

DEPENDENCE AND THE COPULA APPROACH

by

Xin Chen
Bachelor of Science (2005)

PROJECT SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

In the
Faculty
of
Business Administration
Financial Risk Management

© Xin Chen 2006

SIMON FRASER UNIVERSITY



Summer 2006

All rights reserved. This work may not be reproduced in whole or in part,
by photocopy or other means, without permission of the author.

APPROVAL

Name: Xin CHEN

Degree: Master of Arts

Title of Project: Dependence and the Copula Approach

Supervisory Committee:

Geoffrey Poitras
Senior Supervisor
Professor of Finance

Robbie Jones
Supervisor
Professor of Economics

Date Approved:

August 1, 2006



SIMON FRASER
UNIVERSITY library

DECLARATION OF PARTIAL COPYRIGHT LICENCE

The author, whose copyright is declared on the title page of this work, has granted to Simon Fraser University the right to lend this thesis, project or extended essay to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users.

The author has further granted permission to Simon Fraser University to keep or make a digital copy for use in its circulating collection, and, without changing the content, to translate the thesis/project or extended essays, if technically possible, to any medium or format for the purpose of preservation of the digital work.

The author has further agreed that permission for multiple copying of this work for scholarly purposes may be granted by either the author or the Dean of Graduate Studies.

It is understood that copying or publication of this work for financial gain shall not be allowed without the author's written permission.

Permission for public performance, or limited permission for private scholarly use, of any multimedia materials forming part of this work, may have been granted by the author. This information may be found on the separately catalogued multimedia material and in the signed Partial Copyright Licence.

The original Partial Copyright Licence attesting to these terms, and signed by this author, may be found in the original bound copy of this work, retained in the Simon Fraser University Archive.

Simon Fraser University Library
Burnaby, BC, Canada

ABSTRACT

This paper aims to introduce the essence of dependence in modern finance, especially in the field of insurance and credit modelling, and to study the copulas as a tool to model dependence. In particular, the paper assembles the similarities in the statistical properties in actuarial science and credit modelling, and demonstrates some common copula applications in both of the fields. Also an empirical study is conducted to specifically investigate the sensitivity of default risk associated with the first-to-default (FTD) credit swap with respect to the change of copula dependence structure together with other changing parameters. The results reveal important implications for investment activities, risk modelling, and future dependence modelling with copulas.

Keywords: dependence in modern finance; credit modelling; copulas; risk modelling

EXECUTIVE SUMMARY

Traditional portfolio analysis has an emphasis on diversification, and independence can be assumed. However, an increasing number of insurance products and innovative credit derivatives call for the study of correlation. When elliptical distribution and linearity are not guaranteed, the conventional correlation measure fails to provide a complete risk profile. Recently, more studies are directed to study the dependence modelling. Copulas provide a structure to model the multivariate outcomes, and are recognized as an increasingly important tool in dependence study.

Correlation will probably become a notable risk in addition to market risk, credit risk and other significant risks in risk management. Correlation derivatives have been newly emerged and will certainly become a new asset class – one of the major waves in derivatives. It is worthy is to study dependence modelling in order to better understand correlation, which is also important for various fields.

This paper provides intensive discussions on the important role of dependence in the modern credit modelling and insurance applications, various copulas and their initial and current applications in credit modelling and actuarial science, as well as some implications based on the empirical experiment conducted by the author.

In this paper, the focus is modelling credit basket derivatives with copulas. We provide insights from the actuarial science perspective for better understanding in copulas, and for seeking possible benefits that can be brought into the credit industry.

ACKNOWLEDGEMENTS

I want to thank for Dr. Geoffrey Poitras, Dr. Robbie Jones, and Dr. Andrey Pavlov's comments and encouragements for this paper. Also, I owe thanks to the visiting instructor Dr. Anton Theunissen's provision of credit spread data from JP Morgan.

TABLE OF CONTENTS

Approval	ii
Abstract	iii
Executive Summary	iv
Acknowledgements	v
Table of Contents	vi
List of Figures	viii
List of Tables	viii
1 Introduction	1
2 Dependence	4
2.1 Dependence in Insurance and Credit Modeling	4
2.2 Essence of Dependence Structure	5
2.3 Examples—Dependence Matters	7
2.4 Caution about Dependence.....	9
2.5 Techniques to model dependence.....	11
3 Copulas	12
3.1 Brief historical background - Copulas.....	12
3.2 Types of Copulas.....	13
3.2.1 Elliptical Copulas	13
3.2.2 Archimedean Copulas	14
3.2.3 Marshall-Olkin Copulas.....	15
3.2.4 Bounds	15
3.3 Correlation Measurement.....	16
3.4 Mixed copulas	16
4 Applications of Copulas	18
4.1 Some underlying mathematic functions and models.....	18
4.2 Applications of Copulas in Actuarial Science.....	23
4.2.1 Survival of Joint lives	23
4.2.2 Competing Risks	25
4.2.3 Common shock	26
4.3 Applications of Copulas in Credit modeling.....	27
4.3.1 CreditMetrics and KMV	28
4.3.2 First-to-Default Valuation.....	30
4.3.3 Remarks	31
4.4 Fitting Copulas to Data	32
4.5 Desired Properties of Copulas.....	34
4.6 Simulations.....	35

5	Numerical Testing and Implications	36
5.1	Definition and the Calibrated Model.....	36
5.2	Descriptions of Portfolios in Analysis.....	39
5.3	Experiments, Results and Discussions	41
6	Conclusion	52
	Appendices	53
	Appendix A –Simulations for Some Common Copulas	53
	Appendix B – Programming Codes	56
	Reference List	61

LIST OF FIGURES

Figure 2-1	Distribution of Default with different ρ	7
Figure 2-2	The Realization of (X, Y) in the Two Models	10
Figure 5-1	Baskets with <i>varied</i> size when $\rho=0.7$	41
Figure 5-2	Compare <i>green(10)</i> to)portfolio <i>B</i> and with various ρ s.....	42
Figure 5-3	Compare <i>green(40)</i> to) portfolio <i>B</i> and with various ρ s.....	45
Figure 5-4	Mean of the Joint distribution to the left or to the right of the expiry	46
Figure 5-5	Basket <i>B</i> , red(10) and green(10) with <i>various</i> ρ	47
Figure 5-6	Red baskets with <i>various</i> # of references and ρ s.....	48

LIST OF TABLES

Table 2.1	Example 2 Comparison of Cases	9
Table 3.1	Archimedean Copulas	15

1 INTRODUCTION

Credit derivative market has experienced a rapid growth in recent years; more complicated credit derivative products are emerging and are used to manage the most important dimension of financial risk – credit risk. While the most important instrument in the current credit derivatives market is still the credit default swaps, which essentially provides insurance against a single reference credit (Lando 2004), many new products are now associated with a basket of credit risk. For example, First-to-default swaps, or more generally, Nth-to-default swaps, extends a single name swaps to a portfolio level. Correlation must come into play in pricing these portfolio-base credit products, and techniques to model correlation in a credit risk framework have quickly become an issue. J.P. Morgan's CreditMetrics and Credit Suisse Financial Product's CreditRisk⁺ are the two especially influential benchmarks among various new approaches to model basket credit risk (Gordy 1998).

While CreditMetrics provides a “mark-to-market” methodology, CreditRisk⁺, on the other hand, is rather mathematical oriented: it focuses only on the default risk, which is analogical to “death” in an insurance setting, and applies mathematical tools borrowed from insurance theory (Bonilla *et al.* 2000). In fact, insurance theory, or actuarial techniques in particular, has provided valuable insights for credit modelling and the financial risk management as a whole along the history. Tapiero (2004 p.7-8) indicated that actuarial science is “in effect one of the first applications of probability theory and statistics to risk analysis”: Tetens and Barrios, already in 1786 and 1834 respectively, were attempting to characterize the ‘risk’ of life annuities and fire insurance and on that basis establish a foundation for present-day insurance. Indeed, mathematical theory of insurance has laid foundation not solely to an insurance setting, but also

has brought significant benefits to different branches in financial management development. Immunization, for example, which was first introduced by Redington, an actuary in 1950s to balance insurance surplus and liability, has become a widely used technique to control interest risk (Poitras 2005). In credit modelling, reduced-form/intensity models rely on Poisson default time process as the basic structure and they share many features with the common models in the actuarial literature. When it comes to model dependence, copulas, which are first introduced by an actuary named Sklar(1959), serves an increasingly important role to present association between underlying risk variables.

The similarity in statistical properties in life insurance and credit risk modelling, such as the typical skewness in default distributions and the same type of survival function first inspired me the idea of this paper. For dependence modelling in particular, actuarial science might have potential value to add to current credit modelling, while the rapid growth in credit risk management might inject new insights into the actuarial field. This paper will focus on the important role of dependence structure in modern finance, and introduce copulas as a useful measure for dependence. Also, we illustrate some applications of copulas to handle multivariate outcomes in the framework of actuarial science and credit risk modelling.

The paper proceeds as follows: in Section 2, we explain why we should be concerned about dependence from insurance and credit modelling perspective, and why we need to develop advanced dependence models. To make the effect of dependence more intuitive, we construct two simplified examples to demonstrate how correlation can affect the whole picture of credit default distribution and thus the credit product value. We also indicate the most common pitfall in understanding correlation and dependence in modern finance. In Section 3, we focus on copulas as an approach to construct dependence: we define the most common copulas and some related properties. In Section 4, we give some applications of copulas in insurance and credit risk

management with an intention to emphasize the similarities between the two fields. Then, we use an empirical example to show how to fit copulas to real data. In addition, we point out some advantages and prospects of using copulas to model dependence. In Section 5, we conduct an empirical study to reveal some important implications by applying conventional copulas in credit modelling. We make the conclusion in Section Six. Appendix A provides some simulation algorithms to construct copulas we use in our paper; programming codes used to produce to result for the empirical study in Section Five are enclosed in Appendix B.

2 DEPENDENCE

2.1 Dependence in Insurance and Credit Modelling

Dependence in insurance: joint-life or multiple-life insurance or annuity policies are traditional products that draw attentions from actuaries to model dependence. For example, a life insurance written on a couple with children is priced as a function of default dependence rather than the individual survival functions. Empirical studies show that the “broken heart” syndrome or generic factors has significant effect. For instance, Frees et al.(1995) found that accounting for dependency in mortality produced about a 3% to 5% reduction in the joint-and-last-survivor annuity values when compared to standard models that assume independence¹. Also under competing risk model (e.g. a single life contract can be subjected to multiple sources of default such as heart decease, cancer, accident and so on), independence between competing risks is a conventional assumption but its reliability has been questioned by many practitioners and academicians such as Carriere (1994) and Seal (1977); multiple decrement theory has a need to adjust for dependence in competing risk. Moreover, common shock, or event risk, such as hurricane and earthquake, causes further concern in default dependence and is recently an intensive study subject. Innovative insurance products, such as the ones to mitigate operational risk, require understanding in multi-dimensional outcomes from financial systems.

Dependence in credit modelling: dependence modelling in credit risk framework has a bit shorter history than that in actuarial science. It hasn't been intensively studied until the recent years when *credit correlation products*, including synthetic Collateralized debt obligation (CDO), Nth-to-default (NTD) and credit link note (CLN), started to emerge and increase rapidly. Similar to the

¹ Free et al(1995) used Frank's copula to model dependence in mortality

one in actuarial science, dependence structure in credit modelling concerns about the probability of the credit references defaulting together, which is crucial for both investors, who need to be compensated by additional risk, and dealers, who need to properly manage overall portfolio risk.

2.2 Essence of Dependence Structure

There are a number of reasons for us to worry about dependence structure. First, we have witnessed rapid growth in derivatives in terms of both magnitude and complexity. New complex products in insurance and finance result in portfolios with complex dependence structure. CDOs or synthetic CDOs are examples of financial products that require fully understanding in dependence between different tranches for both issuer and investors. Neglecting or misunderstanding the dependence structure can result in underestimation of portfolio risk. For example, JP Morgan used to under-priced its basket credit swaps as a result of improper understanding of risk associations. To make the effect of dependence more intuitive, we create two simplified example in Section 2.3 to illustrate how inter relation between underlying risks can have significant impact on default distribution and returns required different risk class investors.

In addition, the risk-based return required by investors cannot be simply determined by the traditional risk measurement such as correlation matrix. Correlation measure has been serving the central role in portfolio theories for decades, yet, it is not a satisfactory dependence measure to capture the observed advanced dependence structures. In Section 2.4 we include an example suggested by Embrecht *et al* (2002) to illustrate this point in detail. Also in Section 2.4, we clarify the common conceptual pitfall between correlation and dependence.

Furthermore, the multivariate normal return distribution has been used as a conventional dependence structure, due to its analytical tractability and small number of parameters required

(Wang 1997). However, the development in the derivatives market certainly has an increasing need for multivariate models with more flexibility than the multivariate normal distribution. Normality assumption is typically not adequate in modelling life and credit defaults. In insurance, deaths or claims are rare events with long-tailed distributions, and so are credit defaults and other extreme events. Normal distribution does not provide an adequate approximation to these data sets. Similar claims can be found from work by Johnson and Kotz (1973) and Johnson, Kotz and Balakrishnan (1997).

Finally yet importantly, strong future trend in modelling dependent risk is reflected in regulation documents. For example, the recommendation of incorporating insurance to mitigate operational risk in the New Basel Capital Accord signals the potential need to better study dependent risk. Traditionally, diversification is used to guarantee independency. This assumption, however, seems to be further violated as the complexity of insurance and reinsurance products increases. Suppose insurance contracts are written on operational risks for different companies, the tendency that these companies subject to the common exogenous “shocks” thus encounter operational failure simultaneously is significant. Also, a specific contract written on several business lines inside a corporation can induce challenge for issuer because of the great likelihood of failure dependence. Indeed, the inability to clearly define underlying risk variables and quantify their dependence is one of the biggest difficulties for current issuers to write insurance policies on operational risk. For credit risk in the Basel framework, securitization is recognized as a risk management technique to enhance risk diversification but is not yet fully contemplated by the new Accord (BIS, 2003). Securitization is at a relatively early development stage and the nature of it determines the importance to develop advance modelling of dependence structure.

Therefore, more advanced dependence modelling is required to support the development in portfolio products and integrated risk management techniques.

2.3 Examples—Dependence Matters

We construct two simplified examples to show how dependence between the underlying references changes the default distribution, risk allocation and required returns.

Example 1: Suppose $X_1, X_2 \dots X_5$ are five identical Bernoulli variables and it takes value 1 if risky security i ($i=1 \dots 5$) defaults in one year and 0 otherwise. $\text{Prob}(X_i=1) = 0.2$. ρ is the pair-wise correlation and we let ρ take value²: -0.25, 0, 0.5 and 1.

Figure 2-1 Distribution of Default with different ρ

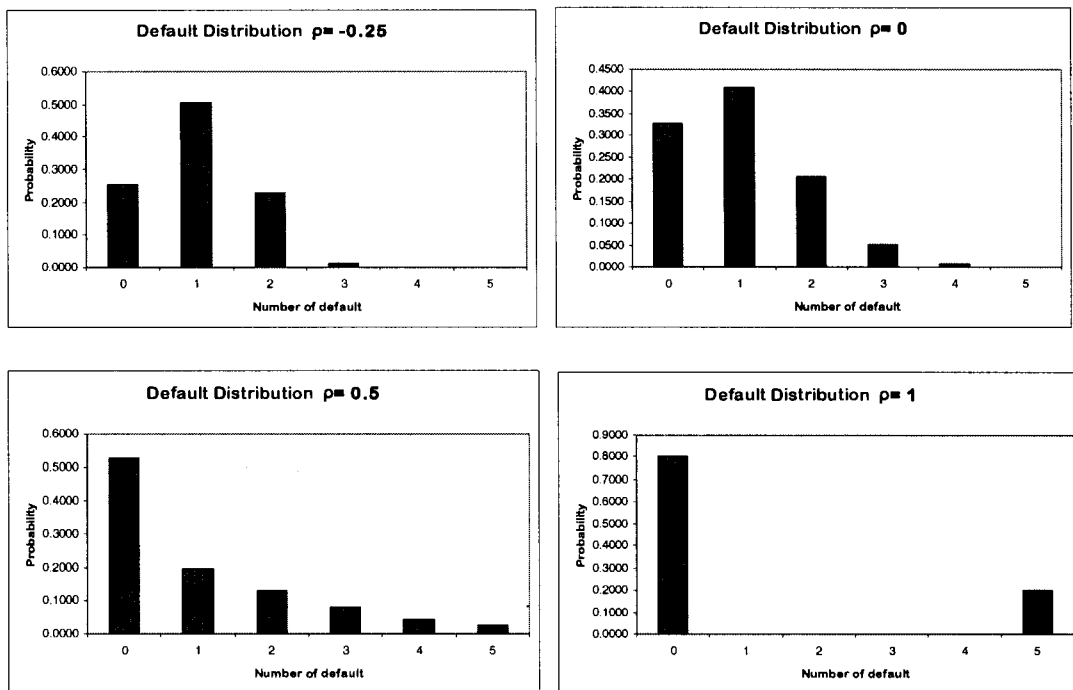


Figure 1 graphically shows the default distribution for the corresponding correlation ρ . One can see that as ρ goes up, the probability that the firms default all at the same time goes up. The use of a discrete distribution here amplifies the effect of the extreme correlation. Notice that when $\rho=1$,

² Note that ρ takes value between a lower bound and an upper bound. The upper bound is 1 and the lower bound is equal to $-p/(1-p)$ which is -0.25 in our example.

the five references either all default together or do not default at all³. Why would this matter?

We provide the second example to mathematically illustrate how this can significantly change the risk allocation and required returns among investors.

Example 2: Suppose an agency combines three of the risky securities in example 1 (each has face value 100) and issues a CDO –Senior, Mezzanine and subordinated one-year bond with face value 140, 90 and 70, respectively to investors with different risk appetite. Assume the continuous interest rate is 0.06 and recovery rate is 0.4, we can calculate the price and yield for the two extreme cases where $\rho=0$ and $\rho=1$ respectively⁴.

Return required by different classes of investors is the “YIELD” shown in Table 1. There are two important implications: 1. When defaults correlation increases from 0 to 1, the yields on the three bonds increase dramatically because the risk profile of each tranche is no longer simply depending on individual defaults – correlation adds risk in each tranche and investors demand higher returns. 2. When correlation increases dramatically, the most risky tranche or Subordinated bonds become relatively less risky and more risks are transferred to higher tranches. Notice that in Case II, the Mezzanine and Subordinated bond holders require almost the same yield. Loosely speaking, if defaults tend to happen at the same time, all the bond holders suffer and the more senior one are not much better-off. Kakodkar *et al*(2003) in their paper formally define investors in Subordinated bonds are *long* correlation and investors in Senior bonds are *short* correlation.

³ In a continuous case, as we can see in our empirical study in Section Five, the result will not be as extreme as this

⁴ For the payoff of Bond I at year 1, we use the formula $Min\left[Max\left(A_T - \sum_{J=1}^{i-1} \tilde{B}_j, 0\right), \tilde{B}_i\right]$

provided by Robert(2006 pp.845) where A_T is the maturity value of the asset pool, \tilde{B}_i is the promised payment of Tranche i. The formula implies that bond i takes the leftover of the claims after the more senior bonds $J=1 \dots i-1$

Table 2.1 Example 2 Comparison of Cases

		Paid-off at time		
# of Default	PROB	1		
		Senior	Mezzazine	Subordinated
0	0.32768	140	90	70
1	0.4096	140	90	10
2	0.2048	140	40	0
3	0.0512	120	0	0
	Risk-neutral Price ⁵	129.99666	70.20590904	25.4592857
	YIELD ⁶	0.0741337	0.248377189	1.011414705

		Paid-off at time		
# of default	PROB	1		
		Senior	Mezzazine	Subordinated
0	0.8	140	90	70
1	0	140	90	10
2	0	140	40	0
3	0.2	120	0	0
	Risk-neutral Price	128.08	67.80705	52.7388
	YIELD	0.088988	0.283144	0.283144

2.4 Caution about Dependence

A caution is worthy being addressed: in the last two examples, we use (linear) correlation as a measure to describe the association between risks. That is, dependence is simplified by using a single statistic summary in our over-simplified examples. In fact, dependence is more than a correlation matrix.

We sometimes see that in some literature, people use the word “correlation” to cover *any* notion of dependence. This is a common mistake in modern finance and insurance. Correlation describes the degree of linearity only, and is an incomplete measure for dependence. For example, a zero correlation doesn’t imply independency – something we are all familiar in a statistical sense. But when it comes to practice in finance, correlation becomes the most

⁵ Risk-neutral Price = $E[\text{payoff} \cdot \text{discount rate}] = \sum_i \text{payoff}(i) \cdot \text{prob}(\text{payoff}(i)) \cdot \text{discount rate}(\text{time } 1)$

⁶ Solved from equation: Risk-neutral Price = face value * $e^{-(\text{FIELD} \cdot 1)}$

misunderstood concept (Embrecht *et al* 2002). One explanation we would suggest is that traditional financial theories such as CAPM and APT have greatly influenced our mind-set and we easily suppose ourselves in a financial world where returns are multivariate normally distributed and linearly related. However, without these assumptions, we can no longer rely on correlation as a measure for dependence.

Embrecht *et al* (1999) demonstrates this pitfall by using two sets of bivariate random variables (X,Y) from two distributions that have *identical* $\text{Gamma}(3,1)$ marginal distributions and *identical* correlation $\rho = 0.7$ but *different* dependence structures⁷.

Figure 2-2 The Realization of (X, Y) in the Two Models

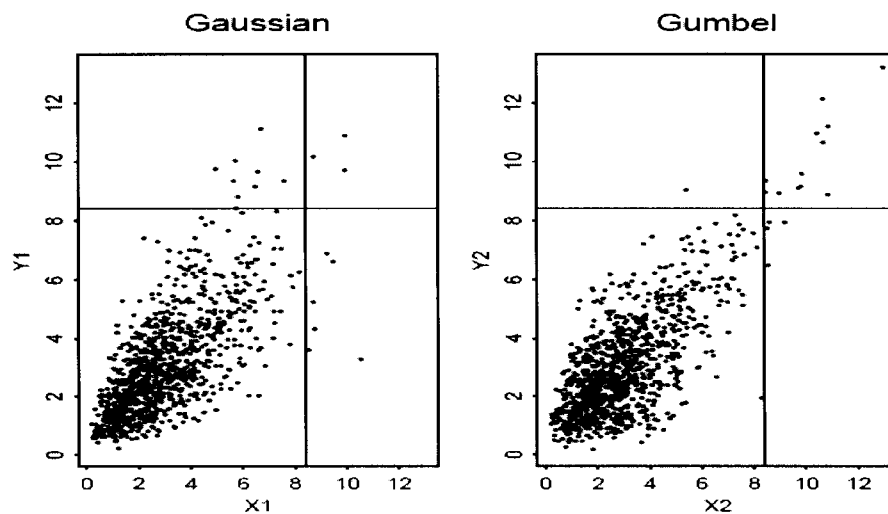


Figure 2⁸ provides a quantitative way of showing the different dependence between X and Y in two models, which implies correlation alone cannot tell such distinction in dependent structure.

To emphasize the important implication of this for practice, Embrecht *et al* (1999) suggest readers consider X and Y are insurance losses for two lines of business, and the second model

⁷ Embrecht *et al.* (1999) use the Gaussian Copula for the first set of (X,Y) and Gumbel Copula for the second set, and these two types of copulas have quite different tail dependence

⁸ The results can be produced by using simulation with Archimedean Copulas (see Embrecht *et al.* (1999) for more details)

reveals more dangerous signal to the insurer as the extreme events have greater tendency to happen together.

The conceptual distinction between correlation and dependence is extremely important for future dependence study. The misuse of correlation to describe dependence is problematic in the case where non-linear derivative products or products associated with non-elliptical distributions are involved.

2.5 Techniques to model dependence

Linear correlation (or Pearson's correlation) is the most popular use in practice as a measure for dependence (Embrecht et al 2001). However, as we just indicated, the invariance property of Pearson's correlation is useful only under strictly increasing *linear* transformation. Knowing that Pearson's correlation fails to summarize dependence for non-elliptical distributions and it is unrealistic to assume elliptical distributions for many financial variables, and that even when jointly elliptical distribution can be safely assumed problems such as heavy tailed data will make the use of linear correlation lack of sense, academics and practioners actively seek for advanced techniques to model dependence. In 1986, the article "The Joy of Copulas" (Genest and MacKay 1986) raised the awareness of the appealing practical usage of copulas and since then, copulas were no longer limited in probability study and have quickly come a main tool by academics and practioners in various industries to model dependence.

3 COPULAS

3.1 Brief historical background - Copulas

A copula is a function representing relationships among multivariate outcomes. To study multivariate outcomes is a basic problem in statistic science and copulas are not the earliest technique. In the late 19th century, Francis Galton initiated the *regression analysis*, which was considered the fundamental contribution to understanding multivariate relationships (Frees *et al.* 1997). Regression analysis opened the study of multivariate analysis and was widely applicable to understanding the effect of explanatory variables. However, a limitation it has is that all the independent variables are regressed on *one* dimension of the outcome (dependent variable), but this kind of relationship is sometimes not the primary interest (such as when two lives are subjected to a failure in actuarial science). Academics tried to understand the distribution of several outcomes – a multivariate distribution. Hoeffding (1940 & 1941) first extended the univariate distribution to the full multivariate one. The name Copula first appeared in 1959 was in fact inherent such an idea to link individual marginal function to a joint distribution.

Abe Sklar (1959) shows that any multivariate distribution F_i can be written as

$$C[F_1(x_1), F_2(x_2), \dots, F_p(x_p)] = F(x_1, x_2, \dots, x_p)$$

The function C is called *copula*. $F_1(x_1), F_2(x_2), \dots, F_p(x_p)$ are marginal functions evaluated at x_1, x_2, \dots, x_p . Note that any arbitrary marginal distribution function is a *uniform* random variable U (i.e. $F_1(x_1) = u_1, \dots, F_p(x_p) = u_p$) so they always fit in a copula representation.

Sklar also shows if the marginal distributions are continuous, then there is a *unique* copula representation

Copula has been studied in the probability literature for about 40 years (Schweizer 1991), so even it as a method to understand multivariate distributions has a relatively short history in the statistic literature (most of the applications have arisen in the last decade (Frees *et al.* 1997) , its well-studied prosperities in the probability literature quickly become recognized as desirable features to model dependence in different branches in risk management. The range of copula applications include management science (e.g. Clemen(1999) suggests copulas to portray uncertain in decision and risk analysis as an alternative to the conventional marginal-and-conditional approach⁹), credit modelling(e.g. Li(1997,1998) et al first introduced copula in correlation trading), failure of paired organs in health science, and human mortality in actuarial science.

3.2 Types of Copulas

For simplicity, we show different types of copula in two-dimensional cases without losing generality. We can always extend a bivariate copula to a multi-dimensional one.

3.2.1 Elliptical Copulas

Gaussian copula is the most common and straightforward type. It has the form:

$$C(u, v; \rho) = \phi_{\rho}(\phi^{-1}(u), \phi^{-1}(v))$$

Where ϕ denotes the standard normal distribution and $\phi_{\rho}(x, y)$ denotes the bivariate normal distribution with correlation ρ between marginal functions.

Similarly, *Student t-copulas* have the form:

$$C(u, v; \rho, \gamma) = t_{\rho, \gamma}(t_{\gamma}^{-1}(u), t_{\gamma}^{-1}(v))$$

⁹ Marginal-and-conditional approach is a short-cut to mean specifying a joint distribution as a products of random variables and conditional distributions

Where t denotes the t-distribution with γ degree of freedom and $t_{\rho,\gamma}(x, y)$ is the bivariate t-distribution with correlation ρ and γ degree of freedom.

The key difference between the Gaussian copula and the Student t-copula is that Gaussian copula has neither upper nor lower tail dependence, while the Student t-copula possesses tail dependent in both tails. More detailed of this aspect can be seen in Section 5 in this paper.

3.2.2 Archimedean Copulas

The class of elliptical copulas provides a rich source of multivariate distributions. However, elliptical copulas do not provide closed form expressions and their symmetry nature is not appropriate for many insurance and finance applications. A useful class of copula functions called *Archimedean copulas*, which has the general form:

$$C_{\phi}(u, v; \alpha) = \phi^{-1}(\phi(u) + \phi(v)) \text{ for } u, v \in (0,1]$$

Where ϕ denotes a *generator* of the copula C_{ϕ} , which is in fact a distribution function (Genest and McKay, 1986); α is the dependence parameter.

There are different choices for the generator ϕ so as to yield different families of copulas to model specific feature of dependence. We can start with the simplest case when $\phi = -\ln t$

And

$$C_{\phi}(u, v) = \phi^{-1}(\phi(u) + \phi(v)) = \exp\{-(\ln(u) - \ln(v))\} = \exp(\ln(uv)) = uv$$

which encounters the joint distribution of u and v when they are *independent*. Indeed, to choose generator $\phi = -\ln t$ gives the independence copula family. Other common generators (summarized in Table 2) might give less intuitive natural probabilistic interpretation

but the inverse of them lead to meaningful distributions to form a copula expression(see last column in Table 2)

Table 3.1 Archimedean Copulas

Family	Generator $\phi(t)$	Bivariate Copula $C_\phi(u, v)$	Inverse Generator (Laplace Transform) $\tau(s) = \phi^{-1}(s)$	Laplace Transform Distribution
C_{GH}	$(-\ln t)^\alpha$	$\exp\left\{-\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{1/\alpha}\right\}$	$\exp(-s^{1/\alpha})$	Positive stable
C_F	$\ln\left(\frac{e^{-\alpha} - 1}{e^{-\alpha} - t}\right)$	$\frac{1}{\alpha} \ln\left(1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^\alpha - 1}\right)$	$\alpha^{-1} \ln[1 + e^s(e^\alpha - 1)]$	Logarithmic series distribution on the positive integers
C_{CCO}	$t^{-\alpha} - 1$	$(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$	$(1 + s)^{-1/\alpha}$	Gamma

GH: Gumbel(1961),Hougaard(1986)

F: Frank(1979)

CCO: Clayton(1978), Cook-Johnson(1981), Oakes(1982)

The above common-used copulas have different emphasis on tail dependence. For example,

Gumbel copula(C_{GH}) can be used to model upper tail dependent and Clayton copula(C_{CCO}) to model lower tail dependence.

3.2.3 Marshall-Olkin Copulas

The bivariate *Marshall-Olkin copula* takes the form

$$C(u, v) = uv \min(u^{-\theta_1}, v^{-\theta_2}) = \min(u^{1-\theta_1}v, uv^{1-\theta_2})$$

This is the bivariate Marshall-Olkin copula, and it can be naturally extended to the Marshall-Olkin n-copula. There are great computation advantages in simulation; algorithms are provided by various related paper (For example, Embrecht et al 2003)

3.2.4 Bounds

Perfect positive and negative dependences, the two extreme cases, would produce simple forms of copula that also provide the lower bound and upper bound for any copula.

$C(u,v)=\min(u,v)$ for perfect positive dependence

$C(u,v)=\max(u+v-1,0)$ for perfect negative dependence

Which are *Frechet (1951) bounds*: $C(u,v)=\max(u+v-1,0) \leq C(u,v) \leq \min(u,v)$. The multivariate extension of Frechet bounds is given by Dall' Aglio(1972).

3.3 Correlation Measurement

Spearman's ρ and Kendall's τ probably can be used as the best alternatives to the linear correlation coefficient. Both can be defined using a copula function only as follows

Spearman's correlation coefficient

$$\begin{aligned}\rho_s &= 12E\{(F_1(x_1)-1/2)(F_2(x_2)-1/2)\} \\ &= 12 \int \int [C(u,v) - uv]dudv\end{aligned}$$

And Kendall's correlation coefficient

$$\begin{aligned}\tau &= \Pr\{(x_1 - x_1^*)(x_2 - x_2^*) > 0\} - \Pr\{(x_1 - x_1^*)(x_2 - x_2^*) < 0\} \\ &= 4 \int \int C(u,v)dC(u,v) - 1\end{aligned}$$

Where (x_1, x_2) and (x_1^*, x_2^*) are independent copies from random variables (X_1, X_2) .¹⁰ Either a common Spearman's ρ or Kendall's τ can be used as a standard base to compare between results using different copula functions.

3.4 Mixed copulas

We can use the mixture of joint distribution to adjust, up or down, the Kendall's tau. For instance, if we feel that a correlation given by a common mixture joint distribution function F

¹⁰ This is defined in the measure of *concordance*

would be too strong, we can mix F with an independent joint distribution function G. if the opposite is the case, we can mix F with a comonotone joint distribution function G'.

Assume that joint distribution functions F_{X_1, \dots, X_k} and G_{X_1, \dots, X_k} have the same marginals

F_{X_1}, \dots, F_{X_k} Then the mixed joint distribution is

$$(1-q)F_{X_1, \dots, X_k}(t_1, \dots, t_k) + qG_{X_1, \dots, X_k}(t_1, \dots, t_k), (0 < q < 1),$$

Also, $\tau(X_i, X_j) = (1-q)\tau^F(X_i, X_j) + q\tau^G(X_i, X_j)$ where τ^F and τ^G denote the Kendall's τ

implied by the joint cdf's F and G, respectively.

4 APPLICATIONS OF COPULAS

In this section, we describe some useful applications of copulas, in particular within insurance and credit risk modelling framework. We focus on these two areas for the following reasons: recently there is an increasing need to model dependence in insurance and credit risk modelling, and actuaries and risk specialists strive to understand multivariate outcomes associated with financial security systems and financial derivatives. Moreover, there are great similarities in fundamental mathematics and statistics in the two areas. Compared to credit risk modelling, actuarial science has a long history modelling dependence. Though not using copula directly, some conventional approaches within actuarial science framework have nice properties that well support copula as a dependent structure. On the other hand, the recent intensive study in modelling credit risk dependence might inject new blood in the traditional insurance approach. We believe that the interactive study of these two should enhance the dependence modelling to both and other areas in risk management.

4.1 Some underlying mathematic functions and models

Actuarial Science and credit risk modelling share notable underlying mathematical principles. It is very important to understand the basic mathematics in order to better develop knowledge in copulas. And we believe that the mathematical base shared by the two areas of study of dependence will help the interactive study in the both areas, which in term, enhances their own empirical applications.

Distribution Functions

Default probability density distribution: $f(t) = \lambda e^{-\lambda t}$ i.e. Exponential Distribution

Default cumulative density distribution: $F(t) = \Pr(X \leq t) = 1 - e^{-\lambda t}$

Survival function: $S(t) = 1 - F(t)$

Force of mortality (Bower et al, 1997, chapter 3) or Hazard rate function (Lando, 2005, chapter

5):
$$h(t) = -\frac{\partial \ln S(t)}{\partial t} = \frac{f(t)}{S(t)}$$

Compounding Method/Mixture Models

Compounding method is considered one of the two main methods for specifying a family of copulas to model multivariate distributions in the current actuarial literature (Free et al. 1997), while mixture models is one of the two underlying models to measure dependence in credit risk applications.

There is a long history of using *compound distributions* for risk classification in actuarial science, particularly in the credibility framework (Free et al. 1997). The basic idea is that for a given underlying variable or a vector of variables, the distributions conditioned on the common variable are independent even their unconditional distributions are not independent. Similarly, in credit risk literature, Frey et al (2001), for example, define *Mixture Models* based on the same principle: Bernoulli default probabilities $Y_i | Q_i$ are conditionally independent given a set of economics factors Q_i .

We use the following example to illustrate the *compounding* method, and will use the result later. In actuarial science and intensity credit modelling, the conventional default mode with a given hazard rate or default intensity γ (i.e. the conditional exponential distribution) is:

$$\Pr(X \leq x | \gamma) = 1 - e^{-\gamma x}$$

And a specified distribution for γ , where $\gamma \sim \text{Gamma}(\alpha, \lambda)$. Then the unconditional default distribution is:

$$\begin{aligned}
F(x) &= \Pr(X \leq x) = \int \Pr(X \leq x | \gamma) \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma} d\gamma \\
&= 1 - \int e^{-\lambda x} \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma} d\gamma = 1 - \left(a + \frac{x}{\lambda}\right)^{-\alpha} \quad (4.1)
\end{aligned}$$

A *Pareto distribution*.

Another common compound distribution is the Poisson mixture distribution: an unconditional distribution of a Poisson distribution with underlying gamma-distributed parameters gives a *negative Binomial distribution*. This is the approach underlying CreditRisk⁺ (Credit-Suisse-Financial-Products 1997). Default correlations in CreditRisk⁺ are assumed to be driven by a vector of K “risk factors” (a set of common economic factors) $\gamma = (\gamma_1, \dots, \gamma_K)'$. Conditional on γ , default of individual obligators are independently Bernoulli-distributed. Further assume a vector of “factor loadings” (scaled factors to represent the sensitivity to each risk factor) $\mathbf{w} = (w_1, \dots, w_K)'$ with $\sum w_i = 1$. The conditional probability $p_i(\gamma)$ of drawing a default for obligor i is a function of the rating class $rc(i)$ of obligator i , risk factors γ and \mathbf{w} :

$$p_i(\gamma) = \overline{p_{rc(i)}}(\gamma, \mathbf{w})$$

Assume that individual default rates are constant, CreditRisk⁺ we have $E[p_i(\gamma)] = \overline{p_{rc(i)}}$. For a single obligator i , the *moment generating function* (pgf)¹¹ of the Bernoulli ($p_i(\gamma)$) is

$$G_i(z | \gamma) = [1 - p_i(\gamma) + p_i(\gamma)z] = 1 + p_i(\gamma)(z - 1)$$

And because $p_i(\gamma)$ is small (default is a rare event), we can write

$$G_i(z | \gamma) = \exp\{\ln[1 + p_i(\gamma)(z - 1)]\} \approx^{12} \exp(p_i(\gamma)(z - 1))$$

The pgf of the sum of obligator defaults is the product of the individual pgfs¹³.

¹¹ CreditRisk+ uses the pgf to calculate defaults, instead of calculating the distribution of default directly.

¹² For small Δ , $\ln(1+\Delta) \approx \Delta$

$$G(z|\gamma) = \prod_i G_i(z|\gamma) \approx \prod_i \exp(\mu(\gamma)(z-1))$$

Where $\mu(\gamma) = \sum_i p_i(\gamma)$

In CreditRisk⁺, the k element in the vector \mathbf{x} (risk factor) has Gamma(1, σ^2_k) distribution. To get the unconditional pgf, we integrate out the \mathbf{x} :

$$\begin{aligned} G(z) &= \int G(z|\gamma) f_{Gamma}(\gamma) d\gamma = \int \sum_{n=0}^{\infty} \exp(\mu(\gamma)) \frac{\mu(\gamma)^n z^n}{n!} f_{Gamma}(\gamma) d\gamma \\ &= \prod_{k=1}^K \left(\frac{1-\delta_k}{1-\delta_k z} \right)^{1/\delta_k^2} \text{ where } \delta_k = \frac{\delta_k \mu_k}{1 + \delta_k^2 \mu_k} \text{ and } \mu_k = \sum_i w_{ik} \overline{p_{rc(i)}} \end{aligned}$$

This form of pgf implies that the total number of defaults is a sum of k independent *negative binomial* variables.

In addition to these two compounding distributions, *Frailty models*, raised within compounding methods, are traditionally used in multivariate survival analysis. Vaupel et al.(1979) defined frailty in terms of the force of mortality h . $h(t)$ is extended to $h(t, Z)$ to include a frailty of Z, which can be used to reflect different individual mortality rates as a result of genders, smoking habits and so on. They also refer to *mixture of power* distributions in current statistical literature (Joe 1997) and are one important type of mixture models.

Consider a specific hazard function of $h(t, Z)$: $h(t, Z) = e^{\beta z} b(t) = \gamma b(t)$; $\gamma = e^{\beta z}$

Where $b(t)$ is the “baseline” hazard function β -vector of regression parameters. By integrating and exponentiating the negative hazard, we can get the famous *Cox(1972) proportional hazards model*:

$$S(t|\gamma) = \exp \left\{ - \int_0^t h(t, z) ds \right\} = \exp \left\{ - \int_0^t \gamma b(t) ds \right\} = \left(\exp \left\{ - \int_0^t b(t) ds \right\} \right)^\gamma = B(t)^\gamma$$

Here,

¹³ This is according to the important property of pgf: if n variable are independent, the pgf of the sum of these n variables are equal to the product of the n pgfs.

$$\left(\exp\left\{- \int b(t)ds\right\} \right) = B(t)$$

is the survival function corresponding to the baseline hazard, and the survival function is condition on γ . γ is called *frailty* because larger γ implies smaller survival function.

For bivariate frailty models¹⁴:

$$\Pr(T_1 \geq t_1, T_2 \geq t_2 | \gamma) = \Pr(T_1 \geq t_1 | \gamma) \Pr(T_2 \geq t_2 | \gamma) = S_1(t_1 | \gamma) S_2(t_2 | \gamma) = B_1(t_1)^\gamma B_2(t_2)^\gamma$$

The (unconditional) bivariate survival function is

$$\begin{aligned} \Pr(T_1 \geq t_1, T_2 \geq t_2) &= \int B_1(t_1)^\gamma B_2(t_2)^\gamma f_\gamma(t) dt \\ &= E_\gamma \{ B_1(t_1)^\gamma B_2(t_2)^\gamma \} \end{aligned} \quad (4.2)$$

This result is important, as Marshall and Olkin(1988) showed that all frailty models of the form (4.2) could be easily written as copulas. More details will be shown in the next section.

Common Shock Model

In both actuarial and intensity credit modelling literature(For example, see Bower chapter9.6 or Joe 1997), one underlying approach is that a default is assumed to be governed by Poisson process which means the default time follows exponential distribution with a default intensity λ . When joint defaults are of interest, different varieties of common “shock” variables that follow independent Poisson process is/are added to the process to introduce independence. This is the idea from Marshall-Olkin(1967).

Consider two references A and B are subject to shocks, and the shocks follow three *independent* Poisson processes with parameters λ_1 , λ_2 and $\lambda_{12} \geq 0$, where each respectively denote the shocks effect only A , only B or both. We know that the Poisson process implies exponential distribution of default time, thus the joint survival time function of x_1 and x_2 is

¹⁴ Same for multivariate frailty models; we use bivariate cases for simplicity

$$\begin{aligned}
S(x_1, x_2) &= \Pr(X_1 \geq x_1, X_2 \geq x_2) = \Pr(X_1 \geq x_1) \Pr(X_2 \geq x_2) \Pr\{Z \geq \max(x_1, x_2)\} \\
&= \exp\{-(\lambda_1 + \lambda_{12})x_1 - (\lambda_2 + \lambda_{12})x_2 + \lambda_{12}\min(x_1, x_2)\}^{15} \\
&= S_1(x_1)S_2(x_1) \min(\exp\{\lambda_{12}x_1\}, \exp\{\lambda_{12}x_2\})
\end{aligned}$$

If we let $\theta_1 = \frac{\lambda_1}{\lambda_1 + \lambda_{12}}$ and $\theta_2 = \frac{\lambda_2}{\lambda_2 + \lambda_{12}}$ then $\exp\{\lambda_{12}x_1\} = S_1(x_1)^{-\theta_1}$ and

$$\exp\{\lambda_{12}x_2\} = S_2(x_2)^{-\theta_2}.$$

Also, we let $C(u, v) = S(x_1, x_2)$ which denotes the survival copula. Then

$$C(u, v) = uv \min(u^{-\theta_1}, v^{-\theta_2}) = \min(u^{1-\theta_1}v, uv^{1-\theta_2})$$

Which is the bivariate *Marshall-Olkin copula* shown previously.

4.2 Applications of Copulas in Actuarial Science

Epidemiological and actuarial sciences have a relatively long history of application of copulas, particularly in joint mortality study, compared to other fields including credit modelling. The first but indirect application of copulas in joint-life models appeared in Clayton's work "A *Model for Association in Bivariate Life Tables*" (1978). The two main methods for specifying a family of copulas, Archimedean approach obtained from frailty models, and compounding approach, are originated in epidemiological and actuarial studies.

4.2.1 Survival of Joint lives

a) Consider two lives X_1 and X_2 that follow exponential distributions with the common underlying parameter γ . Note that their distributions are *not* independent unconditionally, but are

¹⁵ $\text{Max}(x_1, x_2) = x_1 + x_2 - \text{Min}(x_1, x_2)$

independent if conditioned on γ (i.e. $X_1|\gamma$ and $X_2|\gamma$ are independent). From the result (4.1), the joint distribution of X_1 and X_2 is¹⁶

$$F(x_1, x_2) = F_1(x_1) + F_2(x_2) - 1 + \left[(1 - F_1(x_1))^{-1/\alpha} + (1 - F_2(x_2))^{-1/\alpha} - 1 \right]^{-\alpha}$$

Note that the joint distribution $F(x_1, x_2)$ can be written as a function *completely* of $F_1(x_1)$ and $F_2(x_2)$. This feature is appealing because if we let $u_1 = F_1(x_1)$ and $u_2 = F_2(x_2)$ where u_1 and u_2 are realized uniformed random variables, the above equation yields the following copula function:

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1 \right]^{-\alpha}$$

An important implication here is that compounding method is a useful way to build a copula. Also, consider another example in b).

b) For the same pair of lives X_1 and X_2 we can also use Cox proportional hazards function to model conditional survival rates. From result (4.2), the (unconditional) bivariate survival function is

$$\begin{aligned} \Pr(T_1 \geq t_1, T_2 \geq t_2) &= E_r \left\{ B_1(t_1) B_2(t_2) \right\}^\gamma \\ &= E_r \left\{ \exp \left(\ln \left\{ B_1(t_1) B_2(t_2) \right\}^\gamma \right) \right\} \\ &= E_r \left\{ \exp \left(\gamma \ln \left\{ B_1(t_1) B_2(t_2) \right\} \right) \right\} \\ &=^{17} \exp \left(- \left[- \ln \left\{ B_1(t_1) + B_2(t_2) \right\} \right]^\alpha \right) \\ &=^{18} \exp \left(- \left[\left\{ - \ln S_1(t_1) \right\}^{1/\alpha} + \left\{ - \ln S_2(t_2) \right\}^{1/\alpha} \right]^\alpha \right) \end{aligned}$$

¹⁶ $F(x_1, x_2) = \Pr(X_1 < x_1, X_2 < x_2) = 1 - \Pr(X_1 > x_1) - \Pr(X_2 > x_2) + \Pr(X_1 > x_1, X_2 > x_2) = 1 - (1 + x_1/\lambda)^{-\alpha} - (1 + x_2/\lambda)^{-\alpha} + (1 + (x_1 + x_2)/\lambda)^{-\alpha}$

¹⁷ Laplace transformation $E_r \left\{ \exp(-\gamma s) \right\} = \exp(-s^\alpha)$; γ is modelled as a positive “stable distribution”

Compare this to the bivariate Copulas in Table 2, one can recognize this is in form of the *Gumbel Copula*.

In fact, this is a special case¹⁹ for constructing copulas due to Marshall and Olkin(1988). They showed that all frailty models of the form

$$E_r \left\{ B_1(t_1) \cdots B_p(t_p) \right\}^\gamma \quad (4.3)$$

can be easily written as Archimedean copulas by using the Laplace transforms of γ , defined by

$$\tau(s) = E_r e^{-s\gamma}$$

Where is the $\tau(\square)$ is the Laplace transform of γ . Because $\tau[-\ln B_i(t)] = S_i(t)$

and $B_i(t) = \exp\{-\tau^{-1}[S_i(t)]\}$, (3) can be written as

$$\tau\left(\tau^{-1}(S_1(x_1)) + \cdots + \tau^{-1}(S_p(x_p))\right) \quad (4.4)$$

Where τ^{-1} serves as the generator ϕ for an Archimedean copula. Again, we have shown that compounding models can be used generate copulas of interest.

4.2.2 Competing Risks

A theory in actuarial science is called *Multiple Decrement Theory*, which deals with the study of the lifetime distribution of a system subject to several competing causes (see, for example, Bower et al.1997, Chapter.10 and 11). For example, a person dies as a result of one of the following possible causes: heart disease, cancer, and accident and so on. The default time T is $\min(T_1, T_2, \dots, T_p)$ and each element denotes one of p competing causes. It becomes increasing unrealistic to assume T 's are statistically independent, and to account for dependence in this

¹⁸ $s_i(t_i) = \exp\left(-\{-\ln B_i(t_i)\}^\alpha\right)$ for $i=1,2$

¹⁹ By take all latent variables $\gamma_1 = \gamma_2 = \dots = \gamma_p = \gamma$

scenario, one can apply copulas. If we assume that causes of death are independent given a frailty γ , we have

$$\begin{aligned}\Pr(T > t | \gamma) &= \Pr(\min(T_1, T_2, \dots, T_p) > t) \\ &= \Pr(T_1 > t | \gamma) \cdots \Pr(T_p > t | \gamma) \\ &= B_1(t_1)^\gamma \cdots B_p(t_p)^\gamma\end{aligned}$$

By Integrating over γ and taking the Laplace transformation, we can obtain (4.3) thus reach the same result in (4.4).

4.2.3 Common shock

Bowers(1997,p274-275) demonstrates how to use common shock approach to model joint life distribution where the independence between the two lives are not assumed to be independent.

If $T(x)$ and $T(y)$ denote two future lifetime random variables that, if no common shocks, are independent; that is

$$\begin{aligned}S_{T(x)T(y)}(s, t) &= \Pr(T(x) > s \cap T(y) > t) \\ &= S_{T(x)}(s)S_{T(y)}(t)\end{aligned}$$

In addition, if there is a common shock random variable, Z , that can affect the joint distribution of the time-until-death of lives (x) and (y) . This common shock random variable is independent of $T(x)$ and $T(y)$ and has an exponential distribution; that is

$$S_Z(z) = e^{-\lambda z} \quad z > 0, \lambda \geq 0$$

The random variable z can be pictured as some kind of catastrophe events that leads to default. The random variable of interest in building models for life insurance or annuities to (x) and (y) are $T^*(x) = \min [T(x), Z]$ and $T^*(y) = \min [T(y), Z]$. The joint survival function of $T^*(x)$ and $T^*(y)$ is

$$\begin{aligned}
S_{T^*(x)T^*(y)}(s, t) &= \Pr \{ \min [T(x), Z] > s \cap \min [T(y), Z] > t \} \\
&= \Pr \{ [T(x) > s \cap Z > s] \cap [T(y) > t \cap Z > t] \} \\
&= \Pr \{ T(x) > s \cap T(y) > t \cap Z > \max(s, t) \} \\
&= S_{T(x)}(s) S_{T(y)}(t) e^{-\lambda \max(s, t)}
\end{aligned}$$

Note that the final line follows from the independence of $[T(x), T(y), Z]$. Here, the bivariate *Marshall-Olkin copula* can be applied.

4.3 Applications of Copulas in Credit modelling

Previous work in credit modelling literature has been focusing on individual credit risk; however, as credit products rapidly develop at a portfolio level, more efforts are put into the study of dependence. Currently there are two underlying models to measure dependence in credit risk applications: *Mixture Models*, which has already been discussed earlier in this paper, and *Latent variable models*, extended the *firm-value model* (Merton) to a multi-default latent variable models, is specified as follows (Frey et al 2001)

Given random vector $X = (x_1, \dots, x_m)'$ with continuous marginal distributions F_i and thresholds D_1, \dots, D_m define $Y_i := 1_{\{x_i \leq D_i\}}$. Default probability of counterparty i given by

$p_i := \Pr(Y_i = 1) = \Pr(X_i \leq D_i) = F_i(D_i)$ and $(X_i, D_i)_{1 \leq i \leq m}$ denotes a *latent variable model*

The most widely used applications KMV (KMV Corporation 1997) and CreditMetrics (RiskMetrics 1997) are examples of latent variable models.

4.3.1 CreditMetrics and KMV

Model Calibration

In both KMV (KMV-Corporation 1997) and CreditMetrics (RiskMetrics-Group 1997), the latent variables X_i are assumed a linear function of risk factors Θ and idiosyncratic effects ε_i .

$$X_i = \sum_{j=1}^p w_{i,j} \theta_j + \sigma_i \varepsilon_i$$

The p -dimensional random vector Θ are assumed to be $N_p(0, \Omega)$. $\varepsilon_1, \dots, \varepsilon_m$ are assumed iid

$N(0,1)$, which are also *independent* of Θ . $w_{i,j}$ denotes the weights for reference i and factor j .

μ_i, \sum_{ii} and D_i are chosen so that p_i equals average historical default frequency for companies with a similar credit quality²⁰.

The mathematical concept of *exchangeability* can be used to prove that KMV and CreditMetrics are *equivalent* (see, for example Frey et al. 2001 for more details), as are all latent variable models that use the Gaussian dependence structure for latent variables, regardless of how marginals are modelled.

Li(1999) is amongst the first ones to indicate that both the KMV and CreditMetrics are implicitly based on a Gaussian copula. He summarizes how CreditMetrics calculates joint default probability of two credits A and B. He uses the actuarial symbols q_A and q_B to denote the one-year

²⁰ They use different methods to group companies by credit quality :threshold D_i is chosen that p_i equals average default probability of companies with same "distance to default" as company i in KMV-model; D_i is chosen that p_i equals average default probability of companies with same rating class as company i in CreditMetrics model.

default probabilities for A and B respectively (originally denotes as p_i in CreditMetrics and KMV models). Assume q_A and q_B both follow standard normal distribution($Z \sim \text{Normal}(0,1)$), then

$$\begin{aligned} q_A &= Pr[Z < z_A] \\ q_B &= Pr[Z < z_B] \end{aligned}$$

If ρ is the asset correlation, the joint default probability for credit A and B is

$$Pr[Z_A < z_A, Z_B < z_B] = \int_{-\infty}^{z_A} \int_{-\infty}^{z_B} \phi_2(x, y | \rho) dx dy = \phi_2(z_A, z_B, \rho) \quad (*)$$

Where ϕ denotes the standard normal distribution and $\phi_2(x, y)$ denotes the bivariate normal distribution with correlation ρ between marginal functions.

Equation (*) is mathematically equivalent to

$$C(u = q_A, v = q_B, \rho = \gamma) = \phi_2(\phi^{-1}(u), \phi^{-1}(v), \gamma) \quad (**)$$

This is a bivariate normal copula shown in section 3.2. Thus, we can conclude that CreditMetrics uses a bivariate normal copula function with the asset correlation as the correlation. The procedure can be generalized to multiple credit references. That is, using asset correlations, we can construct high dimensional normal copula functions to model the credit portfolio of any size.

Note that we can reach the same result if the margins, q_A and q_B , are exponentially functions(which is the quite often the case in intensity credit modelling). The copula function (**) becomes

$$C(u = q_A, v = q_B, \rho = \gamma) = \phi_2(F^{-1}(u), F^{-1}(v), \gamma)$$

Where $F(\cdot)$ denotes the exponential margin. This is in fact the copula suggested by the Model of Li (CreditMetrics Monitor 1999), where defaults are interpreted in terms of default time instead of some threshold while p_i is chosen in the same way as in the CreditMetrics model. The model of Li can also be seen as equivalent to both CreditMetrics and KMV since the exponentially

distributed survival times in this model are joined together by a Gaussian copula to form a multivariate distribution with exponential margins.

4.3.2 First-to-Default Valuation

Li(1999) suggests some numerical illustrations in his paper “*On Default Correlation: A Copula Function Approach*”. One is to apply the Gaussian copula to the joint default times in the valuation of a first-to-default contract. Assume there are n credits, and the joint distribution of the survival times is

$$F(t_1, t_2, \dots, t_n) = \phi_n \left(\phi^{-1}(F(t_1)), \phi^{-1}(F(t_2)), \dots, \phi^{-1}(F(t_n)) \right)$$

Where ϕ_n is the n dimensional normal cumulative distribution function with correlation coefficient matrix Σ . After simulating a series of n random variables from a u -dimensional normal distribution (via Cholesky decomposition or common factor approach), we can simply invert the exponential function which Li assumes for individual default time function to get the default time t_1, t_2, \dots, t_n . The complete algorithm is given in Appendix A.

The default time of the first-to-default contract is $T = \min(T_1, T_2, \dots, T_n)$, given a fixed parameter. That is, we can write the survival function of this contract

$$\begin{aligned} \Pr(T > t \mid \gamma) &= \Pr(\min(T_1, T_2, \dots, T_n) > t) \\ &= \Pr(T_1 > t \mid \gamma) \cdots \Pr(T_n > t \mid \gamma) \\ &= \exp(-n\gamma t) \end{aligned}$$

Where γ is the default intensity and it is assumed to be the same number for each credit reference by Li. Note that this is exact the same as the case of competing risk in the insurance framework we showed before, except that the (conditional) marginal distributions to model individual default time are different.

We can also apply the common shock approach to model the first-to-default contract, or more generally, the k-to-default contract. Same as above, let the first-to-default time be

$$T = \min(T_1, T_2, \dots, T_n)$$

$$\begin{aligned} \Pr(T > t) &= \exp(\min(T_1, T_2, \dots, T_n) > t) \\ &= \exp(-t\Lambda) \end{aligned}$$

Where $\Lambda = \lambda n + \bar{\lambda} \binom{n}{2}$ if we assume a bivariate symmetric dependence structure²¹ with idiosyncratic shock intensity λ and the joint shock intensity $\bar{\lambda}$.

4.3.3 Remarks

A Gaussian copula provides an easy way to construct high dimensional dependence in terms of simulation procedure and parameter input. However, the occurrence of many joint large movements of defaults is a rare event, and this casts some doubt on the use of Gaussian copula as the dependence structure for latent variable models. A t-copula might be an alternative to a Gaussian one. Moreover, the analogy between mixture models and latent variable models might suggest the more flexible copulas can be applied to model dependence in different credit portfolios. It has been shown that mixture models and latent variable models have very similar underlying mathematical structure despite differences on the surface (Gordy 1998):

CreditMetrics and KMV models can be mapped to mixture models (with some restriction on the former model). Frey and McNeil (2001) has shown that they can be written as a Bernoulli mixture model:

Using the same notations as in CreditMetrics and KMV model calibration, for the mixing factors we take $\Psi = \Theta$,

²¹ Due to the lack of appropriate data available for the estimation of higher order shock intensities, it is often assumed that a model specification with bivariate dependence only, i.e. a model where at most two simultaneous defaults are possible. Another assumption to assume equal intensities for the joint defaults of two firms (referring to bivariate symmetric dependence)

$$\Pr(Y_i = 1 | \Psi) = \Pr(X_i \leq D_i | \Psi) = \Pr\left(\varepsilon_i \leq (D_i - \sum_{j=1}^p w_{i,j} \Psi_j) / \sigma_i\right) = \phi\left((D_i - \sum_{j=1}^p w_{i,j} \Psi_j) / \sigma_i\right)$$

Clearly $Y_i | \Psi \sim \text{Bernoulli}(Q_i)$ where $Q_i = \phi\left((D_i - \sum_{j=1}^p w_{i,j} \Psi_j) / \sigma_i\right)$.

Thus, Q_i has a *probit-normal* distribution: $\phi^{-1}(Q) \sim N(\mu, \sigma^2)$.

We can recall that the Bernoulli mixture model is the structure underlying CreditRisk⁺ as we discussed previous in compounding method/mixture models. Also, it underlies the latent variable as we just showed. As compounding method is considered one of the two main approaches to specify copulas in current actuarial literature to model dependence, further study in mixture models in credit risk framework might introduce more suitable structure to model default correlation additional to the Gaussian standard framework.

4.4 Fitting Copulas to Data

We can use Maximum likelihood Estimation (MLE) to fit copulas to data. There are two stages (i) fitting the marginal distributions and (ii) fitting a copula to the data.

We use an empirical study by Frees and Valdez(1998) to demonstrate how to fit the bivariate copula to real data, which comprise 1,500 general liability random claims provided by Insurance Services Office, Inc.. Note that data is not limited to a bivariate fashion but can contain multivariate variables representing different lines of business, credit entities, market risks and so on.

In this case, each claim contain two components: an indemnity payment (the loss, X_1) and an allocated loss adjustment expense (ALAE, X_2), and some loss X_1 is censored due to the insurance policy limit. The objective is to describe the joint distribution of losses(X_1) and expenses(X_2). The first step is to determine the appropriate marginals. Frees and Valdez(1998) supposed a Pareto marginal distribution and the quality of the fit can be examined with a

graphical comparison of the fitted distribution function against their empirical versions such as Kaplan-Meier empirical distribution.

The second step is to fit a copula to the data. We need to choose a copula function before the parameter estimation. A useful statistic tool is the quantile-quantile (q-q) plot, based which the copula function that gives the closet agreement between nonparametric and parametric quantiles is usually picked. After identifying the best two or three copula function, which is the Gumbel and Frank copulas in this example, we need to fit this copula using MLE. To estimate all the parameters in the marginal and joint functions, we need the following partial derivatives:

$$F_1(x_1, x_2) = \frac{\partial F(x_1, x_2)}{\partial x_1}, \quad F_2(x_1, x_2) = \frac{\partial F(x_1, x_2)}{\partial x_2} \quad \text{and} \quad f(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}$$

Similarly, C_1 and C_2 which are the first partial derivatives for the copula and C_{12} which is the second mixed partial derivative. The log-likelihood function is

$$\log L(x_1, x_2, \delta) = (1 - \delta) \log f(x_1, x_2) + \delta \left\{ \log f_2(x_2) + \log [1 - C_2(F(x_1), F(x_2))] \right\}$$

Where δ takes value of zero if the loss variable(X_1) is not censored and it takes value of one for censored data. The parameter estimates are then determined by maximizing the likelihood for the entire dataset

$$\sum_{i=1}^n \log L(x_{1i}, x_{2i}, \delta)$$

To compare the goodness-of-fit of the two copulas, we can compute Akaike's Information Criteria(AIC):

$$\text{AIC} = 2[-\ln(\text{maximized likelihood})] + 2p$$

Where p is the number of model parameters. The copula function that minimizes the AIC is preferred.

Once we know the copula $C(x_1, x_2)$, i.e. the joint distribution of (x_1, x_2) , we can go on to calculate the expected value of any known (increasing) function $g(x_1, x_2)$, which, for example, can take a specific form for a reinsurer's payment.

4.5 Desired Properties of Copulas

The significant advantage of copulas over the conventional correlation as a dependence measure is that, linear correlation is invariant only under strictly increasing *linear* transformation while copulas preserve the dependence structure in any increasing transformation. For example, if X_1 and X_2 are two random variables and g_1 and g_2 are two arbitrary strictly increasing functions, $\text{correlation}(X_1, X_2)$ will not be necessarily equal to $\text{correlation}(g_1(X), g_2(X_2))$ but there will be $\text{copula}(X_1, X_2) = \text{copula}(g_1(X), g_2(X_2))$. In the latter case, X_1 and X_2 "moved together" is captured by the copula, regardless of the scale in which each variable is measured. This is useful because the increasing complexity of financial products seriously destroy the linear relationship assumption between financial assets, which have made the traditional correlation coefficient less and less appropriate for modern portfolio management. And the increasing emerging portfolio level products and the unrealistic independence assumption are calling for the development of dependence techniques. Copulas provide a much more flexible structure for dependence in many situations.

Also, copulas are nicely incorporated with the association measures Spearman's correlation coefficient and Kendall's correlation coefficient. Schweizer and Wolff (1981) showed that two these two standard nonparametric correlation measures could be expressed *solely* in terms of the copula function (see 3.3 in this paper). This implies that these measures are not affected by nonlinear changes of scale

Furthermore, historically many multivariate distributions have been developed as immediate extensions of univariate distributions, examples being the bivariate Gamma, the bivariate Pareto and so on (Frees et al 1998). Drawbacks of these approaches include: (i) measure of association often appear in the marginal distribution,(ii)extensions to more than just the bivariate case are not clear and (iii) one needs a different family for each marginal distribution. A construction of multivariate distributions based on copulas does not suffer from these drawbacks.

In addition to an intuitively appealing structure, copulas offer analysts some technical convenience. For example, Clemen *et al*(1999) suggests copula as an alternative to the conventional approach in management science and decision making to specify a joint distribution, claiming the benefit that “ it only requires marginal distributions and measures of dependence among variables” . Also, it provides the possibility to build a variety of dependence structure based on existing parametric or non-parametric models of the marginal distributions. Moreover, the relative mathematical simplicity and the natures of copula construction can ease the simulation techniques implemented in a spreadsheet thus enhance the practical attractiveness. For example, copulas can be used to build large simulation models for long time horizons.

4.6 Simulations

Simulation is a widely used tool for summarizing the distribution of stochastic outcomes. The copula construction allows us to simulate outcomes from a multivariate distribution easily. Appendix A provides algorithms for some of the copulas we use in the previous cases.

5 NUMERICAL TESTING AND IMPLICATIONS

In this section, we use the First-to-default (FTD) credit swap as the underlying credit instrument to illustrate the role of correlation, and the choice of dependence structures²² in determining the portfolio risk profile.

5.1 Definition and the Calibrated Model

The credit instrument we use to demonstrate relationship between risk and dependence is the First-to-default (FTD) credit swap. FTD is a type of *basket* credit derivatives. Payment would be made to the swap holder when any one of the references in the basket defaults within the specified period. For example, a 5-year FTD credit swap will default as long as one of its references defaults within 5 years. Note that the meaning of default varies with contents. That is, different credit derivatives have different definitions for default. For example, for Last-to-default credit swaps, default occurs when *all* the references in the basket default before expiry. For CDOs, senior bonds do not necessarily default when bonds in lower tranche defaults.

The comparisons will be made to see the interactive effects due to the change in the number of references in the basket and the change of correlation rho that underlies the copulas within the two copula frameworks. We will construct additional portfolios to comparisons, based one a starting, or benchmark basket. We call it the *base portfolio*, B , which contains 5 similar risky bonds/references²³.

²² We would use Gaussian and T-copula that are easier to implement and are currently widely used in modelling dependence in basket credit derivatives.

²³ Risk level is gauged by the magnitude of the credit spreads data from JP Morgan. That is, the larger the credit spread, the risky the reference/bond would be defined

Model calibration:

1. The credit default function is modelled by the common *exponential* function with *hazard rate* λ : $f(t) = \lambda e^{-\lambda t}$ and $S(t) = e^{-\lambda t}$

2. With the conventional assumptions (risk-neutral, no arbitrage etc.), and assume interest rate is deterministic (i.e. no interest rate risk), we have this equation for a 5-yr first-to-default credit swap

$$E^* [\text{PV}(\text{payment to the swap holder})] = E^* [\text{PV}(\text{continuous payment from the swap holder})]$$

Which equals the risk-neutral price of the swap.

3. To formulate the above equality, we have

$$\int_{t=0}^5 (1-R) * e^{-rt} * f(t) dt = \int_{t=0}^5 \text{spread} * e^{-rt} * S(t) dt$$
$$\int_{t=0}^5 (1-R) * e^{-rt} * \lambda e^{\lambda t} dt = \int_{t=0}^5 \text{spread} * e^{-rt} * e^{\lambda t} dt$$

Take out the constants in the integrals, $(1-R)\lambda \int_{t=0}^5 e^{-rt} * e^{\lambda t} dt = \text{spread} \int_{t=0}^5 e^{-rt} * e^{\lambda t} dt$

Or, $(1-R)\lambda = \text{spread}$

4. When we know the *spread term structure* and the recovery rates, we can extract the *hazard rate* based on the above equation

$$\lambda = \frac{\text{spread}}{(1-R)}$$

5. For the base credit basket, B, we will have a term structure of λ for each of the 5 underlying bonds

6. The survival times of bonds are t_1, t_2, \dots, t_5 . The most widely used dependence structures,

Gaussian copula and T-copula, in the current industry are used in our analysis.

a) Under the *Gaussian Copula* structure²⁴:

$$C(u_1, u_2, \dots, u_5; \rho) = \phi_\rho(\tau_1, \tau_2, \dots, \tau_5)$$

Where $\tau_i = F^{-1}(u_i)$ for $i=1, 2, \dots, 5$. F is the cumulative exponential function.

b) Under the *T-Copula* structure:

$$C(u_1, u_2, \dots, u_5; \rho, \gamma = 5) = t_{\rho,5}(\tau_1, \tau_2, \dots, \tau_5)$$

Where $\tau_i = F^{-1}(u_i)$ for $i=1, 2, \dots, 5$. F is the cumulative exponential function.

And in both cases, the correlation matrix is $\Sigma = \begin{pmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \dots & \rho \\ \dots & \dots & \dots & \dots \\ \rho & \rho & \dots & 1 \end{pmatrix}$

7. Motel-Carol Simulation²⁵ is used to calculate the default probabilities and/or swap prices in each of the cases. As we know, Gaussian and T copulas do not give us close form solution, in order to obtain the expected number of defaults and the expected swap price, we take the average of default probability/swap price on each trial. Programming is written to calculate the prices and default probabilities for all the constructed FTD baskets under the Gaussian and T copula dependence structures. Programming codes are given in Appendix B.

²⁴ This is the model of Li in section 4.3.1

²⁵ Simulations are based on the pre-determined Halton Sequence of random numbers, so for the same number of simulations, the random numbers used are the same. This leads to better comparison between the two dependence structures.

5.2 Descriptions of Portfolios in Analysis

Before going into the copula approach to model the joint distribution thus dependence, we need to make some sense about the individual credit default probability. That is, the marginal function of a single risky reference in *basket B*.

As the bonds in *basket B* are at the similar risk level, we can roughly assume they individually have the same default probability p . The default time is modelled by the exponential function, which is given in the above model calibration part. The default probability within 5 years is $\text{Prob}(T < 5) = 1 - S(5) = 1 - e^{-5\lambda} = 1 - 0.7225 = 0.2775$ ²⁶. Note that this is the individual default probability. In the rest of the paper, the default probability refers to the default of the FTD default, which is $\text{prob}[any(t_1, t_2, \dots, t_n) \text{ if } t_i < 5]$.

If we picture the default of the FTD credit basket as a binomial case *bin* ($n, p = 0.2775$), a single trial would be a bond either defaults or does not default in 5 years; for a basket of 5 *independent* risky bonds (i.e. if an independence assumption is imposed on the *basket B*), $n=5$ and the probability of FTD default in 5 year is 0.803126 ²⁷. If the basket has 40 similar risky *independent* bonds, then $n=40$ and the binomial distribution can be approximated by a normal distribution with 11.1 ($np=0.2775*40$). and variance 8.0198 ($np(1-p)=0.2775*(1-0.2775)*40$). Therefore, the probability of no default occurring in 5 years is approximately:

$$\text{prob}(n < 1) = \phi\left(\frac{1 - np}{\sqrt{np(1-p)}}\right) = \phi\left(\frac{1 - 11.1}{\sqrt{8.0198}}\right) = \phi(-3.5665) \approx 0$$

That is, a FTD credit swap is almost defaulting for sure, given a portfolio that contains 40 similar risky *independent* bonds.

²⁶ Survival probabilities for this typical risky bond over 5 years are: 0.9664, 0.9148, 0.8530, 0.7856, and 0.7225 respectively.

²⁷ $1 - \text{prob}(\text{no default}) = 1 - C(5,0) * (1 - 0.2775)^5 = 0.803126$

Note that this is based on the *independence* assumption, which means rho is zero²⁸. Certainly this is not the focus of our paper, but it is very important for our later comparisons, where we vary the correlations in constructions of Gaussian and T-copulas. We will later find the probability calculated under the dependent assumption gives the minimum survival rate, or the upper bound for default probability paths under Gaussian copula dependence structure. The correlation will “help” to decrease the overall portfolio default risk. From the FTD issuer’s point of view, FTD is *long* correlation²⁹. We will fully demonstrate this point in later analysis.

Next we construct more baskets by adding references to the *basket B*. To ease the recognition of the level of risk associated with different baskets, we use *red baskets* to denote a set of relatively high risky portfolios, and use *green baskets* for the relatively low risk ones.

To construct the *green* baskets, we add 5, 15, 25, and 35 *risky* bonds³⁰ into the based basket *B*; call these baskets, *red(10)*, *red(20)*, *red(30)*, and *red(40)*, respectively. For example, *red(10)*=base basket *B* + 5 *risky* bonds. Also to Construct *red* baskets: : add 5, 15, 25, and 35 *low risk* bonds³¹ into the based basket *B*; call these baskets *green(10)*, *green(20)*, *green(30)*, and *green(40)*, respectively. For example, *green(10)*=base basket *B* + 5 *low risky* bonds.

By imposing Gaussian and T copula dependence structures on two sets of baskets, we have 4 default probability paths. In the following section, we will first look at the four paths together, and then we separately analyses *green* and *red baskets*.

²⁸ Note that independence always lead to zero correlation; however, the other way around is not true unless the multivariate distribution is a normal/Gaussian one.

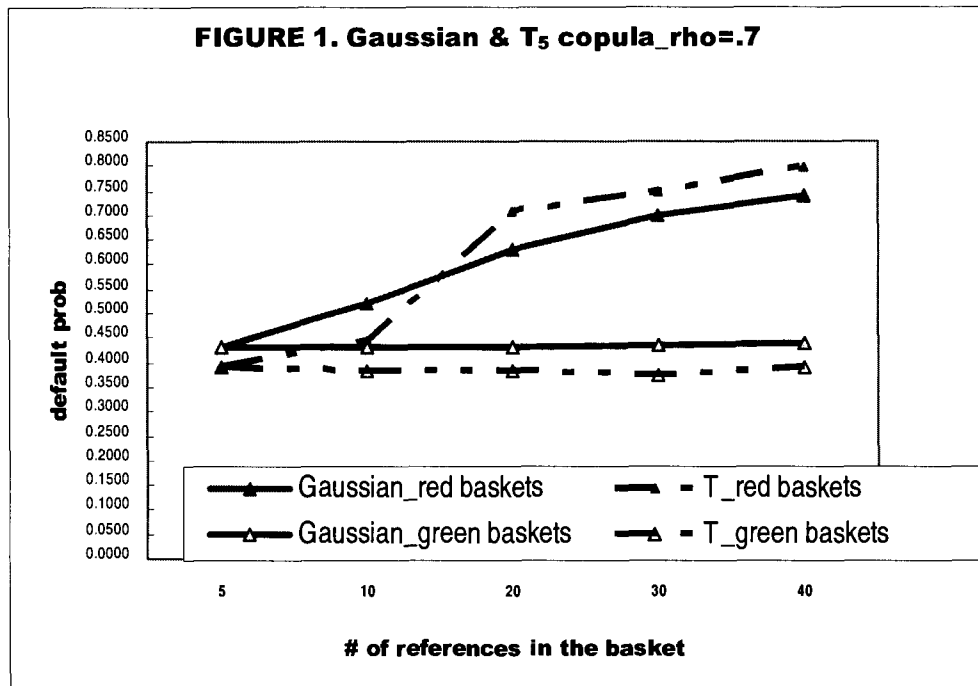
²⁹ Note that this is similar to the CDO equity tranche investors, who are *long* correlation. We mentioned this point in Section 2.3

³⁰ Similar risk level as the references in *basket B*

³¹ Almost default-free in 5 years

5.3 Experiments, Results and Discussions

Figure 5-1 Baskets with varied size when rho=0.7



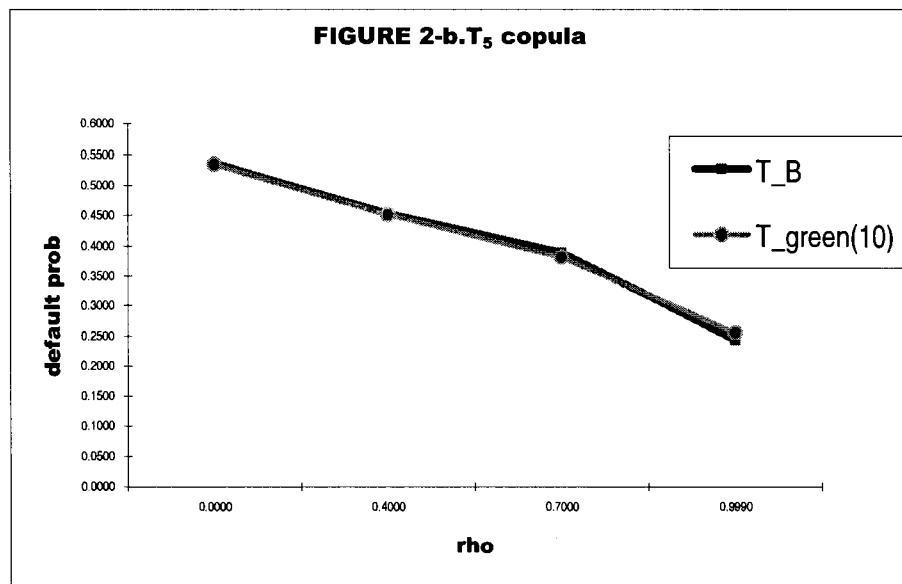
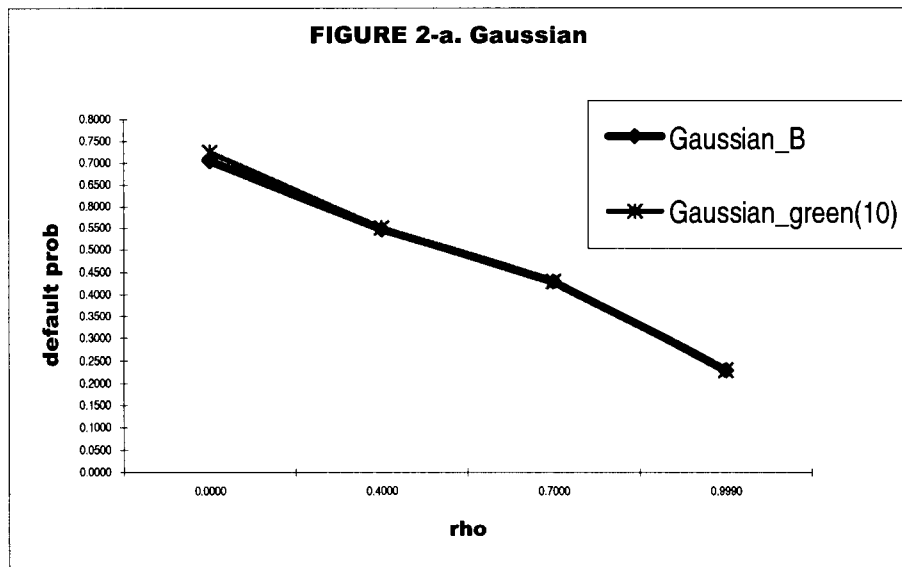
We first focus on the *green baskets*.

a) The solid green line indicates the default probabilities³² under the Gaussian and the dash one under T dependence structure. As we can see, when low risk bonds are added, the risk of the portfolio merely changes regardless the dependence structure. This is important because investors may have an intuition that “they are more likely to get paid by a FTD credit swap based on 40 bonds than one based on 5 bonds.”, which, however, is not true this kind of case. On the other hand, within the company itself, with such illusion, people in the marketing department may be willing to accept a higher selling price of a FTD swap when additional underlying references are included by the pricing department but these bonds actually do not bring in additional default risk thus not requiring for a higher price.

³² Note that in the case of FTD, default probability is an increasing function of the price; in other words, the default probability reflects the same trend graphically as price does.

Note that in this case we have fixed rho and have varied the number of references in the basket to see the effect on the default probability. We can also vary rho and to see how default probabilities associated with *green* portfolios would respond to the changes in rho.

Figure 5-2 Compare *green(10)* to *portfolio B* and with various rhos



b) A very important implication is that, default probabilities are *strictly decreasing* when rho increases from zero to one. And this is always the case in all the scenarios we create in this

paper, *regardless* the number of references, a Gaussian or a T-copula, or the total default risk underlying the portfolios. This fact is quite intuitive. Recall the simplified example 1 in Section 2.3: when there is a perfect correlation (one defaults implies all default in that case), the probability of zero defaults of the five Bernoulli variables becomes very high (0.8), and this “takes away” the probability of defaulting. The higher the correlation, the harder for default to occur, since default would require *all* of the variables default at the same time – when the marginal default rate is low, this has very low chance to happen. Similarly, for a FTD credit swap, higher correlation makes greater chance of no defaults, which is desired for the issuer (he or she does not need to make payments to the swap holder). Therefore, the issuer of the FTD credit swap is in a *long* position of default correlation between the underlying references. This is crucial for pricing and other practical issues: correlation can take significant effect in the determination of risk profile of the whole portfolio, and thus the risk compensated returns. As we will see later, correlation can significantly dominate the whole portfolio’s overall risk, either in the case of a Gaussian or a T-copula. For example, for two portfolios with the same number of similar risky assets, different default correlations between assets can lead to significant deviations in default probabilities. A misunderstanding in correlation can lead to a mis-price in derivatives thus open an arbitrage opportunity to sophisticated traders.

When we compare the *green(10)* basket to the *base* basket *B*, under either of the dependence structure, we see that the two probability paths are almost overlapping. Combining with result we discussed in a), certain **small** number of additional low risk bonds will not increase the default risk to the FTD swap under either of the dependence structures.

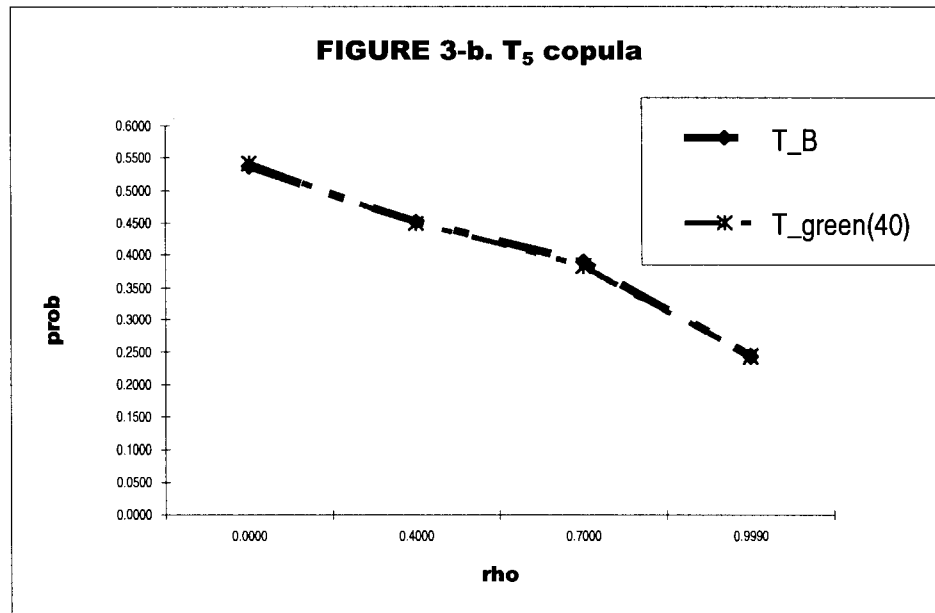
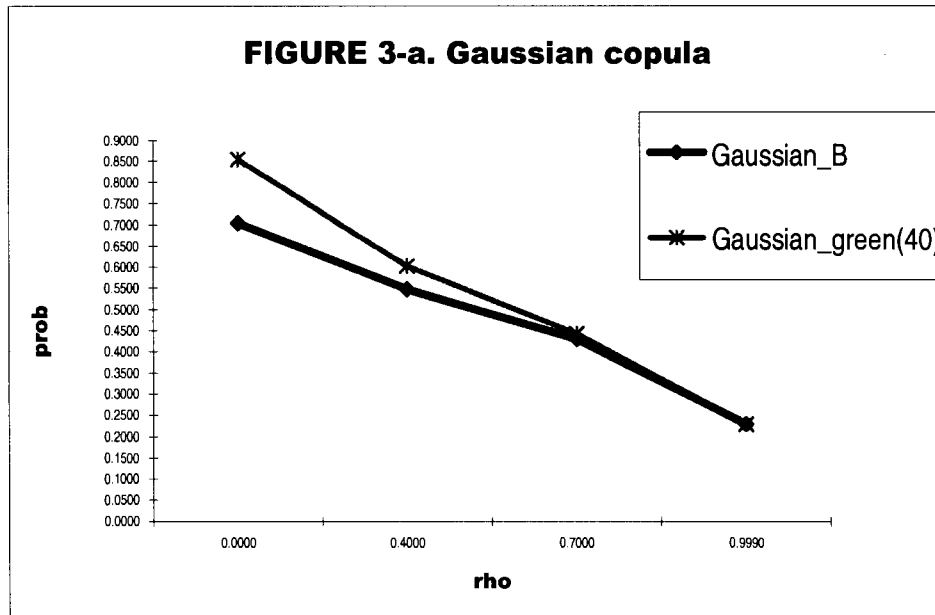
c) The next question would be, what happens if we add a **large** number, say 35, of low risk bonds to the base basket *B*, when rho is changing.

When we compare the *green(40)* basket to the *base* basket B, under the T-copula dependence structure, we see that the two probability paths are almost overlapping (same as in b we just discussed). However, under the Gaussian copula dependence structure, default probabilities start to deviate from about $\rho=0.7$.

For small ρ values, the deviation of default paths under the Gaussian case can be explained by this: Gaussian copula doesn't have tail dependence, and while there is no significant correlation effect that makes the outcome tend to be the two extreme values³³, defaults tend to more evenly possible over five years. This means more random individual defaults take place before expiry, which makes the FTD go default. These additional defaults are not significant in the *green (10)* case because the additional five references are low risk associated. However, when the number of these references, even low risk, increases to 35, their appearance in the basket will contribute effective credit risk to the portfolio. This contribution of credit risk is *off-set* by the correlation effect the additional references bring in when correlation is sufficient, and we can see the two probability paths in Figure 3-a are still overlapping within ρ value between 0.7 and 1. However, as the correlation decreases, the benefit of correlation created by these additional references starts to lose and cannot beat the additional default risk they bring in, and the FTD default risk increases to a higher level.

³³ Quartiles that are about two standard deviation from the expected default time.

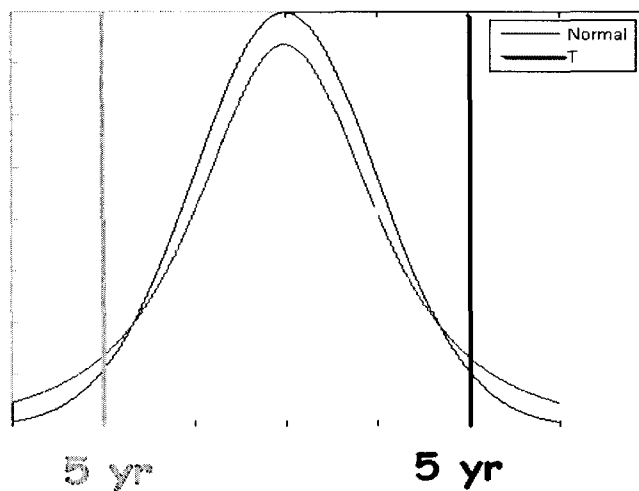
Figure 5-3 Compare *green(40)* to) portfolio B and with various rhos



The probability paths in Figure 3-b are still overlapping, as a T-copula has tail dependence that “keeps” certain portion of extreme probabilities on the two tails, and the upper tail effect decreases the default rates in the range of expiry. The extreme value effect serves somehow like the correlation, which amplifies the probability of no default on the upper tail.

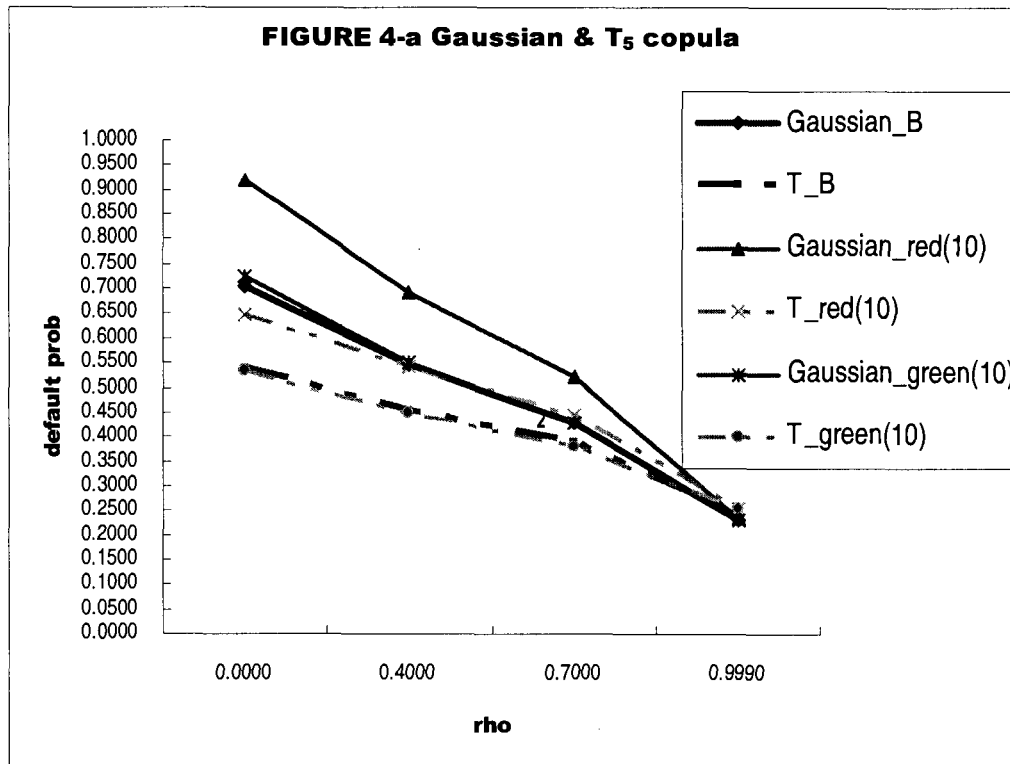
d) We see that default probabilities are generally higher in the Gaussian case from the figures above except when rho is close to one. The reason we would suggest is that, the T-copula has tail dependence, which “moves” more probabilities to the two tails on the joint default distribution; this means greater probabilities of defaulting early, or defaulting later (i.e. no default). Since the base portfolio B has a joint default time distribution whose mean is to the left of the expiry, the greater the probability in the *upper tail* would lead to a smaller expected default rate of a FTD swap expiry (See Figure 5-3 below: the black line indicates the expiry in this case, and it is on the right to the mean of the joint default distribution). Thus, we observe T-copula gives smaller default rates, except for $\rho \approx 1$. When rho is close to 1, the correlation effect is sufficiently large to prevent early defaults, which brings the mean of the joint default time distribution to the right of the expiry (See Figure 5-3 below: the grey line indicates the expiry in this case, and it is on the left to the mean of the joint default distribution). In this case, T-copula has a fatter *lower tail*, which results in slightly higher default rates.

Figure 5-4 Mean of the Joint distribution to the left or to the right of the expiry



Next, let us focus on the *red baskets*.

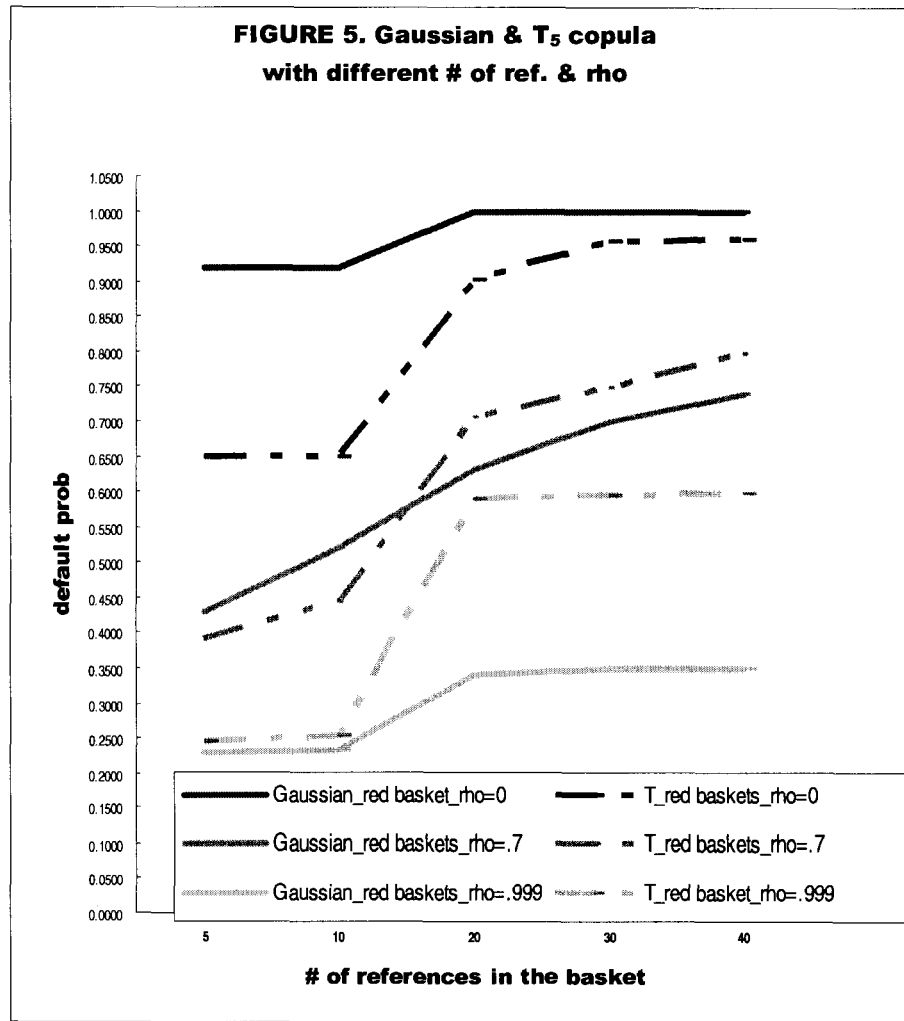
Figure 5-5 Basket B, red(10) and green(10) with various rho



e) The *red(10)* basket gives higher default probabilities compared to the *green(10)* basket: while a small number of additional low risk bonds do not change the risk profile of the FTD swap(as discussed in a)), even 5 additional risky bonds can add significant risk to the basket *B* . This meets our intuition: the more risky bonds included, the higher chance of default occurring.

We also observe that default risks are higher when Gaussian copula is used. Does this apply to all rhos, and all portfolio sizes?

Figure 5-6 Red baskets with various # of references and rhos



This figure reveals several interesting implications.

f) Under both copulas, we see that the paths are relatively flat when rhos are closed to 0 or 1, except a jump between *red (10)* and *red (20)*. This is probably because when there is a perfect correlation, correlation effect dominates and it maintains a low default rate until the default risk is cumulative enough to take the control. After sufficient number of reference added, correlation is brought in by more and more references, and because rho is one, the correlation effect is strong enough to prevent further default rate increment. Therefore, default rates stay

fairly constant after 15 references are added (i.e. *red (20)*). When correlation is zero, the portfolio default probability is dominated by the underlying total credit risks and is not benefited from correlation. After 15 references are added (i.e. *red (20)*), the portfolio is almost default for sure. This reconciles the probability we calculated in Section 5.2, where we use a normal distribution to approximate the probability of no default in a binomial distribution, and this probability is about zero with large baskets. A zero correlation does not imply independency, and this is part of the explanation for why under a T-copula, the default rates do not go to 1 with $\rho=0$. But for Gaussian copula, the construction of dependence with zero ρ seems to have a similar result as independence thus we can get the result as the binomial case.

g) We see that Gaussian copula does not always imply higher default probabilities even when ρ is not close to 1. The default probabilities completely reverse at the two extreme ρ values: Gaussian leads to the highest default rates when correlation is zero, and the lowest one when correlation is 1³⁴. That is, the two paths under Gaussian bound all the possible probability paths when ρ is at the minimum and maximum, respectively. Within the middle, whether or not Gaussian copula gives higher default rates depend on the number of risky references (i.e. the underlying total default risk). The different default probabilities implied by the two copulas can be again explained by the location of the mean of the joint default, seasonings similar to d) above. If correlation is large enough, default risk, even very significant in an absolute sense, will not determine the portfolio risk. Significant correlation dominates the whole and a very positive correlation leads to low expected rate of default of a FTD swap. Given a low mean that is to the left of the expiry, a copula structure with a fatter lower tail will result in higher default rates. In contrast, if correlation effect is not significant, the default risk associated with the portfolio itself will directly determine the portfolio risk. If the references are risky such as our *Basket B*, the

³⁴ We use .999 for the approximation for the perfect correlation, since $\rho=1$ fails to produce a positive definite matrix required by Cholesky factorization in the programming

mean of the default time distribution will locate to the right of the expiry, and a copula that has a fatter upper tail generally imply lower default rates(of a FTD swap).

h) We see bigger variation in default probability paths via Gaussian dependence structure when rho changes. We know in statistics, a wider confidence interval means a greater uncertainty in making the inference to the population parameter. Similarly, a wide range of variation in default paths under Gaussian implies that, if we do not know the correlation rho or make a wrong estimation in it, the mis-calculation part in price is greater in the case of Gaussian, compared to a T-copula . This means a greater modelling risk associated with a Gaussian copula used to model dependence in this case.

To summarize the implications of our experiment: a first-to-default credit swap is *long* correlation from the issuer's point of view. This might somehow seem contradictory to the conventional view on correlation, within which we try to diversify the portfolio so as to eliminate the correlation between assets thus decrease the portfolio variance. But for FTD swap issuer, they gain benefit from higher default correlation between underlying references in the basket. That is, the default probability of FTD swap decreases as correlation goes up, which means the issuer has less chance to make the payment to the swap holders. Thus, correlation is a significant factor in determining the default probability, and in term, in determining the swap price. The study of correlation is important because results can be also applied to other credit instruments. For example, CDOs. The equity tranche bond holders are in the same position as the FTD swap issuers, who are long correlation. On the other hand, the senior tranche bond holders are *short* correlation. Therefore, correlation can do either good or bad to the risk level depending on the content. Both issuers and investors have to properly understand correlation in their risk analysis.

To model correlation, we can use copulas to construct the joint default distribution of the underlying references. From our experiment, we see that the portfolio default risk is sensitive to

the copula dependence structure chosen, and such difference in default probability either can be off-set or be amplified by the change in other factors such as the correlation ρ , number of references included, and the portfolio's default risk level.

Correctly understand the risk nature of the portfolio is the first step, as we can see in our comparison between the *green* and *red* credit baskets. Further analysis based on the different mix of portfolios lead to very different modelling and pricing implications. For example, the tail dependence introduced by the T-copula has two-side effects, based on the underlying default risk and the correlation.

In addition, we need to understand the mutual effect of the tail dependence, the correlation and the portfolio's own default risk. They can together make a compound effect, or they can balance each other. For example, the positive correlation values close to 1 usually have dominative effect in determining default risk. Unless the portfolio itself has very strong default tendency, the strong positive correlation can prevent individual defaults by holding them to default later all together. However, when correlation gets weaker, this effect no longer dominates the true default risk imbedded in the portfolio. The tail effect together with the correlation effect can reverse the default levels implied by a Gaussian and a T-copula.

Therefore, for credit derivatives at a portfolio level, individual default risks are unquestionably the major risk factor. However, it is more important to look at them as a whole, which requires a dependence structure imposed on individual defaults. Risk analysts should be particularly aware of this when manage portfolio risk.

6 CONCLUSION

The increasingly important role of dependence in financial instruments is calling for advanced correlation measure and copulas have quickly become a popular tool for it in various situations. In insurance and credit instrument industries in particular, have shown significant effects in studying copulas and their applications of pricing insurance/reinsurance products, synthetic CDOs and so forth. We have narrowed our focus on these two areas because of their fundamental similarities in terms of some empirical natures and thus the mathematical modelling. Indeed, our paper has shown that they share some basic important properties that allow development in common copula functions. We believe the interactive study of the two will bring benefits to their own future studies and applications. Moreover, the knowledge of copulas will enhance the study in other risk management branches such as value-at-risk, stress testing and dynamic financial analysis. It is certainly that the appealing properties of copulas to model dependence will be more and more recognized and applied in practice.

At the end, this paper provides an interesting experiment to reveal several important implications to managing portfolio risk, which requires careful analysis on different risk factors – bringing in additional risk does not necessarily increase the overall portfolio risk . The choice of copulas with different correlation parameters provides different dependence in conjunction of pricing models. In the current credit instrument industry, elliptical copulas are widely used, or implicitly used. Will we apply more forms of copulas in credit modelling in the future? And as we have shown in Section 5, portfolio risk can be very sensitive to the dependence structure and the parameter inputs. The choice of copulas and the estimation of parameters play an crucial role. We believe the topic of copulas and copulas as a structure to model dependence will no doubt be more hotly discussed in the coming years.

APPENDICES

Appendix A –Simulations for Some Common Copulas

i) Gaussian Copula & Monte Carlo Simulation

Procedure:

Step 1. Specify/estimate the symmetric and positive definite correlation matrix

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \dots & \rho_{1k} \\ \rho_{21} & 1 & \dots & \rho_{2k} \\ \dots & \dots & \dots & \dots \\ \rho_{k1} & \rho_{k2} & \dots & 1 \end{pmatrix}$$

Step 2. Use Cholesky Decomposition to determine A such that $\Sigma = AA^T$

Step 3. Generate a vector of independent standard normal deviates, $z^T = \{z_1, \dots, z_n\}$

Step 4. Impose the dependence structure by calculating the correlated vector of standard normal deviates $\mathbf{x} = Az^T$, $X \sim N(0, \Sigma)$

Step 5. Set $u_i = \Phi(x_i)$. This yields a realization of the uniform random vector,

$\mathbf{U}, \mathbf{u}^T = \{u_1 \dots u_n\}$ with a Gaussian Copula dependence structure.

Step 6. Find the simulated correlated default time for every issuer: $\tau_i = F^{-1}(u_i)$.; F is an arbitrary marginal default time distribution

ii)T-Copula & Monte Carlo Simulation

Procedure:

Step 1-3 are same as above

Step 4. Simulate a random variate s from χ^2_v independent of $\{z_1, \dots, z_n\}$

Step 5. Set $\mathbf{y} = \frac{\sqrt{v}}{\sqrt{s}} \mathbf{x}$

Step 6. Set $u_i = t_v(x_i)$. This yields a realization of the uniform random vector,

$\mathbf{U}, \mathbf{u}^\top = \{u_1 \dots u_n\}$ with a T-Copula dependence structure

Step 6. Find the simulated correlated default time for every issuer: $\tau_i = F^{-1}(u_i)$. F is an arbitrary marginal default time distribution

iii) Exponential dependence Simulation

Procedure:

Step 1. Simulate a vector of independent exponential shock arrival times $\mathbf{t}^T = \{t_1, \dots, t_m\}$ with

given parameter $\{\lambda_1, \dots, \lambda_m\}$ ($\lambda_k > 0$) where $\tau_k = F^{-1}(u_k)$. F is an exponential marginal default time distribution

Step 2. Simulate an n -vector $\{T_1, \dots, T_n\}$ of joint exponential default times by considering, for

each reference $i \in \{1, 2, \dots, n\}$, the minimum of the relevant shock arrival times:

$$T_i = \min\{t_k : 1 \leq k < m, a_{ik} = 1\}$$

Where a_{ik} is a matrix such that $a_{ik} = 1$ if shock $k \in \{1, 2, \dots, m\}$, modelled through the Poisson

process N_k with intensity λ_k , leads to a default of reference $i \in \{1, 2, \dots, n\}$, and

$a_{ik} = 0$ otherwise.

Step 3. Generate a sample $\{v_1, \dots, v_n\}$ from the (survival) default time copula C^r by setting, for

$i \in \{1, 2, \dots, n\}$

$$v_i = \exp \left\{ -T_i \sum_{k=1}^m a_{ik} \lambda_k \right\}$$

Step 4. Find the simulated correlated default time for every issuer: $\tau_i = F^{-1}(v_i)$ F is an arbitrary marginal default time distribution

iv) Multivariate Outcomes from Archimedean Copula Simulation

Procedure:

Step1. Generate a vector of uniform variables $u^T = \{u_1 \dots u_n\}$

Step2. Set $x_1 = F^{-1}(u_1)$

Step3. For $k=2, \dots, p$, recursively calculate the x_k as the solution of

$$u_2 = F_k(x_k | x_1, \dots, x_{k-1}) = \frac{\phi^{-1(k-1)} \{c_{k-1} + \phi[F_k(x_k)]\}}{\phi^{-1(k-1)}(c_{k-1})}$$

For example, in the context of Frank's copula for $p=2$:

$$u_2 = e^{-\alpha u_1} \left(1 + \frac{(e^{-\alpha} - e^{-\alpha F_2(x_2)})}{e^{-\alpha F_2(x_2)} - 1} \right)^{-1}$$

We calculate x_2 as the solution of

$$u_{*2} = \frac{u_2 e^{-\alpha} - e^{-\alpha u_1} (1 - u_{*2})}{u_2 + e^{-\alpha u_1} (1 - u_{*2})}$$

That is, calculate $x_2 = F_2^{-1}(u_{*2})$ where

Genest(1978) gave this algorithm.

Appendix B – Programming Codes

```
%Main Script File
```

```
%Gaussian v.s. T copula
```

```
%5-yr FTD Swap
```

```
clear all;
```

```
load data;
```

```
rho=0.999;
```

```
spreadMatrix=[spreadMatrix(1:5, :, 3)];%
```

```
spreadMatrix=[spreadMatrix(1:5, :, 3);spreadMatrix(:, :, 3);spreadMatrix(1:15, :, 3)];%spreadMatrix(
:,:,1:3);
```

```
%for the third index
```

```
%index 1 denotes low risk port
```

```
%index 2 denotes median risk port
```

```
%index 3 denots high risk port
```

```
simNo=5000;
```

```
assetNo=length(spreadMatrix);
```

```
%INPUTS
```

```
R=.4;% recover rate
```

```
r=0.05;%interest rate
```

```
%to obtain the correlated matrix
```

```
corrMatrix=rho*(1-eye(assetNo,assetNo))+eye(assetNo,assetNo);
```

```
%to obtain the spot lemdas
```

```
%based on the exponential default model
```

```
%lemmda=spread/(1-R)
```

```
spotlemmdaMatrix=spreadMatrix/(1-R);
```

```
%to obtain the forward lemdas
```

```
for i=1:assetNo
```

```
    for t=1:4
```

```
        fwdlemmdaMatrix(i,t)=spotlemmdaMatrix(i,t+1)*(t+1)-spotlemmdaMatrix(i,t)*(t);
```

```
    end
```

```
end
```

```
% *****
```

```
% Gaussian Copula
```

```
% *****
```

```
%to simulate a simNo-by-assetNo matrix of normal/Gaussian variates
```

```
% gevMatrix=randn(simNo,assetNo);
```

```
Quasi_RanSeqce=Halton_vector(simNo,assetNo);
```

```
%to obtain the correlated normal/Gaussian variates by using Cholesky
```

```
%decomposition
```

```
%That is, Gaussian Copula dependence structure
```

```
corrgevMatrix=normcdf(Quasi_RanSeqce*chol(corrMatrix));
```

```
% %for a single reference, marginal default profanity over 5 years
```

```
surProb=exp(-[1,2,3,4,5].*spotlemmdaMatrix(5,:))
```

```
[price1,Prob_default1,vector_FTPtao1,taoMatrix1]=PRICE(corrgevMatrix,spotlemmdaMatrix,fw  
dlemmdaMatrix, simNo,assetNo,r,R);
```

```

price1

Prob_default1

figure(1)

scatter(taoMatrix1(:,1),taoMatrix1(:,2),'.');

title('FIGURE 3-c.Gaussian_B basket');

xlabel('default time-firm1');

ylabel('default time-firm2');

% *****

% T-Copula

% *****

deg=5;

%to simulate a vector of chi-square random variates

S=chi2inv(rand(simNo,1),deg);

for i=1: simNo

    fac(i,1:assetNo)=sqrt(deg./S(i))*ones(1,assetNo));

```

end

```
corr_tMatrix=tcdf(fac.*corrgevMatrix,deg);
```

```
[price2,Prob_default2,vector_FTPTao2,taoMatrix2]=PRICE(corr_tMatrix,spotlemmdaMatrix,fwdl  
emmdaMatrix, simNo,assetNo,r,R);
```

price2

Prob_default2

vector_FTPTao2;

figure(2)

```
scatter(taoMatrix2(:,1),taoMatrix2(:,2),'');
```

```
title('FIGURE 3-c.T_B basket');
```

```
xlabel('default time-firm1');
```

```
ylabel('probability-firm2');
```

REFERENCE LIST

- Bank for International Settlements (2003). Over View of New Basel Accord. Retrieved date: August 1st, 2006, from <http://www.sfu.ca/~rjones/bus864/readings/BISaccord03draftoverview.pdf>
- Bonilla, M., Casasús, T. and Sala, R. (Ed.). (2000). *Financial Modelling*. Heidelberg, NY: Physica-Verl.
- Bowers, N., Gerber, H., Jones, D.A., & Nesbitt, C. (1997). *Actuarial Mathematics*. (2nd ed.). Ithaca, NY: The Society of Actuaries.
- Clayton, D.G. (1978). A model for association bivariate tables and its application in epidemiological studies of family tendency in chronic disease incidence, *Biometrika*, Vol. 65(2), 141-151.
- Clemen, Robert T. and Reilly, Terence (1999). Correlations and Copulas for Decision and Risk Analysis. *Management Science*, Vol.45(2), 208-224
- Credit-Suisse-Financial-Products(1997). CreditRisk+ a Credit Risk Management Framework. Technical document. Retrieved date: August 1st, 2006, from <http://www.csfb.com/institutional/research/assets/creditrisk.pdf>
- Embrecht, P., Mcneil, A. & Straumann, D. (2002). Correlation and Dependence in Risk Management: Properties and Pitfalls. *Risk management: value at risk and beyond*, Dempster M. (ed). Cambridge: Cambridge University Press.
- Frees, Edard W. and Valdez Emiliano A. (1997). Understanding Relationships Using Copulas. *North American Actuarial Journal*, Vol.2(2), 1-25
- Genest C. & MacKay J. (1986). The Joy of Copulas: Bivariate Distributions with Uniform Marginals. *The American Statistician*, Vol. 40(4), 280-283
- Gordy, M., B. (1998). A Comparative Anatomy of Credit Risk Models, Working paper. Washington, DC: Board of Governors of the Federal Reserve System.
- Gordy, M.B. (2000). A comparative anatomy of credit risk models. *Journal of Banking and Finance*, Vol.24, 119-149
- Joe, H. (1997). *Multivariate Models and Dependence Concepts*. London: Chapman and Hall.
- Kakodkar A., Marin, B. & Galiani S. (2003). Correlation trading. New York, NY: Merrill Lynch Global Securities Research & Economics Group Fixed Income Strategy
- KMV-Corporation (1997). Modelling Default Risk” Technical document. Retrieved date: August 1st, 2006, from <http://www.moodyskmv.com/research/files/EDFprimer.ppt#262,1,Measuring%20&%20Managing%20Credit%20Risk:%20Understanding%20the%20EDF%20Credit%20Measure%20for%20Public%20Firms>

- Lando, D. (2004). *Credit Risk Modelling*. Princeton: Princeton University Press.
- Li, X.D. (1999). On Default Correlation: A Copula Function Approach. Working paper. New York, NY: The RiskMetrics Group.
- Marshall, A. W. & Olkin I. (1988). Families of Multivariate Distributions. *Journal of the American Statistical Association*, Vol. 83(403), 834-841
- RiskMetrics-Group (1997). CreditMetrics-Technical Document, Technical document. Retrieved date: August 1st, 2006, from <http://www.riskmetrics.com/cmtdovv.html>
- Robert L. McDonald, 2006, "*Derivatives Markets*". (2nd ed.). Boston, MA: Addison-Wesley
- Poitras, G. (2005). *Security Analysis and Investment Strategy*, Boston, MA: Blackwell Publishing Ltd..
- Schweizer, B. (1991). Thirty Years of Copula. *Advances in Probability Distributions with Given Marginals*. Eds. G. Dall'Aglio, S. Kotz, and G. Salinetti, Kluwer Academic Publishers, Dordrecht, 13-50
- Shemyakiny, A. E., & Younz, H. (2005). Copula Models of Joint Survival Analysis. Submitted to *Applied Stochastic Models in Business and Industry*. Retrieved date: April 21st, 2006, from <http://cam.mathlab.stthomas.edu/pdf/preprints/copula.pdf>
- Sklar, A. (1959). *Fonctions de Repartition a n Dimensions et leurs Marges*, Publ. Inst. Stat. Univ. Paris, Vol. 8, 229-231.
- Tapiero, C., S. (2004). *Risk and Financial management*. Ontario, Canada: John Wiley & Sons Ltd..
- Wang, S. (1997). Aggregation of Correlated Risk Portfolios: Models and Algorithms, Arlington, Virginia: Preprint Casualty Actuarial Society (CAS). Retrieved date: August 1st, 2006, from <http://www.casact.org/pubs/proceed/proceed98/980848.pdf>
- Vaupel, J.W., Manton, K.G. & Stallard E. (1979). The Impact of Heterogeneity in Individual Frailty on the Dynamics of Mortality. *Demography*, Vol 16(3), 439-454