there is a $U_{2,4}$ minor in $M|_{E \setminus \delta(x)}$, so x is committed. Hence if $x \in V(H_1)$ is not committed, the deletion of x must destroy all

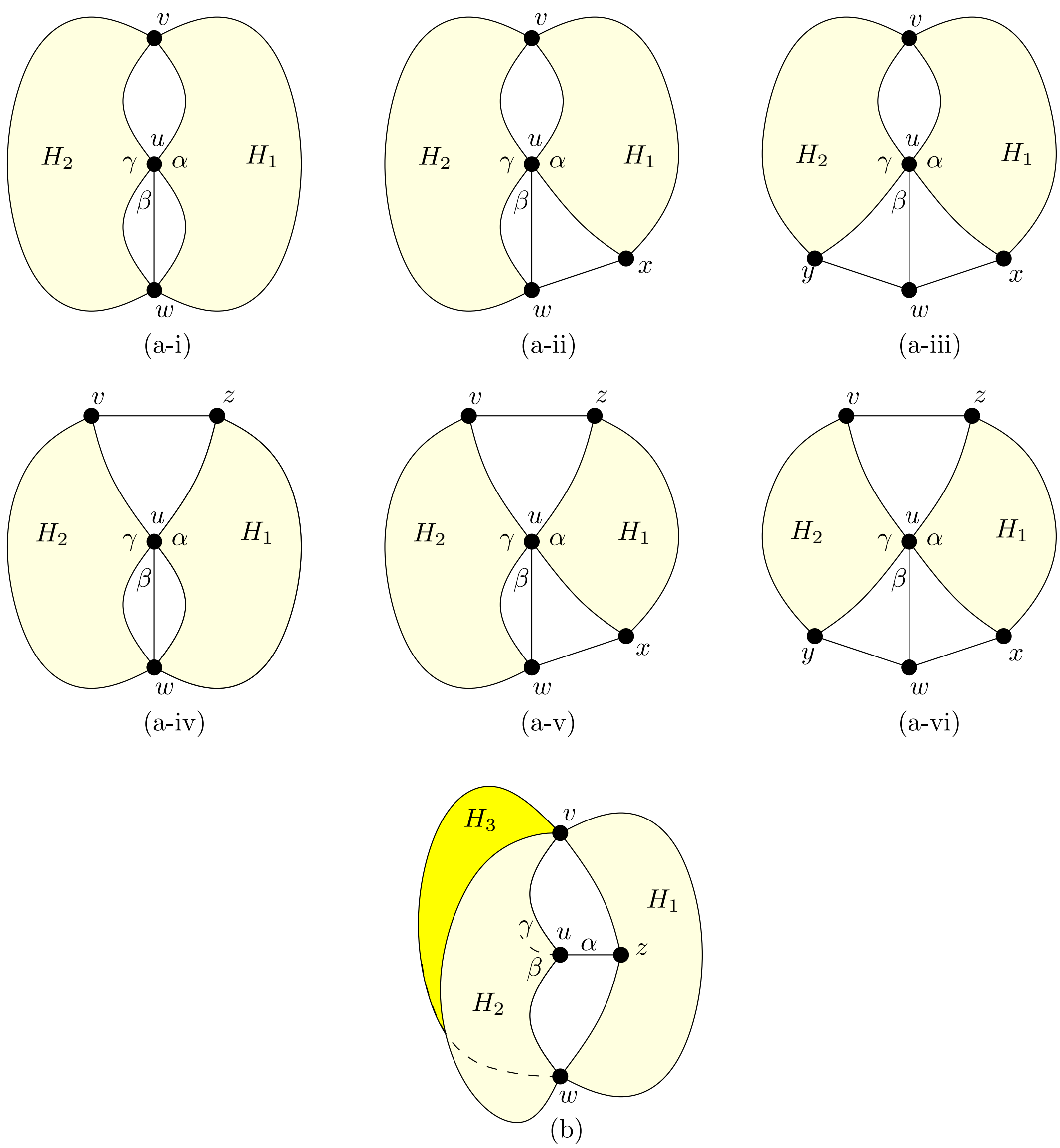


Figure 5.12: Possibilities for lobes in Ω in the case Ω has a balancing class of size one.

such paths. Connectivity implies then that $x = z$. So suppose now $x \in V(H_2) \setminus \{u, v, w\}$. Again, as long as there is either a u-v or a u-w path in H_2 avoiding x, x is committed. Hence if $x \in V(H_2)$ is not committed, there are no such paths in H_2 avoiding x. Connectivity now implies that x is incident to u and that there are no other vertices in H_2 incident to u. But this is a contradiction, as then both the α and β balancing classes in $\delta(u)$ are of size 1, contradicting Lemma 5.15. □

We now consider each of these possibilities for Ω in which all internal vertices of the lobes H_i are committed, in turn: biased graphs (a-i)-(a-vi), and (b) of Figure 5.12.

(a-i) Suppose first Ω is as shown in Figure 5.12 (a-i). Suppose that $V(H_2) \setminus \{u, v, w\}$ is empty. Then without loss of generality we may assume $|H_2| = 2$, since we may assume that a vw edge belongs to H_1 (biases of cycles containing such an edge do not depend on whether it is placed in H_1 or H_2). Since $F(\Omega)$ has rank at least 4 and is non-graphic, $|V(\Omega)| \geq 4$, so $V(H_1) \setminus \{u, v, w\} \neq \emptyset$. Replacing H_1 with a balanced triangle (Figure 5.13), we obtain the biased graph Ψ_0 of Proposition 5.17. Hence, by Proposition 5.17, all biased graphs with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ are in this case those of Figure 5.36.

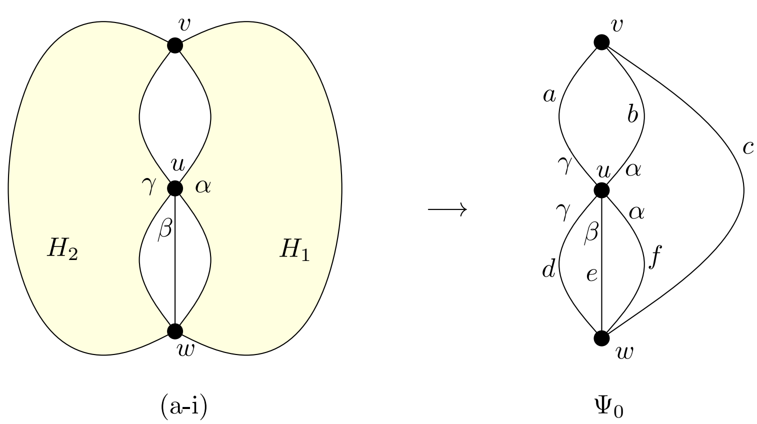
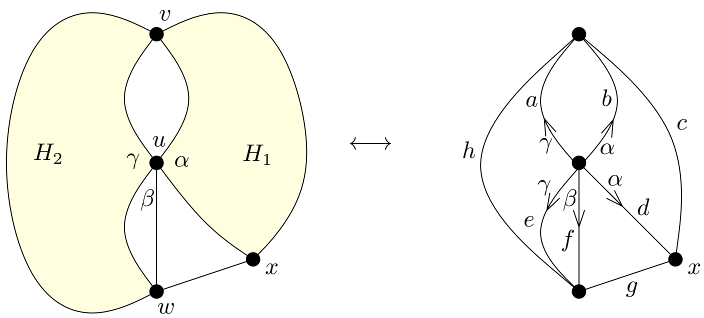


Figure 5.13: Ω and $\mathrm{re}_H(\Omega) = \Psi_0$.

Now suppose Ω is as shown in Figure 5.12 (a-i), and both $V(H_1) \setminus \{u, v, w\}$ and $V(H_2) \setminus \{u, v, w\}$ are non-empty. Replacing both H_1 and H_2 with a balanced triangle, we obtain the biased graph Ψ_0 of Figure 5.13 with an additional edge g in parallel with c. Hence, again by Proposition 5.17, the biased graphs with frame matroids isomorphic to $F(\mathrm{re}_H(\Omega))$ are those obtained by adding an edge g in parallel with edge c to each biased graph of Figure 5.36.

(a-ii) Lobe H_1 has > 2 edges, else vertex x has degree two so $F(\Omega)$ has two elements in series. If $|E(H_2)| = 2$, replacing H_1 with a balanced triangle we obtain the biased graph of Figure 5.39(a). By Proposition 5.18 then, every biased graph with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ either appears in Figure 5.40 or is a roll-up of one of these biased graphs. Otherwise, replacing each of H_1 and H_2 with a balanced triangle we obtain biased graph G_2 of Figure 5.14. This is the biased graph of Figure 5.39(a)



(a-ii) G_2

Figure 5.14: Ω and $\mathrm{re}_H(\Omega)) = G_2$.

extended by an element h. Hence, again by Proposition 5.18, all biased graphs with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ are obtained as an extension of a biased graph of Figure 5.40, or as an extension of a roll-up of one of these, where such an extension is possible. The resulting biased graphs (with dashed edges) and their H-enlargements are those shown in Figure 5.41, along with roll-ups of those having a balancing vertex.

(a-iii) Each of H_1, H_2 has size > 2, else $F(\Omega)$ has two elements in series, a contradiction. Replacing H_1 and H_2 with a balanced triangle we obtain the biased graph shown in Figure 5.15, let us call it G_3. Since G_3 is obtained from G_2 by a coextension, we obtain

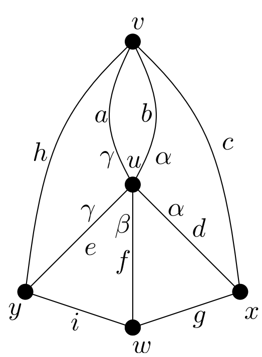


Figure 5.15: $\mathrm{re}_H(\Omega)) = G_3$.

all biased graphs Ψ with $F(\Psi) \cong F(G_3)$ by coextending each biased graph representing $F(G_2)$ in all possible ways by an element in such a way that the resulting biased graph Ψ has $F(\Psi) \cong F(G_3)$. Observe that $(G_3 - x)/\{i, f\} \cong U_{2,4}$, so vertex x is committed. Similarly, y is committed. This reduces the number of possible coextensions we need to examine. The biased graphs representing $F(G_3)$ — ***i.e.***, those with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ in this case — are those shown in Figure 5.42, and those obtained by a roll up of a balancing class of a balancing vertex of one of these biased graphs.

(a-iv) Neither H_1 nor H_2 can have only two edges, else $F(\Omega)$ has two elements in series, a contradiction. Replacing each of H_1 and H_2 with a balanced triangle we obtain the biased graph of Figure 5.16, let us call it G_4. Since G_4 is obtained from Ψ_0 by a

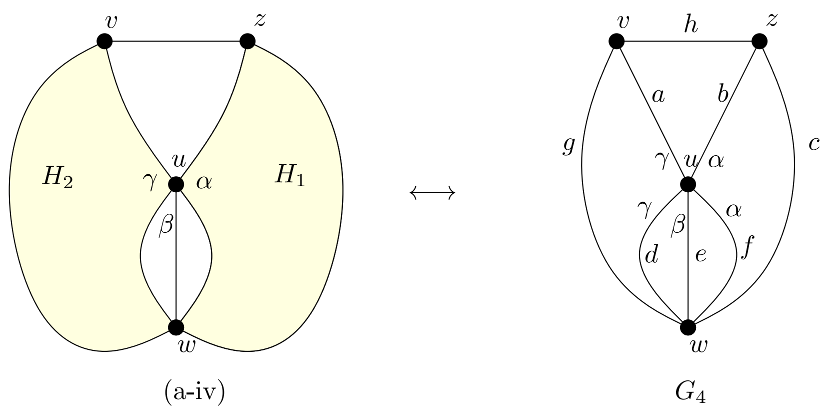


Figure 5.16: Ω and $\mathrm{re}_H(\Omega) = G_4$.

coextension, every biased graphs representing $F(G_4)$ may obtained by a coextension of a biased graph representing $F(\Psi_0)$ (the biased graphs representing $F(\Psi_0)$ are those shown in Figures 5.13 and 5.36 with an edge added in parallel to c). The biased graphs so obtained are shown in Figures 5.43 and 5.44. These are thus the biased graphs with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ in this case.

(a-v) Again, each of H_1, H_2 has at least 3 edges, else $F(\Omega)$ has two elements in series, a contradiction. Replacing each of H_1 and H_2 with a balanced triangle we obtain the biased graph of Figure 5.17, let us call it G_5. Since G_5 is a coextension of G_2 (Figure 5.14), all representations of $F(G_5)$ are obtained by coextensions of biased graphs representing $F(G_2)$. The later are shown in Figure 5.41. The biased graphs representing $F(G_5)$ are shown in Figure 5.17 and 5.45, or are roll ups of a balancing class of a balancing vertex of one of these.

(a-vi) Again, each of H_1, H_2 has at least 3 edges, else $F(\Omega)$ has two elements in series, a contradiction. Replacing each of H_1 and H_2 with a balanced triangle we obtain the biased graph of Figure 5.18, let us call it G_6. Since G_6 is a coextension of G_3 (Figure

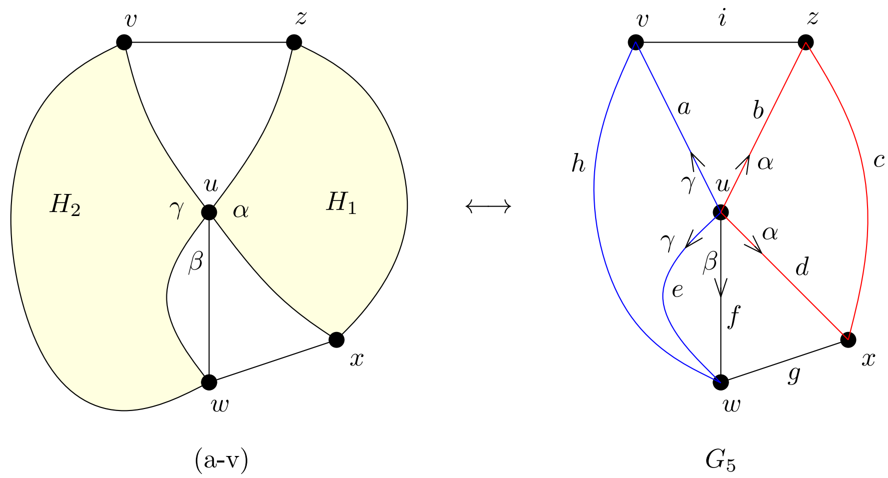


Figure 5.17: Ω and $\mathrm{re}_H(\Omega) = G_5$.

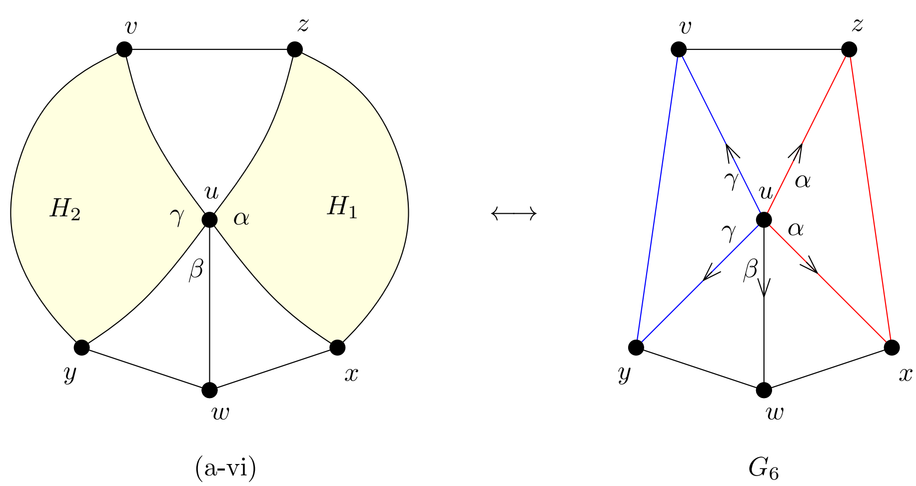


Figure 5.18: Ω and $\mathrm{re}_H(\Omega) = G_6$.

5.15), all representations of $F(G_6)$ are obtained by coextensions of biased graphs representing $F(G_3)$. The later are shown in Figures 5.42. The biased graphs representing $F(G_6)$ are shown in Figures 5.46 and 5.47. (Note that $(G_6 - x)/\{h, i, j\} \cong U_{2,4}$, so x is committed. Hence in every biased graph representing $F(G_6)$ there is a vertex whose incident edges are exactly $\{c, d, g\}$. Similarly, y is committed, so every such biased graph also has a vertex whose incident edges are exactly $\{e, i, h\}$. The greatly reduces the number of coextensions we need to examine.) Hence in this case, Ω' is an H-enlargement of one of these biased graphs.

(b) If all of H_1, H_2, H_3 are of size two, then Ω is the biased graph of Figure 5.39(b). By Proposition 5.19 then, Ω is one of the biased graphs shown in Figure 5.48, or a roll-up of one of these biased graphs. Otherwise, replacing those of H_1, H_2, H_3 of size at least three with a balanced triangle, $\mathrm{re}_H(\Omega)$ is the biased graph shown in Figure 5.19 (deleting some of $\{h, i, j\}$ as appropriate if some H_1, H_2, H_3 has size two); let us call it G_7. Biased graph G_7 is that of Figure 5.39(b) extended by three elements h, i, j

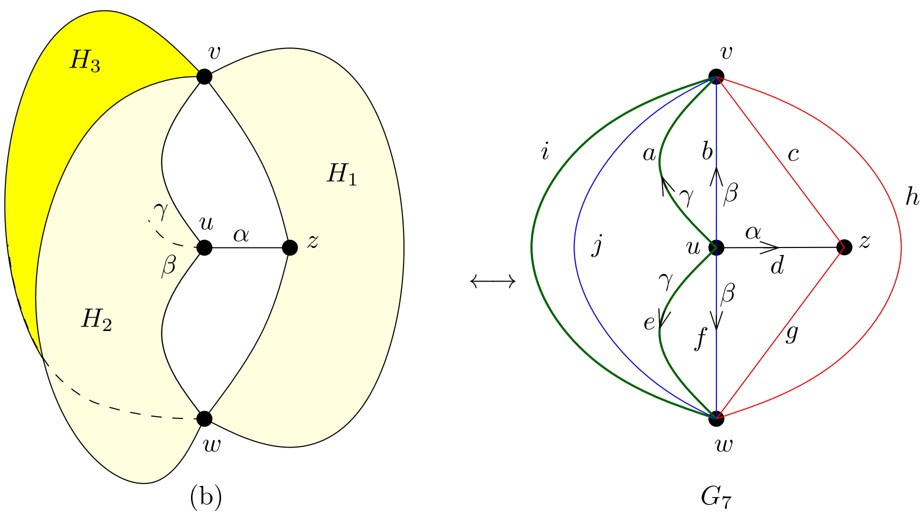


Figure 5.19: Ω and $\mathrm{re}_H(\Omega) = G_7$.

added in parallel. Hence by Proposition 5.19, all biased graphs representing $F(G_7)$ are obtained by adding edges h, i, j in parallel to a biased graph shown in Figure 5.48. The resulting biased graphs obtained from (b), (c), and (d) are shown in Figure 5.49.

B. Each balancing class of size > 1

Now suppose that in Ω each balancing class of $\delta(u)$ has size greater than one. To review, our assumptions in this case are: Ω is 3-connected, $F(\Omega)$ is 3-connected and non-graphic, Ω has a balancing vertex u, there is no unbalanced loop incident to u, there is a vertex $v \neq u$ with $F(\Omega - v)$ binary, and there are exactly three balancing classes in $\delta(u)$ in Ω and in $\Omega - v$, each of which has size > 1 in Ω. Since $F(\Omega - v)$ has no $U_{2,4}$ minor, $\Omega - v$ is a fat theta, and Ω has the form shown at left in Figure 5.20. Let w be the second balancing vertex of the fat theta $\Omega - v$. Together with their edges incident to v, the three lobes of the fat theta $\Omega - v$ are naturally extended to three lobes H_1, H_2, H_3 of G, which meet at $\{v, u, w\}$.

Claim. Each vertex $x \in V(H_i) \setminus \{u, v, w\}$ is committed, $i \in \{1, 2, 3\}$.

Proof. Let $x \in V(\Omega) \setminus \{u, v, w\}$; say $x \in V(H_1)$. We claim that in $H_1 - x$ there is either a u-v path avoiding w or a u-w path avoiding v. For suppose not: then x is a cut vertex of H_1 separating u from $\{v, w\}$. Since u has at least two neighbours in H_1, $\{u, x\}$ determines a 2-separation of G, a contradiction. So suppose P is a u-w path in $H_1 - x$ avoiding v. Since v is not a cut vertex of H_2 or H_3, there are u-w paths P' and P'' avoiding v in H_2 and H_3, respectively. Let Q' be a P'-v path in $H_2 - w$, and Q'' be a P''-v path in $H_3 - w$ (such paths exist, since w is not a cut vertex of H_2 or H_3). Contracting all edges of P, P', and P'' but those incident to w, and all edges of Q', and all edges of Q'' but its edge incident to v yields a biased graph representing $U_{2,4}$ as a minor of $\Omega - x$. Hence every internal vertex $x \in V(H_i) \setminus \{u, v, w\}$ ($i \in \{1, 2, 3\}$) is committed. $\square$

If Ω has rank three, then Ω is obtained from the biased graph of Figure 5.10 by adding a u-v edge g labelled β (not all of H_1, H_2, H_3 have only two edges, since then $F(\Omega)$ would not be 3-connected). If $\text{rank}(F(\Omega)) > 3$, and each lobe H_i has $V(H_i) \setminus \{u, v, w\} \neq \emptyset$, then replacing each H_i with a balanced triangle we obtain the biased graph of Figure 5.20; let us call it G_8. If any of H_1, H_2, H_3 have only two edges, then $\text{re}_H(\Omega)$ is obtained from G_8 by deleting an edge or two from $\{c, h, i\}$. Biased graph G_8 is also an extension of the

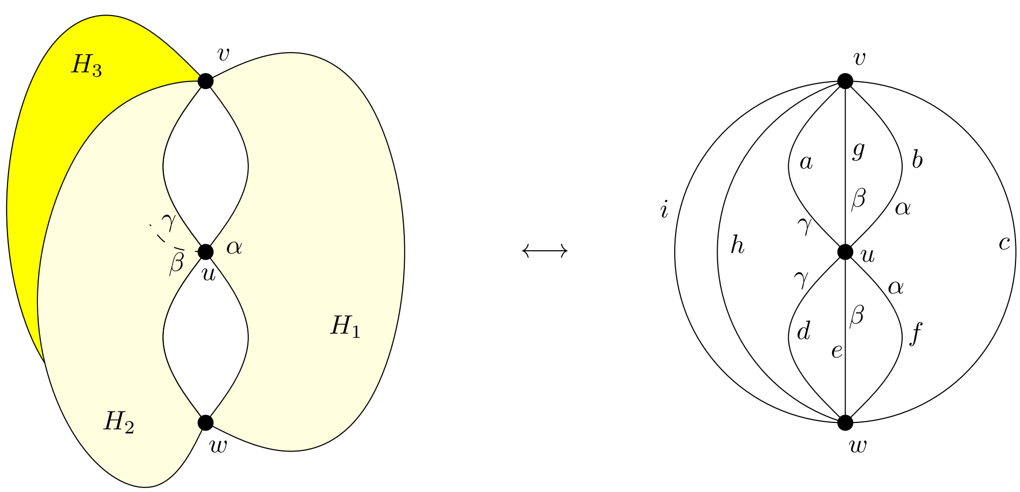


Figure 5.20: Ω and $\text{re}_H(\Omega) = G_8$.

biased graph shown in Figure 5.10, by elements g, h, i, where g is labelled β and h and i are added in parallel with c. Hence by Proposition 5.17, every biased graph representing $F(G_8)$

is obtained by adding edges g, h, i to a biased graph shown in Figure 5.36 appropriately. These biased graphs are shown in Figure 5.50.

ii. $\delta(u)$ has 3 balancing classes in $\Omega - v$, > 3 balancing classes in Ω

To aid the analysis, we now slightly generalise our concept of a lobe: the *lobes* of Ω are the three balanced biased subgraphs H_1, H_2, H_3 of Ω meeting at $\{u, v, w\}$, each of which is obtained from one of the three balanced subgraphs A_1, A_2, A_3 whose union is the fat theta $\Omega - v$, by adding all edges linking v and a vertex in A_i ($i \in \{1, 2, 3\}$. Call a lobe *degenerate* if it contains only 1 edge.

The fact that $F(\Omega)$ is 3-connected forces a balancing class present in Ω but not $\Omega - v$ to be of size 1. Consider first the case that there are exactly four balancing classes in $\delta(u)$ in Ω. For the same reasons as in the previous section, Ω has the form shown shown at left in Figure 5.21, where the edge in the balancing class of $\delta(u)$ not present in $\Omega - v$ is labelled ϵ.

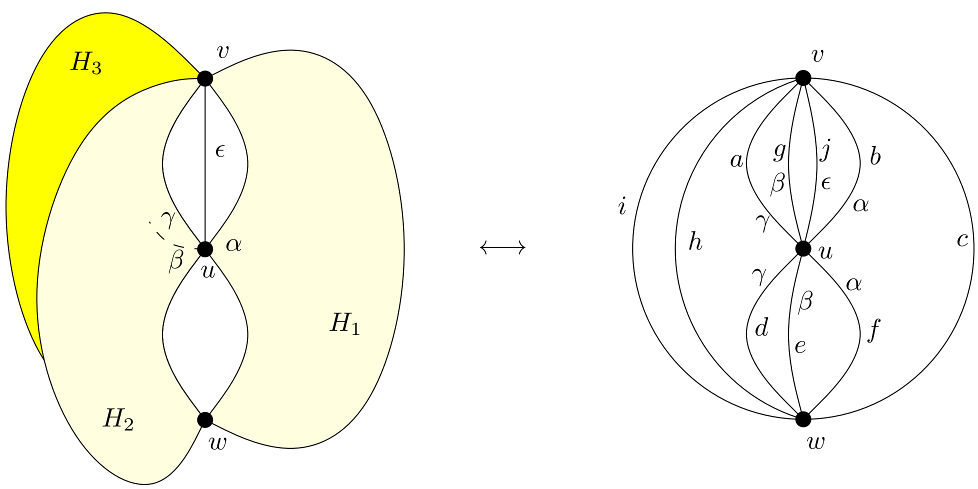


Figure 5.21: Case (b)ii.A. Ω and $\mathrm{re}_H(\Omega) = G_9$

The 3-connectedness of $F(\Omega)$ implies that when there are just four balancing classes in $\delta(u)$ in Ω, not all three lobes are degenerate. The following four sub-cases therefore exhaust the possibilities in the case $\delta(u)$ has three balancing classes in $\Omega - v$ and > 3 balancing classes in Ω:

A. Just 4 balancing classes in $\delta(u)$, no degenerate lobes;

B. Just 4 balancing classes in $\delta(u)$, exactly two degenerate lobes;

C. Just 4 balancing classes in $\delta(u)$, exactly one degenerate lobe;

D. More than 4 balancing classes in $\delta(u)$.

A. Ω has exactly 4 balancing classes, no degenerate lobes

Claim. Every vertex but u and v is committed.

Proof. Let $x \in V(\Omega) \setminus \{u, v\}$. If $x = w$, then since w is not a cut vertex in any of H_1, H_2, H_3, there are u-v paths in each of H_1, H_2, H_3 avoiding w. Together with the u-v edge labelled ϵ, these yield a $U_{2,4}$ minor, so w is committed. If $x \neq w$, suppose without loss of generality $x \in H_1$. Choose u-v paths $P \subseteq H_2$, $P' \subseteq H_3$ avoiding w, a P-w path $Q \subseteq H_2$ avoiding v, and a $P' - w$ path $Q' \subseteq H_3$ avoiding v. Together with the edge labelled ϵ, these yield a $U_{2,4}$ minor in $F(\Omega - x)$, so again x is committed. □

Replacing lobes H_1, H_2, H_3 with balanced cycles bdf, geh, and adi, respectively, we obtain the biased graph at right in Figure 5.21, let us call it G_9; if any of H_1, H_2, H_3 has only two edges, deleting edges in $\{c, h, i\}$, gives $\mathrm{re}_H(\Omega)$ — since these edges are all in parallel, we obtain all biased graphs with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ from G_9 in any case. We recognise G_9 as an extension of G_8 by a single element j. Hence every biased graph representing $F(G_9)$ is obtained by adding a single edge to one of the biased graphs shown in Figure 5.50. Checking, we find that the only other biased graphs representing $F(G_9)$ are roll-ups of G_9.

B. Ω has exactly 4 balancing classes and exactly two degenerate lobes

Suppose there is just one edge in $\delta(u)$ labelled β and just one edge labelled γ. Clearly each of u, v, w are uncommitted. Let $x \in V(\Omega) \setminus \{u, v, w\}$. Then x is committed unless the deletion of x destroys either all u-w paths or all u-v paths. Hence connectivity implies Ω has the form of one of the biased graphs shown in Figure 5.22, where $|E(H_1)| \geq 3$ and every vertex $z \in V(\Omega) \setminus \{u, v, w, x, y\}$ is committed. Replacing balanced lobe H_1 with a balanced

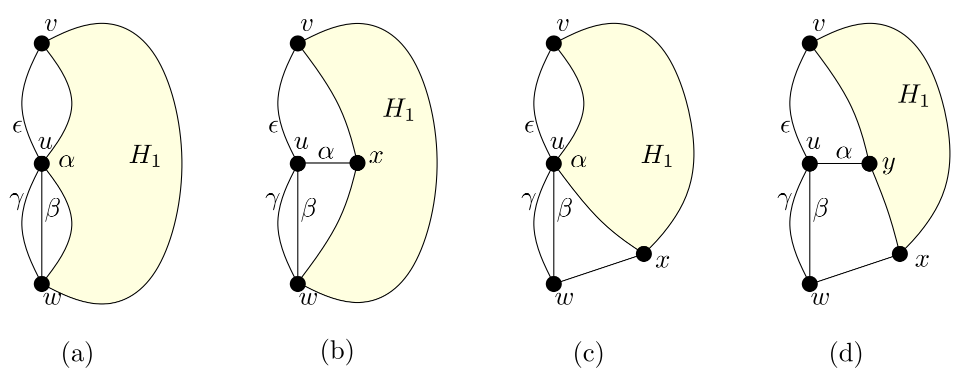


Figure 5.22: Case (b)ii.B. All vertices except u, v, w, x, y are committed.

triangle in each biased graph, we see $\mathrm{re}_H(\Omega)$ is one of the biased graphs shown in Figure 5.23. By Lemma 5.20, if Ψ represents $F(\mathrm{re}_H(\Omega))$ then Ψ is one of the biased graphs of Figures 5.51, 5.52, 5.53, 5.54, 5.55, 5.56, 5.57, or 5.58.

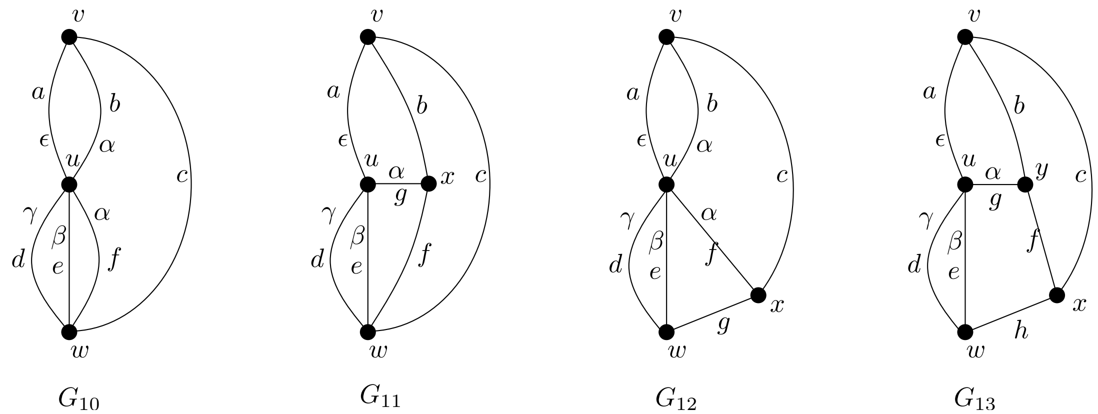


Figure 5.23: $\mathrm{re}_H(\Omega)$ in cases Ω is of the form (a), (b), (c), or (d), resp. of Figure 5.22.

C. Ω has exactly 4 balancing classes and exactly one degenerate lobe

In this case, Ω is as shown in Figure 5.24 (a). Let $x \in V(\Omega) \setminus \{u, v, w\}$. It is easy to see that

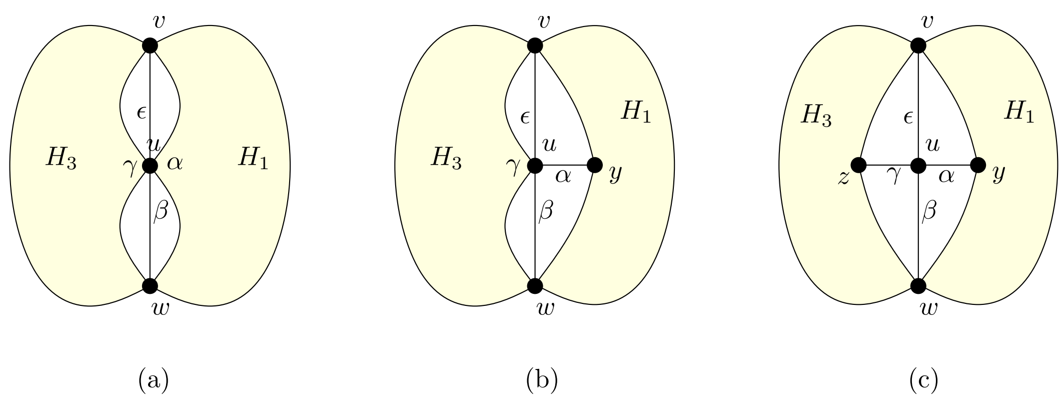


Figure 5.24: Case (b)ii.C. Ω has exactly four balancing glasses and exactly one degenerate lobe.

x is committed unless the deletion of x destroys both all u-v and all u-w paths. Hence Ω has the form of biased graph (a), (b), or (c) of Figure 5.24, where all vertices $x \notin \{u, v, w, y, z\}$ are committed. Replacing each lobe H_1, H_2 with a balanced triangle, we obtain the biased graphs of Figure 5.25, where in each case there are two more edges in parallel with c. (We omit these edges for now, since they just clutter up our pictures.) Let us call these biased graphs G_{14}, G_{15}, G_{16}, respectively.

Assume first $|H_1|, |H_3| > 2$. Observe that $F(G_{14})$ is a single element extension of $F(G_{10})$. Hence we obtain every biased graph representing $F(G_{14})$ by adding an edge h to every biased graph representing $F(G_{10})$ in every possible way such that the result has frame matroid isomorphic to $F(G_{14})$. These are shown in Figure 5.59 (labels correspond to the biased graph of Figures 5.51 or 5.52 to which edge h is added).

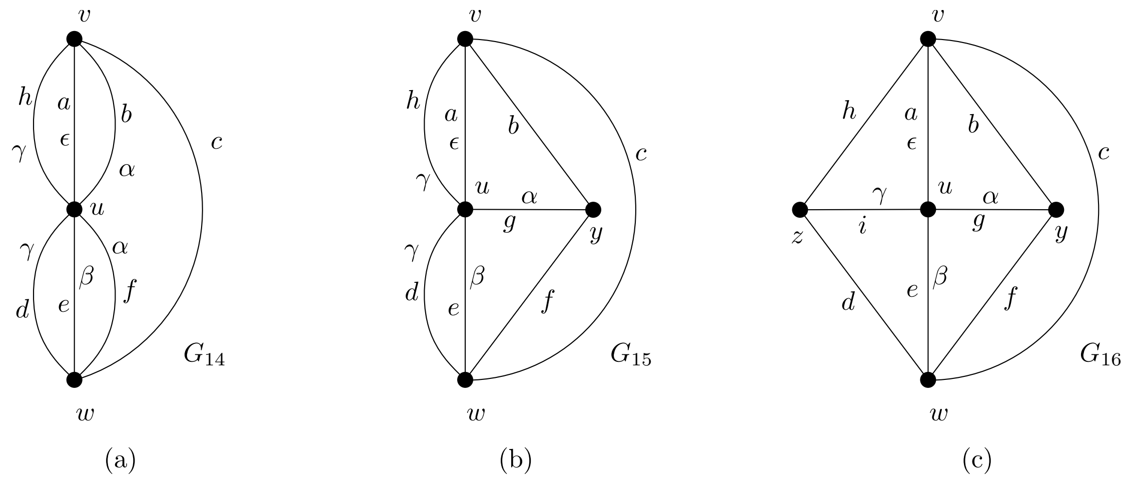


Figure 5.25: $\mathrm{re}_H(\Omega)$ (minus a parallel edge) in cases (a), (b), (c) of Figure 5.24

Similarly, $F(G_{15})$ is a single element extension of $F(G_{11})$, and we obtain every biased graph representing $F(G_{15})$ by adding an edge h to every biased graph representing $F(G_{11})$. These are shown in Figure 5.60 (labels correspond to the biased graph of Figure 5.55 or 5.56 to which edge h is added).

Now consider biased graph G_{16} shown in Figure 5.25 (c); $F(G_{16})$ is a single element coextension of $F(G_{15})$. Hence we obtain every biased graph representing $F(G_{16})$ by uncontracting an edge in each biased graph representing $F(G_{15})$. These are shown in Figure 5.61 (labels correspond to the biased graph of Figure 5.60 in which edge i is uncontracted).

If $|E(H_1)| = |E(H_2)| = 2$, connectivity implies Ω is one of (b) or (c) after deleting edge c, and we may similarly find all biased graphs representing $F(\Omega)$; each is an H-enlargement of a biased graph on four or five vertices. In the case $\Omega \cong G_{15} \setminus c$, the biased graphs with frame matroid isomorphic to $F(\Omega)$ are either roll-ups of Ω or one of the two biased graphs shown in Figure 5.63. In the case $\Omega \cong G_{16} \setminus c$, since $F(G_{16} \setminus c)$ is a coextension by i of $F(G_{15} \setminus c)$, we may obtain all biased graphs representing Ω by considering uncontracting an edge of every biased graph representing $F(G_{15} \setminus c)$. Doing so, we find the only representations of $F(G_{16} \setminus c)$ are the four roll-ups of $G_{16} \setminus c$.

D. Ω has > 4 balancing classes

Suppose now Ω has more than four balancing classes in $\delta(u)$; first assume Ω has just five balancing classes, and so has the form shown at left in Figure 5.26.

Suppose first that none of the lobes H_1, H_2, H_3 is degenerate. Then there is a u-v path avoiding w in each of H_1 and H_2, and so $F(G - w)$ has a $U_{2,4}$ minor, so w is committed. Let $x \in V(\Omega) \setminus \{u, v\}$. Since in each of the two lobes not containing x there is both a u-v path and a u-w path, we find a $U_{2,4}$ minor in $F(\Omega - x)$, so x is committed. Replacing each lobe H_1, H_2, H_3 with a balanced triangle, and removing two of the resulting three parallel edges

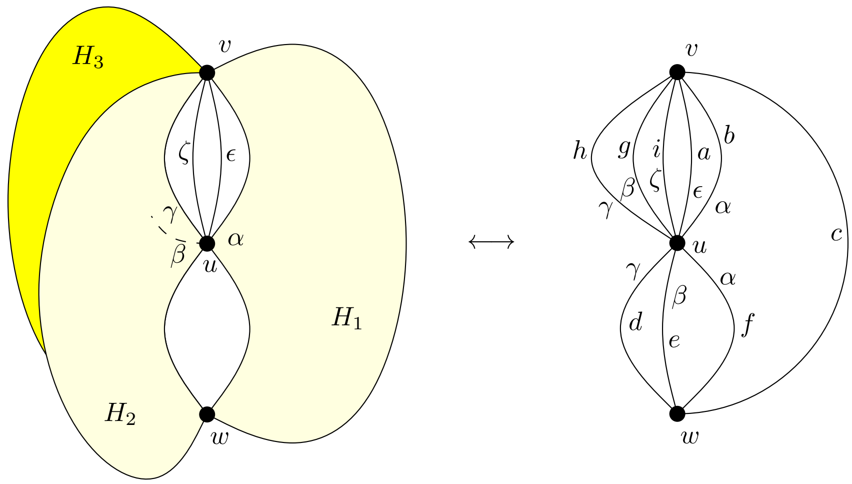


Figure 5.26: Case (b)ii.D. Ω and $\text{re}_H(\Omega) = G_{17}$.

(which for now just clutter up our pictures), we obtain biased graph G_{17} shown at right in Figure 5.26. Since G_{17} is obtained from G_9 by adding a single uv edge in a new balancing class (and removing the edges in parallel with c), so $F(G_{17})$ is a single element extension of $F(G_9)$, we obtain every biased graph representing $F(G_{17})$ by adding a single edge to every biased graph representing $F(G_9)$. The only biased graphs representing $F(G_9)$ are roll-ups of G_9. Checking, we find the only biased graph representations of $F(G_{17})$ are roll-ups of G_{17}.

Hence, in the case none of H_1, H_2, H_3 is degenerate, every biased graph representing $F(\Omega)$ is obtained as a roll-up of Ω. Moreover, this implies that if Ω has more than five balancing classes, then (since the deletion of any of the appropriate number of elements represented by uv edges in Ω yields $F(G_{17})$) any biased graph representing $F(\Omega)$ is obtained as a roll-up of Ω.

Suppose now Ω has a degenerate lobe. Suppose lobe H_2 has size one, both $|H_1|, |H_3| > 1$, and for now assume Ω has just five balancing classes. Let $x \in V(\Omega) \setminus \{u, v\}$. If $x = w$, choose in $G - w$ a u-v path in H_1 and a u-v path in H_3: we find a $U_{2,4}$ minor in $F(G - w)$. If $x \neq w$, choose a u-v and a u-w path in the lobe not containing x: we thus find a $U_{2,4}$ minor in $F(\Omega - x)$. Hence every vertex $x \in V(\Omega) \setminus \{u, v\}$ is committed. Replacing lobes H_1 and H_3 with balanced triangles, we obtain $G_{17} \setminus g$. Let us call this resulting biased graph G_{18} (Figure 5.27). Since G_{18} is obtained from G_{14} by adding a single uv edge i in a new balancing class, ***i.e.***, $F(G_{18})$ is a single element extension of $F(G_{14})$, we obtain every biased graph representing $F(G_{18})$ by adding an edge i to every biased graph representing $F(G_{14})$ in every possible way such that the result has frame matroid isomorphic to $F(G_{18})$. Since w is committed, we need only consider those biased graphs of Figure 5.59 having a vertex w' with $\delta(w') = \{c, d, e, f\}$. Doing this, we find again that the only biased graphs with frame

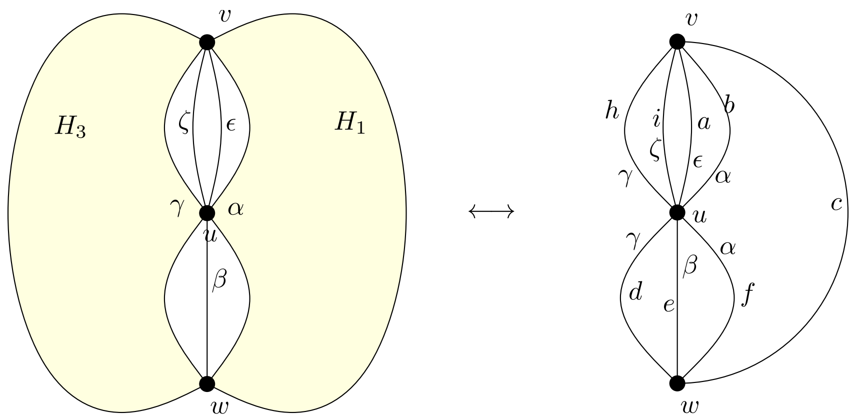
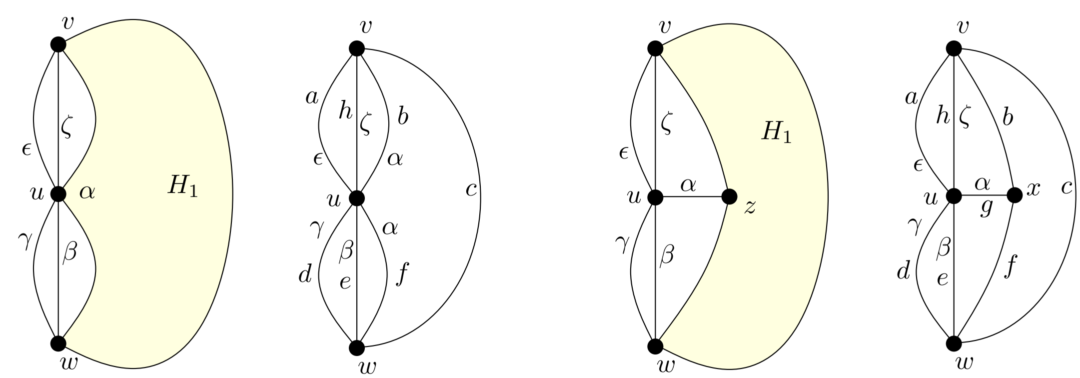


Figure 5.27: Ω and $\mathrm{re}_H(\Omega) = G_{18}$.

matroid isomorphic to $F(G_{18})$ are obtained as roll-ups of G_{18}. Since the same argument applies to extensions of $F(G_{18})$ by an element obtained by adding a new uv edge in a new balancing class to G_{18}, we conclude that the only biased graphs representing $F(\Omega)$ where Ω has more than 4 balancing classes but only 3 balancing classes in $\Omega - v$ and at most one degenerate lobe, are obtained as roll-ups of Ω.

Suppose now Ω has two degenerate lobes, H_2 and H_3, with $|H_1| > 1$, and again let us first assume Ω has exactly five balancing classes (Figure 5.28(a)). Let $x \in V(\Omega) \setminus \{u, v, w\}$.

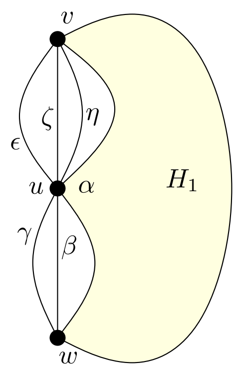


(a) G_{19} (b) G_{20}

Figure 5.28: Ω has exactly 5 balancing classes and two degenerate lobes.

It is easy to see that unless the deletion of x destroys both all u-v and all u-w paths, there is a $U_{2,4}$ minor in $F(\Omega - x)$. Hence Ω has the form of one of biased graphs (a) or (b) in Figure 5.28 where all vertices $x \in V(H_1) \setminus \{u, v, w, x\}$ are committed. Replacing lobe H_1 with a balanced triangle, we obtain biased graphs G_{19} and G_{20} of Figure 5.28. Biased graph G_{19} is a single element extension of G_{10} (Figure 5.51) and G_{20} is a single element extension of G_{11} (Figure 5.55). We therefore obtain every biased graph representing $F(G_{19})$ by adding an edge h to every biased graph representing $F(G_{10})$ in every possible way so that the resulting biased graph has frame matroid isomorphic to $F(G_{19})$. Similarly, we obtain every biased graph representing $F(G_{20})$ by adding an edge h in a similar manner to every biased graph representing $F(G_{11})$. These are shown in Figures 5.62 and 5.64.

Now suppose Ω has two degenerate lobes, H_2 and H_3, with $|H_1| > 1$, and that Ω has more than five balancing classes. Then Ω has the form shown in Figure 5.29(a), possibly with additional uv edges in additional balancing classes. Let $x \in V(\Omega) \setminus \{u, v\}$. If $x = w$,



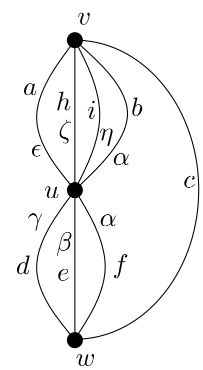


Figure 5.29: Ω and $\mathrm{re}_H(\Omega) = G_{21}$.

then since there is a u-v path in H_1 avoiding w, we find a $U_{2,4}$ minor in $F(G - w)$, so w is committed. Otherwise, the edges labelled β, γ, ϵ, ζ, η form a biased subgraph in $\Omega - x$ containing a $U_{2,4}$ minor, so x is committed. Replacing H_1 with a balanced cycle bcf, we obtain biased graph G_{21} of Figure 5.29. Since $F(G_{21})$ is a single element extension of $F(G_{19})$, we obtain every biased graph representing $F(G_{21})$ by adding an element i to each biased graph representing $F(G_{19})$ such that the result has frame matroid isomorphic to $F(G_{21})$. Since w is committed, we just need consider those biased graphs having a vertex w' with $\delta(w') = \{c, d, e, f\}$. Checking, we find all biased graphs representing $F(G_{21})$ are obtained as a roll-up of G_{21}.

Our remaining case when Ω has more than four balancing classes is that when Ω has all three lobes degenerate. Assuming first that Ω has exactly five balancing classes, Ω is the biased graph of Figure 5.30; let us call it G_{22}. It is straightforward to determine that all biased graphs representing $F(G_{22})$ are those shown in Figure 5.65 along with roll-ups of these which have a balancing vertex. If Ω has more than five balancing classes, then Ω is obtained via extensions of G_{22}. All biased graphs are may therefore be obtained by adding an edge to a biased graph representing $F(G_{22})$ such that the result has frame matroid isomorphic to $F(\Omega)$. Doing so, we find the only biased graphs representing $F(\Omega)$ are those obtained by a roll-up of Ω.

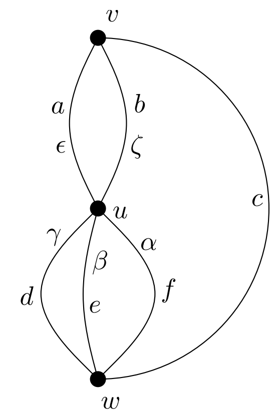


Figure 5.30: G_{22}.

iii. $\delta(u)$ **has** < 3 **balancing classes in** $\Omega - v$

We may assume that Ω does not have an uncommitted vertex z leaving just three balancing classes in $\delta(u)$ in $G - z$, since we have dealt with this case in the previous section. We consider two sub-cases, depending on the number of balancing classes in $\delta(u)$ in $\Omega - v$.

A. There is just one balancing class of $\delta(u)$ *in* $\Omega - v$

In this case, v is also balancing. There must be at least four balancing classes in Ω and at least three u-v edges each in a distinct balancing class, else $F(\Omega)$ would be graphic (Figure 5.31). But then the partition (X, Y) of $E(\Omega)$ in which X consists of the u-v edges is a 2-separation of $F(\Omega)$, a contradiction. Hence this case cannot occur.

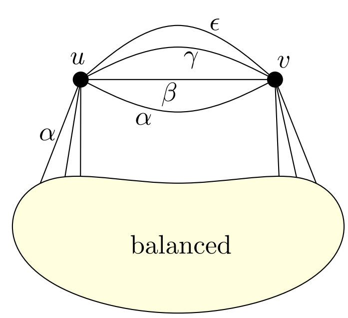


Figure 5.31: Case (b)iii.A. Just one balancing class in $\Omega - v$.

B. There are 2 balancing class in $\Omega - v$

Label the edges in the two balancing classes remaining in $\Omega - v$ with α and β. Let A and B, respectively, be the sets of edges in balancing classes α and β. Since $F(\Omega)$ is non-graphic, there is at least one u-v edge in a balancing class distinct from A and B. Note that since $F(\Omega)$ does not have circuits of size two, no two u-v edges are in the same balancing class.

Let $F = (\delta(u) \cap \delta(v)) \setminus (A \cup B)$. Label the edges in F with group elements γ, ϵ, ζ, ... Then Ω has the form of the biased graph at left in Figure 5.32. Let $C = \delta(v) \setminus \delta(u)$. Let $W = V \setminus \{u, v\}$.

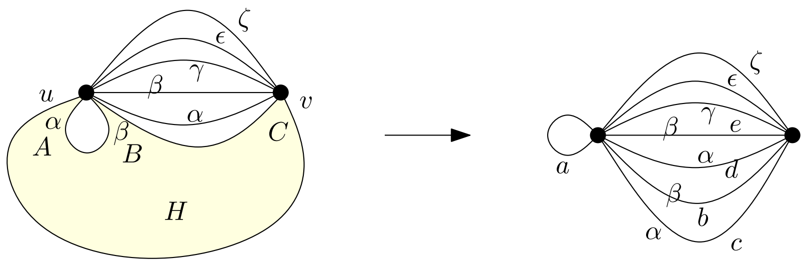


Figure 5.32: Case (b)iii.B. Ω and $\mathrm{re}_H(\Omega)$.

Claim. Every vertex $x \in W$ is committed.

Proof of Claim. If $|F| \geq 3$, then since there is a u-v path via $\Omega[W \setminus x]$ for every $x \in W$, we easily find a $U_{2,4}$ minor in $F(G - x)$.

Suppose $|F| = 2$. If both $|A|$ and $|B|$ are at least two, then again, for every $x \in X$ there is a u-v path through $G[W \setminus x]$, and so a $U_{2,4}$ minor in $F(\Omega - x)$. So suppose one of A or B has size one, say $|A| = 1$. Then taking x to be the endpoint of the edge $e = ux \in A$ in W, we find $\Omega - x$ has three balancing classes remaining in $\delta(u)$, and $\Omega - x$ a fat theta, and so binary, a contradiction.

So suppose now $|F| = 1$. If both $|A|$ and $|B|$ are at least two, then for any $x \in W$, there are three balancing classes in $\Omega - x$, so it must be that x is committed (else we have the contradiction that there are exactly three balancing classes in $\Omega - x$). So finally suppose one of A or B has size one, say, without loss of generality, A. Then the edge in A is in series with the edge in F, a contradiction. $\square$

Replacing H with a pair of tight handcuffs abc, we have that $\mathrm{re}_H(\Omega)$ is the biased graph at right in Figure 5.32. Let d, e be the u-v edges in Ω labelled α and β, respectively, if present. These are in parallel with c and d in $F(\mathrm{re}_H(\Omega))$, so let us temporarily remove them. We have $F(\mathrm{re}_H(\Omega) \setminus \{d, e\} \cong U_{2,m}$, where $m = |F| + 2 - k$, where k is the number of u-v edges in Ω in $A \cup B$. Therefore all biased graphs with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ are obtained from a biased graph representing $U_{2,m}$ by possibly adding edges d, e in parallel with edges b, c.

This completes the proof of Theorem 5.1. $\square$

5.4 Biased graphs representing reductions of Ω

In this section we exhibit the biased graphs representing $F(\mathrm{re}_H(\Omega))$ in the various cases for the particular subgraphs H considered in the proof of Theorem 5.1.

If the case Ω has a second uncommitted vertex v, an unbalanced loop incident to u, just one u-v edge not in balancing classes A or B, and has $|A| = 1$ with $|B| > 1$, $\mathrm{re}_H(\Omega)$ is obtained from L_1 by possibly adding an edge f in parallel with d. Hence every biased graph with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ in this case is obtained from a biased graph with frame matroid isomorphic to $F(L_1)$ after possibly adding an edge in parallel with d. (Two edges of a biased graph are *in parallel* if they in parallel in the associated frame matroid — *i.e.*, if they are links having the same endpoints forming a balanced cycle or are two unbalanced loops incident to the same vertex.) These are the biased graphs shown in Figure 5.33. If also $|B| = 1$, the biased graphs with frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$ are those shown in Figures 5.34 and 5.35.

Proposition 5.17. *Let Ψ_0 be the biased graph shown in Figure 5.10, and suppose Ψ is a biased graph with $F(\Psi) \cong F(\Psi_0)$. Then Ψ is one of the biased graphs shown in Figure 5.36.*

Proof. We consider the circuits of $F(\Psi_0)$, and determine all possible ways these may appear simultaneously as circuits in Ψ, such that no other edge sets of Ψ form circuits of $F(\Psi)$.

Consider first circuit def. In Ψ, the edges representing d, e, and f may be

1. a balanced 3-cycle,
2. an countrabalanced theta,
3. a digon with an unbalanced loop incident to one of the vertices of the digon, or,
4. an edge with an unbalanced loop incident to each endpoint.

In case 1, checking possible choices of endpoints for edges representing elements a, b, and c, we find Ψ must be one of the biased graphs shown in Figure 5.36 (a)-(h). In case 2, we find that Ψ can only be the biased graph of Figure 5.10. In case 3, Ψ must be one of the biased graphs of Figure 5.36 (i)-(m), and in case 4, one of (n)-(p). □

Figure 5.37 shows the H-enlargement of some of the biased graphs of Figure 5.36. These biased graphs represent $F(\Omega)$ in the case that Ω has a second uncommitted vertex v, no loop incident to u, there are exactly three balancing classes in $\delta(u)$ in Ω and in $\Omega - v$, Ω has a balancing class of size one, and Ω has the form shown in Figure 5.12 (a-i), with $V(H_1) \setminus \{u, v, w\}$ non-empty with all internal vertices committed and $V(H_2) \setminus \{u, v, w\} = \emptyset$.

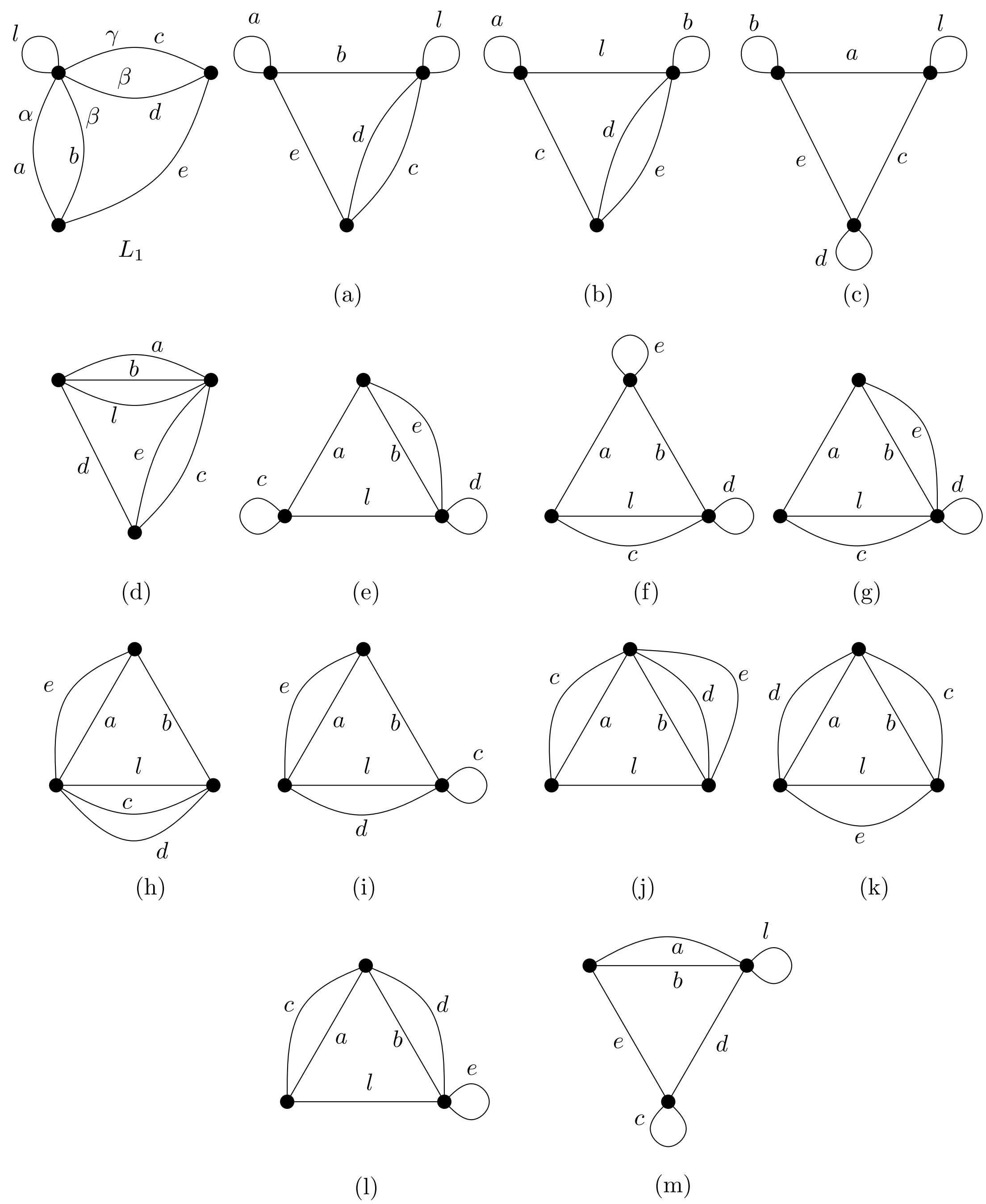


Figure 5.33: $F(L_1)$. All cycles with edges sets $\{a, b, l\}$, $\{c, d, l\}$, or $\{b, d, e\}$ are balanced; all other cycles are unbalanced.

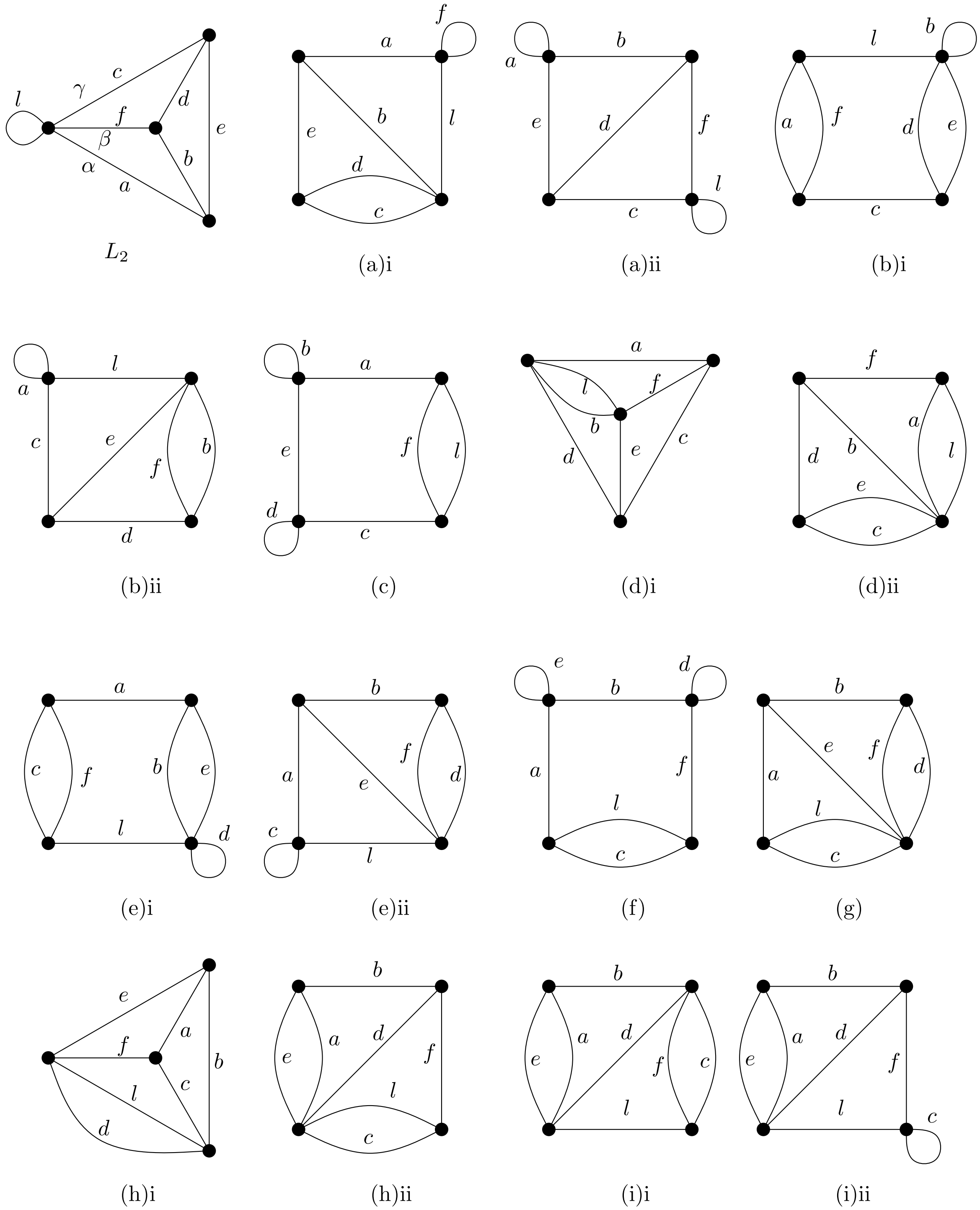


Figure 5.34: $F(L_2)$. All cycles with edge sets $\{b, d, e\}$, $\{a, b, f, l\}$, $\{c, d, f, l\}$, and $\{a, c, e, l\}$ are balanced, all other cycles are unbalanced.

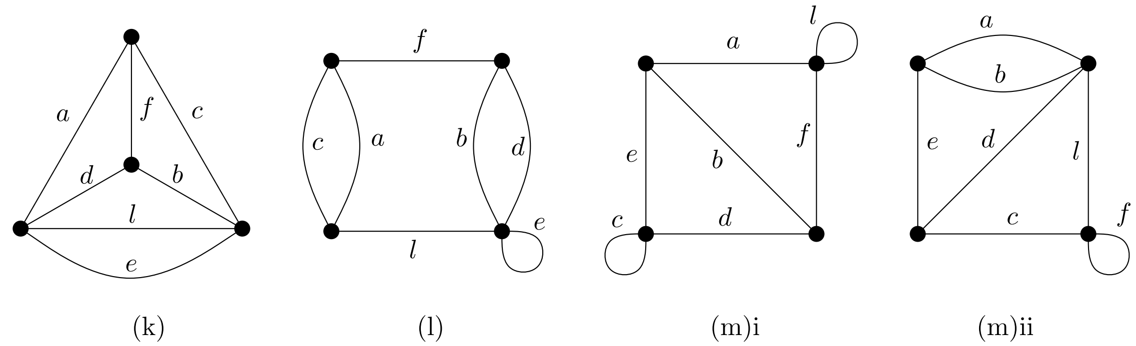


Figure 5.35: $F(L_2)$. All cycles with edge sets $\{b, d, e\}$, $\{a, b, f, l\}$, $\{c, d, f, l\}$, and $\{a, c, e, l\}$ are balanced, all other cycles are unbalanced.

The biased graphs with frame matroid isomorphic to $F(\text{re}_H(\Omega))$ in the case that Ω has a second uncommitted vertex v, no loop incident to u, there are exactly three balancing classes in $\delta(u)$ in Ω and in $\Omega - v$, Ω has a balancing class of size one, and Ω has the form shown in Figure 5.12 (a-i), with both $V(H_1) \setminus \{u, v, w\}$ and $V(H_2) \setminus \{u, v, w\}$ non-empty with all internal vertices committed are those are those obtained by adding an edge g in parallel with edge c to each biased graph of Figure 5.36. Some representations of $F(\Omega)$ obtained as H-enlargements of these are shown in Figure 5.38.

Proposition 5.18. *Let Ω_1 be the biased graph shown in Figure 5.39 (a), and suppose Ω is a biased graph with $F(\Omega) \cong F(\Omega_1)$. Then Ω is one of the biased graphs shown in Figure 5.40 or is obtained by rolling up a balancing class of a balancing vertex of one of these graphs.*

Proof. It is straightforward to check that the frame matroid of each biased graph shown in Figure 5.40 has the same set of bases as $F(\Omega_1)$. To see that these the only such biased graphs, let Ω be a biased graph with $F(\Omega) \cong F(\Omega_1)$. Since $F(\Omega_1)$ has rank 4 and is non-graphic, $|V(\Omega)| = 4$. We consider circuits of $F(\Omega_1)$ and how they may be represented in Ω.

The circuits of $F(\Omega_1)$ are bcd, $abef$, $aceg$, $defg$, and all subsets of size 5 not containing bcd, $abef$, $aceg$, or $defg$. Consider circuit $defg$. There are six possible arrangements of edges d, e, f, g such that $defg$ is a circuit of $F(\Omega)$:

1. a 4-cycle,

2. a path of length two with an unbalanced loop at each end,

3. a digon and a path of length two linking its vertices,

4. a triangle with an unbalanced loop on one vertex,

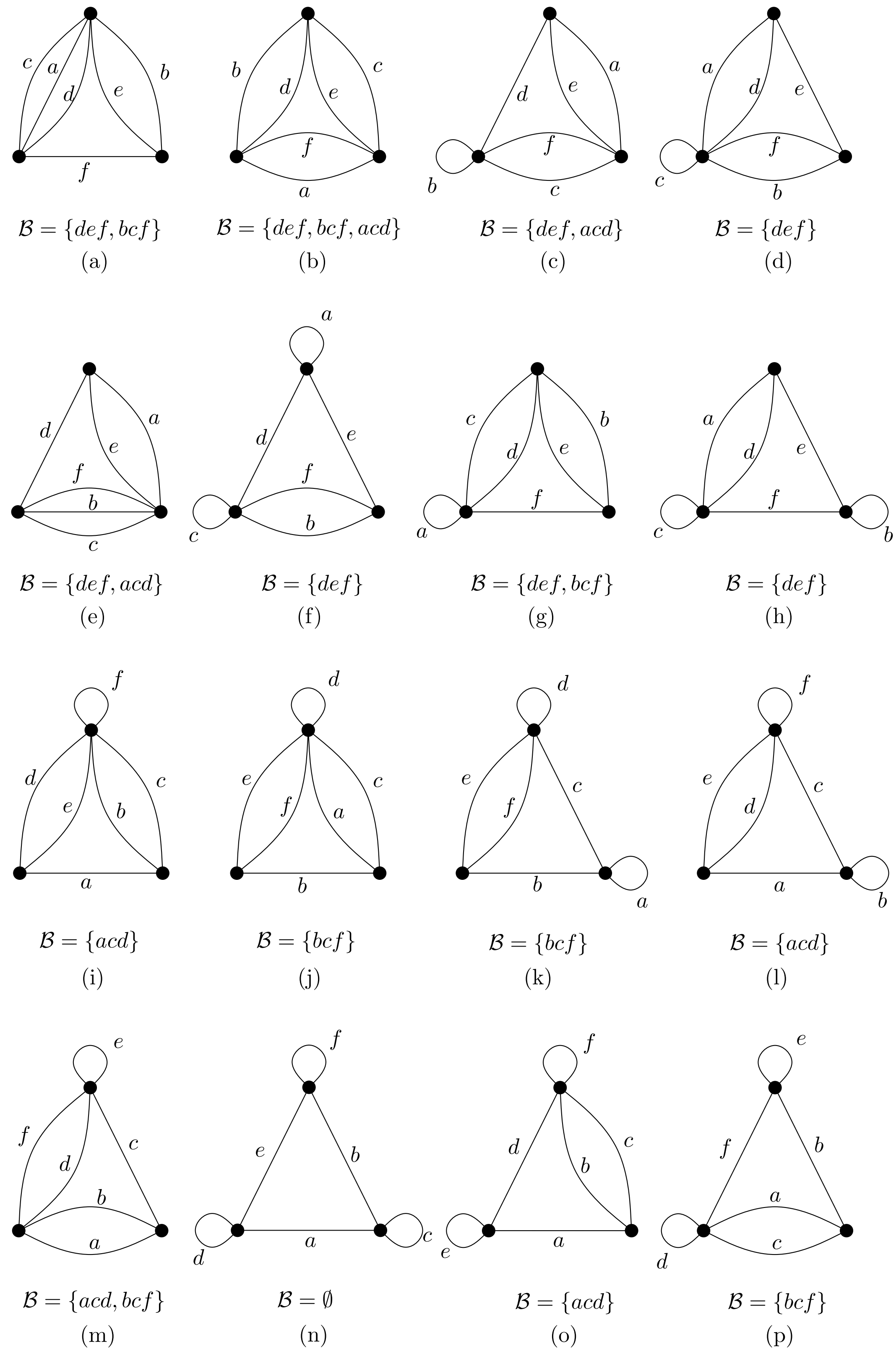


Figure 5.36: Representations of $F(\Psi_0)$

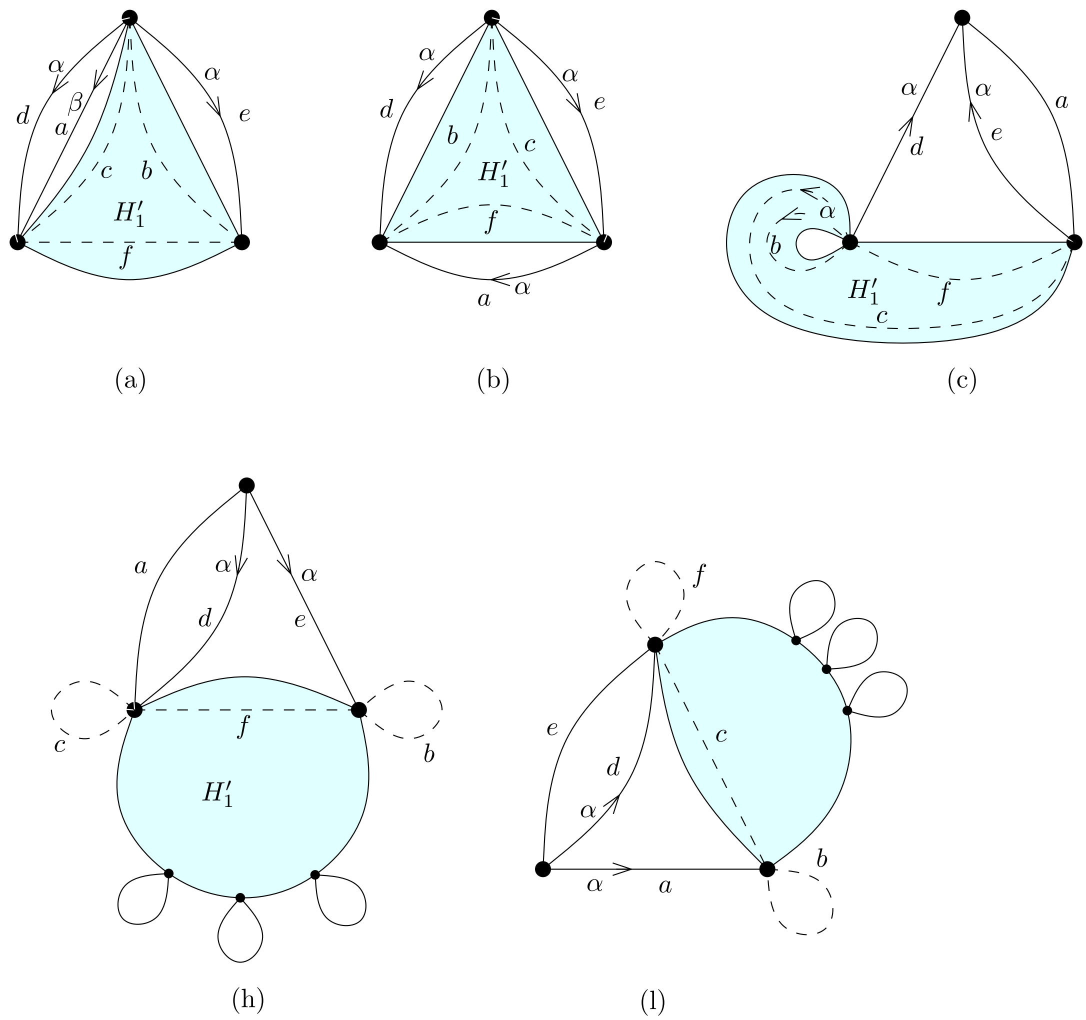


Figure 5.37: Representations of $F(\Omega)$. All loops are unbalanced; biases of other cycles are given by the indicated group-labelling.

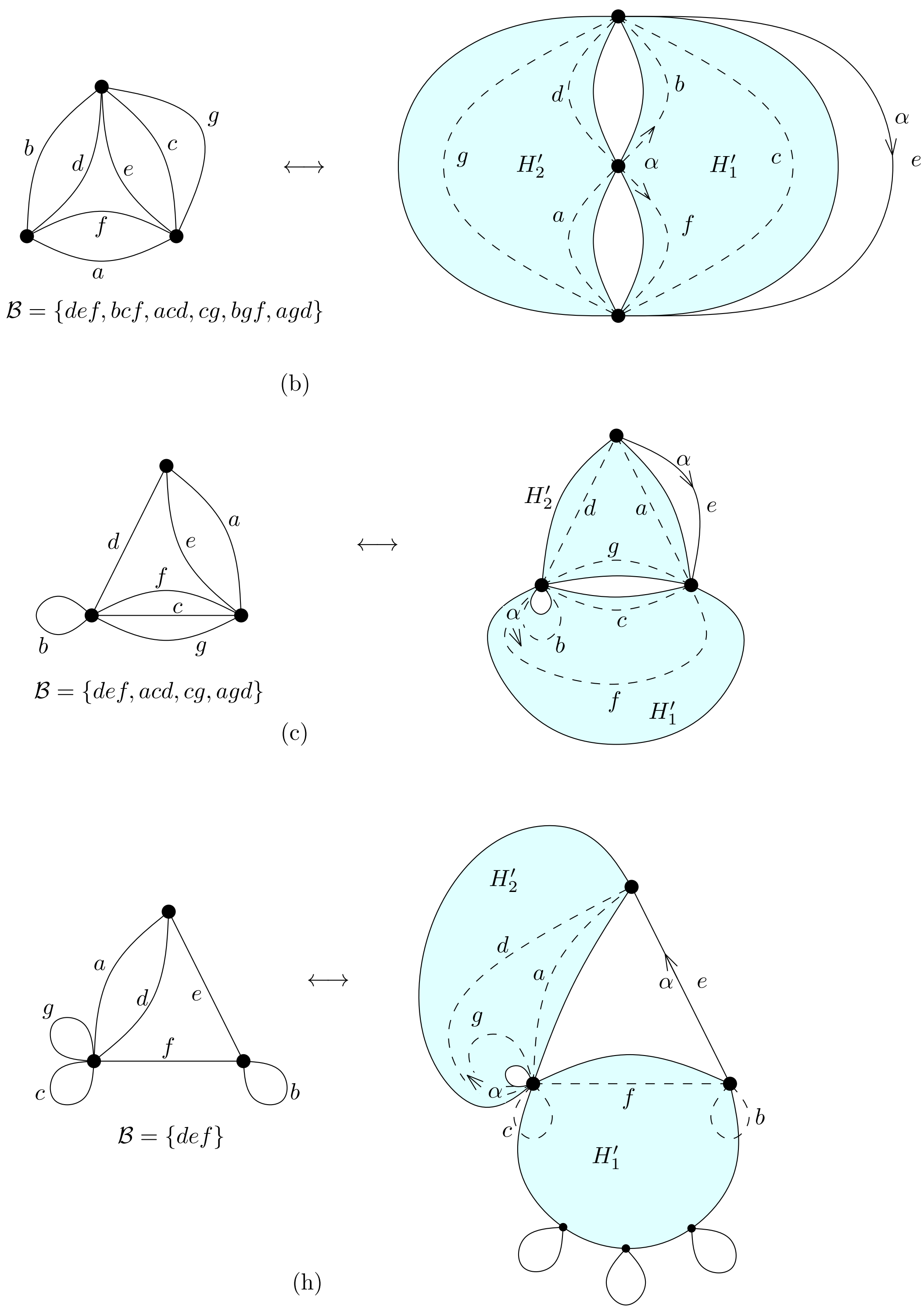
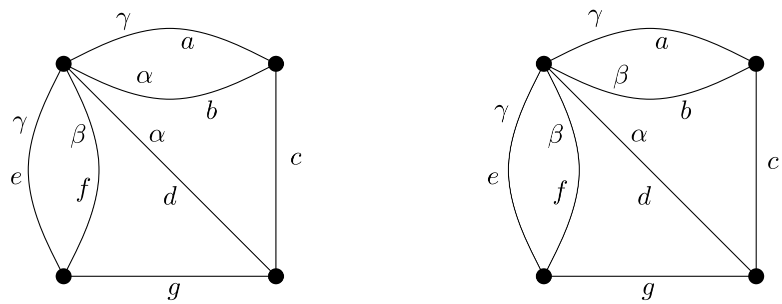


Figure 5.38: Representations of $F(\Omega)$ obtained as H-enlargements of some biased graphs of Figure 5.36 with an edge g added in parallel with c.



(a): Ω_1 (b): Ω_2

Figure 5.39: Biases of cycles are given by the group-labelling.

5. a digon and an unbalanced loop connected by an edge,

6. two digons sharing a vertex.

Suppose first (1) $defg$ is a balanced cycle in Ω. There are three possible cyclic orderings of d, e, f, g in the cycle. Consideration of circuits $aceg$ and bcd determines possibilities for the endpoints of edges a, b, and c. Discarding those biased graphs whose frame matroids contain circuits which are not circuits of $F(\Omega_1)$, we find that Ω is the biased graph shown in Figure 5.40 (b), (c), or (d).

Now suppose $defg$ is a circuit of type (2) in $F(\Omega)$. Consideration of circuits $abef$ and bcd determine possibilities for the endpoints of the remaining edges of Ω. Considering the other circuits of $F(\Omega_1)$, we find that Ω is isomorphic to one of two biased graphs obtained by rolling up an edge of the biased graph shown in Figure 5.40 (f). One of these is obtained by rolling up edge f of biased graph (f), and the other by rolling up edge g of (f).

Suppose $defg$ appears as a set of edges of type (3). Then $defg$ induces a subgraph on three vertices, and since Ω may not have a vertex of degree less than three, each of a, b, c is incident to the fourth vertex of Ω. If $\{e, f\}$ form a digon, then circuit $abef$ implies $\{a, b\}$ forms a path linking its endpoints. But then not both bcd and $aceg$ may be circuits of $F(\Omega)$, a contradiction. Similarly, if $\{d, g\}$ form a digon, circuit $abef$ implies $\{a, b\}$ forms a path linking its endpoints, and again not both bcd and $aceg$ may be circuits of $F(\Omega)$, a contradiction. If $\{d, e\}$ form a digon, circuits bcd and $aceg$ imply Ω is the biased graph of Figure 5.40 (e). If $\{e, g\}$ form a digon, circuits bcd and $aceg$ imply Ω is the biased graph of Figure 5.40 (g).

Next suppose $defg$ is a circuit in $F(\Omega)$ of type (4). If any of d, f, g is the unbalanced loop, we find it not possible that all of bcd, $abef$, and $aceg$ may be circuits of $F(\Omega)$. If e is the unbalanced loop, we find Ω must be the biased graph of Figure 5.40 (f).

Now assume edges $\{d, e, f, g\}$ form a circuit of type (5) in $F(\Omega)$. If $\{e, f\}$ form the digon, then circuit $abef$ implies $\{a, b\}$ is a path linking its endpoints, but then not both bcd

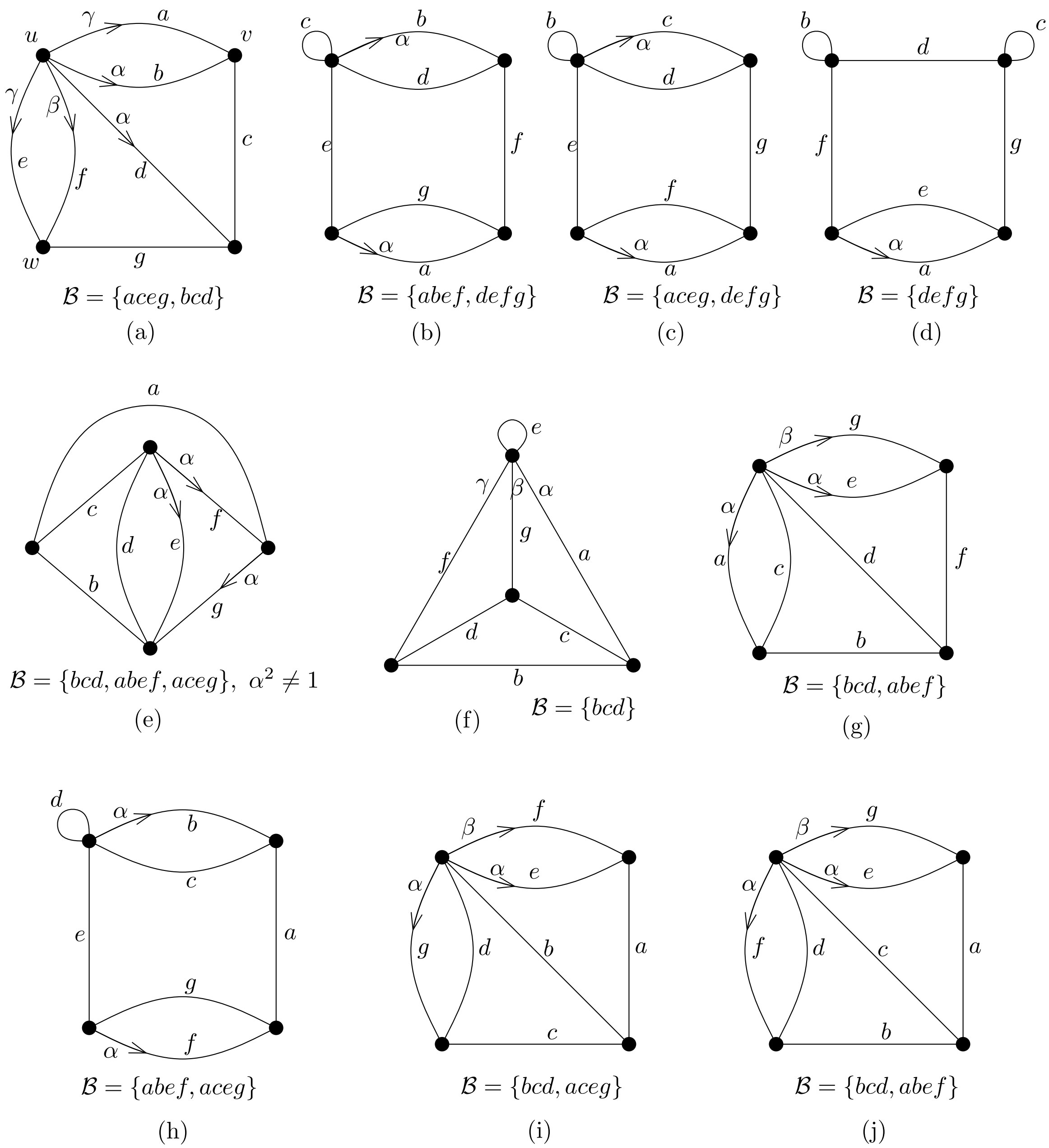


Figure 5.40: Representations of $F(\Omega_1)$; all loops are unbalanced.

and $aceg$ may be circuits in $F(\Omega)$, a contradiction. If $\{e, f\}$ form a path, then $abef$ must be a balanced cycle, and circuits bcd and $aceg$ imply Ω is either obtained by rolling up edge d of biased graph (j) in Figure 5.40 (if e is in the digon with g), or that Ω is obtained by rolling up edge g of biased graph (j) in Figure 5.40 (if f is in the digon with d), or that Ω is biased graph (h) in Figure 5.40 (if f is in the digon with g). The only other possibility in this case is that $\{e, f\}$ are incident to a common vertex and one of e or f is a loop. Then circuits $abef$, bcd and $aceg$ implies Ω is obtained by rolling up edge f of biased graph (i) shown in Figure 5.40.

Finally, suppose circuit $defg$ is type (6) in Ω. The two digons formed by $defg$ induce a subgraph on three vertices; since Ω may not have a vertex of degree one or two, each of $\{a, b, c\}$ are incident to the fourth vertex of Ω. Circuit $abef$ implies edges $\{a, b, e, f\}$ form either an unbalanced theta or a balanced cycle. If the former, then circuit bcd implies Ω is the biased graph shown in Figure 5.40 (i). If the later, then circuit bcd implies Ω is that shown in Figure 5.40 (j). $\square$

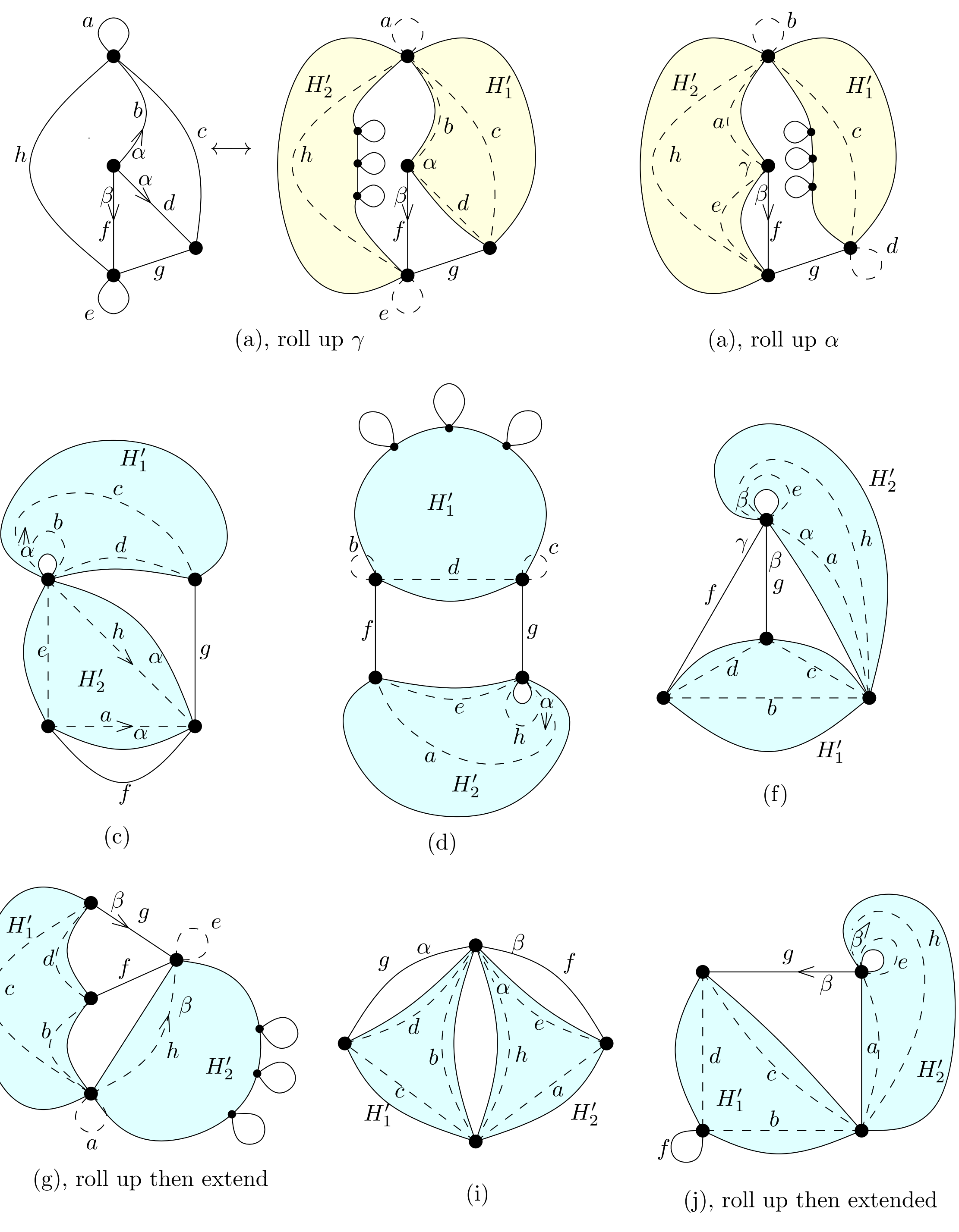


Figure 5.41: H-enlargements of representations of $F(G_2)$.

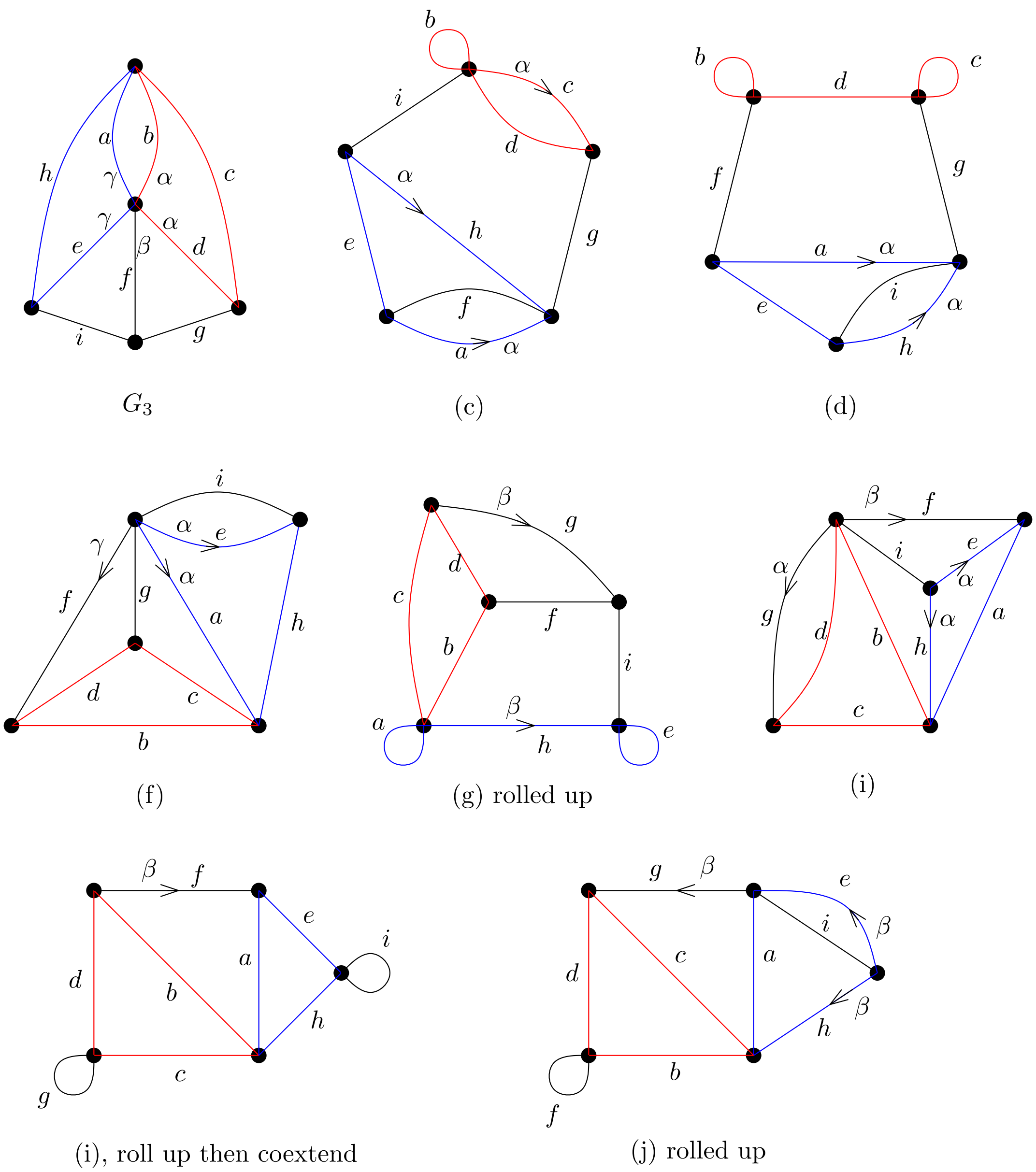


Figure 5.42: Representations of $F(G_3)$.

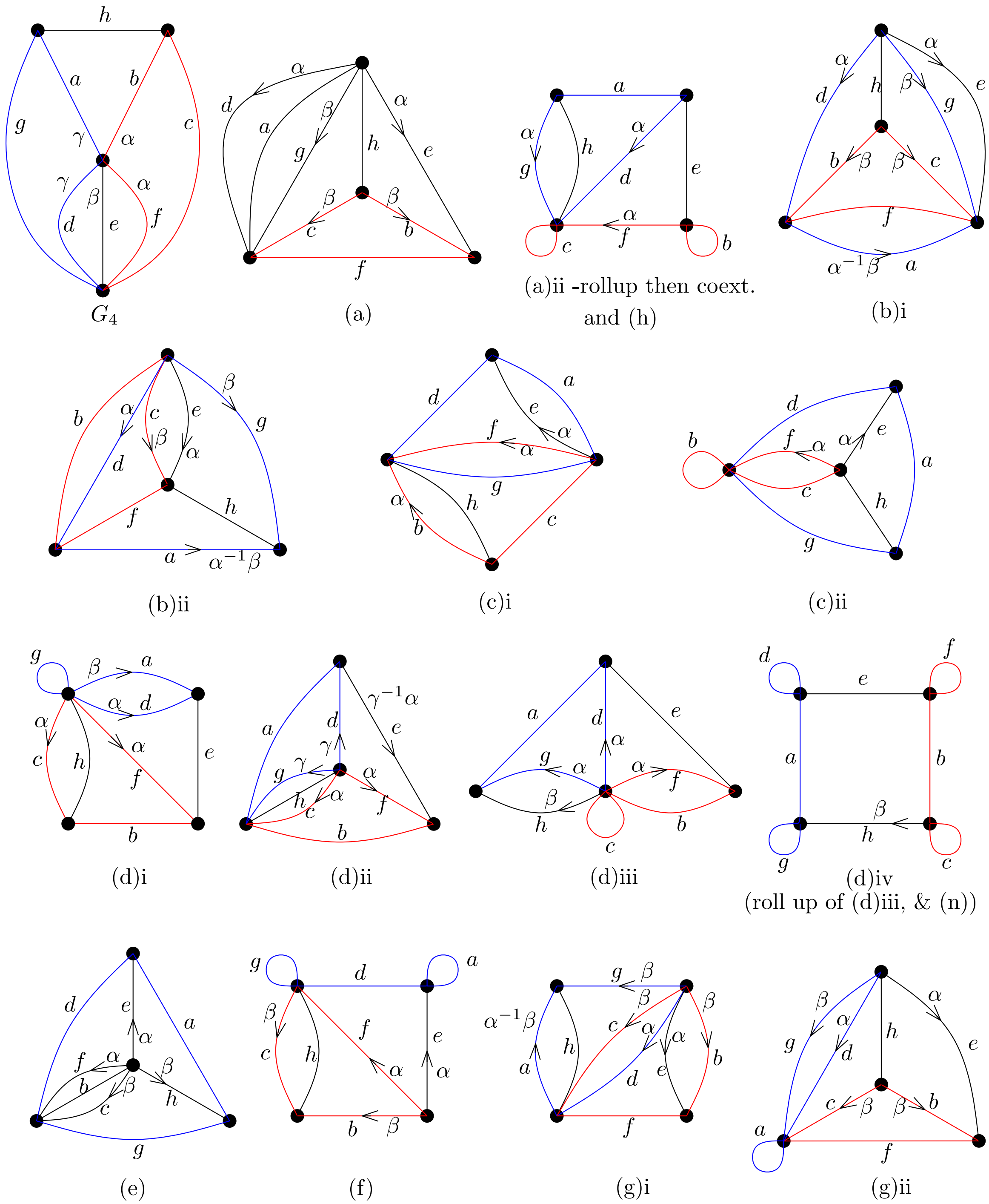


Figure 5.43: Representations of $F(G_4)$. Cycles with edge sets $\{a, d, g\}$, $\{b, c, f\}$, $\{d, e, f\}$, $\{c, g, h\}$, $\{a, e, f, g\}$, $\{b, c, d, e\}$, $\{a, c, d, h\}$, $\{b, f, g, h\}$ are balanced; all other cycles are unbalanced.

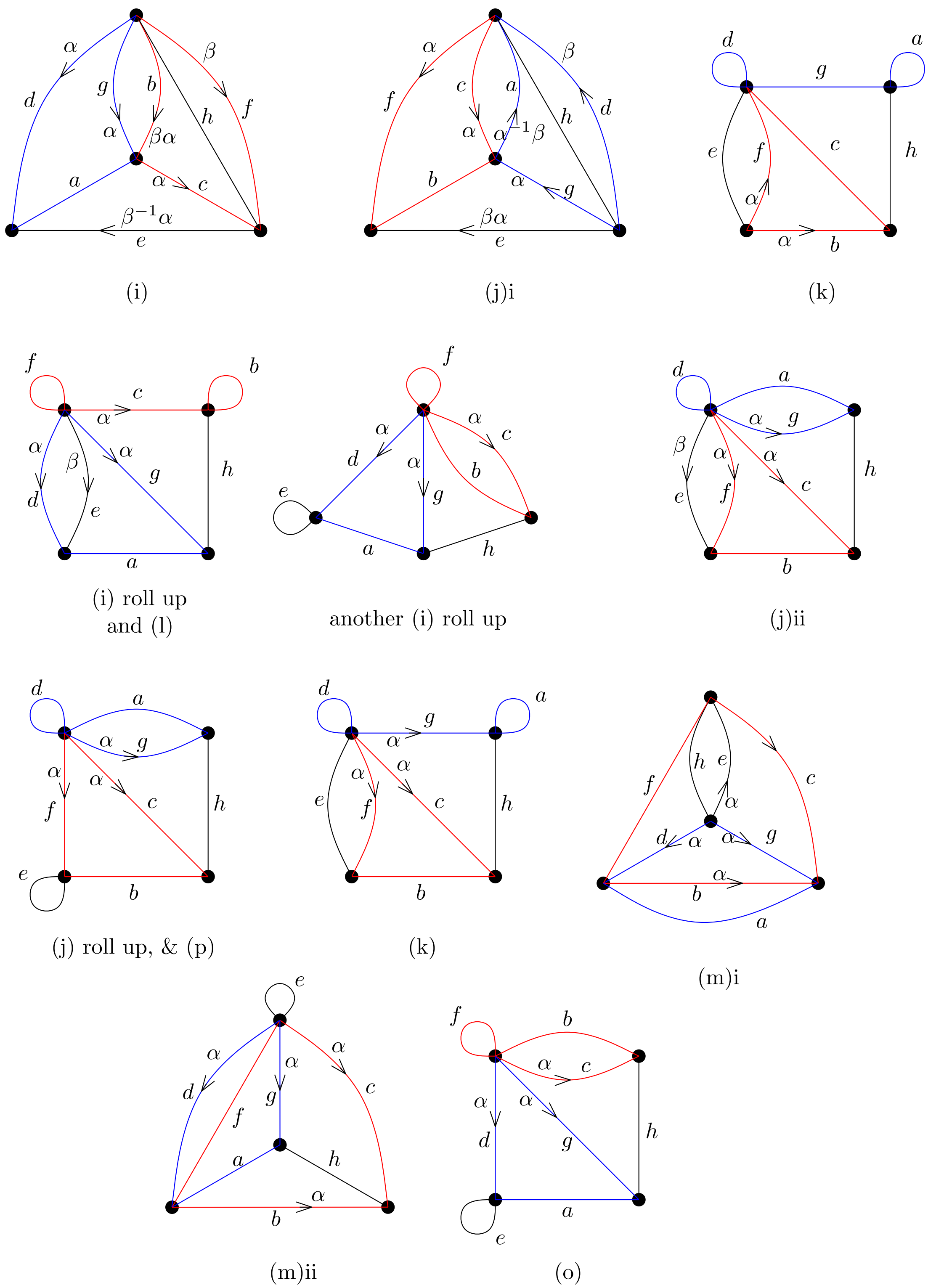


Figure 5.44: More representations of $F(G_4)$.

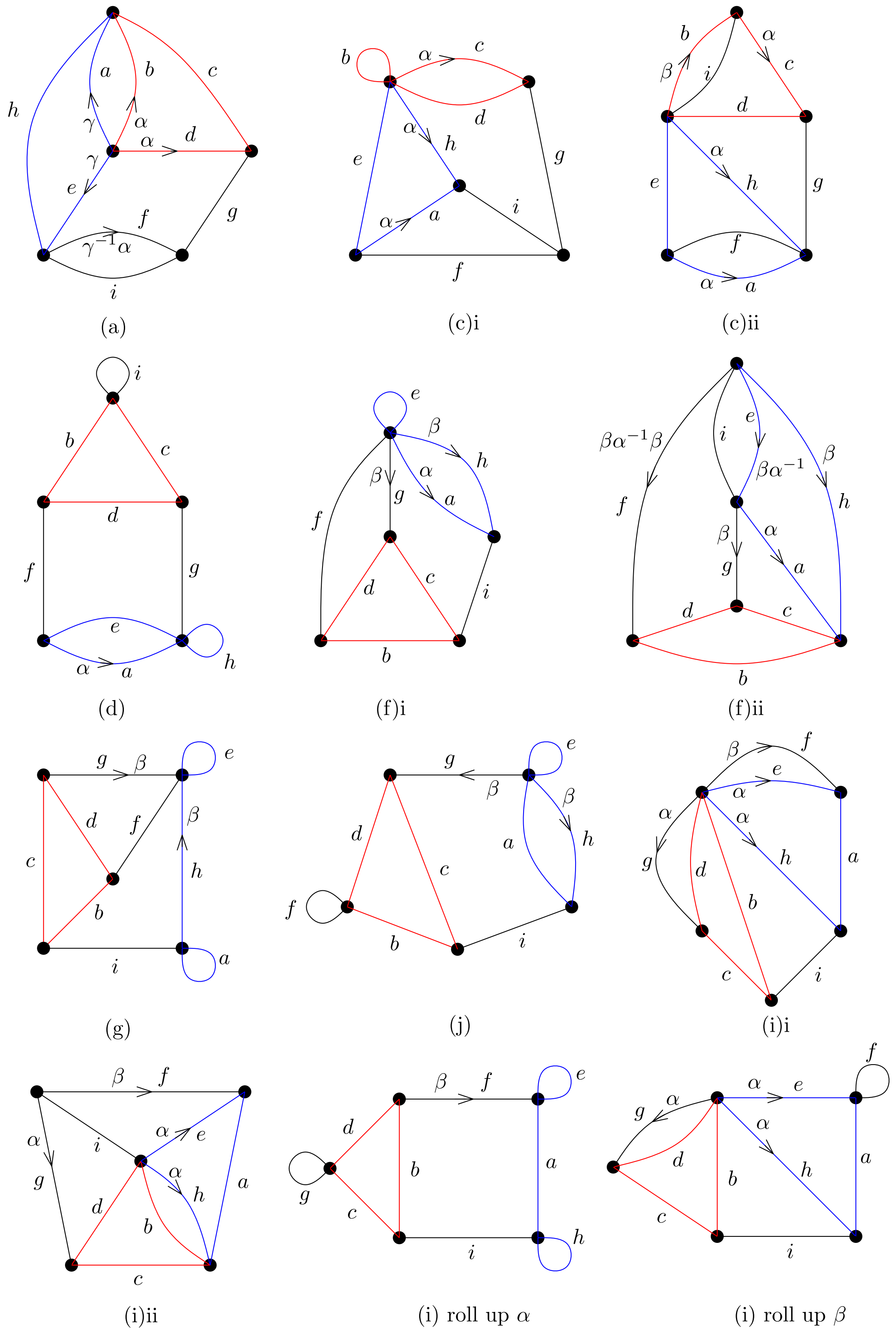


Figure 5.45: Representations of $F(G_5)$.

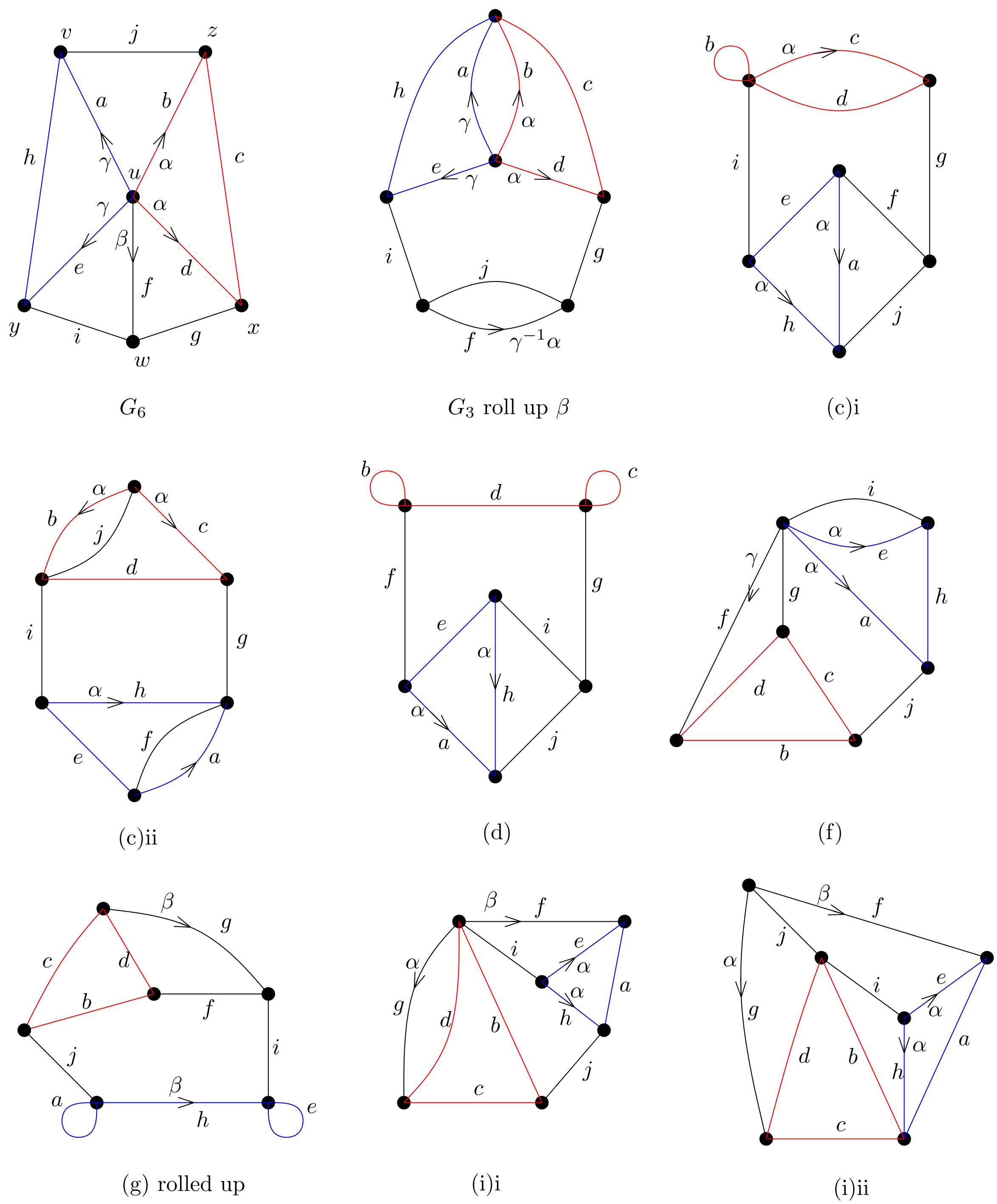
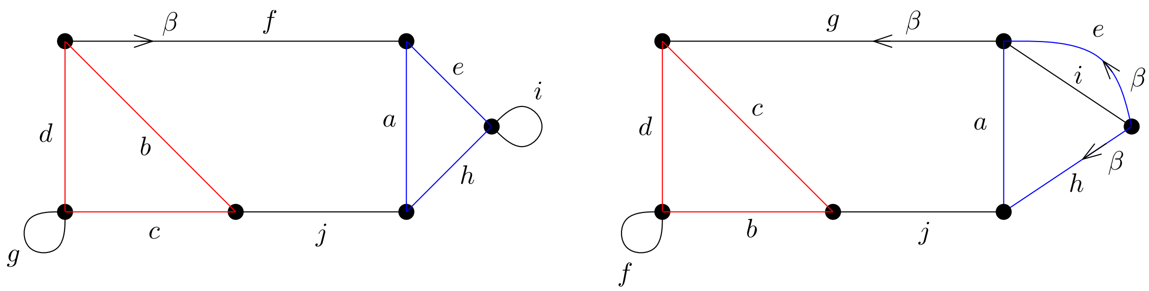


Figure 5.46: Representations of $F(G_6)$.



(i), roll up then coextend

(j) rolled up

Figure 5.47: More representations of $F(G_6)$.

Proposition 5.19. *Let Ω_2 be the biased graph shown in Figure 5.39 (b). Suppose Ω is a biased graph with $F(\Omega) \cong F(\Omega_2)$. Then Ω is one of the biased graphs in Figure 5.48 or is obtained by rolling up a balancing class of the balancing vertex of one of these biased graphs.*

Proof. We proceed along the same lines as the proof of Proposition 5.18. Since $F(\Omega)$ is non-graphic it contains an unbalanced cycle, and so Ω has four vertices. We consider the circuits of $F(\Omega_2)$ and how they may be represented in Ω. The circuits of $F(\Omega_2)$ are $abef$, $abcd$, $aceg$, $defg$, $bcfg$, and every subset of size five not containing one of these circuits of size four. In the following, we find that the circuits of size four are enough to determine the possibilities for biased graphs Ω.

Consider circuit $abef$ of $F(\Omega_2)$. Suppose first $abef$ is a balanced cycle. If $aceg$ is also a balanced cycle, then we find Ω is biased graph (b) of Figure 5.48. If $aceg$ form tight handcuffs, then Ω must be the biased graph shown in Figure 5.48 (c). If $aceg$ is an countrabalanced theta, we find Ω is the biased graph of Figure 5.48 (d), and if $aceg$ form loose handcuffs then Ω must be biased graph (e) of Figure 5.48.

Now suppose $abef$ is an countrabalanced theta in $F(\Omega)$. Then $abef$ induces a subgraph on three vertices. Since $F(\Omega)$ is 3-connected, each of the remaining edges c, d, g must be incident to the fourth vertex of Ω. If $\{a, b\}$, $\{a, f\}$, $\{b, e\}$, or $\{e, f\}$ form a digon, then not both $defg$ and $aceg$ may be circuits of $F(\Omega)$, a contradiction. If $\{a, e\}$ form a digon, Ω must be the biased graph of Figure 5.48 (g). If $\{b, f\}$ form a digon, then Ω must be the biased graph of Figure 5.48 (f).

If $abef$ forms tight handcuffs, we find the other circuits of size four imply Ω must in fact be Ω_2 (Figure 5.48 (a)).

Suppose $abef$ form loose handcuffs with one of the unbalanced cycles being a digon. Since circuit $abef$ induces a subgraph on three vertices and $F(\Omega)$ is 3-connected, each of c, d, and g must be incident to the fourth vertex x of Ω. But for every choice of two of $\{a, b, e, f\}$ in the digon and choice of one of the remaining elements for the unbalanced

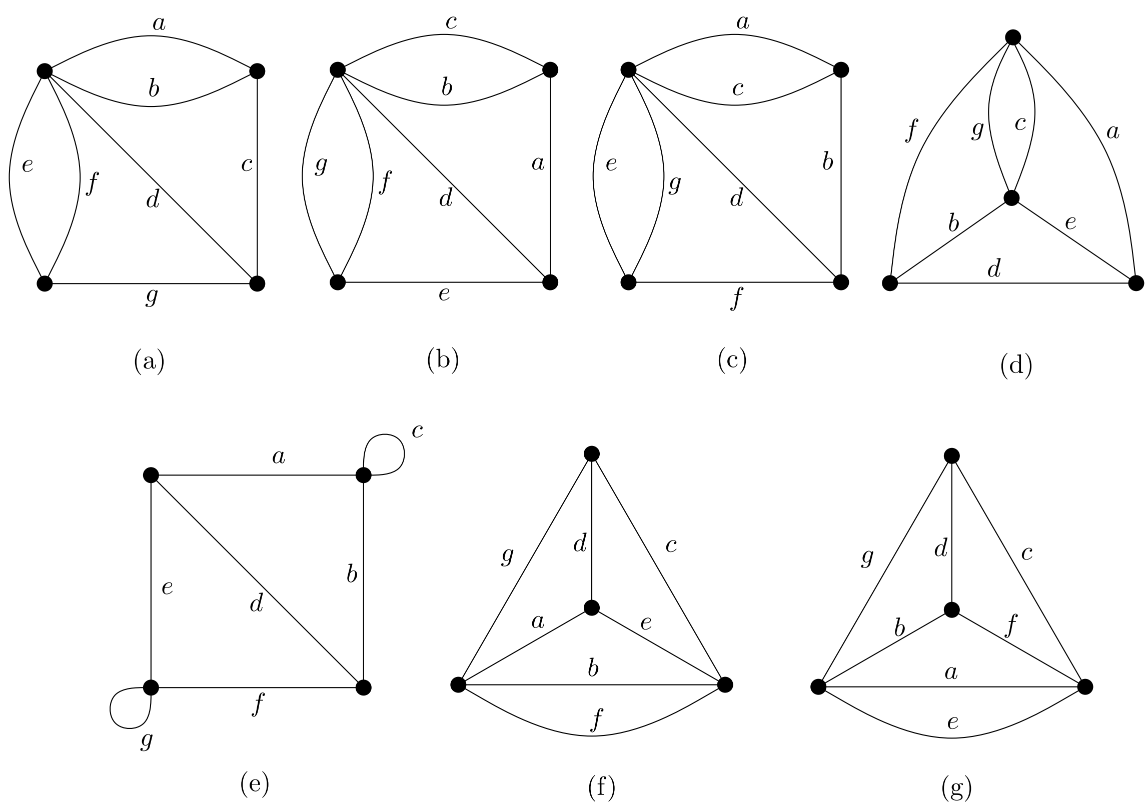


Figure 5.48: Representations of $F(\Omega_2)$. Cycles $aceg$, $bcfg$, $abef$, $abcd$, or $defg$, are balanced; all other cycles are unbalanced.

loop, there is a 4-circuit of $F(\Omega_2)$ containing an edge of the digon and the loop, and such a pair of edges cannot be extended to a circuit of $F(\Omega)$ unless its representing edges avoid x, a contradiction.

Finally, suppose $abef$ form loose handcuffs with two unbalanced loops. Again, $abef$ induces a subgraph on three vertices, so each of $\{c, d, g\}$ must be incident to the fourth vertex of Ω. Checking possible arrangements of elements a, b, e, f in loose handcuffs, we find the only possibilities for Ω are roll-ups of Ω_2 (Figure 5.48 (a)). $\square$

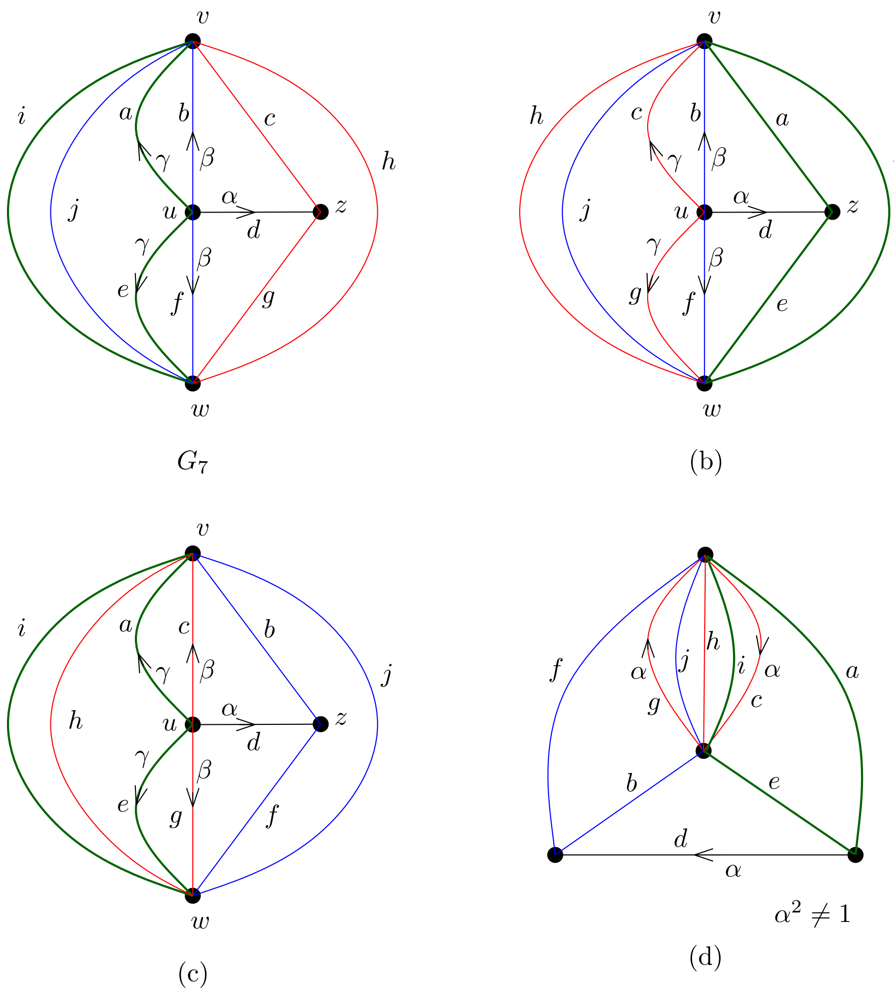


Figure 5.49: Representations of $F(G_7)$. (Note that if a biased graph representation of $F(\Omega)$ is obtained as an H-enlargement of biased graph (d), then Ω has $|H_1| \leq 3$.)

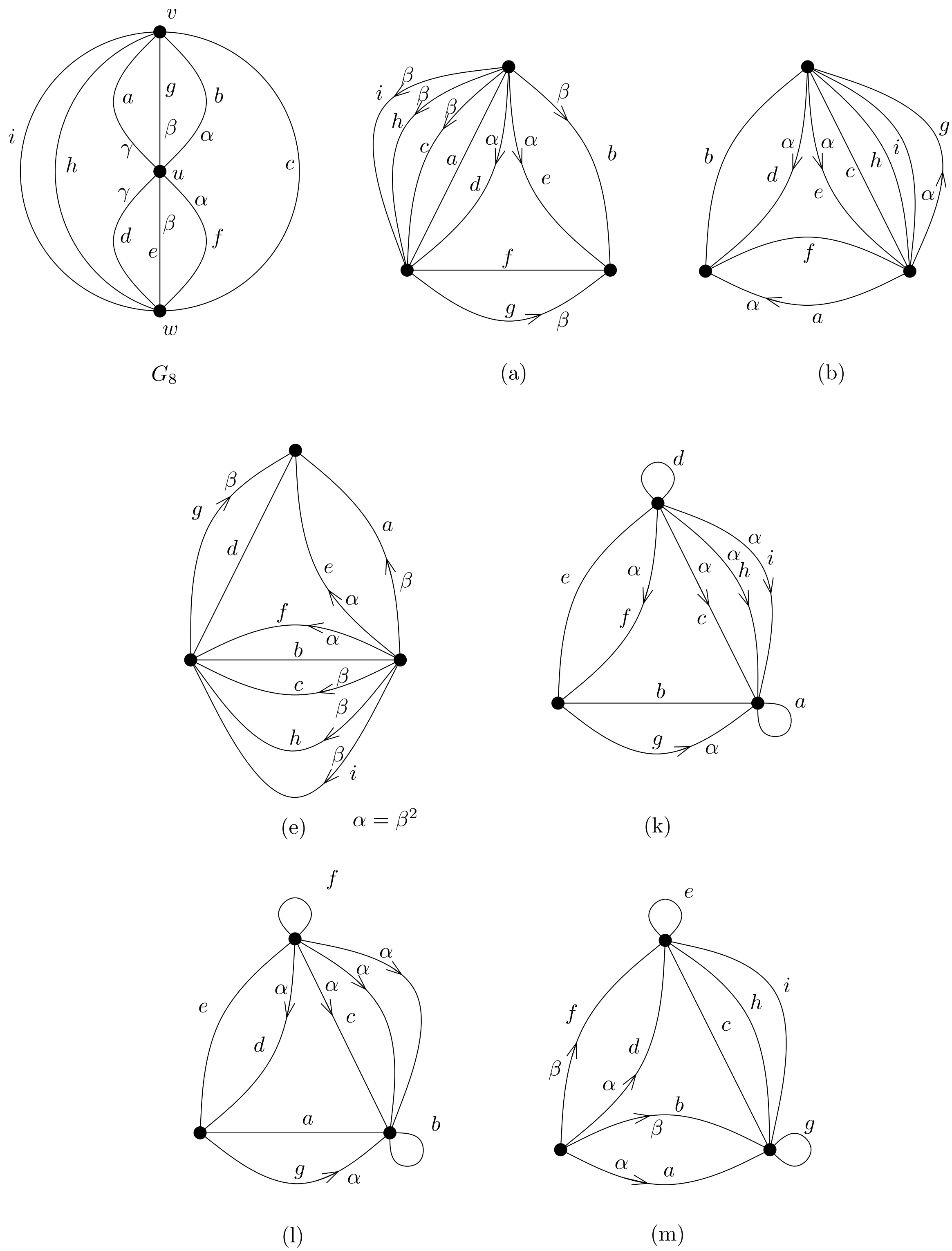


Figure 5.50: Representations of $F(G_8)$.

Lemma 5.20. *Let Ω be a 3-connected biased graph with $F(\Omega)$ non-graphic and 3-connected, with a balancing vertex u with exactly four balancing classes, and a vertex $v \neq u$ such that $F(\Omega - v)$ is binary and there are exactly three balancing classes in $\delta(u)$ in $\Omega - v$. Suppose Ω has exactly two degenerate lobes. If Ψ has frame matroid isomorphic to $F(\mathrm{re}_H(\Omega))$, then Ψ is a biased graph shown in one of Figures 5.51, 5.52, 5.53, 5.54, 5.55, 5.56, 5.57, or 5.58.*

Proof. First consider biased graph G_{10} of Figure 5.23. We determine all biased graphs Ω' whose frame matroid is isomorphic to $F(G_{10})$. These are shown in Figures 5.51 and 5.52. Since $F(G_{10})$ has rank 3 and is non-graphic, every biased graph representing $F(G_{10})$ has three vertices. Furthermore, $F(G_{10})$ is a paving matroid with exactly two circuits of size three (bcf and def), and every other 3-set of elements a basis. Hence when determining if $F(\Omega') \cong F(G_{10})$, we just need check that bcf and def are circuits of $F(\Omega')$ and that bcf and def are the only circuits of $F(\Omega')$. Let Ω' be a biased graph with $F(\Omega') \cong F(G_{10})$.

Consider circuit def; in Ω' it may be:

1. a balanced triangle,
2. loose handcuffs,
3. an unbalanced theta,
4. tight handcuffs.

If def is (1) a balanced triangle, then since ab forms a circuit with each of de, df, and ef, either each of edge a and b is in parallel with a distinct edge of def, or one of a, b is a loop and the other is an edge in parallel with the edge of def not incident to the loop. Suppose first that b is in parallel with an edge of def. If b is not in parallel with f, then since bcf is a circuit, this implies that bcf is a balanced triangle, so c is an edge in parallel with edge d or e. Since a is not in any circuit of size less than four, a is not in parallel with c, nor a loop, and so is an edge in parallel with edge f. Hence in this case Ω' is either biased graph (a) or (b) of Figure 5.51, where in each biased graph cycles bcf and def are balanced and all other cycles are unbalanced. If b is in parallel with f, then since bcf is a circuit, c is either an unbalanced loop incident with a vertex that is an endpoint of b and f or in parallel with b and f. Hence in this case Ω' is one of biased graphs (c), (d), (e), or (f) of Figure 5.51, where cycle def is the only balanced cycle. So suppose now b is an unbalanced loop. Then a is an edge in parallel with the edge of def not sharing an endpoint with b. Since bcf is a circuit, b shares an endpoint with f, and since a is in no circuit of size two or three, c is an edge in parallel with f. Hence Ω' is one of biased graphs (g) or (h) of Figure 5.51, where only cycle def is balanced.

Suppose now def form (2) loose handcuffs. Then two vertices of Ω' are incident to def; since $F(G_{10})$ is 3-connected, each of a, b, c is incident to the third vertex of G'. Since a is in no circuit of size two or three, a is not an unbalanced loop, nor is b or c an unbalanced loop, and nor is b or c in parallel with a. Hence f is an unbalanced loop, and Ω' is one of biased graphs (i) or (j) of Figure 5.51, where all cycles are unbalanced.

Now suppose (3) def is an unbalanced theta. Again, then two vertices of Ω' are incident to def and since $F(G_{10})$ is 3-connected, each of a, b, c is incident to the third vertex of G'. Since bcf is a circuit and f is a link, bc must form a balanced triangle with f. Now a may be a loop sharing an endpoint with both edges b and c or a link in parallel with b or c. Hence Ω' is one of biased graphs (k), (l), or (m) of Figure 5.51, where only cycle bcf is balanced.

Finally, suppose def form (4) tight handcuffs. If f is a link, then since bcf is a circuit, bc must form a balanced cycle with f. Since a is in no circuit of size less than four, a is a link neither of whose endpoints is an endpoint of the loop in def. In this case, Ω' is one of biased graphs (n), (o) of Figure 5.51 or (p), or (q) of Figure 5.52, where the only balanced cycle is bcf. If f is a loop, then since bcf is a circuit either bc form tight handcuffs or loose handcuffs with f, and connectivity implies a is a link not sharing an endpoint with f. Hence Ω' is one of biased graphs (r), (s), or (t) of Figure 5.52, where all cycles are unbalanced.

We next consider biased graph G_{12} of Figure 5.23, and determine now all biased graphs Ω' whose frame matroid is isomorphic to $F(G_{12})$. Since $F(G_{12})/g \cong F(G_{10})$, every biased graph representing $F(G_{12})$ may be obtained from a biased graph representing $F(G_{10})$ by coextending by an element. We therefore consider each of the biased graphs of Figures 5.51 and 5.52 in turn, coextending each in all possible ways such that the frame matroid of the coextension is isomorphic of $F(G_{12})$. These are shown in Figures 5.53 and 5.54.

Observe that $F(G_{12})$ is rank 4, and has all circuits size five except bcf, $defg$, and $abde$. Hence when checking isomorphism of $F(\Omega')$ with $F(G_{12})$, we just need check that bcf, $defg$, and $abde$ are circuits, and that these are the only circuits of size less than five.

- First consider coextensions of G_{10}. G_{12} is obtained by uncontracting an edge at vertex w; the only other possibility, since $defg$ is a circuit, is an uncontraction at u. Then since $defg$ is a circuit, each of d, e, f share an endpoint with g; exactly two of $\{d, e, f\}$ share a common endpoint of g. Since bcf and $abde$ are circuits, Ω' is in this case biased graph (a)i of Figure 5.53, where just cycles bcf and $abde$ are balanced.

- We may consider coextensions of biased graphs (a) and (b) of Figure 5.51 together. Circuits bcf implies that after uncontracting g at a vertex $v \in \{x, y, z\}$, bcf is still a balanced cycle so the two edges of bcf meeting at v are still adjacent. Connectivity implies that the other two edges incident to v are both incident to the other endpoint of g. Since $abde$ is a circuit, the only possibility is uncontracting g at z, and Ω' is the biased graph of Figure 5.53 (a) or (b), where only cycles bcf and $defg$ are balanced.

- coextensions of biased graphs (c) and (d). Since $F(G_{12})$ is 3-connected, we may not uncontract at x. Uncontracting at y, circuit bcf implies edges b and f are incident to a common endpoint of g, and connectivity then implies a and d or e are both incident to the other endpoint of g. But then $abde$ is not a circuit, a contradiction. Uncontracting at z, we find c remaining a loop would violate connectivity, so we obtain Figure 5.53 (c) or (d), where only cycles bcf and $defg$ are balanced.

- coextensions of biased graphs (e) and (f). A coextension at either of vertices x or z is not possible; the former would result in a vertex of degree two, as would the later, since edges b, c, f must remain incident to a common vertex (since bcf is a circuit). But coextending at y, since again edges b, c, f must remain incident to a common endpoint of g, so a is incident to the other endpoint of g, we find $abde$ is not a circuit, a contradiction. So there are no coextensions of these biased graphs with frame matroid isomorphic to $F(G_{12})$.

  Each of (g) and (h) yield a single biased graph with frame matroid $F(G_{12})$ obtained as a coextension. Connectivity prevents uncontracting g at vertex y. Uncontracting at x, we find b cannot remain a loop, else fc must share an endpoint of g, and the other endpoint of g is of degree 2, a contradiction. Hence b must in this case be a edge in parallel with g, and bcf a balanced triangle; but then $abde$ is not a circuit, a contradiction.

  So uncontract g at z: Since bcf is a circuit, edges f, c are both incident to a common endpoint of g, so a and d or e are both incident to the other endpoint of g. We obtain biased graphs (g) and (h) of Figure 5.53 (c) or (d), where only cycle $defg$ is balanced.

- coextensions of biased graphs (i) and (j). Circuit $defg$ implies we must uncontract at x or z, or that g is an unbalanced loop incident to the fourth vertex to which f and d or e is also incident. But in the later case $abde$ would not be a circuit, a contradiction.

  Uncontracting at x, we obtain biased graphs (i) and (j) of Figure 5.53, where cycle $abde$ is the only balanced cycle.

  Uncontracting at z, we find f cannot remain a loop, else b, c must share a common endpoint of g with f (since bcf is a circuit), which leave the other endpoint of g with degree 2, a contradiction. Hence in this case f is an edge in parallel with g, so bcf is a balanced triangle; circuit $abde$ implies we have one of biased graphs (i)ii or (j)ii of Figure 5.53, where only cycle bef is balanced.

- coextensions of biased graphs (k). Since $defg$ is a circuit, g may be uncontracted from vertex x or z. In either case, f must form a balanced cycle with bc, and so d, e are both incident to the endpoint of g not incident to f. Circuit $abde$ implies we have the biased graph of Figure 5.53 (k), where just cycle bcf is balanced.

- (l) and (m). Since $defg$ is a circuit, we may uncontract g at x or z. Uncontracting at z, we find bcf must be a balanced triangle, so d, e must both be incident to the endpoint of g not incident to f. Since $abde$ is a circuit, we obtain biased graph (l)i of Figure 5.53, where bcf is the only balanced cycle.

Uncontracting at x, say $g = x'x''$, we see that bcf must be a balanced triangle, say with $f = zx'$. If edge $a = yx'$, then connectivity implies d and e are both zx'' edges. But then $abcd$ is not a circuit, a contradiction. Hence $a = yx''$. Edges d and e may both have endpoints z, x'', or one may be incident to x' while the other is incident to x''. We obtain biased graphs (l)ii, (l)iii of Figure 5.53 and (m) of Figure 5.54, where in (l)ii and (l)iii only cycles bcf and $abcd$ are balanced and in (m) only cycle bcf is balanced.

- (n), (o), (p), (q). Since $defg$ is a circuit, we may not uncontract at y. Uncontract at x: circuit bcf must remain a balanced cycle, and connectivity forces edges d and e incident to the endpoint of g not incident to f. If d or e remains a loop, circuit $abde$ then implies b is parallel to a, so this yields (n) or (o) of Figure 5.54, where just cycle bcf is balanced. If d or e is an edge in parallel with g, then circuit $abde$ implies c is parallel to a, and we have (p)i or (q)i of Figure 5.54, where just cycles bcf and $abde$ are balanced.

  Uncontracting at z, we find the following. Again circuit bcf must remain a balanced cycle, and connectivity forces edge d or e, as well as edge a, to be incident to the endpoint of g not incident to f. Now circuit $abde$ implies that b is a yx edge. We obtain biased graph (p)ii or (q)ii of Figure 5.54, where only cycle bcf is balanced.

- (r), (s). Again circuit $defg$ implies we may uncontract at vertices x or z, but not y. Uncontracting at x, we find circuit bcf forces f to remain a loop. Connectivity then forces edges d and e incident to the endpoint of g not incident to loop f. Circuit $abde$ implies b must be a loop incident to y. We obtain biased graph (s) of Figure 5.54, where all cycles are unbalanced.

- Finally, consider uncontracting an edge of biased graph (t). Again, circuit $defc$ implies we may not uncontract at y. Uncontracting at z would result in a vertex of degree 2, violating the 3-connectedness of $F(G_{12})$.

  Uncontracting at x, say $g = x'x''$, we find that if f were to remain a loop, say at x', then circuit bcf implies edges b and c are yx' edges, and so connectivity implies d and e are zx'' edges. But then $abde$ would not be a circuit, a contradiction. Hence f is an $x'x''$ edge in parallel with g. Circuit bcf then must be a balanced triangle. If d and e have different endpoints x', x'', then $abde$ would not be a circuit, a contradiction. Hence d and e remain parallel, sharing a common endpoint with edges f, g. Circuit $abde$ now implies that edge b must also share this common endpoint, so we obtain biased graph (t) of Figure 5.54, where just cycle bfc is balanced.

Next consider biased graph G_{11} of Figure 5.23. We determine now all biased graphs Ω' whose frame matroid is isomorphic to $F(G_{11})$. As above, since $F(G_{11})$ is obtained from $F(G_{10})$ by coextension by a single element, we just need uncontract an element g in every possible way in every biased graph representing $F(G_{10})$ such that the resulting biased graph has frame matroid isomorphic to $F(G_{11})$. Since $F(G_{11})$ has just three circuits (bcf, $defg$, $acde$) of size less than five and has rank 4, when checking isomorphism of the frame matroid of a biased graph with $F(G_{11})$, we just need check that the only subsets of edges of sizes 3 and 4 forming circuits are precisely bcf, $defg$, and $acde$. The details are similar to the case above of determining all biased graphs representing $F(G_{12})$, and are omitted. The biased graphs so obtained are shown in Figures 5.55 and 5.56. Cycles with edges sets $\{b, c, f\}$, $\{d, e, f, g\}$, or $\{a, c, d, e\}$ are balanced; all other cycles are unbalanced.

Finally, consider biased graph G_{13} of Figure 5.23. We determine all biased graphs Ω' whose frame matroid is isomorphic to $F(G_{13})$. Since $F(G_{13})$ is a single element coextension of $F(G_{11})$ and a single element coextension of $F(G_{12})$, we may apply the same procedure as above to all the biased graph representations of either of these matroids. Since $F(G_{13})$ has rank 5 and only circuits bcf, $defgh$, $abdeh$, and $acdeg$ of size less than six, we just need check that in any biased graph obtained by uncontracting an edge of a biased graph representing $F(G_{12})$, these edge sets are circuits and that these are the only circuits of size less than six. In this way we obtain the biased graphs shown in Figures 5.57 and 5.58 (letters labelling the biased graphs correspond to the biased graph representing $F(G_{12})$ in Figure 5.53 or 5.54 from which the biased graph is obtained as an uncontraction). $\square$

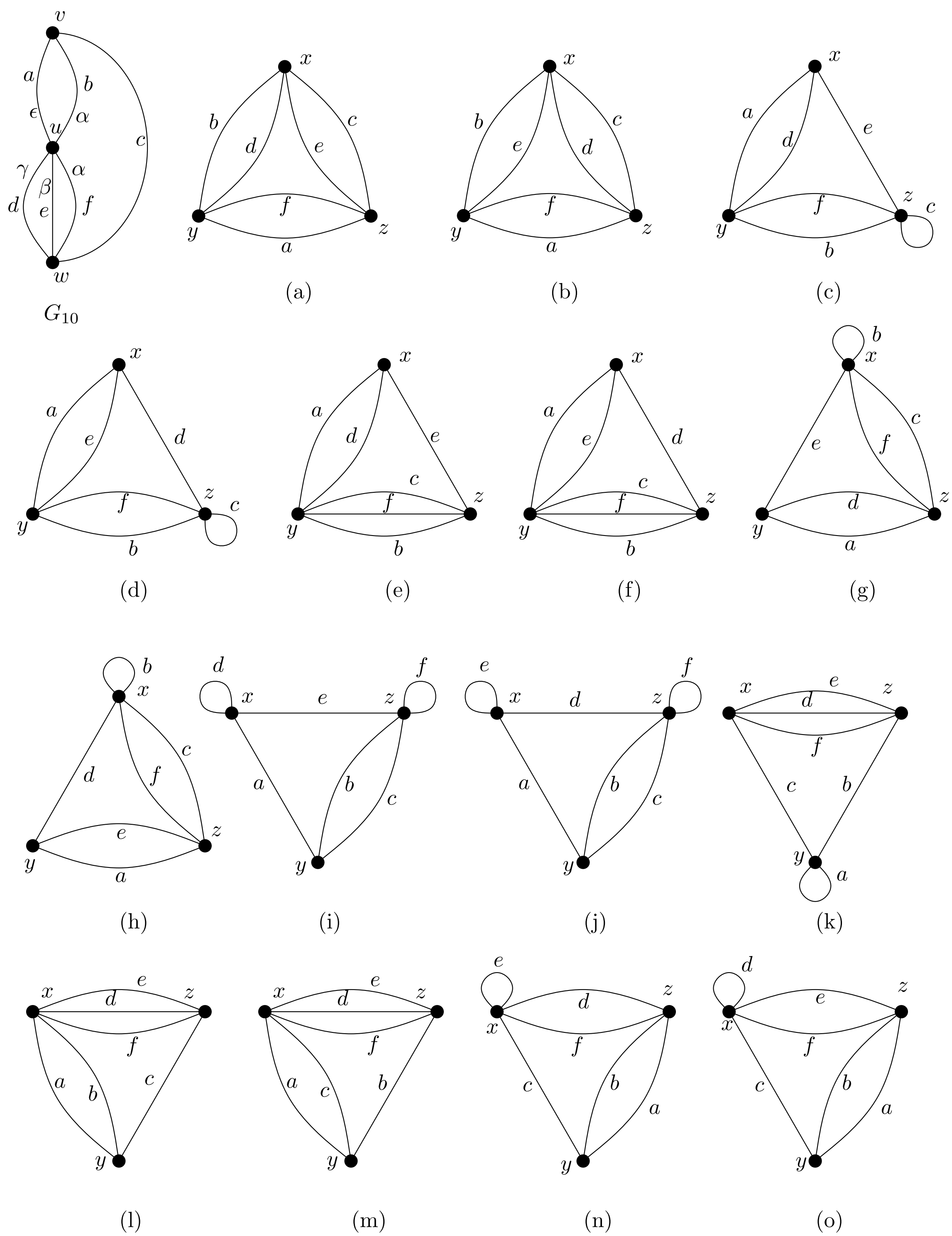


Figure 5.51: Representations of $F(G_{10})$.

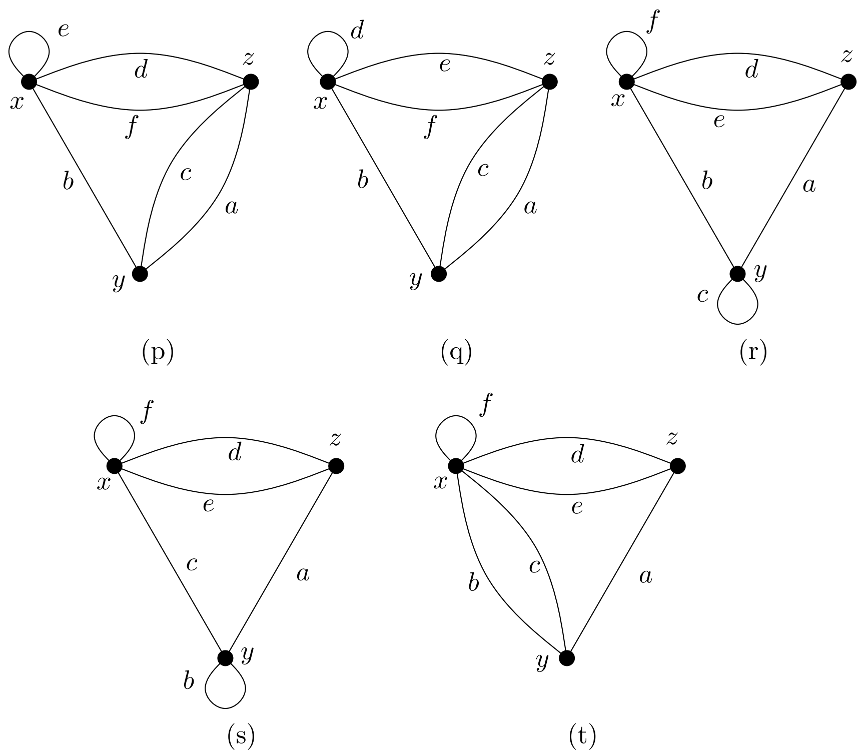


Figure 5.52: More representations of $F(G_{10})$.

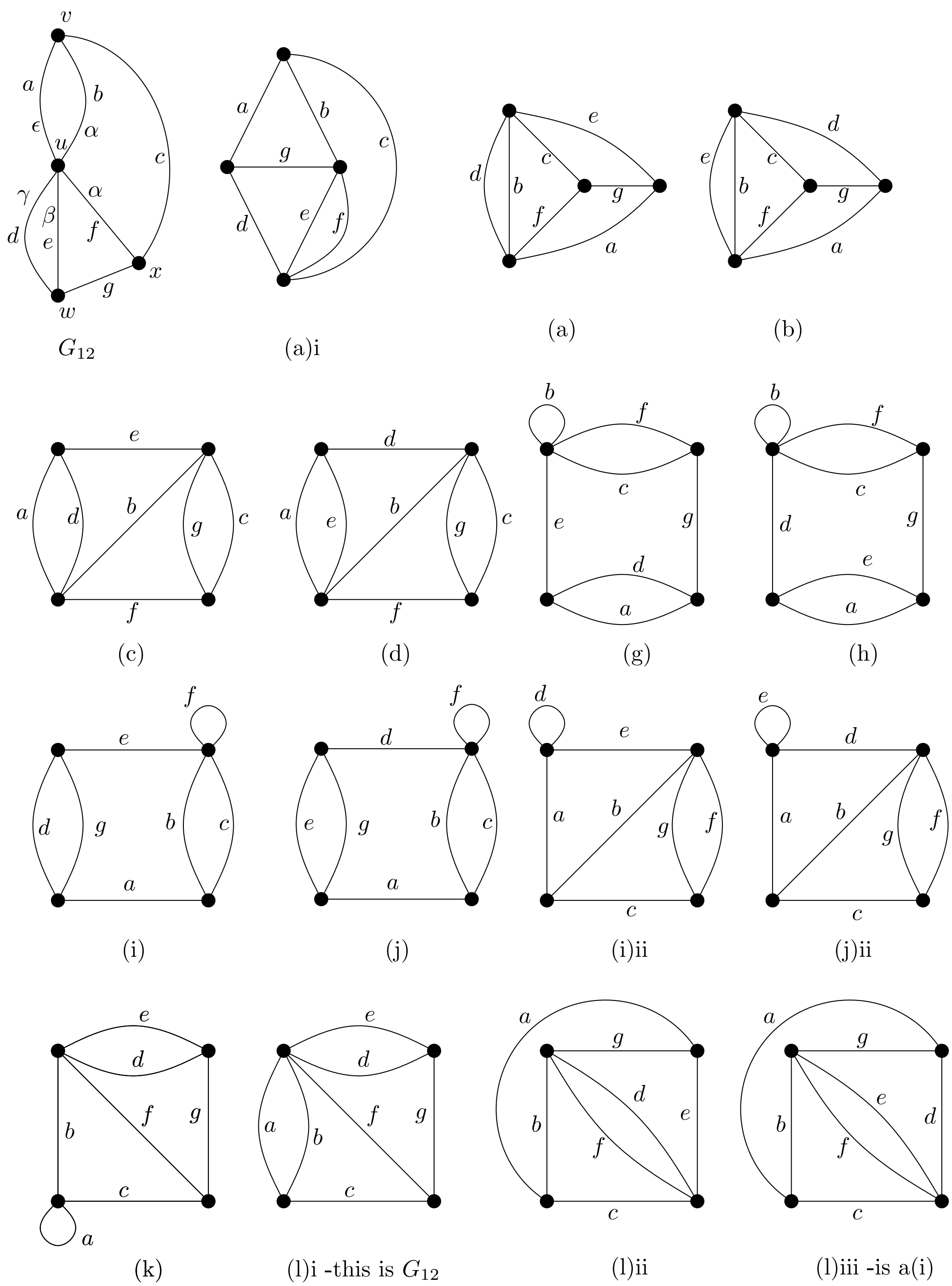


Figure 5.53: Representations of $F(G_{12})$.

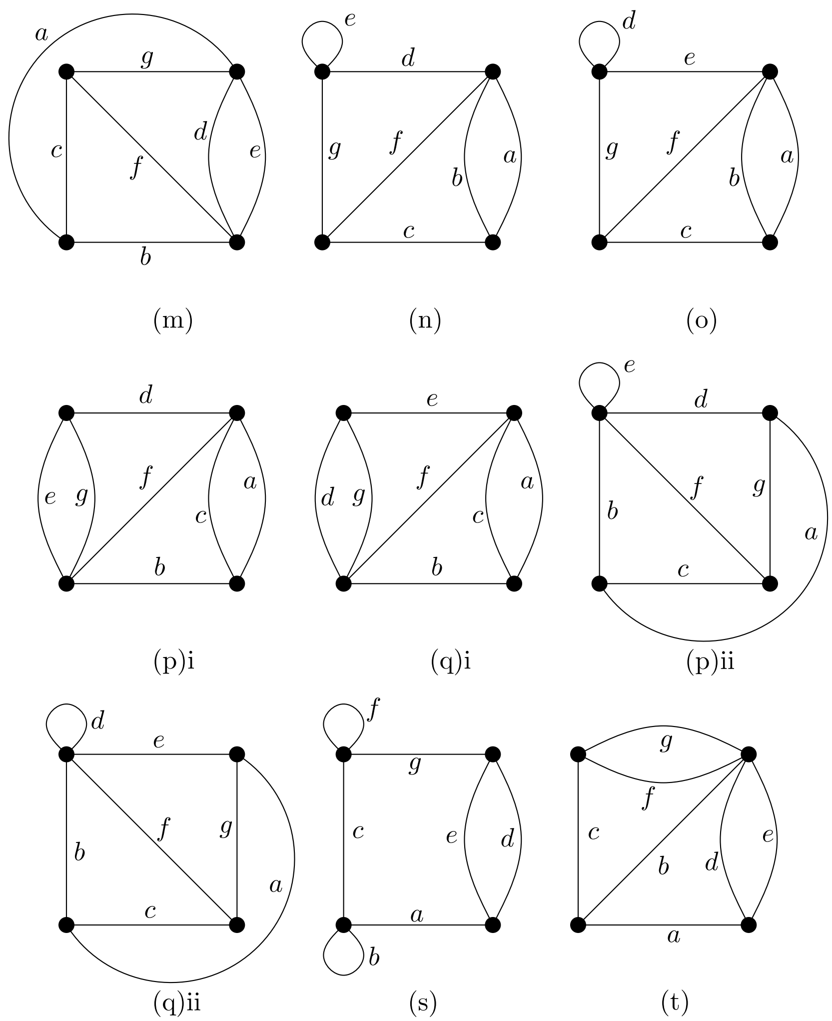


Figure 5.54: More representations of $F(G_{12})$.

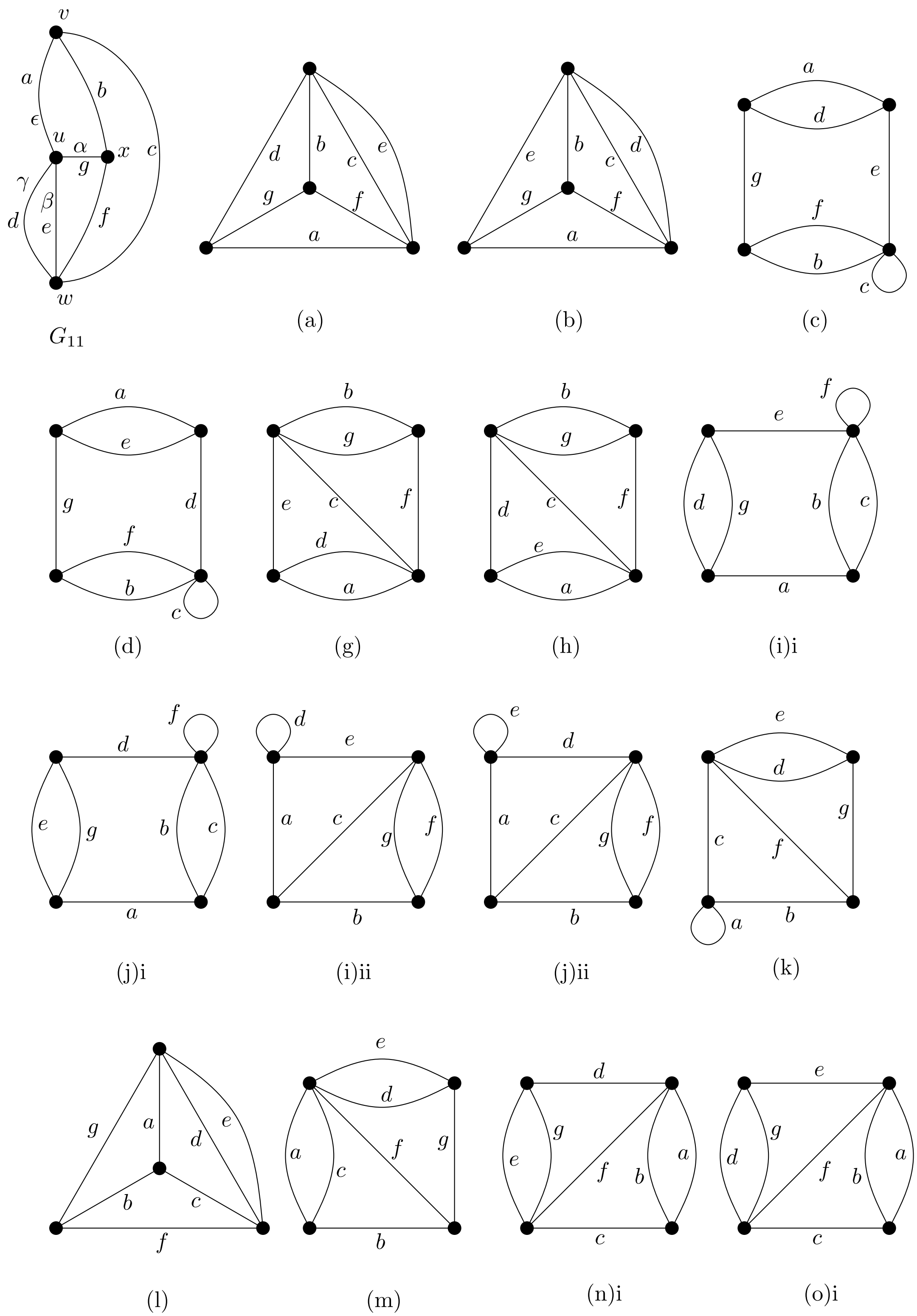


Figure 5.55: $F(G_{11})$. Cycles with edges sets $\{b, c, f\}$, $\{d, e, f, g\}$, or $\{a, c, d, e\}$ are balanced; all other cycles are unbalanced.

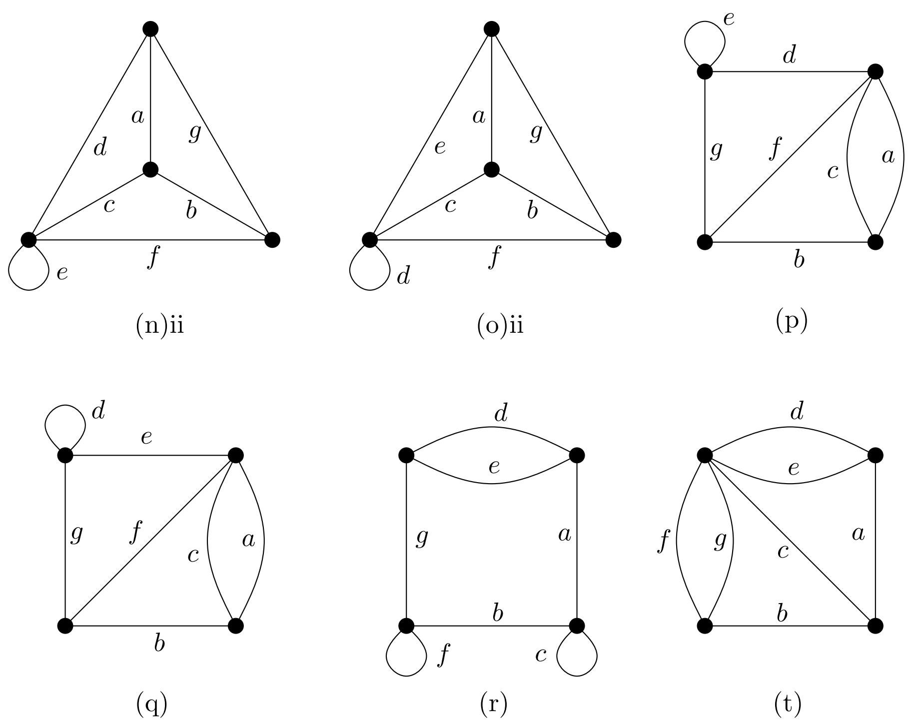


Figure 5.56: $F(G_{11})$. Cycles with edges sets $\{b, c, f\}$, $\{d, e, f, g\}$, or $\{a, c, d, e\}$ are balanced; all other cycles are unbalanced.

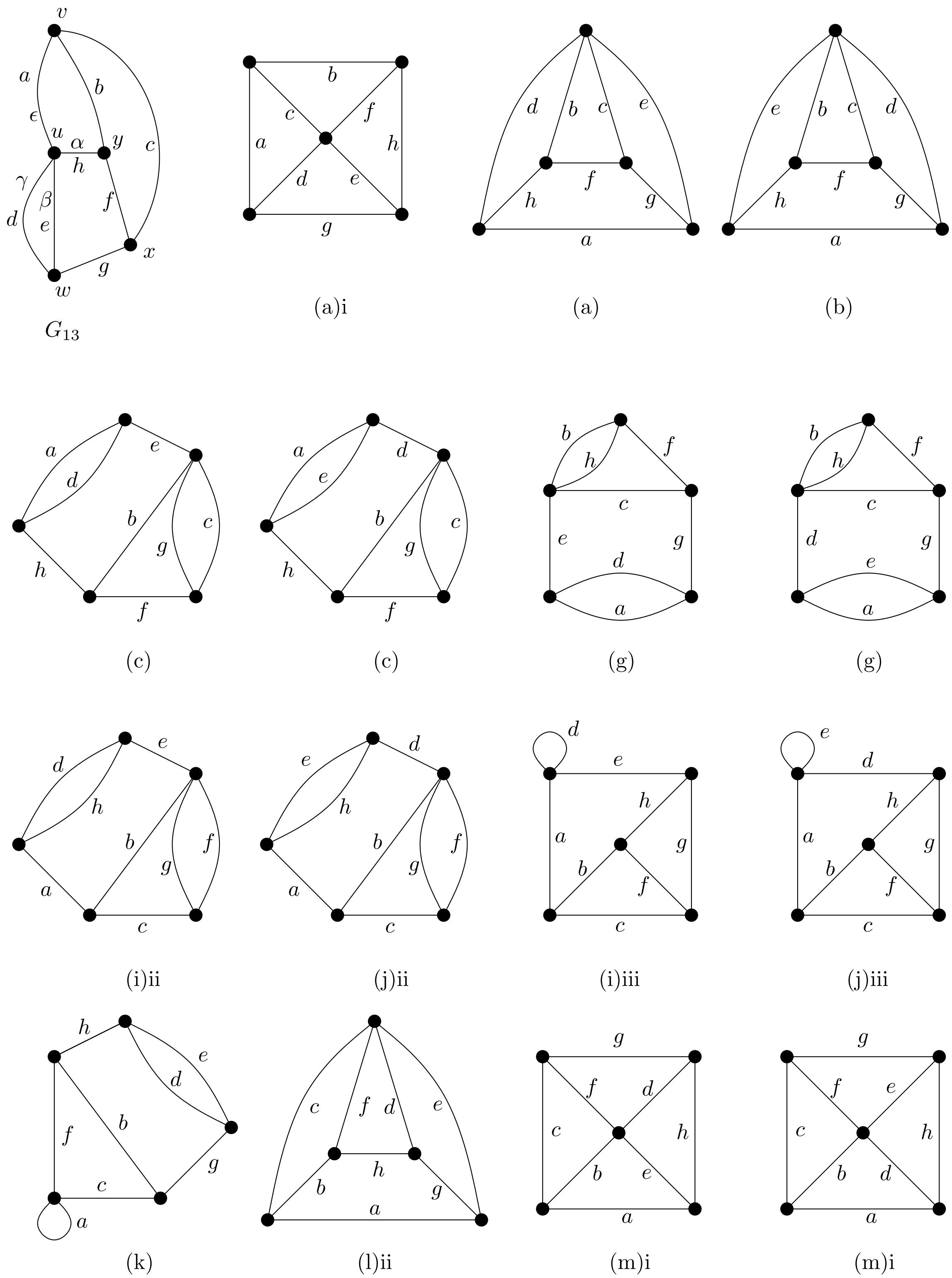


Figure 5.57: $F(G_{13})$. Cycles with edges sets $\{b, c, f\}$, $\{d, e, f, g, h\}$, $\{a, b, d, e, h\}$, or $\{a, c, d, e, g\}$ are balanced; all other cycles are unbalanced.

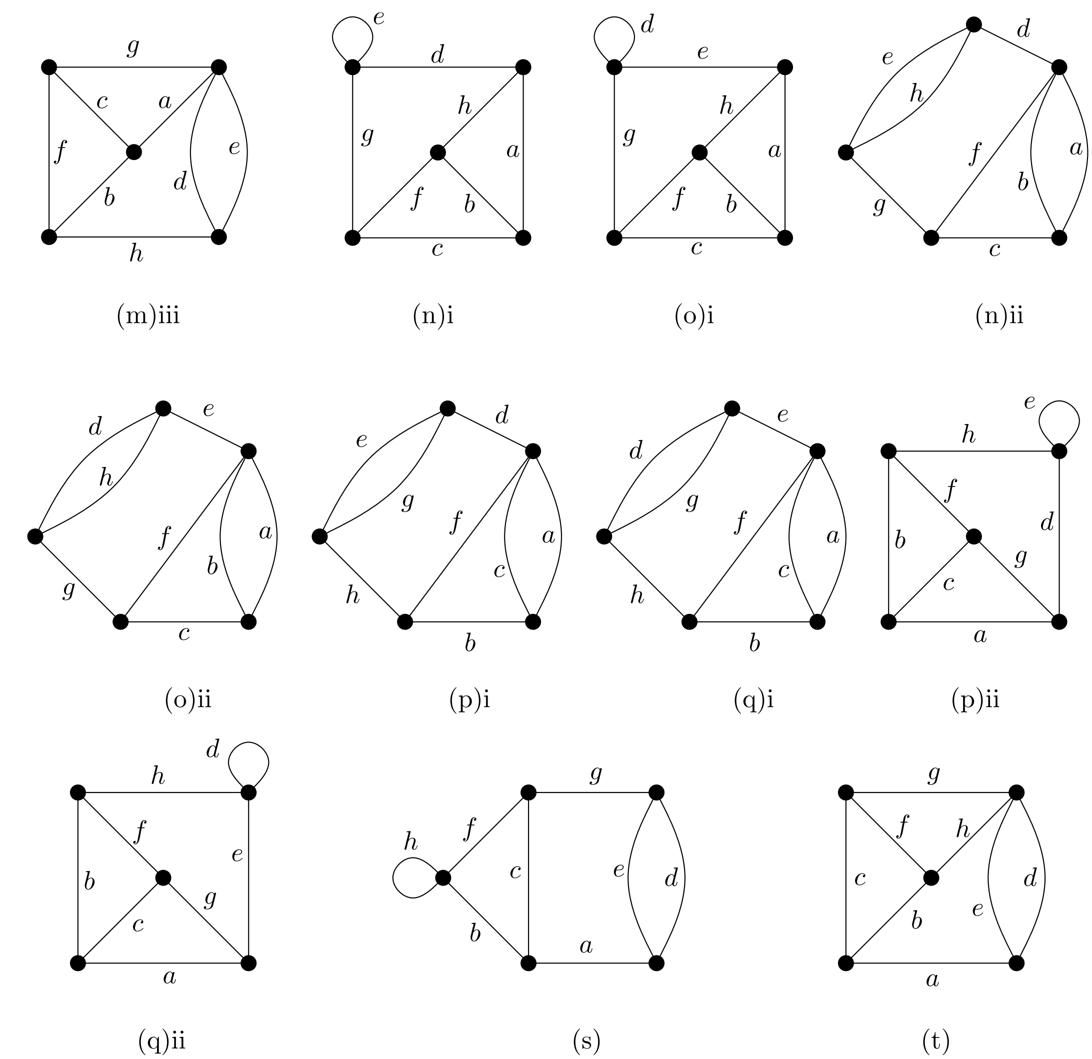


Figure 5.58: $F(G_{13})$. Cycles with edges sets $\{b, c, f\}$, $\{d, e, f, g, h\}$, $\{a, b, d, e, h\}$, or $\{a, c, d, e, g\}$ are balanced; all other cycles are unbalanced.

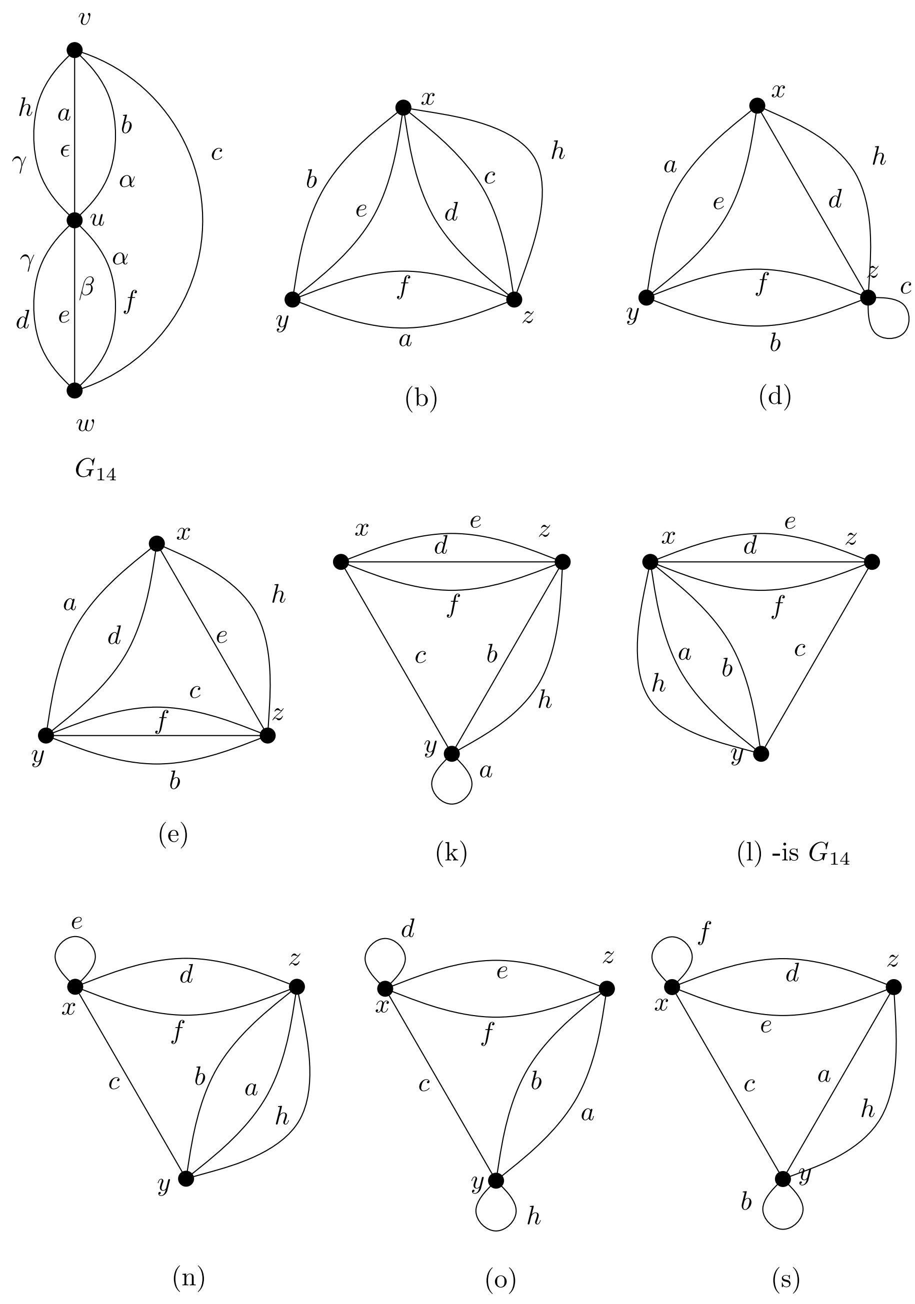


Figure 5.59: $F(G_{14})$. Cycles bcf, def, abg, and cdg are balanced; all other cycles are unbalanced.

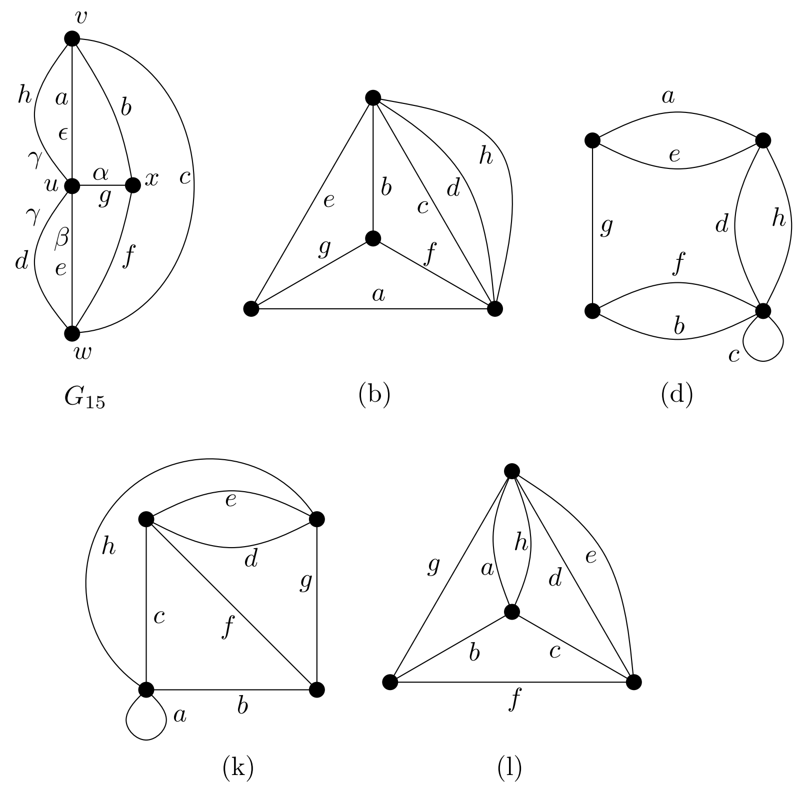


Figure 5.60: $F(G_{15})$. Cycles with edge sets $\{b, c, f\}, \{c, d, h\}, \{a, b, g, h\}$, and $\{b, d, f, h\}$ are balanced; all other cycles are unbalanced.

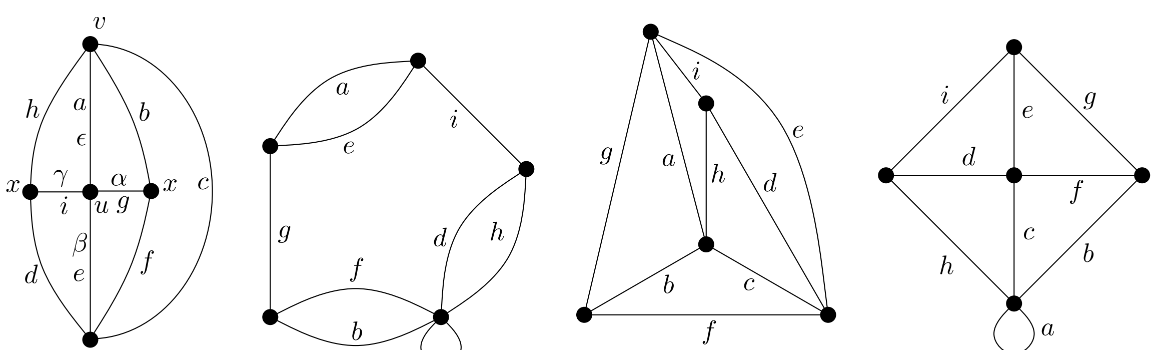


Figure 5.61: $F(G_{16})$. Cycles with edges sets $\{b, c, f\}, \{c, d, h\}, \{a, b, g, h, i\}$, and $\{d, e, f, g, i\}$ are balanced; all other cycles are unbalanced.

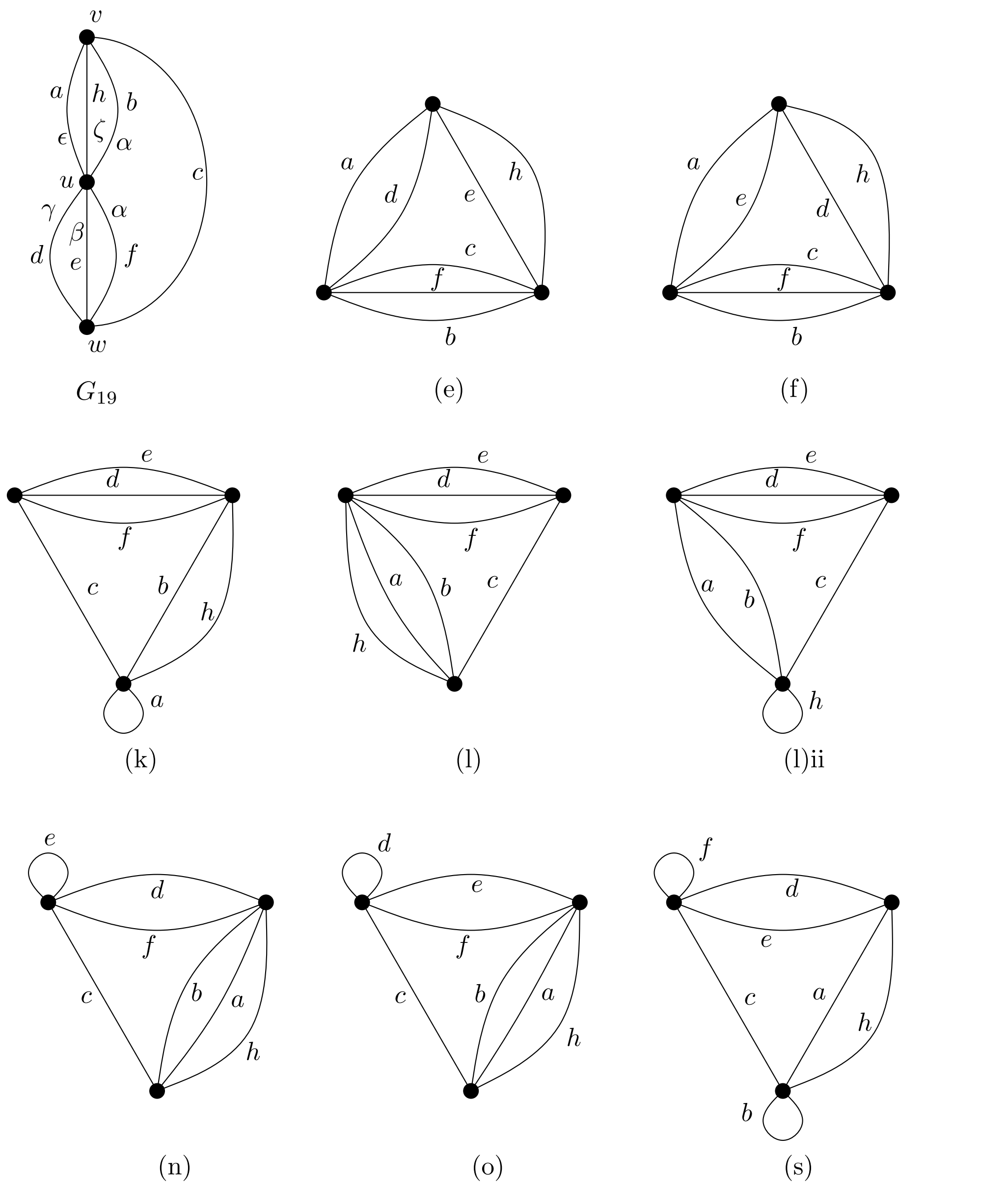


Figure 5.62: $F(G_{19})$. Cycles with edge sets $\{bcf\}$, $\{d, e, f\}$, $\{a, b, h\}$ are balanced; all other cycles are unbalanced.

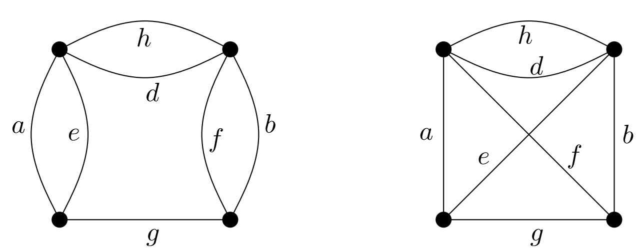


Figure 5.63: Biased graphs with frame matroid isomorphic to $F(G_{15} \setminus c)$; each has set of balanced cycles $\mathcal{B} = \{abgh, defg\}$.

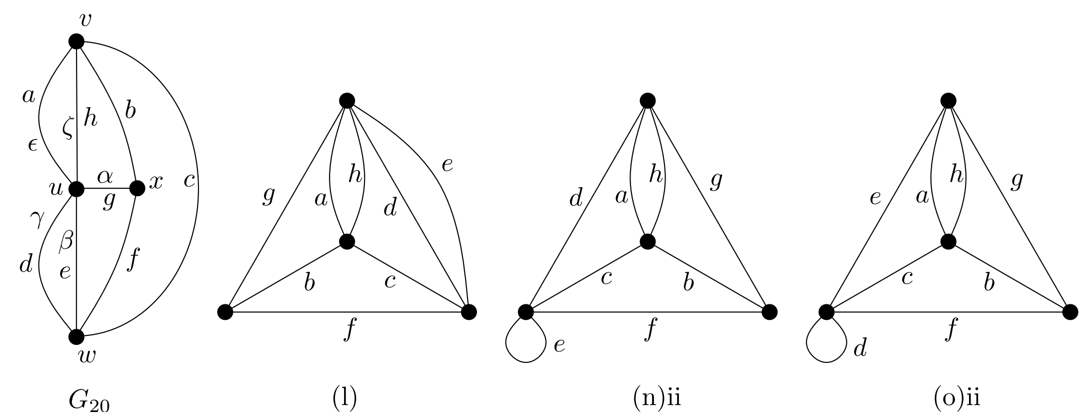


Figure 5.64: $F(G_{20})$. Cycles with edge set $\{b, c, f\}$ is balanced, all other cycles are unbalanced.

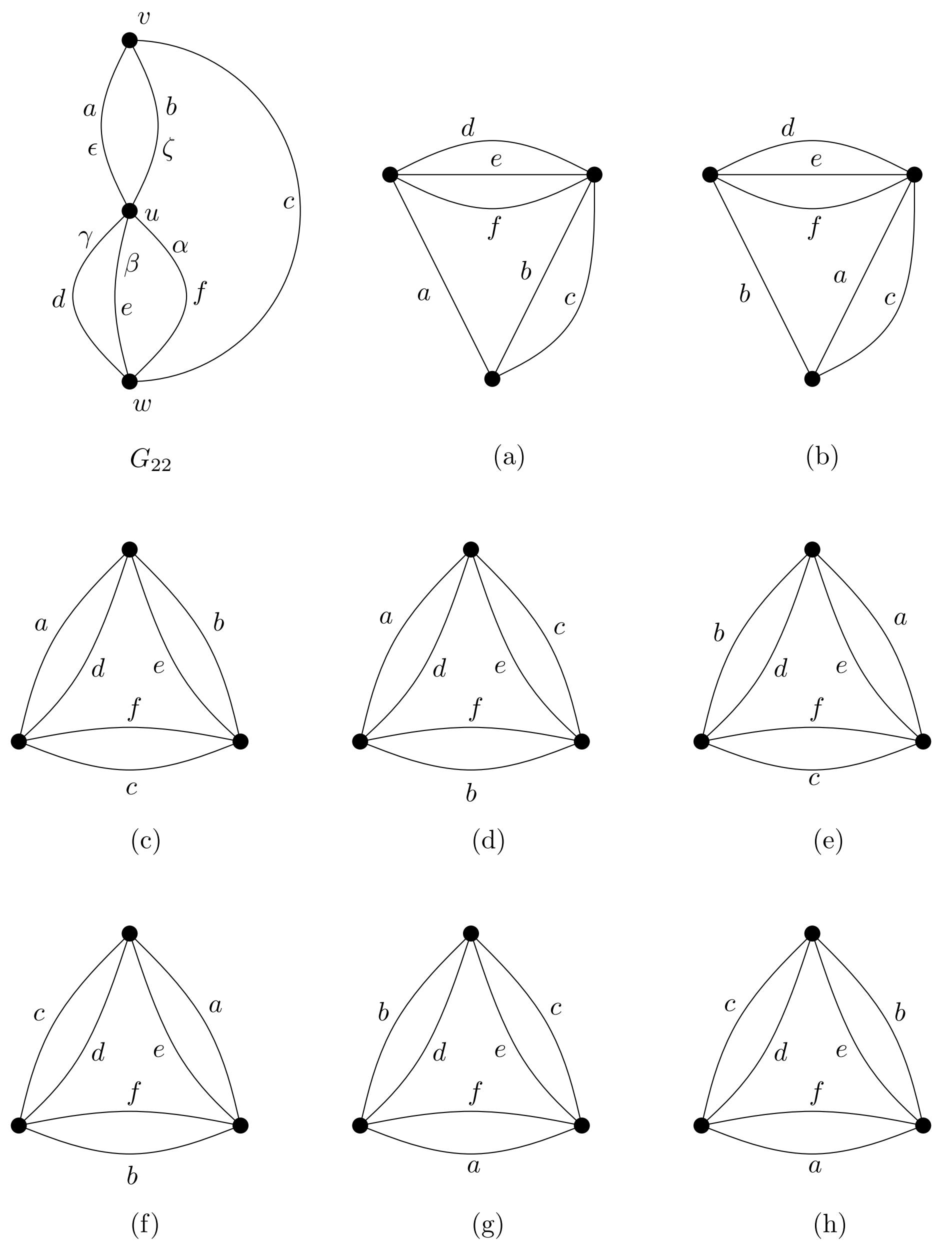


Figure 5.65: $F(G_{22})$ is represented just by these biased graphs and any biased graph obtained as a roll-up of one with a balancing vertex. Cycles def are balanced, all other cycles are unbalanced.

Chapter 6

Outlook

We began this project as a study of excluded minors for the class of frame matroids. For this, we need to understand the possible biased graph representations of a given frame matroid. The notion of a committed vertex (Sections 2.5.1 and 5.2.2) led to a need to understand the structure of biased graphs whose frame matroids are graphic. If all vertices in a biased graph Ω representing a frame matroid M are committed, then Ω uniquely represents M (Observation 2.17). Otherwise, Ω has an uncommitted vertex; in fact, under a mild connectivity assumption its deletion leaves a biased graph whose frame matroid is graphic. With Theorem 3.1, we determine the structure of all such biased graphs.

The most obvious example of a biased graph whose frame matroid is graphic is that of a balanced biased graph. So perhaps the most obvious reason a frame matroid fails to have a unique biased graph representation is that it has a representation with a vertex that is uncommitted for the reason that its deletion leaves a balanced biased graph. A reasonable start to an investigation of representations of frame matroids by biased graphs therefore, seemed to be to determine all biased graphs representing a frame matroid having a representation Ω with a balancing vertex. This we came close to achieving with Theorem 5.1. While ideally we would like to know all biased graphs representing $F(\Omega)$ where Ω is any biased graph with a balancing vertex, when Ω has low connectivity the problem seems especially difficult. However, we would like to drop the assumption in Theorem 5.1 that Ω be 3-connected. This would give us a result on representations which we could apply when considering general 3-connected frame matroids. Equipped with the tools developed to prove Theorem 5.1, this should now be quite straightforward.

Problem 1. Given a 3-connected non-graphic frame matroid $M{=}F(\Omega)$ represented by a biased graph Ω with a balancing vertex, determine all biased graph representations of M.

A solution to Problem 1 would give us an understanding of representations of a 3-connected frame matroid $M{=}F(\Omega)$ in the case Ω has a vertex that is uncommitted for the reason that its deletion leaves a balanced biased graph. Theorem 3.1 lists five other reasons that a vertex of a biased graph Ω representing a frame matroid $M{=}F(\Omega)$ may fail to be committed. What are the possible representations for M in these cases? We have made partial progress toward answering this question. We have hope that the techniques developed to prove Theorem 5.1 will be useful here. In particular, a large and decently connected biased graph of each of the forms (2)-(6) given by Theorem 3.1 has large balanced biased subgraphs. Perhaps a technique similar to that used in Chapter 5 of identifying such subgraphs H and applying H-reductions and H-enlargements may be useful.

Problem 2. Given a 3-connected non-graphic frame matroid M, determine all biased graphs Ω with $M \cong F(\Omega)$.

The motivation for understanding representations of frame matroids is to study excluded minors for the class of frame matroids. While understanding representations is difficult when connectivity is low, Theorem 4.1 already tells us much about excluded minors having low connectivity. Not only does Theorem 4.1 provide a list of 18 excluded minors of connectivity 2, it gives a strong structural description of the remaining excluded minors that are not 3-connected. Hence determining representations for a frame matroid M in the case M is 3-connected would provide significant progress toward our goal.

Theorem 4.1 may be enough to allow us to begin the study of excluded minors under the assumption of 3-connectedness. Nevertheless, it would be nice to complete the list $\mathcal{E}$ of excluded minors of connectivity 2 given by Theorem 4.1. We have determined some twenty excluded minors of the form specified by Theorem 4.1 for an excluded minor of connectivity 2 not in $\mathcal{E}$. Hence we have made significant progress toward the following problem.

Problem 3. Determine the complete list of excluded minors of connectivity 2 for the class of frame matroids.

It is likely that the analysis required to show that such a list is complete will be significantly longer and more technical than that of the proof of Theorem 4.1. We would at least like to show that the list of excluded minors of connectivity 2 is finite. This may be a more tractable problem. The excluded minors we have found thus far all have small rank. Each such excluded minor we find imposes more constraints on possibilities for other excluded minors. This, together with the structure imposed by Theorem 4.1 on excluded minors of this form, suggests that such an excluded minor should not have large rank.

Problem 4. Is there is a positive integer n such that an excluded minor of connectivity 2 for the class of frame matroids does not have rank $\geq n$?

Biased graphs share many characteristics with graphs. We therefore began this project with the feeling that, as is the case for graphic matroids, the class of frame matroids may be characterised by a finite list of excluded minors. Theorem 4.1 provides perhaps some evidence that this is the case, although the fact that we do not yet have answers to Problems 3 or 4 leaves the door open, even for excluded minors of connectivity 2.

When working with a biased graph $(G, \mathcal{B})$, it is often convenient to describe the collection $\mathcal{B}$ of its balanced cycles using a group-labelling. This led us to the question of Chapter 2.

The answer given by Theorem 2.1 has some troubling consequences for the view that biased graphs behave similarly to graphs. While graphs are well-quasi-ordered under the minor relation, we find that biased graphs and frame matroids are emphatically not. We have exhibited many proper minor-closed classes of biased graphs and of frame matroids having rich and wild infinite sets of excluded minors, and found many infinite antichains of biased graphs and frame matroids all of whose members are on a fixed number of vertices and of a fixed rank. Thus we are less sure now than when we started of what we should guess the answer to the following problem may be.

Problem 5. Is the class of frame matroids characterised by a finite list of excluded minors?

This thesis presents significant progress toward solutions to each of problems 1-4. Chapter 2 shows that the class of frame matroids is large and wild. On the other hand, the results of Chapters 3 and 5 show that representations of frame matroids are perhaps not so wild. For instance, while swirls are examples of linear matroids having many non-equivalent matrix representations, their biased graph representations are unique. Corollary 5.2 says that if M is non-graphic and represented by a 3-connected biased graph with a balancing vertex, then up to roll-ups the number of representations for M is less than 27; Corollary 5.3 says that a 4-connected non-graphic frame matroid with such a representation is, up to roll-ups, uniquely represented. It may be that similar results hold for frame matroids having a biased graph representation with a vertex that is uncommitted for the reason that its deletion leaves a member of one of the other five families of Theorem 3.1. It may be that despite the rich, wild nature of frame matroids, the class as a whole is characterised by a finite list of excluded minors. Complete solutions to Problems 2-4 would provide significant progress toward a solution to Problem 5.

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